

Skew Knörrer’s periodicity theorem

Yang Liu, Yuan Shen, and Xin Wang

Abstract. In this paper, we introduce a class of twisted matrix algebras of $M_2(E)$ and twisted direct products of $E \times E$ for an algebra E . Let A be a noetherian Koszul Artin–Schelter regular algebra, $z \in A_2$ be a regular central element of A and $B = A_P[y_1, y_2; \sigma]$ be a graded double Ore extension of A . We use the Clifford deformation $C_{A^\dagger}(z)$ of Koszul dual $A^\dagger$ to study the noncommutative quadric hypersurface $B/(z + y_1^2 + y_2^2)$. We prove that the stable category of graded maximal Cohen–Macaulay modules over $B/(z + y_1^2 + y_2^2)$ is equivalent to certain bounded derived categories, which involve a twisted matrix algebra of $M_2(C_{A^\dagger}(z))$ or a twisted direct product of $C_{A^\dagger}(z) \times C_{A^\dagger}(z)$ depending on the values of P . These results are presented as skew versions of Knörrer’s periodicity theorem. Moreover, we show $B/(z + y_1^2 + y_2^2)$ may not be a noncommutative graded isolated singularity even if $A/(z)$ is.

1. Introduction

Throughout the paper, $\mathbb{k}$ is an algebraically closed field of characteristic 0. All vector spaces and algebras are over $\mathbb{k}$. Unless otherwise stated, the tensor product $\otimes$ means $\otimes_{\mathbb{k}}$.

Knörrer’s periodicity theorem is a powerful tool in the study of the Cohen–Macaulay representation theory and also the singularities of (commutative) quadric hypersurfaces (see [7]). It implies that the stable category $\underline{\text{mcm}} S$ of maximal Cohen–Macaulay modules over a quadric hypersurface $S = \mathbb{k}[x_1, \dots, x_n]/(f)$ is equivalent to $\underline{\text{mcm}} S^{\# \#}$ where $f \in (x_1, \dots, x_2)^2$ and $S^{\# \#} = \mathbb{k}[x_1, \dots, x_n][u, v]/(f + u^2 + v^2)$ is the second double-branched cover of S , which helps us to reduce the computation of the Cohen–Macaulay representation of a quadric hypersurface.

A noetherian Koszul Artin–Schelter regular algebra A is regarded as a noncommutative polynomial and a coordinate ring of a class of noncommutative projective spaces. Let $z \in A_2$ be a regular central element of A . The quotient algebra $A/(z)$ is an analogue of (commutative) quadric hypersurface, usually called a *noncommutative quadric hypersurface*. There have been many studies on noncommutative quadric hypersurfaces over the past few years (e.g., [1–6, 9, 10, 16, 18, 19]). To study graded Cohen–Macaulay modules of a noncommutative quadric hypersurface $A/(z)$, there are two useful tools. One is a finite-dimensional algebra $C(A/(z))$ introduced by Smith and van den Bergh in [16],

and the other one is a Clifford deformation $C_{A^\dagger}(z)$ of the Koszul dual $A^\dagger$ of A introduced by He and Ye in [2]. The Clifford deformation $C_{A^\dagger}(z)$ is a strongly $\mathbb{Z}_2$ -graded algebra and $C_{A^\dagger}(z)_0 \cong C(A/(z))$ (see [2]), and so $C_{A^\dagger}(z)$ may provide more information about $A/(z)$.

Similar to the commutative case, the noncommutative Knörrer's periodicity theorem, that is, $\underline{\text{mcm}} A/(z)$ and $\underline{\text{mcm}}(A/(z))^{\#\#}$ are equivalent, was proved for a noncommutative quadric hypersurface $A/(z)$ in [1, 2, 10]. He, Ma and Ye proved a generalized version of Knörrer's periodicity theorem for tensor products of two noncommutative quadric hypersurfaces with the help of Clifford deformations (see [3, Theorem 2.15]).

In a noncommutative setting, it is reasonable to consider noncommutative versions of double-branched covers and twisted tensor products of noncommutative quadric hypersurfaces. For example, $T = A[u; \sigma_1][v; \sigma_2]/(z + u^2 + v^2)$ defined by two-step iterated Ore extensions for some graded automorphisms σ_1 and σ_2 seems to make more sense as a class of second noncommutative double-branched covers, and a natural question is how we can describe $\underline{\text{mcm}} T$ by $\underline{\text{mcm}} A/(z)$. If $\sigma_1 = \sigma_2|_A$ and $\sigma_2(u) = u$, it is a special case of the noncommutative Knörrer's periodicity theorem given by Mori and Ueyama (see [10, Theorem 1.3]), and this case can also reduce to the second double-branched cover $(A/(z))^{\#\#}$ by a Zhang twist (see [20]). Nevertheless, we are curious about how to describe $\underline{\text{mcm}} T$ in general.

On the other hand, Hu, Matsuno and Mori classified noncommutative conics $A/(z)$ in Calabi–Yau quantum projective planes up to isomorphism of noncommutative projective schemes, that is to say, A is a 3-dimensional Calabi–Yau quantum polynomial algebra (see [6]). It inspires us to discuss the classification of noncommutative quadric hypersurfaces obtained by 4-dimensional noetherian Koszul Artin–Schelter regular algebras. To achieve this goal, it is necessary to compute the Cohen–Macaulay representation of such noncommutative quadric hypersurfaces. Note that Zhang and Zhang introduced an approach called double Ore extension to obtain a classification of 4-dimensional noetherian Koszul Artin–Schelter regular algebras (see [21, 22]). Also, double Ore extensions include two-step iterated Ore extensions. Therefore, it is reasonable to study relations between the graded maximal Cohen–Macaulay modules over noncommutative quadric hypersurfaces $A/(z)$ and $A_P[y_1, y_2; \sigma]/(z + y_1^2 + y_2^2)$ where $A_P[y_1, y_2; \sigma]$ is a double Ore extension of A .

To this end, we introduce twisted algebraic structures on a matrix algebra $M_2(E)$ and a direct product $E \times E$ for an algebra E . The twisted matrix algebra of $M_2(E)$ (see Theorem 3.7) comes from a twisting system Θ (see Definition 3.6) consisting of two linear maps $\theta^{(i)} : E \rightarrow M_2(E)$ for $i \in \mathbb{Z}_2$ and an invertible $\mathbb{Z}_2$ -graded basis of $M_2(\mathbb{k})$, where $M_2(\mathbb{k})_0$ consists of diagonal matrices and $M_2(\mathbb{k})_1$ consists of anti-diagonal matrices. Denote the new twisted matrix algebra by ${}^\Theta M_2(E)$, and it is a $\mathbb{Z}_2$ -graded algebra. This twisted algebra can be viewed as a generalization of Zhang twists (see [20]) for graded algebras, but they have different properties. Furthermore, taking the subalgebra ${}^\Theta M_2(E)_0$ of ${}^\Theta M_2(E)$ provides an approach to obtaining a twisted algebraic structure on $E \times E$ (see Corollary 3.14), and denote the new twisted algebra by $\theta_\times(E \times E)$ where the twisting system $\theta_\times$ consists of a linear map θ from E to $M_2(E)$ and an invertible basis of

$\mathbb{k} \times \mathbb{k}$. If E is a $\mathbb{Z}_2$ -graded algebra, then ${}^\Theta M_2(E)$ is $\mathbb{Z}_2 \times \mathbb{Z}_2$ -graded and also $\mathbb{Z}_2$ -graded by taking the total degrees, and ${}^{\theta \times} (E \times E)$ is $\mathbb{Z}_2$ -graded.

Let A be a noetherian Koszul Artin–Schelter regular algebra, $z \in A_2$ be a regular central element and $B = A_P[y_1, y_2; \sigma]$ be a $\mathbb{Z}$ -graded double Ore extension of A where $P = \{p_{11}, p_{12}\} \subseteq \mathbb{k}$. The main object we study is the noncommutative quadric hypersurface $B/(z + y_1^2 + y_2^2)$. By the deformation method introduced by He–Ye, $\underline{\text{mcm}} B/(z + y_1^2 + y_2^2)$ is totally determined by the Clifford deformation $C_{B^!}(z + y_1^2 + y_2^2)$ of $B^!$ [2, Theorem 1.1]. We aim to describe $C_{B^!}(z + y_1^2 + y_2^2)$ by the Clifford deformation $C_{A^!}(z)$ of $A^!$. It should be noted that the semi-trivial extension (see Theorem 2.6) is a useful construction for our goal. To make the noncommutative quadric hypersurface $B/(z + y_1^2 + y_2^2)$ well defined, it forces $p_{12} = \pm 1$. We divide it into two cases.

If $p_{12} = 1$, p_{11} should be 0. In this case, the Clifford deformation $C_{B^!}(z + y_1^2 + y_2^2)$ is a twisted matrix algebra of $M_2(C_{A^!}(z))$, and $\underline{\text{mcm}}(B/(z + y_1^2 + y_2^2))$ can be described by a semi-trivial extension.

Theorem 1.1 (Theorems 5.4 and 5.8). *Let $B = A_{\{1,0\}}[y_1, y_2; \sigma]$ be a $\mathbb{Z}$ -graded double Ore extension of a noetherian Koszul Artin–Schelter regular algebra A with $\deg y_1 = \deg y_2 = 1$ and $z \in A_2$ be a regular central element of A . Suppose B is noetherian and $z + y_1^2 + y_2^2$ is a central element of B . Then*

- (1) *there exists a twisting system Θ of $M_2(C_{A^!}(z))$ such that*

$$C_{B^!}(z + y_1^2 + y_2^2) \cong {}^\Theta M_2(C_{A^!}(z)),$$

as $\mathbb{Z}_2$ -graded algebras;

- (2) *there exists a semi-trivial extension $\Lambda = S \ltimes_\psi M(1)$ of a $\mathbb{Z}_2$ -graded subalgebra S of $C_{A^!}(z)$ and a $\mathbb{Z}_2$ -graded S -bimodule M (a subspace of $C_{A^!}(z)$) through some map $\psi : M(1) \otimes_S M(1) \rightarrow S$ such that*

$$\begin{aligned} \underline{\text{mcm}}(B/(z + y_1^2 + y_2^2)) &\cong D^b(\text{gr}_{\mathbb{Z}_2} {}^\Theta M_2(C_{A^!}(z))) \cong D^b(\text{mod } {}^\Theta M_2(C_{A^!}(z))_0) \\ &\cong D^b(\text{gr}_{\mathbb{Z}_2} \Lambda), \end{aligned}$$

as triangulated categories.

If $p_{12} = -1$, we can reduce p_{11} to be 0 in case $p_{11} \neq \pm 2\sqrt{-1}$. Then one obtains a twisted direct product $\Gamma = {}^{\theta \times} (C_{A^!}(z) \times C_{A^!}(z))$ for some twisting system $\theta \times$, and the Clifford deformation $C_{B^!}(z + y_1^2 + y_2^2)$ is a semi-trivial extension constructed by Γ . In particular, the 0-th component of $C_{B^!}(z + y_1^2 + y_2^2)$ is exactly a Zhang-twisted algebra (see [20]) of Γ .

Theorem 1.2 (Theorem 6.9). *Let $B = A_{\{-1,0\}}[y_1, y_2; \sigma]$ be a $\mathbb{Z}$ -graded double Ore extension of a noetherian Koszul Artin–Schelter regular algebra A with $\deg y_1 = \deg y_2 = 1$ and $z \in A_2$ be a regular central element of A . Suppose B is noetherian and $z + y_1^2 + y_2^2$ is a central element of B . Then*

- (1) *there exist a twisted direct product Γ of $C_{A^!}(z) \times C_{A^!}(z)$ and a semi-trivial extension $\Gamma \ltimes_{\psi} ({}_{\mu}\Gamma(1))$ of $\mathbb{Z}_2$ -graded algebra Γ and $\mathbb{Z}_2$ -graded Γ -bimodule ${}_{\mu}\Gamma(1)$ through some map ψ where μ is a $\mathbb{Z}_2$ -graded automorphism of Γ such that*

$$C_{B^!}(z + y_1^2 + y_2^2) \cong \Gamma \ltimes_{\psi} ({}_{\mu}\Gamma(1)),$$

as $\mathbb{Z}_2$ -graded algebras;

- (2) *there are equivalences of triangulated categories*

$$\underline{\text{mcm}}(B/(z + y_1^2 + y_2^2)) \cong D^b(\text{gr}_{\mathbb{Z}_2} \Gamma \ltimes_{\psi} ({}_{\mu}\Gamma(1))) \cong D^b(\text{mod } {}^v\Gamma),$$

where ${}^v\Gamma$ is a Zhang-twisted algebra of Γ for the left Zhang-twisting system $v = \{v_0 = \text{id}, v_1 = \mu\}$ of $\mathbb{Z}_2$ -graded algebra Γ .

In particular, Theorem 1.1 reduces to the noncommutative Knörrer's periodicity theorem if σ is a diagonal map. Theorems 1.1 and 1.2 can be viewed as skew versions of Knörrer's periodicity theorem.

For the exceptional case $p_{12} = -1$ and $p_{11} = \pm 2\sqrt{-1}$, we have that any noncommutative quadric hypersurface $A_{\{-1, p_{11}\}}[y_1, y_2; \sigma]/(z + y_1^2 + y_2^2)$ is not a noncommutative graded isolated singularity in the sense of [17] (see Proposition 6.1) or whose noncommutative projective scheme is not smooth. So we are also interested in whether $A_{\{\pm 1, 0\}}[y_1, y_2; \sigma]/(z + y_1^2 + y_2^2)$ is a noncommutative graded isolated singularity in case $A/(z)$ is. We give a negative answer to this question (see Proposition 6.12). It should be emphasized that this result is a totally different phenomenon from the usual second double-branched covers and tensor products of noncommutative quadric hypersurfaces, which preserve the property of noncommutative graded isolated singularity (see [2, Theorem 1.4] and [3, Theorem 2.5]).

This paper is organized as follows. In Section 2, we recall some definitions and results, including noncommutative quadric hypersurfaces, Clifford deformations and semi-trivial extensions. In Section 3, we introduce a class of twisted matrix algebras and a class of twisted direct products. In Section 4, we recall the definition of double Ore extensions and show some basic facts about the Clifford deformations for noncommutative quadric hypersurfaces obtained by double Ore extensions. We devote Section 5 and Section 6 to proving Theorem 1.1 and Theorem 1.2, respectively.

2. Preliminaries

Let $C = \bigoplus_{g \in G} C_g$ be a G -graded algebra where G is an abelian group. Write $\text{Gr}_G C$ for the category of G -graded right C -modules with morphisms of C -module homomorphisms preserving grading and $\text{gr}_G C$ for the full subcategory of $\text{Gr}_G C$ consisting of finitely generated G -graded right C -modules. Let M be a G -graded right C -module, then $M(h)$ stands for the shifting G -graded right C -module where $M(h)_g = M_{g+h}$ for

any $g \in G$ and some $h \in G$. For a G -graded automorphism μ of C , the twisted G -graded right C -module M_μ is the abelian group M with the right C -action $m \cdot c = m\mu(c)$ for any $m \in M, c \in C$. Similarly, we can define shifting and twisted G -graded left C -modules (or C -bimodules).

For $M, M' \in \text{Gr}_G C$, write

$$\underline{\text{Hom}}_C(M, M') = \bigoplus_{g \in G} \text{Hom}_{\text{Gr}_G C}(M, M'(g)),$$

and $\underline{\text{Ext}}_C^i$ for the i -th derived functor of $\underline{\text{Hom}}_C$. In particular, $\mathbb{k}$ is always a G -graded algebra concentrated in degree 0, and denote the G -graded linear dual functor $\underline{\text{Hom}}_{\mathbb{k}}(-, \mathbb{k})$ by $(-)^*$. For any nonzero $k \in \mathbb{k}$, define a G -graded algebra automorphism ξ_k of C by $\xi_k(c) = k^{\deg c} c$ for any homogeneous element c of C .

A set of G -graded linear automorphisms $\nu = \{\nu_g \mid g \in G\}$ of C is called a *left Zhang-twisting system* of C if $\nu_l(\nu_h(x)y) = \nu_{hl}(x)\nu_l(y)$ for all $g, h, l \in G$ and $x \in C_g, y \in C_h$. If $\nu = \{\nu_g \mid g \in G\}$ is a left Zhang-twisting system of C , the graded vector space $C = \bigoplus_{g \in G} C_g$ with the multiplication $x \diamond y = \nu_h(x)y$ for all $x \in C_g, y \in C_h$ is a G -graded algebra, called a *Zhang-twisted algebra* of A by ν , denoted by ${}^\nu A$ (see [20, Section 4]).

Definition 2.1. Let C be a G -graded algebra (resp. G -graded vector space) and $\sigma = (\sigma_{ij})$ and $\varphi = (\varphi_{ij})$ be two algebra homomorphisms (resp. linear maps) from C to $M_2(C)$ where σ_{ij} and φ_{ij} are G -graded linear transformations of C for any $i, j = 1, 2$. If

$$\sum_{k=1}^2 \sigma_{ki} \varphi_{kj} = \delta_{ij} \text{id}_C \quad \text{and} \quad \sum_{k=1}^2 \varphi_{jk} \sigma_{ik} = \delta_{ij} \text{id}_C,$$

where δ_{ij} is the Kronecker symbol for any $i, j = 1, 2$, we say σ is *invertible*, φ is *t -invertible*, φ is an *inverse* of σ and σ is a *t -inverse* of φ .

In this case, it is not hard to obtain that the inverse of σ and the t -inverse of φ are both unique.

Remark 2.2. Definition 2.1 is a graded version of [21, Definition 1.8]. Note that the invertible form and t -invertible form are not the same. The letter “ t ” of t -invertible stands for the word “transpose” since the transpose of φ has the same invertible property as σ .

A non-negative $\mathbb{Z}$ -graded algebra C with $C_0 = \mathbb{k}$ and $\dim C_i < \infty$ for any $i \geq 1$ is called a *locally finite connected $\mathbb{Z}$ -graded algebra*. A locally finite connected $\mathbb{Z}$ -graded algebra A is called a *Koszul algebra* (see [13]) if the trivial module $\mathbb{k}_A$ has a free resolution

$$\cdots \rightarrow P_n \rightarrow \cdots \rightarrow P_1 \rightarrow P_0 \rightarrow \mathbb{k} \rightarrow 0,$$

where P_i is a $\mathbb{Z}$ -graded free right A -module generated in degree i for any $i \geq 0$. Then $A \cong T(V)/(R)$ for some finite-dimensional vector space V with degree 1 and some vector space $R \subseteq V \otimes V$. The *Koszul dual* of A is the quadratic algebra $A^! = T(V^*)/(R^\perp)$,

where V^* is the dual vector space of V and $R^\perp \subseteq V^* \otimes V^*$ is the orthogonal complement of R . The Koszul dual $A^!$ is also Koszul.

Definition 2.3. A locally finite connected $\mathbb{Z}$ -graded algebra D is called an *Artin–Schelter Gorenstein algebra* of injective dimension d if

- (1) $\text{injdim}_D D = \text{injdim } D_D = d < \infty$;
- (2) $\underline{\text{Ext}}_D^i(\mathbb{k}_D, D_D) = \underline{\text{Ext}}_D^i({}_D \mathbb{k}, {}_D D) = 0$ if $i \neq d$;
- (3) $\underline{\text{Ext}}_D^d(\mathbb{k}_D, D_D), \underline{\text{Ext}}_D^d({}_D \mathbb{k}, {}_D D)$ are both 1 dimensional.

In addition, if D has finite global dimension, then D is called an *Artin–Schelter regular algebra* of dimension d .

Let $A = T(V)/(R)$ be a noetherian Koszul Artin–Schelter regular algebra of dimension d and $z \in A_2$ be a regular central element of A . The quotient algebra $S = A/(z)$ is a noncommutative quadric hypersurface, and it is a noetherian Koszul Artin–Schelter Gorenstein algebra of injective dimension $d - 1$ by [16, Lemma 5.1 (1)] and [2, Lemma 2.2].

The noncommutative projective scheme $\text{qgr } S = \text{gr}_{\mathbb{Z}} S / \text{tors}_{\mathbb{Z}} S$ where $\text{tors}_{\mathbb{Z}} S$ is the full subcategory of $\text{gr}_{\mathbb{Z}} S$ consisting of finite-dimensional $\mathbb{Z}$ -graded right S -modules. We say S is a *noncommutative graded isolated singularity* if $\text{qgr } S$ has finite global dimension (see [17]).

A finitely generated $\mathbb{Z}$ -graded right S -module N is called a *maximal Cohen–Macaulay module* if the local cohomology

$$H_{\mathfrak{m}}^i(N) = \lim_{n \rightarrow \infty} \underline{\text{Ext}}_S^i(S/S_{\geq n}, N) = 0 \quad \text{if } i \neq d - 1,$$

where $\mathfrak{m} = S_{\geq 1}$. Denote the full subcategory of $\text{gr}_{\mathbb{Z}} S$ consisting of all the maximal Cohen–Macaulay modules by $\text{mcm } S$ and the stable category of $\text{mcm } S$ by $\underline{\text{mcm}} S$ which is a triangulated category.

In this paper, we use the Clifford deformation introduced by He–Ye in [2] to discuss $\underline{\text{mcm}}(S)$. Recall some results in [2] here. By [15, Proposition 5.10], $A^!$ is a Koszul Frobenius algebra. Choose $\hat{z} \in V^{\otimes 2}$ as a lift of z . There is a Clifford map

$$\begin{array}{ccc} \theta_z : & R^\perp & \rightarrow \quad \mathbb{k} \\ & f & \mapsto f(\hat{z}), \end{array}$$

satisfying $(\theta_z \otimes 1 - 1 \otimes \theta_z)(V^* \otimes R^\perp \cap R^\perp \otimes V^*) = 0$. Note that the map θ_z is independent of the choice of $\hat{z}$. Then we have an algebra

$$C_{A^!}(z) = T(V^*)/(f - \theta_z(f) : f \in R^\perp),$$

which is called a *Clifford deformation* of $A^!$. The $\mathbb{Z}$ -graded algebra $T(V^*)$ is also a $\mathbb{Z}_2$ -graded algebra if we set the subspace $\bigoplus_{n \geq 0} (V^*)^{\otimes 2n}$ with degree 0 and $\bigoplus_{n \geq 0} (V^*)^{\otimes 2n+1}$ with degree 1. Then $C_{A^!}(z)$ is also a $\mathbb{Z}_2$ -graded algebra induced by the $\mathbb{Z}_2$ -grading of $T(V^*)$. The following important properties of $C_{A^!}(z)$ can be referred to [2, Propositions 4.5 and 4.6, Theorems 1.1 and 1.2].

Theorem 2.4. *Retain the notations above. Then*

- (1) $C_{A^!}(z)$ is a strongly $\mathbb{Z}_2$ -graded Frobenius algebra;
- (2) S is a noncommutative graded isolated singularity if and only if $C_{A^!}(z)$ is a $\mathbb{Z}_2$ -graded semisimple algebra;
- (3) there are equivalences of triangulated categories

$$\underline{\text{mcm}} S \cong D^b(\text{gr}_{\mathbb{Z}_2} C_{A^!}(z)) \cong D^b(\text{mod } C_{A^!}(z)_0).$$

Let $\sigma = (\sigma_{ij}) : A \rightarrow M_2(A)$ be an algebra homomorphism such that each σ_{ij} is a $\mathbb{Z}$ -graded linear transformation of A for any $i, j = 1, 2$. Write $\sigma|_V = (\sigma_{ij}|_V) : V \rightarrow M_2(V)$ for the restriction of σ on $A_1 = V$. One can extend $\sigma|_V$ to an algebra homomorphism

$$\tilde{\sigma} = (\tilde{\sigma}_{ij}) : T(V) \rightarrow M_2(T(V)) \quad (2.1)$$

such that

$$\tilde{\sigma}(R) \subseteq M_2(R),$$

where each $\tilde{\sigma}_{ij}$ is a $\mathbb{Z}$ -graded linear transformation of $T(V)$ for any $i, j = 1, 2$. Applying the $\mathbb{Z}$ -graded linear dual functor $(-)^*$ to each component of $\tilde{\sigma}$, one obtains an algebra homomorphism

$$\tilde{\sigma}^* = (\tilde{\sigma}_{ij}^*) : T(V^*) \rightarrow M_2(T(V^*)). \quad (2.2)$$

For any $f \in R^\perp$, $r \in R$ and $i, j = 1, 2$, we have $\tilde{\sigma}_{ij}^*(f)(r) = f(\tilde{\sigma}_{ij}(r)) = 0$. It implies

$$\tilde{\sigma}^*(R^\perp) \subset M_2(R^\perp). \quad (2.3)$$

Suppose $\sigma_{ij}(z) = \delta_{ij}z$ for any $i, j = 1, 2$. Then the lift map $\tilde{\sigma} = (\tilde{\sigma}_{ij})$ defined in (2.1) satisfies

$$\tilde{\sigma}_{ij}(\hat{z}) - \delta_{ij}\hat{z} \in R$$

for any $i, j = 1, 2$. So the dual map $\tilde{\sigma}^* = (\tilde{\sigma}_{ij}^*) : T(V^*) \rightarrow M_2(T(V^*))$ defined in (2.2) satisfies

$$\begin{aligned} \tilde{\sigma}_{ij}^*(f - f(\hat{z})) &= \tilde{\sigma}_{ij}^*(f) - \delta_{ij}f(\hat{z}) = \tilde{\sigma}_{ij}^*(f) - f(\tilde{\sigma}_{ij}(\hat{z})) \\ &= \tilde{\sigma}_{ij}^*(f) - \tilde{\sigma}_{ij}^*(f)(\hat{z}) \in (g - g(\hat{z}) \mid g \in R^\perp) \end{aligned}$$

for any $f \in R^\perp$ and $i, j = 1, 2$, where the first equality holds since $\tilde{\sigma}^*$ is an algebra homomorphism and $f(\hat{z}) \in \mathbb{k}$, and the final belonging follows from (2.3). Hence, $\tilde{\sigma}^*$ induces an algebra homomorphism

$$\sigma^! = (\sigma_{ij}^!) : C_{A^!}(z) \rightarrow M_2(C_{A^!}(z)), \quad (2.4)$$

where each $\sigma_{ij}^!$ is a $\mathbb{Z}_2$ -graded linear transformation of $C_{A^!}(z)$ for any $i, j = 1, 2$.

Lemma 2.5. *If σ is invertible, then $\sigma^!$ is t -invertible.*

Proof. Write φ for the inverse of σ . We have $\delta_{ij}z = \sum_{k=1}^2 \varphi_{ik}\sigma_{jk}(z) = \varphi_{ij}(z)$ for $i, j = 1, 2$. Hence, $\varphi^\dagger$ is well defined by (2.4) and is the t -inverse of $\sigma^\dagger$. ■

We conclude this section by introducing a class of extensions of algebras, which is called a *semi-trivial extension* in [12].

Theorem 2.6. *Let E be an algebra and M be an E -bimodule. Let $\psi : M \otimes_E M \rightarrow E$ be an E -bimodule homomorphism such that $m\psi(m' \otimes m'') = \psi(m \otimes m')m''$ for any $m, m', m'' \in M$. Then the vector space $E \oplus M$ with the multiplication*

$$(x, m) \odot (x', m') = (xx' + \psi(m \otimes m'), xm' + mx') \quad \forall x, x' \in E, m, m' \in M$$

is an algebra, called a semi-trivial extension of E by M through ψ , denoted by $E \ltimes_\psi M$.

Proof. It is clear $(1, 0)$ is the identity. Since ψ is an E -bimodule homomorphism and $m\psi(m' \otimes m'') = \psi(m \otimes m')m''$ for any $m, m', m'' \in M$, the associativity of this multiplication holds. ■

Remark 2.7. The semi-trivial extension is a generalization of the trivial extension and also admits some natural graded structures.

- (1) The semi-trivial extension $E \ltimes_\psi M$ is the trivial extension $E \ltimes M$ if $\psi = 0$.
- (2) A semi-trivial extension $E \ltimes_\psi M$ can be viewed as a $\mathbb{Z}_2$ -graded algebra (or a superalgebra) if $\deg(x, 0) = 0, \deg(0, m) = 1$ for any $x \in E, m \in M$.
- (3) Suppose E is a G -graded algebra, M is a G -graded E -bimodule and ψ is a G -graded map where G is an abelian group. Then $E \ltimes_\psi M$ is a $\mathbb{Z}_2 \times G$ -graded algebra whose homogeneous space $(E \ltimes_\psi M)_{(0,g)} = E_g$ and $(E \ltimes_\psi M)_{(1,g)} = M_g$ for any $g \in G$ and also a G -graded algebra by forgetting the first $\mathbb{Z}_2$ -degree. In particular, if $G = \mathbb{Z}_2$, we always view $\mathbb{Z}_2 \times \mathbb{Z}_2$ -graded algebra $E \ltimes_\psi M$ as a $\mathbb{Z}_2$ -graded algebra by forgetting the first $\mathbb{Z}_2$ -degree in the sequel, that is,

$$(E \ltimes_\psi M)_0 = E_0 \oplus M_0, \quad (E \ltimes_\psi M)_1 = E_1 \oplus M_1.$$

Example 2.8. Here we present two examples of semi-trivial extensions, which will be useful in subsequent sections.

- (1) Let E be an algebra and $\psi : E \otimes_E E \rightarrow E$ be the multiplication of E . Then one obtains a semi-trivial extension $E \ltimes_\psi E$.
- (2) Let E be a $\mathbb{Z}_2$ -graded algebra and μ be a $\mathbb{Z}_2$ -graded algebra automorphism of E such that $\mu^2 = \text{id}$. Then

$$\begin{array}{ccc} \psi : & {}_\mu E(1) \otimes_E {}_\mu E(1) & \rightarrow E \\ & a \otimes b & \mapsto \mu(a)b \end{array}$$

is a $\mathbb{Z}_2$ -graded E -bimodule homomorphism satisfying $\psi(a \otimes b) \cdot c = a\psi(b \otimes c)$ for any $a, b, c \in {}_\mu E(1)$. Hence, we have a semi-trivial extension $E \ltimes_\psi ({}_\mu E(1))$

which is a $\mathbb{Z}_2$ -graded algebra (see Remark 2.7 (3) when $G = \mathbb{Z}_2$). More precisely, $(E \ltimes_{\psi} (\mu E(1)))_0 = E_0 \oplus E_1$ and $(E \ltimes_{\psi} (\mu E(1)))_1 = E_1 \oplus E_0$ with multiplication

$$(a, b) \odot (a', b') = (aa' + \mu(b)b', \mu(a)b' + ba')$$

for any homogeneous elements $(a, b), (a', b') \in E \ltimes_{\psi} (\mu E(1))$.

3. Twisted matrix algebras and twisted direct products

In this section, we introduce a class of twisting systems for the matrix algebra $M_2(E)$ over some algebra E and a class of twisted matrix algebras. As an application, there is a twisted structure on the direct product $E \times E$ by choosing a subalgebra of a twisted matrix algebra.

3.1. Twisted matrix algebras

Let $M_2(\mathbb{k})$ be a $\mathbb{Z}_2$ -graded algebra whose $\mathbb{Z}_2$ -grading is given by

$$M_2(\mathbb{k})_0 = \begin{pmatrix} \mathbb{k} & 0 \\ 0 & \mathbb{k} \end{pmatrix}, \quad M_2(\mathbb{k})_1 = \begin{pmatrix} 0 & \mathbb{k} \\ \mathbb{k} & 0 \end{pmatrix}.$$

Let $\mathbb{I}_1^{(0)}, \mathbb{I}_2^{(0)}$ be invertible and a basis of $M_2(\mathbb{k})_0$, and let $\mathbb{I}_1^{(1)}, \mathbb{I}_2^{(1)}$ be invertible and a basis of $M_2(\mathbb{k})_1$. We say $\mathbb{I} = \{\mathbb{I}_j^{(i)} \mid i \in \mathbb{Z}_2, j = 1, 2\}$ is an invertible $\mathbb{Z}_2$ -graded basis of $M_2(\mathbb{k})$. Let $\gamma_1, \gamma_2 \in \mathbb{k}$ such that

$$\gamma_1 \mathbb{I}_1^{(0)} + \gamma_2 \mathbb{I}_2^{(0)} = I_2,$$

where I_2 is the 2×2 identity matrix.

For any $i \in \mathbb{Z}_2$ and $j = 1, 2$, it is clear that $\{\mathbb{I}_j^{(i)} \mathbb{I}_1^{(i')}, \mathbb{I}_j^{(i)} \mathbb{I}_2^{(i')}\}$ is also a basis of $M_2(\mathbb{k})_{i+i'}$ since $\mathbb{I}_j^{(i)}$ is invertible for any $i' \in \mathbb{Z}_2$. Then there exists a set $\{l_{s jj'}^{(ii')} \in \mathbb{k} \mid s, j' = 1, 2\}$ for any $i, i' \in \mathbb{Z}_2, j = 1, 2$ such that

$$\begin{aligned} \mathbb{I}_j^{(i)} \mathbb{I}_1^{(i')} &= \mathbb{I}_1^{(i+i')} l_{1j1}^{(ii')} + \mathbb{I}_2^{(i+i')} l_{2j1}^{(ii')}, \\ \mathbb{I}_j^{(i)} \mathbb{I}_2^{(i')} &= \mathbb{I}_1^{(i+i')} l_{1j2}^{(ii')} + \mathbb{I}_2^{(i+i')} l_{2j2}^{(ii')}, \end{aligned} \quad (3.1)$$

and we write

$$L_j^{(ii')} = \begin{pmatrix} l_{1j1}^{(ii')} & l_{1j2}^{(ii')} \\ l_{2j1}^{(ii')} & l_{2j2}^{(ii')} \end{pmatrix} \in M_2(\mathbb{k}). \quad (3.2)$$

In fact, each matrix $L_j^{(ii')}$, which is a kind of transition matrix, defines a linear automorphism of the vector space $M_2(\mathbb{k})_{i+i'}$, and it is invertible for any $i, i' \in \mathbb{Z}_2$ and $j = 1, 2$.

Lemma 3.1. Let the set $\{l_{sjj'}^{ii'} \in \mathbb{k} \mid i, i' \in \mathbb{Z}_2, s, j, j' = 1, 2\}$ be defined as in (3.1).

(1) For any $i, i', i'' \in \mathbb{Z}_2$ and $j, j', j'', t = 1, 2$,

$$\sum_{s=1}^2 l_{tjs}^{(i, i' + i'')} l_{sj'j''}^{(i' i'')} = \sum_{s=1}^2 l_{tsj''}^{(i + i', i'')} l_{sjj'}^{(i i')}.$$

(2) For any $i \in \mathbb{Z}_2$ and $s, j = 1, 2$, $\sum_{t=1}^2 l_{sjt}^{(i0)} \gamma_t = \delta_{sj}$.

(3) For any $i \in \mathbb{Z}_2$ and $s, t = 1, 2$, $\sum_{j=1}^2 l_{sjt}^{(0i)} \gamma_j = \delta_{st}$.

Proof. (1) It follows from the associativity of $\mathbb{I}_j^{(i)} \mathbb{I}_{j'}^{(i')} \mathbb{I}_{j''}^{(i'')}$.

(2) By (3.1) with $i' = 0$,

$$\sum_{s=1}^2 \mathbb{I}_s^{(i)} \left(\sum_{t=1}^2 l_{sjt}^{(i0)} \gamma_t \right) = \sum_{t=1}^2 \mathbb{I}_j^{(i)} \mathbb{I}_t^{(0)} \gamma_t = \mathbb{I}_j^{(i)}.$$

(3) By (3.1) with $i = 0$,

$$\sum_{s=1}^2 \mathbb{I}_s^{(i')} \left(\sum_{j=1}^2 \gamma_j l_{sjt}^{(0i')} \right) = \sum_{j=1}^2 \gamma_j \mathbb{I}_j^{(0)} \mathbb{I}_t^{(i')} = \mathbb{I}_t^{(i')}.$$

■

Let E be an algebra. The basis $\mathbb{I}$ of $M_2(\mathbb{k})$ provides a decomposition for $\mathbb{Z}_2$ -graded vector space $M_2(E)$ as follows: for any $i \in \mathbb{Z}_2$,

$$M_2(E)_i = \{\mathbb{I}_1^{(i)} x \mid x \in E\} \oplus \{\mathbb{I}_2^{(i)} x \mid x \in E\}.$$

To introduce a twisted system of the matrix algebra $M_2(E)$, we consider a class of linear maps from E to the matrix algebra $M_2(E)$.

Proposition 3.2. Let $\theta^{(i)} = (\theta_{jj'}^{(i)}) : E \rightarrow M_2(E)$ be a linear map for $i \in \mathbb{Z}_2$. Then the following are equivalent:

(1) For any $x, x' \in E$, $i, i', i'' \in \mathbb{Z}_2$ and $j, j', j'', v = 1, 2$,

$$\sum_{s,t,u=1}^2 l_{vtu}^{(i, i' + i'')} l_{tjs}^{(i i')} \theta_{uj''}^{(i'')} (\theta_{sj'}^{(i')}(x) x') = \sum_{s,t,u=1}^2 l_{vjs}^{(i, i' + i'')} l_{tj'u}^{(i' i'')} \theta_{st}^{(i' + i'')}(x) \theta_{uj''}^{(i'')}(x').$$

(2) For any $x, x' \in E$, $i, i', i'' \in \mathbb{Z}_2$ and $j, j', j'', v = 1, 2$,

$$\sum_{s,t,u=1}^2 l_{vjt}^{(i, i' + i'')} l_{tsu}^{(i' i'')} \theta_{uj''}^{(i'')} (\theta_{sj'}^{(i')}(x) x') = \sum_{s,t,u=1}^2 l_{vjs}^{(i, i' + i'')} l_{tj'u}^{(i' i'')} \theta_{st}^{(i' + i'')}(x) \theta_{uj''}^{(i'')}(x').$$

(3) For any $x, x' \in E$, $i', i'' \in \mathbb{Z}_2$ and $j', j'', p = 1, 2$,

$$\sum_{s,u=1}^2 l_{psu}^{(i' i'')} \theta_{uj''}^{(i'')} (\theta_{sj'}^{(i')}(x) x') = \sum_{t,u=1}^2 l_{tj'u}^{(i' i'')} \theta_{pt}^{(i' + i'')}(x) \theta_{uj''}^{(i'')}(x').$$

Proof. (1) $\Leftrightarrow$ (2) follows from Lemma 3.1 (1).

(2) $\Rightarrow$ (3) Write $(L_j^{(ii')})^{-1} = (g_{jj''}^{(ii')})$ for any $i, i' \in \mathbb{Z}_2$ and $j, j', j'' = 1, 2$. Multiply on both sides of the equation in (2) by $g_{p j v}^{(i, i' + i'')}$ and taking the sum over v from 1 to 2 for any $p = 1, 2$, one obtains (3).

(3) $\Rightarrow$ (2) Multiply on both sides of the equation in (3) by $l_{v j p}^{(i, i' + i'')}$ for any $j, v = 1, 2$ and take the sum over p from 1 to 2. ■

Lemma 3.3. Let $\theta^{(i)} = (\theta_{jj'}^{(i)}) : E \rightarrow M_2(E)$ be a linear map for $i \in \mathbb{Z}_2$. Suppose $\theta^{(0)}$ and $\theta^{(1)}$ are t -invertible with the t -inverses $\varphi^{(0)}$ and $\varphi^{(1)}$, respectively, and satisfy the equivalent conditions of Proposition 3.2. Then

$$\sum_{u, j=1}^2 l_{pqu}^{(ii')} \theta_{uj'}^{(i')} (x \varphi_{rj}^{(i)}(x')) = \sum_{t, j=1}^2 l_{tjr}^{(ii')} \theta_{pt}^{(i+i')} (\varphi_{qj}^{(i)}(x)) x'$$

for any $x, x' \in E, i, i' \in \mathbb{Z}_2$ and $p, q, r = 1, 2$.

Proof. Rewrite the indexes of Proposition 3.2 (3):

$$\sum_{s, u=1}^2 l_{psu}^{(ii')} \theta_{uj'}^{(i')} (\theta_{sj}^{(i)}(x) x') = \sum_{t, u=1}^2 l_{tju}^{(ii')} \theta_{pt}^{(i+i')} (x) \theta_{uj'}^{(i')} (x')$$

for any $x, x' \in E, i, i' \in \mathbb{Z}_2$ and $j, j', p = 1, 2$.

Replacing x and x' by $\varphi_{qj}^{(i)}(x)$ and $\varphi_{rj'}^{(i')}(x')$, respectively, and taking the sum over j, j' both from 1 to 2, the result follows. ■

Lemma 3.4. Retain the assumptions of Lemma 3.3. Then $\theta^{(0)}(1) \in \text{GL}_2(\mathbb{k})$ if and only if $\theta^{(1)}(1) \in \text{GL}_2(\mathbb{k})$. In this case, $\varphi^{(i)}(1) \in \text{GL}_2(\mathbb{k})$ for $i \in \mathbb{Z}_2$.

Proof. Suppose $\theta^{(0)}(1) \in \text{GL}_2(\mathbb{k})$, $(\varphi^{(0)}(1))^T \theta^{(0)}(1) = ((\varphi^{(0)})^T \theta^{(0)})(1) = I_2$. Clearly, $\varphi^{(0)}(1) \in \text{GL}_2(\mathbb{k})$.

Taking $x = x' = 1$ and $i = 0, i' = 1$ in Lemma 3.3, we have

$$l_{pqr}^{(01)} = \sum_{t, j=1}^2 l_{tjr}^{(01)} \theta_{pt}^{(1)} (\varphi_{qj}^{(0)}(1)) = \sum_{t, j=1}^2 l_{tjr}^{(01)} \theta_{pt}^{(1)}(1) \varphi_{qj}^{(0)}(1)$$

for any $p, q, r = 1, 2$. Write those equations in the form of matrices:

$$\bar{L}_{1,2;r}^{(01)} = \theta^{(1)}(1) \bar{L}_{1,2;r}^{(01)} (\varphi^{(0)}(1))^T,$$

where $\bar{L}_{1,2;r}^{(01)} = \begin{pmatrix} l_{11r}^{(01)} & l_{12r}^{(01)} \\ l_{21r}^{(01)} & l_{22r}^{(01)} \end{pmatrix}$ for any $r = 1, 2$. By (3.1), the elements of $\bar{L}_{1,2;r}^{(01)}$ satisfy

$$\begin{aligned} \mathbb{I}_1^{(0)} \mathbb{I}_r^{(1)} &= \mathbb{I}_1^{(1)} l_{11r}^{(01)} + \mathbb{I}_2^{(1)} l_{21r}^{(01)}, \\ \mathbb{I}_2^{(0)} \mathbb{I}_r^{(1)} &= \mathbb{I}_1^{(1)} l_{12r}^{(01)} + \mathbb{I}_2^{(1)} l_{22r}^{(01)}, \end{aligned}$$

and $\{\mathbb{I}_1^{(0)}\mathbb{I}_r^{(1)}, \mathbb{I}_2^{(0)}\mathbb{I}_r^{(1)}\}$ is also a basis of $M_2(\mathbb{k})_1$ for any $r = 1, 2$. Hence, the matrix $\bar{L}_{1,2;r}^{(01)}$ defines a linear automorphism of the vector space $M_2(\mathbb{k})_1$, and it is invertible. Then

$$\theta^{(1)}(1) = \bar{L}_{1,2;r}^{(01)} \theta^{(0)}(1) (\bar{L}_{1,2;r}^{(01)})^{-1} \in \mathrm{GL}_2(\mathbb{k}).$$

Similar to the invertibility of $\varphi^{(0)}(1)$, one obtains that $\varphi^{(1)}(1) \in \mathrm{GL}_2(\mathbb{k})$.

Similarly, if $\theta^{(1)}(1) \in \mathrm{GL}_2(\mathbb{k})$, then $\varphi^{(1)}(1) \in \mathrm{GL}_2(\mathbb{k})$, and

$$\theta^{(0)}(1) = \bar{L}_{1,2;r}^{(11)} \theta^{(1)}(1) (\bar{L}_{1,2;r}^{(11)})^{-1} \in \mathrm{GL}_2(\mathbb{k}),$$

where $\bar{L}_{1,2;r}^{(11)} = \begin{pmatrix} l_{11r}^{(11)} & l_{12r}^{(11)} \\ l_{21r}^{(11)} & l_{22r}^{(11)} \end{pmatrix}$ is invertible for any $r = 1, 2$. ■

Lemma 3.5. *Retain the assumptions of Lemma 3.3, and suppose $\theta^{(0)}(1) \in \mathrm{GL}_2(\mathbb{k})$ (equivalently, $\theta^{(1)}(1) \in \mathrm{GL}_2(\mathbb{k})$).*

(1) *For any $i, i' \in \mathbb{Z}_2$ and $q, r, s = 1, 2$, then*

$$\sum_{p=1}^2 l_{pqr}^{(ii')} \varphi_{ps}^{(i+i')}(1) = \sum_{j=1}^2 l_{sjr}^{(ii')} \varphi_{qj}^{(i)}(1).$$

(2) *For any $x \in E$ and $v = 1, 2$, we have $\sum_{r,j=1}^2 \theta_{vj}^{(0)}(x \varphi_{rj}^{(0)}(1)) \gamma_r = x \gamma_v$.*

(3) *For any $i \in \mathbb{Z}_2$ and $q, s = 1, 2$, we have $\sum_{t,j,u=1}^2 \gamma_u l_{sjt}^{(0i)} \varphi_{uj}^{(0)}(1) \theta_{tq}^{(i)}(1) = \delta_{sq}$.*

Proof. (1) One obtains that

$$\begin{aligned} \sum_{p=1}^2 l_{pqr}^{(ii')} \varphi_{ps}^{(i+i')}(1) &= \sum_{p,u,j=1}^2 l_{pqu}^{(ii')} \varphi_{ps}^{(i+i')}(1) \theta_{uj}^{(i')}(1) \varphi_{rj}^{(i')}(1) \\ &= \sum_{p,u,j=1}^2 l_{pqu}^{(ii')} \theta_{uj}^{(i')}(1) \varphi_{rj}^{(i')}(1) \varphi_{ps}^{(i+i')}(1) \\ &= \sum_{p,t,j=1}^2 l_{tjr}^{(ii')} \theta_{pt}^{(i+i')}(1) \varphi_{qj}^{(i)}(1) \varphi_{ps}^{(i+i')}(1) \\ &= \sum_{p,t,j=1}^2 l_{tjr}^{(i')}(1) \varphi_{qj}^{(i)}(1) \theta_{pt}^{(i+i')}(1) \varphi_{ps}^{(i+i')}(1) \\ &= \sum_{p,t,j=1}^2 l_{tjr}^{(i')}(1) \varphi_{qj}^{(i)}(1) \varphi_{ps}^{(i+i')}(1) \theta_{pt}^{(i+i')}(1) \\ &= \sum_{j=1}^2 l_{sjr}^{(ii')} \varphi_{qj}^{(i)}(1), \end{aligned}$$

where the second, fourth and fifth equalities follow from Lemma 3.4 and the fact that $\theta^{(i)}$ and $\varphi^{(i)}$ are linear, and the third equality follows by taking $x = 1$, $x' = \varphi_{ps}^{(i+i')}(1)$ in Lemma 3.3 and the sum over p from 1 to 2.

(2) Taking $i' = 0$ and $x' = 1$ in Lemma 3.3, one obtains that

$$\sum_{u,j=1}^2 l_{pqu}^{(i0)} \theta_{uj}^{(0)}(x \varphi_{rj}^{(0)}(1)) = \sum_{t,j=1}^2 l_{tjr}^{(i0)} \theta_{pt}^{(i)}(\varphi_{qj}^{(i)}(x))$$

for any $p, q, r = 1, 2$. By Lemma 3.1 (2), we have

$$\sum_{r,u,j=1}^2 l_{pqu}^{(i0)} \theta_{uj}^{(0)}(x \varphi_{rj}^{(0)}(1)) \gamma_r = \sum_{r,t,j=1}^2 l_{tjr}^{(i0)} \theta_{pt}^{(i)}(\varphi_{qj}^{(i)}(x)) \gamma_r = \sum_{j=1}^2 \theta_{pj}^{(i)}(\varphi_{qj}^{(i)}(x)) = \delta_{pq} x,$$

and then multiply the equation above by $g_{vp}^{(i0)}$ which is the (v, p) -th element of $(L_q^{(i0)})^{-1}$ for any $v, p, q = 1, 2$. The result follows.

(3) By Lemma 3.1 (3), Lemma 3.4 and taking $i = 0$ in (1), one obtains

$$\begin{aligned} \sum_{t,j,u=1}^2 \gamma_u l_{sjt}^{(0i)} \varphi_{uj}^{(0)}(1) \theta_{tq}^{(i)}(1) &= \sum_{p,t,u=1}^2 \gamma_u l_{put}^{(0i)} \varphi_{ps}^{(i)}(1) \theta_{tq}^{(i)}(1) \\ &= \sum_{t=1}^2 \varphi_{ts}^{(i)}(1) \theta_{tq}^{(i)}(1) = \delta_{sq}. \end{aligned} \quad \blacksquare$$

Definition 3.6. Let $\theta^{(i)} = (\theta_{jj'}^{(i)}) : E \rightarrow M_2(E)$ be a t -invertible linear map for $i \in \mathbb{Z}_2$ and $\mathbb{I} = \{\mathbb{I}_j^{(i)} \mid i \in \mathbb{Z}_2, j = 1, 2\}$ be an invertible $\mathbb{Z}_2$ -graded basis of $M_2(\mathbb{k})$. The pair $\Theta = (\{\theta^{(i)}\}_{i \in \mathbb{Z}_2}, \mathbb{I})$ is called a *twisting system* of $M_2(E)$ if

- (1) $\theta^{(0)}, \theta^{(1)}$ satisfy equivalent conditions of Proposition 3.2;
- (2) $\theta^{(1)}(1) \in \text{GL}_2(\mathbb{k})$ (or equivalently, $\theta^{(0)}(1) \in \text{GL}_2(\mathbb{k})$).

Theorem 3.7. Let $\Theta = (\{\theta^{(i)}\}_{i \in \mathbb{Z}_2}, \mathbb{I})$ be a twisting system of $M_2(E)$. Then the $\mathbb{Z}_2$ -graded vector space $M_2(E)$ with the multiplication is as follows: for any $x, x' \in E, i, i' \in \mathbb{Z}_2$ and $j, j' = 1, 2$,

$$\mathbb{I}_j^{(i)} x \star \mathbb{I}_{j'}^{(i')} x' = \sum_{s=1}^2 \mathbb{I}_j^{(i)} \mathbb{I}_s^{(i')} \theta_{sj'}^{(i')}(x) x' = \sum_{s,t=1}^2 \mathbb{I}_t^{(i+i')} l_{tjs}^{(i')} \theta_{sj'}^{(i')}(x) x'$$

is a $\mathbb{Z}_2$ -graded algebra with the identity $\sum_{j,s=1}^2 \gamma_s (\theta_{sj}^{(0)})^{-1}(1) \mathbb{I}_j^{(0)}$, called this new algebra the twisted matrix algebra of $M_2(E)$ by Θ , denoted by ${}^\Theta M_2(E)$.

Proof. It is straightforward to check that the multiplication $\star$ satisfies the associative law by Proposition 3.2 (1).

Write $\varphi^{(0)}$ for the t -inverse of $\theta^{(0)}$ and $\mathbb{I}^{(0)} = \sum_{j,s=1}^2 \gamma_s \varphi_{sj}^{(0)}(1) \mathbb{I}_j^{(0)}$. By Lemma 3.5 (2) and Lemma 3.1 (2), for any $x \in E, i \in \mathbb{Z}_2$ and $j = 1, 2$,

$$\mathbb{I}_j^{(i)} x \star \mathbb{I}^{(0)} = \sum_{s,t,j',u=1}^2 \mathbb{I}_t^{(i)} l_{tjs}^{(i0)} \theta_{sj'}^{(0)}(x) \gamma_u \varphi_{uj'}^{(0)}(1) = \sum_{s,t=1}^2 \mathbb{I}_t^{(i)} l_{tjs}^{(i0)} \gamma_s x = \mathbb{I}_j^{(i)} x.$$

On the other hand, by Lemma 3.5 (3), for any $x \in E$, $i \in \mathbb{Z}_2$ and $j = 1, 2$,

$$\mathbb{I}^{(0)} \star \mathbb{I}_j^{(i)} x = \sum_{s,t,j',u=1}^2 \mathbb{I}_t^{(i)} \gamma_u \varphi_{uj'}^{(0)}(1) l_{tj's}^{(0i)} \theta_{sj}^{(i)}(1) x = \mathbb{I}_j^{(i)} x.$$

Hence, $\mathbb{I}^{(0)}$ is the identity. ■

The last theorem provides a twisted algebraic structure on the matrix algebra $M_2(E)$, but the identity seems complicated. It is natural to find a twisting system such that the identity matrix is the identity of the associated twisted matrix algebra. We have an analogous result of [20, Proposition 2.4].

Theorem 3.8. *Let $\Theta = (\{\theta^{(i)}\}_{i \in \mathbb{Z}_2}, \mathbb{I})$ be a twisting system of $M_2(E)$. Let $\Upsilon = (v^{(0)}, v^{(1)})$ be a pair of linear maps from E to $M_2(E)$ satisfying*

$$v^{(i)}(x) = \theta^{(i)}(x)(\varphi^{(i)}(1))^T,$$

where $\varphi^{(i)}$ is the t -inverse of $\theta^{(i)}$ for any $i \in \mathbb{Z}_2$ and $x \in E$. Then

(1) $v^{(i)}$ is t -invertible with the t -inverse map from E to $M_2(E)$ defined by

$$x \mapsto \varphi^{(i)}(x)(\theta^{(i)}(1))^T$$

for any $i \in \mathbb{Z}_2$;

(2) $\Upsilon = (\{v^{(i)}\}_{i \in \mathbb{Z}_2}, \mathbb{I})$ is a twisting system of $M_2(E)$;

(3) the identity of the twisted matrix algebra ${}^\Upsilon M_2(E)$ is the identity matrix I_2 ;

(4) ${}^\Theta M_2(E) \cong {}^\Upsilon M_2(E)$ as $\mathbb{Z}_2$ -graded algebras.

Proof. (1) It is straightforward.

(2) It is clear that $v^{(i)}(1) = \theta^{(i)}(1)(\varphi^{(i)}(1))^T = I_2 \in \text{GL}_2(\mathbb{k})$ for any $i \in \mathbb{Z}_2$ by Lemma 3.4. For any $x, x' \in E$, $i, i', i'' \in \mathbb{Z}_2$ and $j, j', j'' = 1, 2$,

$$\begin{aligned} & \sum_{s,t,u=1}^2 l_{vtu}^{(i+i',i'')} l_{tjs}^{(ii')} v_{uj''}^{(i'')} (v_{sj'}^{(i')}(x)x') \\ &= \sum_{s,t,u,p,q=1}^2 l_{vtu}^{(i+i',i'')} l_{tjs}^{(ii')} \theta_{up}^{(i'')} (\theta_{sq}^{(i')}(x)x') \varphi_{j'q}^{(i')}(1) \varphi_{j''p}^{(i'')}(1) \\ &= \sum_{s,t,u,p,q=1}^2 l_{vjs}^{(i,i'+i'')} l_{tqu}^{(i'i'')} \theta_{st}^{(i'+i'')}(x) \theta_{up}^{(i'')}(x') \varphi_{j'q}^{(i')}(1) \varphi_{j''p}^{(i'')}(1) \\ &= \sum_{s,t,u,p,q=1}^2 l_{vjs}^{(i,i'+i'')} l_{qj'u}^{(i'i'')} \theta_{st}^{(i'+i'')}(x) \theta_{up}^{(i'')}(x') \varphi_{qt}^{(i'+i'')}(1) \varphi_{j''p}^{(i'')}(1) \\ &= \sum_{s,u,q=1}^2 l_{vjs}^{(i,i'+i'')} l_{qj'u}^{(i'i'')} v_{sq}^{(i'+i'')}(x) v_{uj''}^{(i'')}(x'), \end{aligned}$$

where the second equality follows from Proposition 3.2 (1) and the third equality follows from Lemma 3.5 (1).

(3) By Theorem 3.7, the identity of ${}^{\Upsilon}M_2(E)$ is

$$\sum_{j,s,t=1}^2 \gamma_s \varphi_{st}^{(0)}(1) \theta_{jt}^{(0)}(1) \mathbb{I}_j^{(0)} = \sum_{j,s,t=1}^2 \gamma_s \theta_{jt}^{(0)}(\varphi_{st}^{(0)}(1)) \mathbb{I}_j^{(0)} = \sum_{j=1}^2 \gamma_j \mathbb{I}_j^{(0)} = I_2.$$

(4) Define two $\mathbb{Z}_2$ -graded linear maps

$$\begin{aligned} F : {}^{\Theta}M_2(E) &\rightarrow {}^{\Upsilon}M_2(E) & G : {}^{\Upsilon}M_2(E) &\rightarrow {}^{\Theta}M_2(E) \\ \mathbb{I}_j^{(i)} x &\mapsto \sum_{s=1}^2 \mathbb{I}_s^{(i)} x \theta_{sj}^{(i)}(1), & \mathbb{I}_j^{(i)} x &\mapsto \sum_{s=1}^2 \mathbb{I}_s^{(i)} x \varphi_{js}^{(i)}(1). \end{aligned}$$

It is straightforward to verify that F and G are inverses of each other. Moreover, F is an algebra homomorphism, since for any $i, i' \in \mathbb{Z}_2$ and $j = 1, 2$,

$$\begin{aligned} F(\mathbb{I}_j^{(i)} x) \star F(\mathbb{I}_{j'}^{(i')} x') &= \sum_{p=1}^2 \mathbb{I}_p^{(i)} x \theta_{pj}^{(i)}(1) \star \sum_{q=1}^2 \mathbb{I}_q^{(i')} x' \theta_{qj'}^{(i')}(1) \\ &= \sum_{p,q,s,t=1}^2 \mathbb{I}_t^{(i+i')} l_{tps}^{(ii')} \upsilon_{sq}^{(i')}(x \theta_{pj}^{(i)}(1)) \theta_{qj'}^{(i')}(1) x' \\ &= \sum_{p,q,s,t,u=1}^2 \mathbb{I}_t^{(i+i')} \theta_{pj}^{(i)}(1) \theta_{qj'}^{(i')}(1) l_{tps}^{(ii')} \theta_{su}^{(i')}(x \varphi_{qu}^{(i')}(1)) x' \\ &= \sum_{p,t=1}^2 \mathbb{I}_t^{(i+i')} \theta_{pj}^{(i)}(1) \left(\sum_{q,v,w=1}^2 l_{wvq}^{(ii')} \theta_{tw}^{(i+i')}(1) \varphi_{pv}^{(i)}(x) \theta_{qj'}^{(i')}(1) \right) x' \\ &= \sum_{p,t=1}^2 \mathbb{I}_t^{(i+i')} \theta_{pj}^{(i)}(1) \left(\sum_{q,u,j''=1}^2 l_{tpu}^{(ii')} \theta_{uj''}^{(i')}(1) \varphi_{qj''}^{(i')}(x) \theta_{qj'}^{(i')}(1) \right) x' \\ &= \sum_{p,u,t=1}^2 \mathbb{I}_t^{(i+i')} \theta_{pj}^{(i)}(1) l_{tpu}^{(ii')} \theta_{uj'}^{(i')}(x) x' \\ &= \sum_{u,s,t=1}^2 \mathbb{I}_t^{(i+i')} l_{tsju}^{(ii')} \theta_{ts}^{(i+i')}(1) \theta_{uj'}^{(i')}(x) x' \\ &= F(\mathbb{I}_j^{(i)} x \star \mathbb{I}_{j'}^{(i')} x'), \end{aligned}$$

where the fourth and the fifth equalities are by Lemma 3.3 and the seventh equality is by Proposition 3.2 (3). ■

Before that, the twisting systems on matrix algebras depend on an invertible $\mathbb{Z}_2$ -graded basis of $M_2(\mathbb{k})$. In fact, we can fix an invertible $\mathbb{Z}_2$ -graded basis of $M_2(\mathbb{k})$ to obtain twisted matrix algebras.

Theorem 3.9. Let $\Theta = (\{\theta^{(i)}\}_{i \in \mathbb{Z}_2}, \mathbb{I})$ be a twisting system of $M_2(E)$. Suppose $\mathbb{J} = \{\mathbb{J}_j^{(i)} \mid i \in \mathbb{Z}_2, j = 1, 2\}$ be an invertible $\mathbb{Z}_2$ -graded basis of $M_2(\mathbb{k})$. Then there exists a twisting system $\Omega = (\{\omega^{(i)}\}_{i \in \mathbb{Z}_2}, \mathbb{J})$ of $M_2(E)$ such that ${}^\Theta M_2(E) \cong {}^\Omega M_2(E)$ as $\mathbb{Z}_2$ -graded algebras.

Proof. Since $\{\mathbb{I}_1^{(i)}, \mathbb{I}_2^{(i)}\}$ and $\{\mathbb{J}_1^{(i)}, \mathbb{J}_2^{(i)}\}$ are both bases of $M_2(E)_i$, there exists an invertible matrix $U^{(i)} = (u_{st}^{(i)}) \in M_2(\mathbb{k})$ such that $\mathbb{I}_j^{(i)} = \sum_{s=1}^2 \mathbb{J}_s^{(i)} u_{sj}^{(i)}$ for any $i \in \mathbb{Z}_2, j = 1, 2$. Also invertible matrices $U^{(0)}$ and $U^{(1)}$ define a $\mathbb{Z}_2$ -graded linear automorphism η of the $\mathbb{Z}_2$ -graded vector space $M_2(E)$.

Write $\Omega = (\omega^{(0)}, \omega^{(1)})$, where

$$\begin{aligned} \omega^{(i)} : E &\rightarrow M_2(E) \\ x &\mapsto U^{(i)} \theta^{(i)}(x) (U^{(i)})^{-1} \end{aligned}$$

for any $i \in \mathbb{Z}_2$. It is straightforward to check that $\Omega = (\{\omega^{(i)}\}_{i \in \mathbb{Z}_2}, \mathbb{J})$ is a twisting system of $M_2(E)$ by Definition 3.6.

It is also not hard to obtain that the $\mathbb{Z}_2$ -graded linear isomorphism η is also a $\mathbb{Z}_2$ -graded algebra isomorphism between ${}^\Theta M_2(E)$ and ${}^\Omega M_2(E)$. ■

The subsequent result follows from Theorems 3.8 and 3.9 immediately.

Corollary 3.10. Let $\Theta = (\{\theta^{(i)}\}_{i \in \mathbb{Z}_2}, \mathbb{I})$ be a twisting system of $M_2(E)$. Suppose $\mathbb{J} = \{\mathbb{J}_j^{(i)} \mid i \in \mathbb{Z}_2, j = 1, 2\}$ be an invertible $\mathbb{Z}_2$ -graded basis of $M_2(\mathbb{k})$. Then there exists a twisting system $\Omega = (\{\omega^{(i)}\}_{i \in \mathbb{Z}_2}, \mathbb{J})$ of $M_2(E)$ such that ${}^\Theta M_2(E) \cong {}^\Omega M_2(E)$ as $\mathbb{Z}_2$ -graded algebras and the identity of ${}^\Omega M_2(E)$ is I_2 .

This corollary implies that we can always fix an invertible $\mathbb{Z}_2$ -graded basis of $M_2(\mathbb{k})$ and assume the identity of a twisted matrix algebra is I_2 .

We turn to graded cases. Let E be a G -graded algebra where G is an abelian group. Suppose $\Theta = (\{\theta^{(i)}\}_{i \in \mathbb{Z}_2}, \mathbb{I})$ is a twisting system of $M_2(E)$ where each $\theta_{jj'}^{(i)}$ is a G -graded linear map for any $i \in \mathbb{Z}_2$ and $j, j' = 1, 2$. Then the twisted matrix algebra ${}^\Theta M_2(E)$ is a $\mathbb{Z}_2 \times G$ -graded algebra, where each component is

$${}^\Theta M_2(E)_{(i,g)} = \{\mathbb{I}_1^{(i)} x \mid x \in E_g\} \oplus \{\mathbb{I}_2^{(i)} x \mid x \in E_g\}$$

for any $i \in \mathbb{Z}_2$ and $g \in G$.

Remark 3.11. In particular, if E is a $\mathbb{Z}_2$ -graded algebra, then $\mathbb{Z}_2 \times \mathbb{Z}_2$ -graded algebra ${}^\Theta M_2(E)$ has the following two approaches to becoming a $\mathbb{Z}_2$ -graded algebra:

(1) By taking the total degrees, ${}^\Theta M_2(E)$ becomes a $\mathbb{Z}_2$ -graded algebra where

$${}^\Theta M_2(E)_0 = \begin{pmatrix} E_0 & E_1 \\ E_1 & E_0 \end{pmatrix}, \quad {}^\Theta M_2(E)_1 = \begin{pmatrix} E_1 & E_0 \\ E_0 & E_1 \end{pmatrix}.$$

(2) By forgetting the first degree, ${}^{\Theta}M_2(E)$ becomes a $\mathbb{Z}_2$ -graded algebra where

$${}^{\Theta}M_2(E)_0 = \begin{pmatrix} E_0 & E_0 \\ E_0 & E_0 \end{pmatrix}, \quad {}^{\Theta}M_2(E)_1 = \begin{pmatrix} E_1 & E_1 \\ E_1 & E_1 \end{pmatrix}.$$

These are the only two good $\mathbb{Z}_2$ -gradings on $M_2(E)$ (see [11]).

Remark 3.12. Let E be a $\mathbb{Z}_2$ -graded algebra. By taking the total degrees, ${}^{\Theta}M_2(E)$ and ${}^{\Theta'}M_2(E)$ become $\mathbb{Z}_2$ -graded for some twisting systems Θ and Θ' of $M_2(E)$. In general, $\text{gr}_{\mathbb{Z}_2} {}^{\Theta}M_2(E)$ and $\text{gr}_{\mathbb{Z}_2} {}^{\Theta'}M_2(E)$ are not equivalent and are not even derived equivalent (see Example 5.9). It is different from the one of Zhang twists (see [20, Theorem 3.1]).

3.2. Twisted direct products

Let E be an algebra and $\Theta = (\{\theta^{(i)}\}_{i \in \mathbb{Z}_2}, \mathbb{I})$ be a twisting system of $M_2(E)$. The 0-th homogeneous space

$$\begin{pmatrix} E & 0 \\ 0 & E \end{pmatrix}$$

of $\mathbb{Z}_2$ -graded algebra ${}^{\Theta}M_2(E)$ provides a subalgebra of ${}^{\Theta}M_2(E)$, which is isomorphic to $E \times E$ as vector spaces. It implies that there is a kind of twisted algebraic structure on $E \times E$ induced by that of $M_2(E)$. It is analogous to Definition 3.6 and Theorem 3.7; we just list them.

Let $\varepsilon_1 = (a_1, a_2), \varepsilon_2 = (b_1, b_2)$ be an invertible $\mathbb{k}$ -basis of $\mathbb{k} \times \mathbb{k}$ where $a_1 a_2 b_1 b_2 \neq 0$ and $\{l_{p;jj'} \mid p, j, j' = 1, 2\} \subseteq \mathbb{k}$ such that $\varepsilon_j \varepsilon_{j'} = \sum_{p=1}^2 \varepsilon_p l_{p;jj'}$ for any $j, j' = 1, 2$.

Definition 3.13. Let $\theta = (\theta_{jj'}) : E \rightarrow M_2(E)$ be a t -invertible linear map and $\varepsilon = \{\varepsilon_1, \varepsilon_2\}$ be an invertible basis of $\mathbb{k} \times \mathbb{k}$. We say $\theta_{\times} = (\theta, \varepsilon)$ is a *twisting system* of $E \times E$ if $\theta(1) \in \text{GL}_2(\mathbb{k})$ and

$$\sum_{s,u=1}^2 l_{p;su} \theta_{uj'}(\theta_{sj}(x)x') = \sum_{t,u=1}^2 l_{t;ju} \theta_{pt}(x) \theta_{uj'}(x')$$

for any $x, x' \in E$ and $j, j', p = 1, 2$.

Corollary 3.14. Let $\theta_{\times} = (\theta, \varepsilon)$ be a twisting system of $E \times E$. Then the vector space $E \times E$ with the multiplication

$$\varepsilon_j x \star \varepsilon_{j'} x' = \sum_{s,t=1}^2 \varepsilon_t l_{t;j's} \theta_{sj'}(x)x' \quad \forall x, x' \in E, j, j' = 1, 2$$

is an algebra, called the twisted direct product of $E \times E$ by $\theta_{\times}$, denoted by $\theta_{\times}(E \times E)$.

In particular, if $\theta : E \rightarrow M_2(E)$ is the diagonal map, then $\theta_{\times} = (\theta, \varepsilon)$ is always a twisting system of $E \times E$ for any invertible $\mathbb{k}$ -basis ε and $\theta_{\times}(E \times E)$ is just the usual direct product $E \times E$.

Remark 3.15. To distinguish from the Zhang-twisting system, we add a subscript “ $\times$ ” for θ to represent a twisting system of a direct product.

Let E be a G -graded algebra where G is an abelian group. Suppose $\theta_\times = (\theta, \varepsilon)$ is a twisting system of $E \times E$, where each $\theta_{jj'}$ is a G -graded linear map for any $j, j' = 1, 2$. Then the twisted direct product ${}^{\theta_\times}(E \times E)$ is a G -graded algebra, where for any $g \in G$,

$${}^{\theta_\times}(E \times E)_g = E_g \oplus E_g.$$

4. Double Ore extensions

In this section, we recall the definition of double Ore extensions in [21] and show some basic properties of noncommutative quadric hypersurfaces defined by double Ore extensions.

Definition 4.1. We say B is a $\mathbb{Z}$ -graded right double Ore extension of a $\mathbb{Z}$ -graded algebra A if B is generated by A and two homogeneous variables y_1, y_2 satisfying

- (1) a homogeneous relation $y_2 y_1 = p_{12} y_1 y_2 + p_{11} y_1^2 + \tau_1 y_1 + \tau_2 y_2 + \tau_0$ for some $p_{12}, p_{11} \in \mathbb{k}$ and homogeneous elements $\tau_0, \tau_1, \tau_2 \in A$;
- (2) B is a $\mathbb{Z}$ -graded free left A -modules with basis $\{y_1^i y_2^j \mid i, j \geq 0\}$;
- (3) $y_1 A + y_2 A \subseteq Ay_1 + Ay_2 + A$.

Similarly, one can define a $\mathbb{Z}$ -graded left double Ore extension. An algebra B is called a $\mathbb{Z}$ -graded double Ore extension of A if it is a $\mathbb{Z}$ -graded left and a $\mathbb{Z}$ -graded right double Ore extension of A with the same generating set $\{y_1, y_2\}$.

Condition (3) of the definition of $\mathbb{Z}$ -graded right double Ore extension is equivalent to the existence of an algebra homomorphism $\sigma = (\sigma_{ij}) : A \rightarrow M_2(A)$ and a σ -derivation $\delta = (\delta_i) : A \rightarrow A^{\oplus 2}$ where all σ_{ij} and δ_i are $\mathbb{Z}$ -graded linear transformations of A for $i, j = 1, 2$ such that

$$y_i a = \sigma_{i1}(a) y_1 + \sigma_{i2}(a) y_2 + \delta_i(a)$$

for any $a \in A$ and $i = 1, 2$. We write $A_P[y_1, y_2; \sigma, \delta, \tau]$ for a $\mathbb{Z}$ -graded right double Ore extension where $P = \{p_{12}, p_{11}\}$ and $\tau = \{\tau_0, \tau_1, \tau_2\}$, and write $A_P[y_1, y_2; \sigma]$ for convenience if $\tau = \{0, 0, 0\}$ and $\delta = 0$.

We will focus on the trimmed case $A_P[y_1, y_2; \sigma]$ in the sequel. By [21, Lemma 1.10], σ in the trimmed case $A_P[y_1, y_2; \sigma]$ satisfies following conditions:

$$\begin{aligned} \sigma_{21}\sigma_{11} &= p_{12}\sigma_{11}\sigma_{21}, \\ \sigma_{22}\sigma_{12} &= p_{12}\sigma_{12}\sigma_{22}, \\ \sigma_{21}\sigma_{12} + p_{12}\sigma_{22}\sigma_{11} &= p_{12}\sigma_{11}\sigma_{22} + p_{12}^2\sigma_{12}\sigma_{21}. \end{aligned} \tag{4.1}$$

There is a conjecture about the relation between the invertibility of σ and the condition that a right double Ore extension is a double Ore extension (see [21]). It is true in the connected graded case by [21, Lemma 1.9, Proposition 1.13].

Lemma 4.2. *Let A be a locally finite connected $\mathbb{Z}$ -graded algebra and $B = A_P[y_1, y_2; \sigma]$ be a connected $\mathbb{Z}$ -graded right double Ore extension of A where $p_{12} \neq 0$. Then B is a $\mathbb{Z}$ -graded double Ore extension of A if and only if σ is invertible.*

Let $A = T(V)/(R)$ be a noetherian Koszul Artin–Schelter regular algebra and $B = A_P[y_1, y_2; \sigma]$ be a $\mathbb{Z}$ -graded double Ore extension of A with $p_{12} \neq 0$ and $\deg y_1 = \deg y_2 = 1$. By [21, Theorem 0.2] and [23, Theorem 2.1] (see also [14, Proposition 2.10]), B is a Koszul Artin–Schelter regular algebra. Clearly, $J = \mathbb{k}\langle y_1, y_2 \rangle / (y_2 y_1 - p_{12} y_1 y_2 - p_{11} y_1^2)$ is a $\mathbb{Z}$ -graded subalgebra of B and a noetherian Koszul Artin–Schelter regular algebra. We identify elements of J as the elements of B and write R_J for the vector space spanned by $y_2 y_1 - p_{12} y_1 y_2 - p_{11} y_1^2$ in the sequel. The following result is clear.

Lemma 4.3. *Let $A^!$, $B^!$ and $J^!$ be Koszul duals of A , B and J , respectively.*

(1) $B^! = T(V^* \oplus \mathbb{k}y_1^* \oplus \mathbb{k}y_2^*) / (R^\perp \oplus R_J^\perp \oplus R_\tau)$, where

$$R_J^\perp = \text{Span}_{\mathbb{k}}\{(y_2^*)^2, y_1^* y_2^* + p_{12} y_2^* y_1^*, (y_1^*)^2 + p_{11} y_2^* y_1^*\},$$

$$R_\tau = \text{Span}_{\mathbb{k}}\{v^* y_i^* + y_1^* \sigma_{1i}^*(v^*) + y_2^* \sigma_{2i}^*(v^*) \mid v^* \in V^*, i = 1, 2\}.$$

(2) $A^!$ and $J^!$ are $\mathbb{Z}$ -graded subalgebras of $B^!$.

(3) $B^!$ is a $\mathbb{Z}$ -graded free $A^!$ -module and a $\mathbb{Z}$ -graded free $J^!$ -module on both sides.

(4) $B^!$ has a $\mathbb{k}$ -basis $\{e, y_1^* e, y_2^* e, y_1^* y_2^* e \mid e \in \mathcal{B}_A\}$ where $\mathcal{B}_A$ is a $\mathbb{k}$ -basis of $A^!$.

Lemma 4.4. *Retain the notations above. Suppose B is noetherian and $z + h$ is a regular central element of B for some $z \in A_2$ and $h \in J_2$. Write $C_{A^!}(z) = T(V^*) / (\widetilde{R^\perp})$ and $C_{J^!}(h) = T(\mathbb{k}y_1^* \oplus \mathbb{k}y_2^*) / (\widetilde{R_J^\perp})$, where*

$$\widetilde{R^\perp} = \{f - f(\widehat{z}) \mid f \in R^\perp\}, \quad \widetilde{R_J^\perp} = \{g - g(\widehat{h}) \mid g \in R_J^\perp\},$$

and $\widehat{z} \in V^{\otimes 2}$ and $\widehat{h} \in (\mathbb{k}y_1 \oplus \mathbb{k}y_2)^{\otimes 2}$ are lifts of z and h , respectively. Then

(1) $C_{B^!}(z + h) \cong T(V^* \oplus \mathbb{k}y_1^* \oplus \mathbb{k}y_2^*) / (\widetilde{R^\perp} \oplus \widetilde{R_J^\perp} \oplus R_\tau)$ as $\mathbb{Z}_2$ -graded algebras;

(2) $C_{A^!}(z)$ and $C_{J^!}(h)$ are $\mathbb{Z}_2$ -graded subalgebras of $C_{B^!}(z + h)$;

(3) $C_{B^!}(z + h)$ is a $\mathbb{Z}_2$ -graded free $C_{A^!}(z)$ -module and a $\mathbb{Z}_2$ -graded free $C_{J^!}(h)$ -module on both sides;

(4) $\text{gldim}_{\mathbb{Z}_2} C_{B^!}(z + h) \geq \max\{\text{gldim}_{\mathbb{Z}_2} C_{A^!}(z), \text{gldim}_{\mathbb{Z}_2} C_{J^!}(h)\}$.

Proof. Clearly, $z \in A_2$ and $h \in J_2$ are both central elements of A and J , respectively.

(1) and (2) follow from the definition of Clifford deformations and Lemma 4.3.

(3) Since $C_{A^!}(z)$, $C_{J^!}(h)$ and $C_{B^!}(z + h)$ are Clifford deformations, a class of PBW deformations, they have the same forms of $\mathbb{k}$ -bases as $A^!$, $J^!$ and $B^!$, respectively. By Lemma 4.3 (4), the map of multiplication

$$C_{J^!}(h) \otimes C_{A^!}(z) \rightarrow C_{B^!}(z + h)$$

is an isomorphism as left $\mathbb{Z}_2$ -graded $C_{J^!}(h)$ -modules and right $\mathbb{Z}_2$ -graded $C_{A^!}(z)$ -modules.

Similarly, one can check that the natural map $C_{A^!}(z) \otimes C_{J^!}(h) \rightarrow C_{B^!}(z + h)$ is an isomorphism as right $\mathbb{Z}_2$ -graded $C_{J^!}(h)$ -modules and left $\mathbb{Z}_2$ -graded $C_{A^!}(z)$ -modules.

(4) It follows from a graded version of [8, Theorem 7.2.6]. ■

Our main goal is to present a skew version of the classical Knörrer's periodicity theorem. To this end, we consider a noncommutative quadric hypersurface having the form $B/(z + y_1^2 + y_2^2)$ in the rest of this paper where $z \in A_2$ is a regular central element. To make such a class of noncommutative quadric hypersurfaces well defined, $y_1^2 + y_2^2$ should be a central element of B and also of J . By an easy calculation, we have the following result.

Lemma 4.5. *Suppose the element $y_1^2 + y_2^2$ is central in B , then $P = \{1, 0\}$ or $P = \{-1, p_{11}\}$.*

With respect to the values of p_{12} , we discuss in two cases in the following.

5. A skew version of Knörrer's periodicity theorem with $p_{12} = 1$

Let $B = A_{\{1,0\}}[y_1, y_2; \sigma]$ be a $\mathbb{Z}$ -graded double Ore extension of a noetherian Koszul Artin–Schelter regular algebra $A = T(V)/(R)$ with $\deg y_1 = \deg y_2 = 1$ and $z \in A_2$ be a regular central element of A . It is not hard to obtain the following conditions for σ .

Lemma 5.1. *The element $z + y_1^2 + y_2^2$ is a central element of B if and only if*

- (1) $\sigma_{11}^2 + \sigma_{21}^2 = \sigma_{12}^2 + \sigma_{22}^2 = \text{id}_A$;
- (2) $\sigma_{11}\sigma_{12} + \sigma_{21}\sigma_{22} + \sigma_{12}\sigma_{11} + \sigma_{22}\sigma_{21} = 0$;
- (3) $\sigma(z) = \begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix}$.

By Lemma 5.1 (3) and construction (2.4), we have the following algebra homomorphism:

$$\sigma^! = (\sigma_{ij}^!) : C_{A^!}(z) \rightarrow M_2(C_{A^!}(z)),$$

where any $\sigma_{ij}^!$ is a $\mathbb{Z}_2$ -graded map for $i, j = 1, 2$.

Corollary 5.2. *If $z + y_1^2 + y_2^2$ is a central element of B , then the following properties of the map $\sigma^!$ hold.*

- (1) $(\sigma_{11}^!)^2 + (\sigma_{21}^!)^2 = (\sigma_{12}^!)^2 + (\sigma_{22}^!)^2 = \text{id}_{C_{A^!}(z)}$.
- (2) $\sigma_{12}^!\sigma_{11}^! + \sigma_{22}^!\sigma_{21}^! + \sigma_{11}^!\sigma_{12}^! + \sigma_{21}^!\sigma_{22}^! = 0$.
- (3) $\sigma_{11}^!\sigma_{21}^! = \sigma_{21}^!\sigma_{11}^!$.
- (4) $\sigma_{12}^!\sigma_{22}^! = \sigma_{22}^!\sigma_{12}^!$.

$$(5) \quad \sigma_{22}^! \sigma_{11}^! - \sigma_{12}^! \sigma_{21}^! = \sigma_{11}^! \sigma_{22}^! - \sigma_{21}^! \sigma_{12}^!.$$

(6) $\sigma^!$ is t -invertible.

Proof. Conditions (1)–(5) follow from Lemma 5.1 and (4.1), and condition (6) follows from Lemmas 2.5 and 4.2. ■

From now on, we construct a twisting system on $M_2(C_{A^!}(z))$. We choose an invertible $\mathbb{Z}_2$ -graded basis $\mathbb{I} = \{\mathbb{I}_1^{(0)}, \mathbb{I}_2^{(0)}, \mathbb{I}_1^{(1)}, \mathbb{I}_2^{(1)}\}$ of $M_2(\mathbb{k})$ as follows:

$$\begin{aligned} \mathbb{I}_1^{(0)} &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, & \mathbb{I}_2^{(0)} &= \begin{pmatrix} -\sqrt{-1} & 0 \\ 0 & \sqrt{-1} \end{pmatrix}, \\ \mathbb{I}_1^{(1)} &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, & \mathbb{I}_2^{(1)} &= \begin{pmatrix} 0 & \sqrt{-1} \\ -\sqrt{-1} & 0 \end{pmatrix}. \end{aligned} \quad (5.1)$$

Define linear maps $\theta^{(0)}$ and $\theta^{(1)}$ from $C_{A^!}(z)$ to $M_2(C_{A^!}(z))$ by

$$\theta^{(0)} = (\theta_{jj'}^{(0)}) = \begin{pmatrix} \text{id} & \sigma_{12}^! \sigma_{11}^! + \sigma_{22}^! \sigma_{21}^! \\ 0 & \sigma_{22}^! \sigma_{11}^! - \sigma_{12}^! \sigma_{21}^! \end{pmatrix}, \quad \theta^{(1)} = (\theta_{jj'}^{(1)}) = \sigma^! \circ \xi_{-1} = (\sigma_{ij}^! \circ \xi_{-1}). \quad (5.2)$$

Recall that ξ_{-1} maps each homogeneous element c to $(-1)^{\deg c} c$. Clearly, $\theta_{jj'}^{(i)}$ is a $\mathbb{Z}_2$ -graded linear map for any $i \in \mathbb{Z}_2, j, j' = 1, 2$. Those two linear maps satisfy the following properties.

Lemma 5.3. *Retain the notations above. Then*

- (1) $\theta_{12}^{(0)}(aa') = a\theta_{12}^{(0)}(a') + \theta_{12}^{(0)}(a)a'$ for any $a, a' \in C_{A^!}(z)$;
- (2) $\theta^{(0)}$ and $\theta^{(1)}$ are algebra homomorphisms;
- (3) $\theta_{12}^{(0)}\theta_{22}^{(0)} + \theta_{22}^{(0)}\theta_{12}^{(0)} = 0$;
- (4) $(\theta_{22}^{(0)})^2 - (\theta_{12}^{(0)})^2 = \text{id}_{C_{A^!}(z)}$.

Proof. The results follow from Corollary 5.2 and the fact that $\sigma^!$ is an algebra homomorphism. ■

Theorem 5.4. *Let $\theta^{(0)}$ and $\theta^{(1)}$ be linear maps from $C_{A^!}(z)$ to $M_2(C_{A^!}(z))$ defined in (5.2) and $\mathbb{I}$ be the basis of $M_2(\mathbb{k})$ defined in (5.1). Then*

- (1) $\Theta = (\{\theta^{(i)}\}_{i \in \mathbb{Z}_2}, \mathbb{I})$ is a twisting system of $M_2(C_{A^!}(z))$;
- (2) $C_{B^!}(z + y_1^2 + y_2^2) \cong {}^\Theta M_2(C_{A^!}(z))$ as $\mathbb{Z}_2$ -graded algebras where the $\mathbb{Z}_2$ -grading of ${}^\Theta M_2(C_{A^!}(z))$ comes from the total degrees (see Remark 3.11 (1)).

Proof. (1) Write φ for the inverse of σ . By Lemma 2.5 and its proof, $\varphi^! = (\varphi_{ij}^!)$ exists and is the t -inverse of $\sigma^!$. Then one obtains that $\theta^{(0)}$ and $\theta^{(1)}$ are t -invertible with the t -inverses

$$\begin{pmatrix} \text{id} & 0 \\ \varphi_{21}^! \varphi_{11}^! + \varphi_{22}^! \varphi_{12}^! & \varphi_{22}^! \varphi_{11}^! - \varphi_{21}^! \varphi_{12}^! \end{pmatrix}, \quad \varphi^! \circ \xi_{-1},$$

respectively. Clearly, $\theta^{(1)}(1) = I_2 \in \text{GL}_2(\mathbb{k})$, and the invertible matrices defined in (3.2) are $L_1^{(ii')} = I_2$ for any $i, i' \in \mathbb{Z}_2$, and

$$L_2^{(00)} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad L_2^{(01)} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad L_2^{(10)} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad L_2^{(11)} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Then by Corollary 5.2 and Lemma 5.3, it is straightforward but tedious to check that Θ satisfies the following simplified equations of Proposition 3.2 (3):

$$\begin{aligned} \theta_{pj'}^{(i')}(\theta_{11}^{(i)}(x)x') + l_{p2u}^{(ii')}\theta_{uj'}^{(i')}(x)x' &= \theta_{p1}^{(i+i')}(x)\theta_{1j'}^{(i')}(x') + \theta_{p2}^{(i+i')}(x)\theta_{2j'}^{(i')}(x'), \\ \theta_{pj'}^{(i')}(\theta_{12}^{(i)}(x)x') + l_{p2u}^{(ii')}\theta_{uj'}^{(i')}(x)x' &= l_{122}^{(ii')}\theta_{p1}^{(i+i')}(x)\theta_{2j'}^{(i')}(x') \\ &\quad + l_{221}^{(ii')}\theta_{p2}^{(i+i')}(x)\theta_{1j'}^{(i')}(x'), \end{aligned}$$

where $u \in \{1, 2\} \setminus \{p\}$ for any $x, x' \in E$, $i, i' \in \mathbb{Z}_2$ and $p = 1, 2$.

(2) By Lemma 4.4 (1), $C_{B^1}(z + y_1^2 + y_2^2)$ is generated by V^* , y_1^* , y_2^* subject to the relations consisting of the relations of $C_{A^1}(z)$, and

$$\begin{aligned} (y_1^*)^2 - 1, (y_2^*)^2 - 1, y_1^*y_2^* + y_2^*y_1^*, v^*y_i^* + y_1^*\sigma_{1i}^*(v^*) + y_2^*\sigma_{2i}^*(v^*) \\ \forall v^* \in V^*, i = 1, 2. \end{aligned}$$

We list the precise multiplication of ${}^\Theta M_2(C_{A^1}(z))$ in Table 1 for any $x, x' \in C_{A^1}(z)$.

Then one obtains that there exists a $\mathbb{Z}_2$ -graded algebra isomorphism from $C_{B^1}(z + y_1^2 + y_2^2)$ to ${}^\Theta M_2(C_{A^1}(z))$ mapping

$$y_1^* \mapsto \mathbb{I}_1^{(1)}, \quad y_2^* \mapsto \mathbb{I}_2^{(1)}, \quad v^* \mapsto \mathbb{I}_1^{(0)}v^* \quad \forall v^* \in V^*. \quad \blacksquare$$

Proposition 5.5. *Retain the assumptions of Theorem 5.4. The homogeneous element $e = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ is a full idempotent of ${}^\Theta M_2(C_{A^1}(z))$.*

As a consequence, $\text{gr}_{\mathbb{Z}_2} {}^\Theta M_2(C_{A^1}(z))$ and $\text{gr}_{\mathbb{Z}_2} e({}^\Theta M_2(C_{A^1}(z)))e$ are equivalent.

Proof. Note that $e = 2^{-1}(\mathbb{I}_1^{(0)} + \sqrt{-1}\mathbb{I}_2^{(0)})$. By Table 1, one obtains

$$2^{-1}(\mathbb{I}_1^{(i)} + \sqrt{-1}\mathbb{I}_2^{(i)}) \star e = 2^{-1}(\mathbb{I}_1^{(i)} + \sqrt{-1}\mathbb{I}_2^{(i)}),$$

and

$$\begin{aligned} \mathbb{I}_1^{(i)}x &= 2^{-1}(\mathbb{I}_1^{(i)} + \sqrt{-1}\mathbb{I}_2^{(i)}) \star e \star \mathbb{I}_1^{(0)}x + 2^{-1}(\mathbb{I}_1^{(i+1)} + \sqrt{-1}\mathbb{I}_2^{(i+1)}) \star e \star \mathbb{I}_1^{(1)}x, \\ \mathbb{I}_2^{(i)}x &= 2^{-1}(\mathbb{I}_1^{(i+1)} + \sqrt{-1}\mathbb{I}_2^{(i+1)}) \star e \star \sqrt{-1}\mathbb{I}_1^{(1)}x \\ &\quad - 2^{-1}(\mathbb{I}_1^{(i)} + \sqrt{-1}\mathbb{I}_2^{(i)}) \star e \star \sqrt{-1}\mathbb{I}_1^{(0)}x \end{aligned}$$

for any $i \in \mathbb{Z}_2$ and $x \in C_{A^1}(z)$. Hence, e is a full idempotent. $\blacksquare$

	$\mathbb{I}_1^{(0)} x'$	$\mathbb{I}_2^{(0)} x'$	$\mathbb{I}_1^{(1)} x'$	$\mathbb{I}_2^{(1)} x'$
$\mathbb{I}_1^{(0)} x$	$\mathbb{I}_1^{(0)} x x'$	$\mathbb{I}_1^{(0)} \theta_{12}^{(0)} (x) x' + \mathbb{I}_2^{(0)} \theta_{22}^{(0)} (x) x'$	$\mathbb{I}_1^{(1)} \theta_{11}^{(1)} (x) x' + \mathbb{I}_2^{(1)} \theta_{21}^{(1)} (x) x'$	$\mathbb{I}_1^{(1)} \theta_{12}^{(1)} (x) x' + \mathbb{I}_2^{(1)} \theta_{22}^{(1)} (x) x'$
$\mathbb{I}_2^{(0)} x$	$\mathbb{I}_2^{(0)} x x'$	$-\mathbb{I}_1^{(0)} \theta_{22}^{(0)} (x) x' + \mathbb{I}_2^{(0)} \theta_{12}^{(0)} (x) x'$	$\mathbb{I}_1^{(1)} \theta_{21}^{(1)} (x) x' - \mathbb{I}_2^{(1)} \theta_{11}^{(1)} (x) x'$	$\mathbb{I}_1^{(1)} \theta_{22}^{(1)} (x) x' - \mathbb{I}_2^{(1)} \theta_{12}^{(1)} (x) x'$
$\mathbb{I}_1^{(1)} x$	$\mathbb{I}_1^{(1)} x x'$	$\mathbb{I}_1^{(1)} \theta_{12}^{(0)} (x) x' + \mathbb{I}_2^{(1)} \theta_{22}^{(0)} (x) x'$	$\mathbb{I}_1^{(0)} \theta_{11}^{(1)} (x) x' + \mathbb{I}_2^{(0)} \theta_{21}^{(1)} (x) x'$	$\mathbb{I}_1^{(0)} \theta_{12}^{(1)} (x) x' + \mathbb{I}_2^{(0)} \theta_{22}^{(1)} (x) x'$
$\mathbb{I}_2^{(1)} x$	$\mathbb{I}_2^{(1)} x x'$	$-\mathbb{I}_1^{(1)} \theta_{22}^{(0)} (x) x' + \mathbb{I}_2^{(1)} \theta_{12}^{(0)} (x) x'$	$\mathbb{I}_1^{(0)} \theta_{21}^{(1)} (x) x' - \mathbb{I}_2^{(0)} \theta_{11}^{(1)} (x) x'$	$\mathbb{I}_1^{(0)} \theta_{22}^{(1)} (x) x' - \mathbb{I}_2^{(0)} \theta_{12}^{(1)} (x) x'$

Table 1. Multiplication table of ${}^\Theta M_2(C_{A_1}(z))$.

To understand the multiplication of ${}^{\Theta}M_2(C_{A^1}(z))$ further, we introduce four linear transformations of $C_{A^1}(z)$:

$$\begin{aligned}\Xi_1 &= 2^{-1}(\text{id} + \sqrt{-1}\theta_{12}^{(0)} + \theta_{22}^{(0)}), \\ \Xi_2 &= 2^{-1}(\text{id} + \sqrt{-1}\theta_{12}^{(0)} - \theta_{22}^{(0)}), \\ \Phi_1 &= 2^{-1}(\theta_{11}^{(1)} - \sqrt{-1}\theta_{12}^{(1)} + \sqrt{-1}\theta_{21}^{(1)} + \theta_{22}^{(1)}), \\ \Phi_2 &= 2^{-1}(\theta_{11}^{(1)} - \sqrt{-1}\theta_{12}^{(1)} - \sqrt{-1}\theta_{21}^{(1)} - \theta_{22}^{(1)}).\end{aligned}$$

We show some parts of the multiplication of ${}^{\Theta}M_2(C_{A^1}(z))$ in the usual matrix forms involving the linear transforms listed above:

$$\begin{aligned}\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} * \begin{pmatrix} a' & 0 \\ 0 & 0 \end{pmatrix} &= \begin{pmatrix} \Xi_1(a)a' & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} * \begin{pmatrix} 0 & a' \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \Phi_1(a)a' \\ 0 & 0 \end{pmatrix}, \\ \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} * \begin{pmatrix} a' & 0 \\ 0 & 0 \end{pmatrix} &= \begin{pmatrix} 0 & \Xi_2(a)a' \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} * \begin{pmatrix} 0 & a' \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \Phi_2(a)a' & 0 \\ 0 & 0 \end{pmatrix}.\end{aligned}\tag{5.3}$$

Here are some useful properties of those four linear transformations.

Lemma 5.6. *The following properities of Ξ_1, Ξ_2, Φ_1 and Φ_2 hold.*

- (1) $\Xi_1^2 = \Xi_1, \Xi_2^2 = \Xi_2.$
- (2) $\Xi_1(ab) = a\Xi_2(b) + \Xi_1(a)\theta_{22}^{(0)}(b) = a\Xi_1(b) + \Xi_1(a)\theta_{22}^{(0)}(b) - a\theta_{22}^{(0)}(b)$ for any $a, b \in C_{A^1}(z).$
- (3) $\Xi_2(ab) = a\Xi_2(b) + \Xi_2(a)\theta_{22}^{(0)}(b) = a\Xi_1(b) + \Xi_2(a)\theta_{22}^{(0)}(b) - a\theta_{22}^{(0)}(b)$ for any $a, b \in C_{A^1}(z).$
- (4) $\Xi_1\Phi_1 = \theta_{22}^{(0)}\Phi_1, \Phi_1\Xi_1 = \Phi_1, \Phi_1\Xi_2 = \Phi_1(\sqrt{-1}\theta_{12}^{(0)}), \Xi_2\Phi_1 = 0.$
- (5) $\Xi_1\Phi_2 = 0, \Phi_2\Xi_1 = \Phi_2(\sqrt{-1}\theta_{12}^{(0)}), \Phi_2\Xi_2 = \Phi_2, \Xi_2\Phi_2 = -\theta_{22}^{(0)}\Phi_2.$
- (6) For any $a, b \in C_{A^1}(z),$

$$\begin{aligned}\Phi_1(\Xi_1(a)b) &= \Phi_1(a)\Phi_1(b), \quad \Phi_1(a\Xi_1(b)) = \Phi_1(a)\Phi_1(\Xi_1(b)), \\ \Phi_1(\Phi_2(a)\Xi_2(b)) &= \Xi_2(a)\Phi_2(b).\end{aligned}$$

- (7) $\Phi_2(\Xi_2(a)b) = \Phi_2(a)\Phi_1(b), \Phi_2(\Phi_1(a)\Xi_2(b)) = \Xi_1(a)\Phi_2(b)$ for any $a, b \in C_{A^1}(z).$
- (8) $(\theta_{11}^{(1)} - \sqrt{-1}\theta_{21}^{(1)})\Phi_1 = (\theta_{22}^{(1)} + \sqrt{-1}\theta_{12}^{(1)})\Phi_1 = \Xi_1.$
- (9) $(\theta_{11}^{(1)} + \sqrt{-1}\theta_{21}^{(1)})\Phi_2 = (\sqrt{-1}\theta_{12}^{(1)} - \theta_{22}^{(1)})\Phi_2 = \Xi_2.$

Proof. All follow from Corollary 5.2 and Lemma 5.3. ■

By Lemma 5.6 (1), we write S for the $\mathbb{Z}_2$ -graded eigenspace for the eigenvalue 1 of Ξ_1 and M for the $\mathbb{Z}_2$ -graded eigenspace for the eigenvalue 1 of Ξ_2 .

Lemma 5.7. *The $\mathbb{Z}_2$ -graded spaces S and M have the following properties:*

- (1) *The $\mathbb{Z}_2$ -graded eigenspace S is a $\mathbb{Z}_2$ -graded subalgebra of $C_{A^1}(z)$ and $S \cong \Phi_1(S)$ as $\mathbb{Z}_2$ -graded algebras.*
- (2) *The $\mathbb{Z}_2$ -graded eigenspace M is a $\mathbb{Z}_2$ -graded S -bimodule with the S -action defined by*

$$s \cdot m \cdot s' = \Phi_1(s)ms'$$

for any $s, s' \in S$ and $m \in M$.

- (3) *For any $m, m' \in M$, $\Phi_2(m)m' \in S$.*
- (4) *The map*

$$\begin{array}{ccc} \psi : M(1) \otimes_S M(1) & \rightarrow & S \\ m \otimes m' & \mapsto & \Phi_2(m)m' \end{array}$$

is a $\mathbb{Z}_2$ -graded S -bimodule homomorphism satisfying $m\psi(m' \otimes m'') = \psi(m \otimes m') \cdot m''$ for any homogeneous elements $m, m', m'' \in M$.

Proof. (1) Clearly, $\Xi_1(1) = 1 \in S$. By Lemma 5.6 (2), for any $s, s' \in S$,

$$\Xi_1(ss') = s\Xi_1(s') + \Xi_1(s)\theta_{22}^{(0)}(s') - s\theta_{22}^{(0)}(s') = ss'.$$

Hence, S is a $\mathbb{Z}_2$ -graded subalgebra of $C_{A^1}(z)$.

By Lemma 5.6 (6), the restriction map $\Phi_{1|S} : S \rightarrow \Phi_1(S)$ is a surjective $\mathbb{Z}_2$ -graded algebra homomorphism. If $s \in \text{Ker } \Phi_{1|S}$, then $s = \Xi_1(s) = (\theta_{11}^{(1)} - \sqrt{-1}\theta_{21}^{(1)})\Phi_1(s) = 0$ by Lemma 5.6 (8). So $\Phi_{1|S}$ is an isomorphism.

(2) By Lemma 5.6 (3) and (4),

$$\begin{aligned} \Xi_2(\Phi_1(s)ms') &= \Phi_1(s)\Xi_2(ms') + \Xi_2(\Phi_1(s))\theta_{22}^{(0)}(ms') \\ &= \Phi_1(s)(m\Xi_1(s') + \Xi_2(m)\theta_{22}^{(0)}(s') - m\theta_{22}^{(0)}(s')) \\ &= \Phi_1(s)ms' \end{aligned}$$

for any $s, s' \in S$ and $m \in M$. It implies that M is a $(\Phi_1(S), S)$ -bimodule. By (1), the result follows.

(3) By Lemma 5.6 (2) and (5), for any $m, m' \in M$,

$$\Xi_1(\Phi_2(m)m') = \Phi_2(m)\Xi_2(m') + \Xi_1(\Phi_2(m))\theta_{22}^{(0)}(m') = \Phi_2(m)m'.$$

(4) For any $s \in S$ and $m, m' \in M(1)$,

$$\psi(ms \otimes m') = \Phi_2(\Xi_2(m)s)m' = \Phi_2(m)\Phi_1(s)m' = \Phi_2(m)(s \cdot m') = \psi(m \otimes s \cdot m')$$

by Lemma 5.6 (7). Combining with (3), ψ is well defined.

For any $s, s' \in S$ and $m, m' \in M(1)$,

$$\psi(s \cdot m \otimes m' \cdot s') = \Phi_2(\Phi_1(s)\Xi_2(m))m's' = \Xi_1(s)\Phi_2(m)m's' = s\psi(m \otimes m')s'$$

by Lemma 5.6 (7). ■

The last lemma allows us to construct a semi-trivial extension

$$\Lambda = S \ltimes_{\psi} M(1),$$

which is a $\mathbb{Z}_2 \times \mathbb{Z}_2$ -graded algebra and also a $\mathbb{Z}_2$ -graded algebra by forgetting the first $\mathbb{Z}_2$ -grading (see Remark 2.7 (3)). In general, Λ is not strongly $\mathbb{Z}_2$ -graded (see Example 5.9).

Theorem 5.8. *Let $B = A_{\{1,0\}}[y_1, y_2; \sigma]$ be a $\mathbb{Z}$ -graded double Ore extension of a noetherian Koszul Artin–Schelter regular algebra A with $\deg y_1 = \deg y_2 = 1$ and $z \in A_2$ be a regular central element of A . Suppose B is noetherian and $z + y_1^2 + y_2^2$ is a central element of B . Then there are equivalences of triangulated categories*

$$\begin{aligned} \underline{\text{mcm}}(B/(z + y_1^2 + y_2^2)) &\cong D^b(\text{gr}_{\mathbb{Z}_2} {}^{\ominus} M_2(C_{A^!}(z))) \cong D^b(\text{mod } {}^{\ominus} M_2(C_{A^!}(z))_0) \\ &\cong D^b(\text{gr}_{\mathbb{Z}_2} \Lambda). \end{aligned}$$

Proof. By Theorems 2.4 and 5.4, one obtains that there are equivalences of triangulated categories

$$\underline{\text{mcm}}(B/(z + y_1^2 + y_2^2)) \cong D^b(\text{gr}_{\mathbb{Z}_2} {}^{\ominus} M_2(C_{A^!}(z))) \cong D^b(\text{mod } {}^{\ominus} M_2(C_{A^!}(z))_0).$$

By Proposition 5.5, $\text{gr}_{\mathbb{Z}_2} {}^{\ominus} M_2(C_{A^!}(z))$ is equivalent to $\text{gr}_{\mathbb{Z}_2} e({}^{\ominus} M_2(C_{A^!}(z)))e$. By Table 1, one obtains

$$\begin{aligned} e \star \mathbb{I}_1^{(0)} x \star e &= 2^{-1}(\mathbb{I}_1^{(0)} \Xi_1(x) + \sqrt{-1} \mathbb{I}_2^{(0)} \Xi_1(x)), \\ e \star \mathbb{I}_2^{(0)} x \star e &= -2^{-1} \sqrt{-1}(\mathbb{I}_1^{(0)} \Xi_1(x) + \sqrt{-1} \mathbb{I}_2^{(0)} \Xi_1(x)), \\ e \star \mathbb{I}_1^{(1)} x \star e &= 2^{-1}(\mathbb{I}_1^{(1)} \Xi_2(x) - \sqrt{-1} \mathbb{I}_2^{(0)} \Xi_2(x)), \\ e \star \mathbb{I}_2^{(1)} x \star e &= 2^{-1} \sqrt{-1}(\mathbb{I}_1^{(1)} \Xi_2(x) - \sqrt{-1} \mathbb{I}_2^{(0)} \Xi_2(x)) \end{aligned}$$

for any $x \in C_{A^!}(z)$. It implies

$$e({}^{\ominus} M_2(C_{A^!}(z)))e = \left\{ \begin{pmatrix} \Xi_1(x) & \Xi_2(x') \\ 0 & 0 \end{pmatrix} \mid x, x' \in C_{A^!}(z) \right\} = \begin{pmatrix} S & M \\ 0 & 0 \end{pmatrix}.$$

Note that $e({}^{\ominus} M_2(C_{A^!}(z)))_0 e = \begin{pmatrix} S_0 & M_1 \\ 0 & 0 \end{pmatrix}$ and $e({}^{\ominus} M_2(C_{A^!}(z)))_1 e = \begin{pmatrix} S_1 & M_0 \\ 0 & 0 \end{pmatrix}$. By (5.3), we have $e({}^{\ominus} M_2(C_{A^!}(z)))e \cong \Lambda$ as $\mathbb{Z}_2$ -graded algebras.

Hence, there is an equivalence of triangulated categories

$$D^b(\text{gr}_{\mathbb{Z}_2} {}^{\ominus} M_2(C_{A^!}(z))) \cong D^b(\text{gr}_{\mathbb{Z}_2} \Lambda). \quad \blacksquare$$

Example 5.9. Let A be a noetherian Koszul Artin–Schelter regular algebra. Let σ_1 and σ_2 be graded automorphisms of A satisfying

$$\sigma_1^2 = \sigma_2^2 = \text{id}_A, \quad \sigma_1 \sigma_2 = \sigma_2 \sigma_1.$$

Naturally, σ_2 can be extended to be a graded automorphism of $A[y_1; \sigma_1]$ which preserves y_1 , still denoted by σ_2 . Then we have that the Ore extension $A[y_1; \sigma_1][y_2; \sigma_2]$ is isomorphic to the double Ore extension $B = A_{\{1,0\}}[y_1, y_2; \sigma]$, where

$$\sigma = \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix} : A \rightarrow M_2(A).$$

Let $z \in A_2$ be a regular central element of A such that $\sigma_1(z) = \sigma_2(z) = z$. By Lemma 5.1, $z + y_1^2 + y_2^2$ is a central element of B . In this case, the linear maps in the twisting system are

$$\theta^{(0)} = \begin{pmatrix} \text{id} & 0 \\ 0 & \sigma_2^! \sigma_1^! \end{pmatrix}, \quad \theta^{(1)} = \begin{pmatrix} \sigma_1^! & 0 \\ 0 & \sigma_2^! \end{pmatrix} \circ \xi_{-1},$$

and the linear transformations of $C_{A^!}(z)$ are

$$\begin{aligned} \Xi_1 &= 2^{-1}(\text{id} + \sigma_2^! \sigma_1^!), & \Xi_2 &= 2^{-1}(\text{id} - \sigma_2^! \sigma_1^!), \\ \Phi_1 &= 2^{-1}(\sigma_1^! + \sigma_2^!) \xi_{-1}, & \Phi_2 &= 2^{-1}(\sigma_1^! - \sigma_2^!) \xi_{-1}. \end{aligned}$$

So

$$S = \{x \in C_{A^!}(z) \mid \sigma_1^!(x) = \sigma_2^!(x)\}, \quad M = \{x \in C_{A^!}(z) \mid \sigma_1^!(x) = -\sigma_2^!(x)\},$$

and $\Lambda = S \ltimes_{\psi} M(1)$ with multiplication

$$(s, m) \odot (s', m') = (ss' + \sigma_1^! \xi_{-1}(m)m', \sigma_1^! \xi_{-1}(s)m' + ms')$$

for any homogeneous elements $s, s' \in S$ and $m, m' \in M$. By Theorem 5.8, there is an equivalence of triangulated categories

$$\underline{\text{mcm}}(B/(z + y_1^2 + y_2^2)) \cong D^b(\text{gr}_{\mathbb{Z}_2} \Lambda).$$

There are some special cases.

- (1) Suppose $\sigma_1 = \sigma_2 = \sigma$. Then $S = C_{A^!}(z)$, $M = 0$ and $\Lambda = C_{A^!}(z)$. Hence, there are equivalences of triangulated categories

$$\underline{\text{mcm}}(A[y_1; \sigma][y_2; \sigma]/(z + y_1^2 + y_2^2)) \cong D^b(\text{gr}_{\mathbb{Z}_2} C_{A^!}(z)) \cong \underline{\text{mcm}}(A/(z)).$$

It is exactly the noncommutative Knörrer's periodicity theorem for noncommutative quadric hypersurfaces (see [10, Theorem 1.3] or [2, Theorem 1.4]).

- (2) Suppose $\sigma_1 = \xi_{-1}$ and $\sigma_2 = \text{id}$. Then $S_0 = C_{A^!}(z)_0$, $S_1 = 0$, $M_0 = 0$, $M_1 = C_{A^!}(z)_1$. So $\mathbb{Z}_2$ -graded algebra Λ is concentrated in degree 0 with $\Lambda_0 = C_{A^!}(z)$. Hence, there is an equivalence of triangulated categories

$$\underline{\text{mcm}}(A[y_1; \xi_{-1}][y_2]/(z + y_1^2 + y_2^2)) \cong D^b(\text{mod } C_{A^!}(z))^{\times 2}.$$

It is equivalent to [10, Theorem 1.4] (up to Zhang twist) which provides a tool called two-point reduction for computing a class of noncommutative quadric hypersurfaces defined by (± 1) -skew polynomial algebras (see [10, Lemma 6.18]).

We will also present another approach to [10, Theorem 1.4] in Example 6.10.

Example 5.10. Let $A = k_{-1}[x_1, x_2]$ and the regular central element $z = x_1^2 + x_2^2$. Let $B = A_{\{1,0\}}[y_1, y_2; \sigma]$ be a $\mathbb{Z}$ -graded double Ore extension of A where $\sigma = (\sigma_{ij}) : A \rightarrow M_2(A)$ satisfies

$$\sigma(x_1) = h \begin{pmatrix} x_1 & x_2 \\ x_2 & -x_1 \end{pmatrix}, \quad \sigma(x_2) = h \begin{pmatrix} x_2 & x_1 \\ x_1 & -x_2 \end{pmatrix},$$

where $2h^2 = 1$. It is a special case of the class $\mathbb{Z}$ with $f = 1$ in [22]. By [22, Proposition 5.16], B is noetherian. It is not hard to obtain that σ satisfies Lemma 5.1, so $x_1^2 + x_2^2 + y_1^2 + y_2^2$ is a central element of B . In this case, the linear maps in the twisting system of $M_2(C_{A^!}(z))$ are

$$\theta^{(0)} = \begin{pmatrix} \text{id} & 0 \\ 0 & \xi_{-1} \end{pmatrix}, \quad \theta^{(1)} = \begin{pmatrix} \sigma_{11}^1 & \sigma_{12}^1 \\ \sigma_{21}^1 & \sigma_{22}^1 \end{pmatrix} \circ \xi_{-1},$$

and two linear transformations of $C_{A^!}(z)$ are

$$\Xi_1 = 2^{-1}(\text{id} + \xi_{-1}), \quad \Xi_2 = 2^{-1}(\text{id} - \xi_{-1}).$$

So the $\mathbb{Z}_2$ -graded algebra S equals $C_{A^!}(z)_0$ concentrated in degree 0 and the $\mathbb{Z}_2$ -graded S -module M equals $C_{A^!}(z)_1$ concentrated in degree 1. The other linear transformations are $\Phi_{1|S} = \text{id}_S$ and Φ_2 mapping

$$x_1^* \mapsto -2h(x_1^* - \sqrt{-1}x_2^*), \quad x_2^* \mapsto 2\sqrt{-1}h(x_1^* + \sqrt{-1}x_2^*).$$

Then the semi-trivial extension $\Lambda = S \ltimes_{\psi} M(1)$ is a $\mathbb{Z}_2 \times \mathbb{Z}_2$ -graded algebra where $\Lambda_{(0,0)} = C_{A^!}(z)_0$, $\Lambda_{(1,0)} = C_{A^!}(z)_1$ and $\Lambda_{(0,1)} = \Lambda_{(1,1)} = 0$ and with multiplication

$$(s, m) \odot (s', m') = (ss' + \Phi_2(m)m', sm' + ms')$$

for any $s, s' \in C_{A^!}(z)_0$, $m, m' \in C_{A^!}(z)_1$. Then Λ is a $\mathbb{Z}_2$ -graded algebra concentrated in degree 0 by forgetting the first grading and Λ_0 is also a $\mathbb{Z}_2$ -graded algebra. In fact, one can find a $\mathbb{Z}_2$ -graded algebra isomorphism between Λ_0 and $C_{A^!}(z)$. But we present another way to discuss here.

We view Λ_0 as a $\mathbb{Z}_2$ -graded algebra. It is not hard to know that $\Phi_{2|M_1}$ is a linear automorphism of M_1 by Lemma 5.6 (5). We define a left Zhang-twisting system $\nu = \{\nu_0, \nu_1\}$ of $\mathbb{Z}_2$ -graded algebra $C_{A^!}(z)$ by

$$\nu_0 = \text{id}, \quad \nu_1|_{C_{A^!}(z)_0} = \text{id}, \quad \nu_1|_{C_{A^!}(z)_1} = \Phi_{2|M_1}.$$

So $\Lambda_0 \cong {}^{\nu}C_{A^!}(z)$ as $\mathbb{Z}_2$ -graded algebras. By Theorem 5.8, there are equivalences of triangulated categories

$$\underline{\text{mcm}}(B/(z + y_1^2 + y_2^2)) \cong D^b(\text{gr}_{\mathbb{Z}_2} \Lambda) \cong D^b(\text{mod } \Lambda_0)^{\times 2} \cong D^b(\text{mod } {}^{\nu}C_{A^!}(z))^{\times 2}.$$

Moreover, $C_{A^!}(z) \cong \mathbb{k}\mathbb{Z}_2 \times \mathbb{k}\mathbb{Z}_2$ as $\mathbb{Z}_2$ -graded algebras, so the algebra ${}^{\nu}C_{A^!}(z)$ is $\mathbb{Z}_2$ -graded semisimple by [20, Corollary 4.4] and ${}^{\nu}C_{A^!}(z) \cong \mathbb{k}\mathbb{Z}_2 \times \mathbb{k}\mathbb{Z}_2 \cong C_{A^!}(z)$ since

${}^v C_{A^!}(z)$ is commutative by an easy computation. It implies that $B/(z + y_1^2 + y_2^2)$ is a noncommutative graded isolated singularity, and there is an equivalence of triangulated categories

$$\underline{\text{mcm}}(B/(z + y_1^2 + y_2^2)) \cong D^b(\text{mod } \mathbb{k})^{\times 8}.$$

6. A skew version of Knörrer's periodicity theorem with $p_{12} = -1$

Let $B = A_{\{-1, p_{11}\}}[y_1, y_2; \sigma]$ be a $\mathbb{Z}$ -graded double Ore extension of a noetherian Koszul Artin–Schelter regular algebra $A = T(V)/(R)$ with $\deg y_1 = \deg y_2 = 1$. In this case, p_{11} can be nonzero. We will always assume $p_{11} = 0$ except for the following special values of p_{11} .

Proposition 6.1. *Let $B = A_{\{-1, p_{11}\}}[y_1, y_2; \sigma]$ be a $\mathbb{Z}$ -graded double Ore extension of a noetherian Koszul Artin–Schelter regular algebra A and $z \in A_2$ be a regular central element of A . Suppose B is noetherian and $z + y_1^2 + y_2^2$ is a regular central element of B . If $p_{11} = \pm 2\sqrt{-1}$, then $B/(z + y_1^2 + y_2^2)$ is not a noncommutative graded isolated singularity.*

Proof. Write

$$\sigma' = \begin{pmatrix} \sigma_{11} + \frac{p_{11}}{2}\sigma_{12} & \sigma_{12} \\ \sigma_{21} + \frac{p_{11}}{2}\sigma_{22} - \frac{p_{11}}{2}\sigma_{11} - \frac{p_{11}^2}{4}\sigma_{12} & \sigma_{22} - \frac{p_{11}}{2}\sigma_{12} \end{pmatrix} : A \rightarrow M_2(A)$$

and

$$\varphi' = \begin{pmatrix} \varphi_{11} + \frac{p_{11}}{2}\varphi_{21} & \varphi_{12} + \frac{p_{11}}{2}\varphi_{22} - \frac{p_{11}}{2}\varphi_{11} - \frac{p_{11}^2}{4}\varphi_{21} \\ \varphi_{21} & \varphi_{22} - \frac{p_{11}}{2}\varphi_{21} \end{pmatrix} : A \rightarrow M_2(A),$$

where $\varphi = (\varphi_{ij})$ is the inverse of σ .

It is routine to check that σ' is an invertible algebra homomorphism with the inverse φ' and $A_{\{-1, 0\}}[y_1, y_2; \sigma']$ is a $\mathbb{Z}$ -graded double Ore extension by Lemma 4.2. Then the algebra homomorphism from B to $A_{\{-1, 0\}}[y_1, y_2; \sigma']$ mapping

$$y_1 \mapsto y_1, \quad y_2 \mapsto y_2 + \frac{p_{11}}{2}y_1, \quad a \mapsto a \quad \forall a \in A$$

is an isomorphism, and the image of $z + y_1^2 + y_2^2$ is $z + y_2^2$ since $p_{11} = \pm 2\sqrt{-1}$. So

$$B/(z + y_1^2 + y_2^2) \cong A_{\{-1, 0\}}[y_1, y_2; \sigma']/(z + y_2^2).$$

It is well known that $k_{-1}[y_1, y_2]/(y_2^2)$ is not a noncommutative graded isolated singularity, and so

$$\text{gldim}_{\mathbb{Z}_2} C_{k_{-1}[y_1, y_2]^!}(y_2^2) > 0$$

by Theorem 2.4 (2). Write $D = A_{\{-1,0\}}[y_1, y_2; \sigma']$. By Lemma 4.4 (4),

$$\text{gldim}_{\mathbb{Z}_2} C_{D^!}(z + y_2^2) \geq \text{gldim}_{\mathbb{Z}_2} C_{k_{-1}[y_1, y_2]^!}(y_2^2) > 0.$$

The result follows from Theorem 2.4 (2). $\blacksquare$

Lemma 6.2. *Let $B = A_{\{-1, p_{11}\}}[y_1, y_2; \sigma]$ be a $\mathbb{Z}$ -graded double Ore extension of A where $p_{11} \neq \pm 2\sqrt{-1}$. Suppose the element $y_1^2 + y_2^2$ is a central element in B , then there exists a $\mathbb{Z}$ -graded double Ore extension $A_{\{-1,0\}}[y_1, y_2; \sigma']$ such that*

$$B/(a + y_1^2 + y_2^2) \cong A_{\{-1,0\}}[y_1, y_2; \sigma']/(a + y_1^2 + y_2^2)$$

for any $a \in A$.

Proof. Write

$$\sigma' = \begin{pmatrix} \sigma_{11} + \frac{p_{11}}{2}\sigma_{12} & c\sigma_{12} \\ c^{-1}(\sigma_{21} + \frac{p_{11}}{2}\sigma_{22} - \frac{p_{11}}{2}\sigma_{11} - \frac{p_{11}^2}{4}\sigma_{12}) & \sigma_{22} - \frac{p_{11}}{2}\sigma_{12} \end{pmatrix} : A \rightarrow M_2(A)$$

for some $c \in \mathbb{k}$ satisfying $c^2 = 1 + \frac{p_{11}^2}{4}$. It is similar to the proof of Proposition 6.1 to obtain that $A_{\{-1,0\}}[y_1, y_2; \sigma']$ is a $\mathbb{Z}$ -graded double Ore extension. There is a $\mathbb{Z}$ -graded algebra isomorphism ρ from B to $A_{\{-1,0\}}[y_1, y_2; \sigma']$ mapping

$$y_1 \mapsto c^{-1}y_1, \quad y_2 \mapsto y_2 + \frac{p_{11}}{2}c^{-1}y_1, \quad x \mapsto x \quad \forall x \in A.$$

Since $p_{11} \neq \pm 2\sqrt{-1}$, then $\rho(a + y_1^2 + y_2^2) = a + y_1^2 + y_2^2$. $\blacksquare$

In the rest of this section, $B = A_{\{-1,0\}}[y_1, y_2; \sigma]$ be a $\mathbb{Z}$ -graded double Ore extension of a noetherian Koszul Artin–Schelter regular algebra $A = T(V)/(R)$ with $\deg y_1 = \deg y_2 = 1$, and $z \in A_2$ be a regular central element of A . There are results similar to Lemma 5.1 and Corollary 5.2.

Lemma 6.3. *The element $z + y_1^2 + y_2^2$ is a regular central element of B if and only if*

- (1) $\sigma_{11}^2 + \sigma_{21}^2 = \sigma_{12}^2 + \sigma_{22}^2 = \text{id}_A$;
- (2) $\sigma_{11}\sigma_{12} + \sigma_{21}\sigma_{22} = \sigma_{12}\sigma_{11} + \sigma_{22}\sigma_{21}$;
- (3) $\sigma(z) = \begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix}$.

By Lemma 6.3 (3), construction (2.4) and Lemmas 2.5 and 4.2, we have the following t -invertible algebra homomorphism:

$$\sigma^! = (\sigma_{ij}^!) : C_{A^!}(z) \rightarrow M_2(C_{A^!}(z)).$$

Corollary 6.4. *If $z + y_1^2 + y_2^2 \in B$ is a regular central element, then*

- (1) $(\sigma_{11}^!)^2 + (\sigma_{21}^!)^2 = (\sigma_{12}^!)^2 + (\sigma_{22}^!)^2 = \text{id}_{C_{A^!}(z)}$;
- (2) $\sigma_{12}^! \sigma_{11}^! + \sigma_{22}^! \sigma_{21}^! = \sigma_{11}^! \sigma_{12}^! + \sigma_{21}^! \sigma_{22}^!$;
- (3) $\sigma_{11}^! \sigma_{21}^! + \sigma_{21}^! \sigma_{11}^! = 0$;
- (4) $\sigma_{12}^! \sigma_{22}^! + \sigma_{22}^! \sigma_{12}^! = 0$;
- (5) $\sigma_{22}^! \sigma_{11}^! + \sigma_{12}^! \sigma_{21}^! = \sigma_{11}^! \sigma_{22}^! + \sigma_{21}^! \sigma_{12}^!$.

Now we introduce a linear map

$$\theta = (\theta_{ij}) = \begin{pmatrix} \text{id} & \sigma_{12}^! \sigma_{11}^! + \sigma_{22}^! \sigma_{21}^! \\ 0 & \sigma_{22}^! \sigma_{11}^! + \sigma_{12}^! \sigma_{21}^! \end{pmatrix} : C_{A^!}(z) \rightarrow M_2(C_{A^!}(z)), \quad (6.1)$$

where each θ_{ij} is a $\mathbb{Z}_2$ -graded map for any $i, j = 1, 2$. Using the fact that $\sigma^!$ is an algebra homomorphism and by Corollary 6.4, one obtains the following properties of θ .

Lemma 6.5. *Retain the notations above. Then*

- (1) θ is an algebra homomorphism;
- (2) $\theta_{12} \theta_{22} + \theta_{22} \theta_{12} = 0$;
- (3) $\theta_{22}^2 + \theta_{12}^2 = \text{id}_{C_{A^!}(z)}$.

For the $\mathbb{Z}_2$ -graded algebra $\mathbb{k} \times \mathbb{k}$, we choose an invertible $\mathbb{k}$ -basis $\varepsilon = \{\varepsilon_1 = (1, 1), \varepsilon_2 = (1, -1)\}$.

Theorem 6.6. *Let θ be the algebra homomorphism from $C_{A^!}(z)$ to $M_2(C_{A^!}(z))$ defined in (6.1). Then $\theta_\times = (\theta, \varepsilon)$ is a twisting system of $C_{A^!}(z) \times C_{A^!}(z)$.*

Proof. The linear map

$$\begin{pmatrix} \text{id} & 0 \\ -\varphi_{21}^! \varphi_{11}^! - \varphi_{22}^! \varphi_{12}^! & \varphi_{22}^! \varphi_{11}^! + \varphi_{21}^! \varphi_{12}^! \end{pmatrix} : C_{A^!}(z) \rightarrow M_2(C_{A^!}(z))$$

is the t -inverse of θ . Clearly, $\theta(1) = I_2 \in \text{GL}_2(\mathbb{k})$.

In this case, $l_{1;jj'} = \delta_{jj'}$ and $l_{2;jj'} = \delta_{jj'} + 1$ for any $j, j' = 1, 2$. It is straightforward but tedious to check that the following simplified conditions of Definition 3.13 hold:

$$\begin{aligned} \theta_{1j'}(\theta_{1j}(x)x') + \theta_{2j'}(\theta_{2j}(x)x') &= x\theta_{jj'}(x') + \theta_{12}(x)\theta_{uj'}(x'), \\ \theta_{1j'}(\theta_{2j}(x)x') + \theta_{2j'}(\theta_{1j}(x)x') &= \theta_{22}(x)\theta_{uj'}(x'), \end{aligned}$$

where $u \in \{1, 2\} - j$ for any $j, j' = 1, 2$ and $x, x' \in C_{A^!}(z)$. ■

Hence, we have a $\mathbb{Z}_2$ -graded algebra $\Gamma = {}^{\theta_\times}(C_{A^!}(z) \times C_{A^!}(z))$ by Corollary 3.14, and the multiplication is as follows:

$$\begin{aligned} \varepsilon_1 x \star \varepsilon_1 x' &= \varepsilon_1 x x', & \varepsilon_1 x \star \varepsilon_2 x' &= \varepsilon_1 \theta_{12}(x)x' + \varepsilon_2 \theta_{22}(x)x', \\ \varepsilon_2 x \star \varepsilon_1 x' &= \varepsilon_2 x x', & \varepsilon_2 x \star \varepsilon_2 x' &= \varepsilon_1 \theta_{22}(x)x' + \varepsilon_2 \theta_{12}(x)x' \end{aligned} \quad (6.2)$$

for any $x, x' \in C_{A^!}(z)$.

Lemma 6.7. *The following map*

$$\begin{aligned}\mu : \quad \Gamma &\rightarrow \Gamma \\ \varepsilon_1 a &\mapsto \varepsilon_1 \sigma_{11}^! \xi_{-1}(a) + \varepsilon_2 \sigma_{21}^! \xi_{-1}(a), \\ \varepsilon_2 a &\mapsto \varepsilon_1 \sigma_{21}^! \xi_{-1}(a) + \varepsilon_2 \sigma_{11}^! \xi_{-1}(a)\end{aligned}$$

is a $\mathbb{Z}_2$ -graded algebra isomorphism satisfying $\mu^2 = \text{id}$.

Proof. It is straightforward to check that μ is an algebra homomorphism by (6.2). For any $a \in C_{A^!}(z)$, one obtains that

$$\begin{aligned}\mu^2(\varepsilon_1 a) &= \varepsilon_1((\sigma_{11}^!)^2 + (\sigma_{21}^!)^2)(a) + \varepsilon_2(\sigma_{21}^! \sigma_{11}^! + \sigma_{11}^! \sigma_{21}^!)(a) = \varepsilon_1 a, \\ \mu^2(\varepsilon_2 a) &= \varepsilon_1(\sigma_{11}^! \sigma_{21}^! + \sigma_{21}^! \sigma_{11}^!)(a) + \varepsilon_2((\sigma_{21}^!)^2 + (\sigma_{11}^!)^2)(a) = \varepsilon_2 a.\end{aligned}$$

Hence, $\mu^2 = \text{id}$, and μ is an isomorphism. ■

By Example 2.8 (2), the following map

$$\begin{aligned}\psi : \quad {}_{\mu} \Gamma(1) \otimes_{\Gamma} {}_{\mu} \Gamma(1) &\rightarrow \Gamma \\ \varepsilon_i a \otimes \varepsilon_{i'} a' &\mapsto \mu(\varepsilon_i a) \varepsilon_{i'} a'\end{aligned}$$

is a $\mathbb{Z}_2$ -graded Γ -bimodule homomorphism, and we have a semi-trivial extension

$$\Gamma \ltimes_{\psi} ({}_{\mu} \Gamma(1)).$$

By Remark 2.7, it is $\mathbb{Z}_2 \times \mathbb{Z}_2$ -graded and also $\mathbb{Z}_2$ -graded by forgetting the first degree of $\mathbb{Z}_2 \times \mathbb{Z}_2$ -degrees. So, viewing this semi-trivial extension as a $\mathbb{Z}_2$ -graded algebra, the 0-th homogeneous space is also a $\mathbb{Z}_2$ -graded algebra.

Lemma 6.8. *Retain the notations above. Then*

- (1) $v = \{v_0 = \text{id}, v_1 = \mu\}$ is a left Zhang-twisting system of $\mathbb{Z}_2$ -graded algebra Γ ;
- (2) ${}^v \Gamma \cong (\Gamma \ltimes_{\psi} ({}_{\mu} \Gamma(1)))_0$ as $\mathbb{Z}_2$ -graded algebras.

Proof. It is straightforward to check that the results hold by the definitions of left Zhang-twisting systems and Zhang-twisted algebras, and the construction of the twisted direct product Γ , the semi-trivial extension $\Gamma \ltimes_{\psi} ({}_{\mu} \Gamma(1))$ and the map μ . ■

Theorem 6.9. *Let $B = A_{\{-1,0\}}[y_1, y_2; \sigma]$ be a $\mathbb{Z}$ -graded double Ore extension of a noetherian Koszul Artin–Schelter regular algebra A and $z \in A_2$ be a regular central element of A . Suppose B is noetherian and $z + y_1^2 + y_2^2$ is a central element of B . Then*

- (1) $C_{B^!}(z + y_1^2 + y_2^2) \cong \Gamma \ltimes_{\psi} ({}_{\mu} \Gamma(1))$ as $\mathbb{Z}_2$ -graded algebras;
- (2) *there are equivalences of triangulated categories*

$$\underline{\text{mcm}}(B/(z + y_1^2 + y_2^2)) \cong D^b(\text{gr}_{\mathbb{Z}_2} \Gamma \ltimes_{\psi} ({}_{\mu} \Gamma(1))) \cong D^b(\text{mod } {}^v \Gamma),$$

where $v = \{v_0 = \text{id}, v_1 = \mu\}$ is a left Zhang-twisting system of $\mathbb{Z}_2$ -graded algebra Γ .

Proof. (1) By Lemma 4.4 (1), $C_{B^!}(z + y_1^2 + y_2^2)$ is generated by V^* , y_1^* , y_2^* subject to the relations consisting of the relations of $C_{A^!}(z)$ and

$$(y_1^*)^2 - 1, (y_2^*)^2 - 1, y_1^* y_2^* - y_2^* y_1^*, v^* y_i^* + y_1^* \sigma_{1i}^*(v^*) + y_2^* \sigma_{2i}^*(v^*) \\ \forall v^* \in V^*, i = 1, 2.$$

Comparing with the multiplication list of $\Gamma \ltimes_{\psi} (\mu \Gamma(1))$ (see (6.2)), one obtains that the $\mathbb{Z}_2$ -graded algebra homomorphism from $C_{B^!}(z + y_1^2 + y_2^2)$ to $\Gamma \ltimes_{\psi} (\mu \Gamma(1))$ mapping

$$y_1^* \mapsto (0, \varepsilon_1), \quad y_2^* \mapsto (0, \varepsilon_2), \quad v^* \mapsto (\varepsilon_1 v^*, 0) \quad \forall v^* \in V^*$$

is an isomorphism. So $\Gamma \ltimes_{\psi} (\mu \Gamma(1))$ is strongly $\mathbb{Z}_2$ -graded by Theorem 2.4 (1).

(2) follows from Theorem 2.4 and Lemma 6.8. ■

Example 6.10. Let A be a noetherian Koszul Artin–Schelter regular algebra. Let σ_1 and σ_2 be graded automorphisms of A satisfying

$$\sigma_1^2 = \sigma_2^2 = \text{id}_A, \quad \sigma_1 \sigma_2 = \sigma_2 \sigma_1.$$

We extend σ_2 to be a graded automorphism $\widetilde{\sigma}_2$ of $A[y_1; \sigma_1]$ such that $\widetilde{\sigma}_2|_A = \sigma_2$ and $\widetilde{\sigma}_2(y_1) = -y_1$. Then we have that the Ore extension $A[y_1; \sigma_1][y_2; \widetilde{\sigma}_2]$ is isomorphic to the double Ore extension $B = A_{\{-1, 0\}}[y_1, y_2; \sigma]$, where

$$\sigma = \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix} : A \rightarrow M_2(A).$$

Let $z \in A_2$ be a regular central element of A such that $\sigma_1(z) = \sigma_2(z) = z$. By Lemma 6.3, $z + y_1^2 + y_2^2$ is a central element of B . In this case, the linear map θ in the twisting system $\theta_{\times}$ is

$$\theta = \begin{pmatrix} \text{id} & 0 \\ 0 & \sigma_2^! \sigma_1^! \end{pmatrix} : C_{A^!}(z) \rightarrow M_2(C_{A^!}(z)).$$

Suppose $\sigma_1 = \sigma_2 = \sigma$, then θ is just the diagonal map. So $\Gamma = {}^{\theta \times} (C_{A^!}(z) \times C_{A^!}(z)) = C_{A^!}(z) \times C_{A^!}(z)$, and one obtains that $\bar{v} = \{\bar{v}_0 = \text{id}, \bar{v}_1 = \sigma^! \xi_{-1}\}$ is a left Zhang-twisting system of $\mathbb{Z}_2$ -graded algebra $C_{A^!}(z)$ such that

$${}^v \Gamma \cong {}^{\bar{v}} C_{A^!}(z) \times {}^{\bar{v}} C_{A^!}(z)$$

as $\mathbb{Z}_2$ -graded algebras, where $v = \{v_0 = \text{id}, v_1 = \mu\}$ is a left Zhang-twisting system of $\mathbb{Z}_2$ -graded algebra Γ . By Theorem 6.9, there is an equivalence of triangulated categories

$$\underline{\text{mcm}}(B/(z + y_1^2 + y_2^2)) \cong D^b(\text{mod } {}^v C_{A^!}(z))^{\times 2}.$$

In particular, if $\sigma = \xi_{-1}$, then ${}^{\bar{v}} C_{A^!}(z) = C_{A^!}(z)$, and so

$$\underline{\text{mcm}}(B/(z + y_1^2 + y_2^2)) \cong D^b(\text{mod } C_{A^!}(z))^{\times 2}.$$

It is exactly [10, Theorem 1.4] (also see Example 5.9).

Example 6.11. Let $A = k_{-1}[x_1, x_2]$ and $z = x_1^2 + x_2^2$ be the central element of A . Let $B = A_{\{-1,0\}}[y_1, y_2; \sigma]$ be a $\mathbb{Z}$ -graded double Ore extension of A where $\sigma = (\sigma_{ij}) : A \rightarrow M_2(A)$ satisfies

$$\sigma(x_1) = h \begin{pmatrix} -x_1 + x_2 & x_1 + x_2 \\ x_1 + x_2 & x_1 - x_2 \end{pmatrix}, \quad \sigma(x_2) = h \begin{pmatrix} x_1 - x_2 & x_1 + x_2 \\ x_1 + x_2 & -x_1 + x_2 \end{pmatrix},$$

where $4h^2 = 1$. It is a special case of $\mathbb{T}$ in [22]. One obtains that B is noetherian by [22, Proposition 5.16] and σ satisfies Lemma 6.3. So $x_1^2 + x_2^2 + y_1^2 + y_2^2$ is a central element of B . In this case, the linear map θ in the twisting system $\theta_\times$ is

$$\theta = (\theta_{ij}) = \begin{pmatrix} \text{id} & 0 \\ 0 & \theta_{22} \end{pmatrix} : C_{A^!}(z) \rightarrow M_2(C_{A^!}(z)),$$

where $\theta_{22}(x_1^*) = x_2^*$, $\theta_{22}(x_2^*) = x_1^*$ and $\theta_{22}(x_1^*x_2^*) = x_1^*x_2^*$. Let $\Gamma = {}^{\theta_\times}(C_{A^!}(z) \times C_{A^!}(z))$ be the twisted direct product, μ be the $\mathbb{Z}_2$ -graded algebra isomorphism defined in Lemma 6.7 and ${}^v\Gamma$ be a Zhang-twisted algebra where $v = \{v_0 = \text{id}, v_1 = \mu\}$ is a left Zhang-twisting system of $\mathbb{Z}_2$ -graded algebra Γ .

Write T_1, T_2, T_3, T_4 and T_5 for right ${}^v\Gamma$ -modules generated by $\{(1 - x_1^*x_2^*, 0), (x_1^* - x_2^*, 0)\}, \{\sqrt{-2h}(1 + x_1^*x_2^*, 0) + (x_1^* + x_2^*, 0)\}, \{\sqrt{-2h}(1 + x_1^*x_2^*, 0) - (x_1^* + x_2^*, 0)\}, \{\sqrt{2h}(0, 1 + x_1^*x_2^*) + (0, x_1^* + x_2^*)\}$ and $\{\sqrt{2h}(0, 1 + x_1^*x_2^*) - (0, x_1^* + x_2^*)\}$, respectively. It is routine to obtain that T_1, T_2, T_3, T_4, T_5 are all simple and non-isomorphic to each other and ${}^v\Gamma \cong T_1^{\oplus 2} \oplus T_2 \oplus T_3 \oplus T_4 \oplus T_5$ as right ${}^v\Gamma$ -modules. So ${}^v\Gamma$ is semisimple, and

$${}^v\Gamma \cong M_2(\mathbb{k}) \times \mathbb{k}^{\times 4}.$$

By Theorem 6.9, there is an equivalence of triangulated categories

$$\underline{\text{mcm}}(B/(z + y_1^2 + y_2^2)) \cong D^b(\mathbb{k})^{\times 5}.$$

Finally, we use a special case of the class $\mathbb{R}$ constructed in [22] to show that noncommutative quadric hypersurfaces obtained by double Ore extension do not preserve the property of noncommutative graded isolated singularity. This behavior does not occur in the usual (second) double-branched covers and tensor products of noncommutative quadric hypersurfaces.

Proposition 6.12. Let $A = k_{-1}[x_1, x_2]$ and $B = A_{\{-1,0\}}[y_1, y_2; \sigma]$ be a $\mathbb{Z}$ -graded double Ore extension of A where $\sigma = (\sigma_{ij}) : A \rightarrow M_2(A)$ satisfies

$$\sigma(x_1) = \begin{pmatrix} x_1 + x_2 & x_1 \\ x_2 & 0 \end{pmatrix}, \quad \sigma(x_2) = \begin{pmatrix} 0 & x_1 \\ -x_2 & -x_1 + x_2 \end{pmatrix}.$$

Then $B/(x_1^2 + x_2^2 + y_1^2 + y_2^2)$ is not a noncommutative graded isolated singularity.

Proof. By [21, Proposition 0.5], B is noetherian. It is straightforward to check that σ satisfies Lemma 6.3, and so $x_1^2 + x_2^2 + y_1^2 + y_2^2$ is a central element of B . So the noncommutative quadric hypersurface $B/(x_1^2 + x_2^2 + y_1^2 + y_2^2)$ is well defined. Write $z = x_1^2 + x_2^2$.

Note that $C_{A^!}(z)$ is generated by x_1^*, x_2^* subject to relations $x_1^*x_2^* - x_2^*x_1^*, (x_1^*)^2 - 1, (x_2^*)^2 - 1$. Write $\varepsilon = \{\varepsilon_1 = (1, 1), \varepsilon_2 = (1, -1)\}$. The linear map θ in the twisting system $\theta_{\times} = (\theta, \varepsilon)$ of $C_{A^!}(z) \times C_{A^!}(z)$ defined in (6.1) is

$$\theta = (\theta_{ij}) = \begin{pmatrix} \text{id} & \theta_{12} \\ 0 & \theta_{22} \end{pmatrix} : C_{A^!}(z) \rightarrow M_2(C_{A^!}(z)),$$

where

$$\begin{aligned} \theta_{12}(x_1^*) &= x_1^* + x_2^*, & \theta_{12}(x_2^*) &= x_1^* - x_2^*, & \theta_{12}(x_1^*x_2^*) &= 2, \\ \theta_{22}(x_1^*) &= -x_2^*, & \theta_{22}(x_2^*) &= x_1^*, & \theta_{22}(x_1^*x_2^*) &= -x_1^*x_2^*. \end{aligned}$$

Let $\Gamma = {}^{\theta_{\times}}(C_{A^!}(z) \times C_{A^!}(z))$ be the twisted direct product, μ be the $\mathbb{Z}_2$ -graded algebra isomorphism defined in Lemma 6.7 and ${}^{\nu}\Gamma$ be a Zhang-twisted algebra of Γ where $\nu = \{\nu_0 = \text{id}, \nu_1 = \mu\}$ is a left Zhang-twisting system of $\mathbb{Z}_2$ -graded algebra Γ . We list partial multiplication of ${}^{\nu}\Gamma$:

$$\begin{aligned} (1, 0) \diamond (a, b) &= (a, 0), & (x_1^*, 0) \diamond (0, b) &= (0, 0), \\ (x_1^*x_2^*, 0) \diamond (1, 0) &= (1, 0), & (x_1^*x_2^*, 0) \diamond (x_1^*x_2^*, 0) &= (x_1^*x_2^*, 0), \\ (x_1^*x_2^*, 0) \diamond (x_1^*, 0) &= (x_1^*, 0), & (x_1^*x_2^*, 0) \diamond (x_2^*, 0) &= (x_2^*, 0), \\ (x_1^*x_2^*, 0) \diamond (0, 1) &= (x_1^*x_2^* - 1, 0), & (x_1^*x_2^*, 0) \diamond (0, x_1^*x_2^*) &= (1 - x_1^*x_2^*, 0), \\ (x_1^*x_2^*, 0) \diamond (0, x_1^*) &= (x_1^* - x_2^*, 0), & (x_1^*x_2^*, 0) \diamond (0, x_2^*) &= (x_2^* - x_1^*, 0), \\ (x_1^*, 0) \diamond (1, 0) &= (x_1^*, 0), & (x_1^*, 0) \diamond (x_1^*x_2^*, 0) &= (x_2^*, 0), \\ (x_1^*, 0) \diamond (x_1^*, 0) &= (x_1^*, 0), & (x_1^*, 0) \diamond (x_2^*, 0) &= (x_2^*, 0), \\ (x_2^*, 0) \diamond (1, 0) &= (x_1^*, 0), & (x_2^*, 0) \diamond (x_1^*x_2^*, 0) &= (x_2^*, 0), \\ (x_2^*, 0) \diamond (x_1^*, 0) &= (-1, 0), & (x_2^*, 0) \diamond (x_2^*, 0) &= (-x_1^*x_2^*, 0), \\ (x_2^*, 0) \diamond (0, 1) &= (x_2^* - x_1^*, 0), & (x_2^*, 0) \diamond (x_1^*x_2^*) &= (x_1^* - x_2^*, 0), \\ (x_2^*, 0) \diamond (0, x_1^*) &= (x_1^*x_2^* - 1, 0), & (x_2^*, 0) \diamond (0, x_2^*) &= (1 - x_1^*x_2^*, 0) \end{aligned}$$

for any $a, b \in C_{A^!}(z)$.

One obtains that the right ${}^{\nu}\Gamma$ -module generated by $(1 - x_1^*x_2^*, 0)$ is nilpotent. So ${}^{\nu}\Gamma$ is not semisimple and $B/(x_1^2 + x_2^2 + y_1^2 + y_2^2)$ is not a noncommutative graded isolated singularity by Lemma 6.8 and Theorems 2.4 and 6.9. ■

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References

- [1] A. Conner, E. Kirkman, W. F. Moore, and C. Walton, [Noncommutative Knörrer periodicity and noncommutative Kleinian singularities](#). *J. Algebra* **540** (2019), 234–273 Zbl [1490.16063](#) MR [4007194](#)
- [2] J. He and Y. Ye, [Clifford deformations of Koszul Frobenius algebras and noncommutative quadrics](#). *Algebra Colloq.* **31** (2024), no. 1, 63–82 Zbl [1562.16045](#) MR [4717568](#)
- [3] J.-W. He, X.-C. Ma, and Y. Ye, Generalized Knörrer’s periodicity theorem. 2021, [arXiv:2107.06438v2](#)
- [4] A. Higashitani and K. Ueyama, [Combinatorial study of stable categories of graded Cohen–Macaulay modules over skew quadric hypersurfaces](#). *Collect. Math.* **73** (2022), no. 1, 43–54 Zbl [1505.16021](#) MR [4358245](#)
- [5] H. Hu, [Classification of noncommutative conics associated to symmetric regular superpotentials](#). *J. Algebra Appl.* **22** (2023), no. 6, article no. 2350136, 21 pp. Zbl [1543.16040](#) MR [4590303](#)
- [6] H. Hu, M. Matsuno, and I. Mori, [Noncommutative conics in Calabi–Yau quantum projective planes](#). *J. Algebra* **620** (2023), no. 2, 194–224 Zbl [1545.16027](#) MR [4531545](#)
- [7] H. Knörrer, [Cohen–Macaulay modules on hypersurface singularities I](#). *Invent. Math.* **88** (1987), no. 1, 153–164 Zbl [0617.14033](#) MR [0877010](#)
- [8] J. C. McConnell and J. C. Robson, [Noncommutative Noetherian rings](#). Revised edn., Grad. Stud. Math. 30, American Mathematical Society, Providence, RI, 2001, 636 pp. Zbl [0980.16019](#) MR [1811901](#)
- [9] I. Mori and K. Ueyama, [Noncommutative matrix factorizations with an application to skew exterior algebras](#). *J. Algebra* **586** (2021), no. 3, 1053–1087 Zbl [1487.16008](#) MR [4296228](#)
- [10] I. Mori and K. Ueyama, [Noncommutative Knörrer’s periodicity theorem and noncommutative quadric hypersurfaces](#). *Algebra Number Theory* **16** (2022), no. 2, 467–504 Zbl [1528.16028](#) MR [4412580](#)
- [11] C. Năstăsescu and F. Van Oystaeyen, [Methods of graded rings](#). Lecture Notes in Math. 1836, Springer, Berlin, 2004, 306 pp. Zbl [1043.16017](#) MR [2046303](#)
- [12] I. Palmér, [The global homological dimension of semi-trivial extensions of rings](#). *Math. Scand.* **37** (1975), no. 2, 223–256 Zbl [0327.16018](#) MR [0404342](#)
- [13] S. B. Priddy, [Koszul resolutions](#). *Trans. Amer. Math. Soc.* **152** (1970), no. 1, 39–60 Zbl [0261.18016](#) MR [0265437](#)
- [14] Y. Shen, G.-S. Zhou, and D.-M. Lu, [Nakayama automorphisms of twisted tensor products](#). *J. Algebra* **504** (2018), 445–478 Zbl [1420.16006](#) MR [3784825](#)
- [15] S. P. Smith, Some finite-dimensional algebras related to elliptic curves. In *Representation theory of algebras and related topics (Mexico City, 1994)*, pp. 315–348, CMS Conf. Proc. 19, American Mathematical Society, Providence, RI, 1996 Zbl [0856.16009](#) MR [1388568](#)
- [16] S. P. Smith and M. Van den Bergh, [Noncommutative quadric surfaces](#). *J. Noncommut. Geom.* **7** (2013), no. 3, 817–856 Zbl [1290.14003](#) MR [3108697](#)

- [17] K. Ueyama, Graded maximal Cohen–Macaulay modules over noncommutative graded Gorenstein isolated singularities. *J. Algebra* **383** (2013), 85–103 Zbl 1314.16016 MR 3037969
- [18] K. Ueyama, On Knörrer periodicity for quadric hypersurfaces in skew projective spaces. *Canad. Math. Bull.* **62** (2019), no. 4, 896–911 Zbl 1490.16045 MR 4028496
- [19] K. Ueyama, Skew graded (A_∞) hypersurface singularities. *C. R. Math. Acad. Sci. Paris* **361** (2023), no. G2, 521–534 Zbl 1519.16008 MR 4548167
- [20] J. J. Zhang, Twisted graded algebras and equivalences of graded categories. *Proc. Lond. Math. Soc.* (3) **72** (1996), no. 2, 281–311 Zbl 0852.16005 MR 1367080
- [21] J. J. Zhang and J. Zhang, Double Ore extensions. *J. Pure Appl. Algebra* **212** (2008), no. 12, 2668–2690 Zbl 1157.16009 MR 2452318
- [22] J. J. Zhang and J. Zhang, Double extension regular algebras of type (14641). *J. Algebra* **322** (2009), no. 2, 373–409 Zbl 1181.16026 MR 2529094
- [23] C. Zhu, F. Van Oystaeyen, and Y. Zhang, Nakayama automorphisms of double Ore extensions of Koszul regular algebras. *Manuscripta Math.* **152** (2017), no. 3–4, 555–584 Zbl 1394.16032 MR 3608305

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Yang Liu

Department of Mathematics, Zhejiang Sci-Tech University, No. 928, Second Avenue, Xiasha Higher Education Zone, 310018 Hangzhou, P. R. China; 202120102045@mails.zstu.edu.cn, wxly120@gmail.com

Yuan Shen (corresponding author)

Department of Mathematics, Zhejiang Sci-Tech University, No. 928, Second Avenue, Xiasha Higher Education Zone, 310018 Hangzhou, P. R. China; yuanshen@zstu.edu.cn

Xin Wang

School of Science, Shandong Jianzhu University, No. 1000 Fengming Road, Lingang Development Zone, 250101 Jinan, P. R. China; wangxin19@sdjzu.edu.cn