

The quantum Perron–Frobenius space

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Abstract. We introduce the concept of $\mathfrak{F}$ -positive elements in planar algebras and establish a Perron–Frobenius theorem for such elements. We investigate the existence and uniqueness of the Perron–Frobenius eigenspace. When the eigenspace is not one dimensional, we characterize its multiplicative structure. Furthermore, we interpret the Perron–Frobenius eigenspace as the space of logical qubits for mixed states in quantum information theory. We also explore the connection between these mathematical results and quantum error correction, particularly in relation to the Knill–Laflamme theorem.

1. Introduction

The Perron–Frobenius (PF) theorem [11–13, 32], proved in the early 1900s, is a fundamental result for positive matrices, with applications in many branches of mathematics, including probability theory [40], dynamical systems [6], graph theory [5], etc., as well as in science and engineering [27, 36]: power control of wireless networks, commodity pricing models, and the ranking of web pages. The PF theorem also leads to analyticity in some statistical mechanical systems, showing uniqueness for ground states and the absence of phase transitions [38].

Krein and Rutman generalized the PF theorem to positive, compact operators on ordered Banach spaces [26]. Glimm and Jaffe established a PF theorem for certain positive, non-compact operators and used this to prove uniqueness of the ground state for quantum field theory Hamiltonians [14]. Evans and Høegh-Krohn established a PF theorem for positive maps and studied multiplicative properties of the spectrum of positive (not necessarily completely positive) maps [10]. In quantum information, a quantum channel is a completely positive, trace-preserving (CPTP) linear map. The PF theorem for quantum channels is called the quantum PF theorem.

In this paper, we interpret the quantum PF theorem for completely positive maps in terms of tensor networks, a diagrammatic language in quantum information; this provides unity between the classic PF theorem and a quantum version. Furthermore, we establish the PF theorem on quantum symmetries in terms of Jones’ subfactor planar

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algebras [20–22]. The tensor network case becomes a special case of spin-model planar algebras.

We call an element $\mathfrak{F}$ -positive if its Fourier transform is a positive operator. This $\mathfrak{F}$ -positivity is a generalization of the complete positivity of a quantum channel in the tensor network approach to quantum information theory. We elaborate on the relation of this concept to the Perron–Frobenius theorem for planar algebras. Moreover, we give several, easy-to-check, sufficient conditions for irreducibility of an $\mathfrak{F}$ -positive element. We obtain a Collatz–Wielandt formula for planar algebras. We provide a self-contained proof of the PF theorem for (not necessarily compact) positive linear maps on an ordered Banach space.

When the PF eigenvector is not unique, we study its PF eigenspace $\mathcal{E}$. Inspired by quantum error correction, we consider an $\mathfrak{F}$ -positive element as a quantum channel, arising from the composition of an error channel, a recovery map, and an encoding map. We apply the PF eigenspace to encode logical qubits which are preserved by the quantum channel, up to a scalar. The size of logical qubits depends on the multiplicative structure of density matrices, so it is insufficient to simply count the dimension of the PF eigenspace. This is our major motivation to study the multiplicative structure of the PF eigenspace, corresponding to the matrix product of density matrices. The multiplicative structure on the PF eigenspace that we study is different from the earlier approach of Choi and Effros [8].

Density matrices of logical qubits are $*$ -invariant. We show that the maximal $*$ -invariant subspace of the PF eigenspace is decomposed as the multiplication of a maximally supported PF eigenvector ζ with a C^* -algebra $\mathcal{C}$. In quantum information, we could regard ζ as a mixed state of ancillary qubits and a subalgebra $\mathcal{D} \cong M_{2^k}(\mathbb{C})$ as k logical qubits. This PF theoretical approach to quantum error corrections suggests us to encode logical information by mixed states.

The paper is organized as follows. In Section 2, we recall the classic PF theorem and quantum PF theorem in the tensor network language. In Section 3, we provide a framework to study quantum channels on subfactors and review Fourier transforms on subfactors. Moreover, we also recall basic notions of Jones’ planar algebras, leading to our study in Section 4 of PF theory in the planar algebra context. In Section 5, we give a self-contained proof of the approximate existence and uniqueness results for PF eigenvectors of (non-compact) positive operators on infinite-dimensional, ordered Banach spaces. In Section 6, we characterize the multiplicative structures of the PF eigenspaces of $\mathfrak{F}$ -positive elements.

In Section 7, we relate our results to quantum information, and in particular to quantum error correction in a mixed state. In Theorem 7.1, we state the quantum error correction for mixed states in the PF eigenspace. Moreover, we contrast our approach to the widely known Knill–Laflamme theory. In our approach, we choose the recovery map as the pseudoinverse of the error map similar to the ones in the Knill–Laflamme theory. The resulting quantum error correcting codes (QECCs) of mixed states have a close connection to those codes of pure states in Knill–Laflamme theory. It would be interesting to explore other types of recovery maps and to study the PF eigenspaces to encode logical information.

2. Tensor networks and quantum channels

In this section, we recall the classic PF theorem for positive matrices and interpret the quantum PF theorem for completely positive maps in tensor networks.

2.1. Tensor networks and classic PF theorem

In quantum information, tensor networks make it convenient to compute vectors, matrices, and high-rank tensors. Let us recall some tensor network notations and operations. A complex vector $v \in \mathbb{C}^n$ is a 1-tensor with label j , which means the j th component can be represented pictorially as

$$v_j = \boxed{v} \overset{j}{\text{---}}. \quad (2.1)$$

A matrix $A \in M_n(\mathbb{C})$ is a 2-tensor with two labels i and j , which means the i th row and j th column are represented pictorially as

$$A_{ij} = \overset{j}{\text{---}} \boxed{A} \overset{i}{\text{---}}. \quad (2.2)$$

A matrix $A \in M_n(\mathbb{C})$ acting on a vector v is the vertical composition:

$$(Av)_i = \sum_j A_{ij} v_j = \boxed{v} \text{---} \boxed{A} \overset{i}{\text{---}}. \quad (2.3)$$

Similarly, the multiplication for two matrices $A, B \in M_n(\mathbb{C})$ is the picture:

$$(AB)_{ij} = \sum_k A_{ik} B_{kj} = \overset{j}{\text{---}} \boxed{B} \text{---} \boxed{A} \overset{i}{\text{---}}. \quad (2.4)$$

For the trace of A , one can connect the output to the input as

$$\text{Tr}(A) = \sum_i A_{ii} = \left(\boxed{A} \right). \quad (2.5)$$

We can easily see the cyclic property, which proves $\text{Tr}(AB) = \text{Tr}(BA)$, namely

$$\text{Tr}(AB) = \left(\boxed{B} \text{---} \boxed{A} \right) = \left(\begin{array}{c} \boxed{A} \\ \boxed{B} \end{array} \right) = \left(\begin{array}{c} \boxed{B^T} \\ \boxed{A} \end{array} \right) = \left(\boxed{A} \text{---} \boxed{B} \right) = \text{Tr}(BA). \quad (2.6)$$

The equality of the first and second closed diagrams is because they are orientation-preserving differentially homeomorphic. Since B_{ij} is a complex number, the flip of B_{ij} is still B_{ij} . Hence

$$g_{ji} = \overset{i}{\text{---}} \boxed{g} \overset{j}{\text{---}} = \overset{j}{\text{---}} \boxed{B} \overset{i}{\text{---}} = B_{ij},$$

which implies that the flip of B becomes the transpose of B . We refer the readers to, for example, [4] for more details of tensor networks.

Recall that a matrix $A \in M_n(\mathbb{C})$ is called pointwise positive if $A_{ij} \geq 0$ and is called strictly pointwise positive if $A_{ij} > 0$. The definition of pointwise positive is basis dependent. A positive matrix means it is pointwise positive, not necessarily positive semidefinite. A positive matrix A is irreducible if $(I + A)^{n-1}$ is strictly pointwise positive, where positive matrix here means pointwise positive, not positive semidefinite. Now we restate the classic PF theorem in tensor networks.

Theorem 2.1 (The classic PF theorem [11–13, 32]). *Suppose $A \in M_n(\mathbb{C})$ is a pointwise-positive matrix, that is, $A_{ij} \geq 0$. Then there exists a nonzero positive vector v , that is, $v_j \geq 0$, such that $Av = r(A)v$, where $r(A)$ is the spectral radius of A . If A is irreducible, then there exists a unique (up to a scalar) strictly positive vector v , that is, $v_j > 0$, satisfying equation (2.7) and $r(A)$ is a simple eigenvalue.*

The equation $Av = r(A)v$ can be depicted as

$$\boxed{v} - \boxed{A} - = r(A) \boxed{v} - \quad (2.7)$$

in tensor network with 2-tensors. In Section 2.2, we interpret the quantum PF theorem in tensor network with 4-tensors, and the classic PF theorem can be included as an example. Let w be the normalized left PF eigenvector of A :

$$- \boxed{A} - \boxed{w} = r(A) - \boxed{w} . \quad (2.8)$$

Let v be the normalized right PF eigenvector in Theorem 2.1. Define

$$P_{ij} = \overset{j}{-} \boxed{P} \overset{i}{-} = \overset{j}{-} \boxed{w} \boxed{v} \overset{i}{-} = v_i w_j. \quad (2.9)$$

Then P is the PF projection, and we have

$$- \boxed{P} - \boxed{A} - = - \boxed{A} - \boxed{P} - = r(A) - \boxed{P} - . \quad (2.10)$$

2.2. The quantum PF theorem: Completely positive maps

Let $\text{Hom}(M_n(\mathbb{C}), M_n(\mathbb{C}))$ be the set of all linear maps on $M_n(\mathbb{C})$, and let $\Phi \in \text{Hom}(M_n(\mathbb{C}), M_n(\mathbb{C}))$. Graphically, we represent Φ as the following 4-tensor:

$$\begin{array}{c} \text{---} \\ \text{---} \end{array} \boxed{\Phi} \begin{array}{c} \text{---} \\ \text{---} \end{array}, \quad (2.11)$$

and the action of Φ on $D \in M_n(\mathbb{C})$ as

$$\Phi(D) = \boxed{D} \begin{array}{c} \text{---} \\ \text{---} \end{array} \boxed{\Phi} \begin{array}{c} \text{---} \\ \text{---} \end{array}. \quad (2.12)$$

Up to a system of matrix basis $\{E_{ij}\}_{i,j=1}^n$, we have

$$\begin{array}{c} \text{---} \quad \text{---} \end{array} \boxed{\Phi} \begin{array}{c} \text{---} \quad \text{---} \end{array} = \sum_{i,j=1}^n \begin{array}{c} \text{---} \quad \text{---} \end{array} \boxed{E_{ij}} \boxed{\Phi(E_{ij})} \begin{array}{c} \text{---} \quad \text{---} \end{array} . \quad (2.13)$$

The Fourier transform $\mathcal{F}: \text{Hom}(M_n(\mathbb{C}), M_n(\mathbb{C})) \rightarrow M_n(\mathbb{C}) \otimes M_n(\mathbb{C})$ is naturally represented as

$$\begin{array}{c} \text{---} \quad \text{---} \end{array} \boxed{\Phi} \begin{array}{c} \text{---} \quad \text{---} \end{array} \xrightarrow{\mathcal{F}} \begin{array}{c} \text{---} \quad \text{---} \end{array} \boxed{\Phi} \begin{array}{c} \text{---} \quad \text{---} \end{array} . \quad (2.14)$$

We denote the Fourier transform as

$$\begin{array}{c} \text{---} \quad \text{---} \end{array} \boxed{\hat{\Phi}} \begin{array}{c} \text{---} \quad \text{---} \end{array} = \begin{array}{c} \text{---} \quad \text{---} \end{array} \boxed{\Phi} \begin{array}{c} \text{---} \quad \text{---} \end{array} = \sum_{i,j=1}^n \begin{array}{c} \text{---} \quad \text{---} \end{array} \boxed{E_{ij}} \boxed{\Phi(E_{ij})} \begin{array}{c} \text{---} \quad \text{---} \end{array} . \quad (2.15)$$

We reinterpret $\Phi(D)$ as a convolution $D * \hat{\Phi}$ in the Fourier dual:

$$\begin{array}{c} \text{---} \end{array} \boxed{D} \begin{array}{c} \text{---} \end{array} \boxed{\hat{\Phi}} \begin{array}{c} \text{---} \end{array} . \quad (2.16)$$

Using the generalized Choi–Kraus representation (see, e.g., [37, Corollary 3.12]), there exists a finite set of matrices $(V_i)_{i=1}^m$ and $(U_i)_{i=1}^m$, $1 \leq m \leq n^2$, such that

$$\Phi(D) = \sum_{i=1}^m V_i D U_i = \sum_{i=1}^m \begin{array}{c} U_i \\ | \\ D \\ | \\ V_i \end{array} = \sum_{i=1}^m \begin{array}{c} \text{---} \\ | \\ D \\ | \\ \text{---} \end{array} \begin{array}{c} U_i \\ | \\ V_i \end{array}. \quad (2.17)$$

Note that

$$\begin{aligned} \hat{\Phi} &\in \text{span} \left\{ \begin{array}{c} \text{---} \\ | \\ U \\ | \\ \text{---} \end{array} : U, V \in M_n(\mathbb{C}) \right\} \\ &= \text{span} \left\{ \begin{array}{c} \text{---} \\ | \\ U \\ | \\ \text{---} \end{array} : U, V \in M_n(\mathbb{C}) \right\}. \end{aligned} \quad (2.18)$$

A quantum channel Φ on $M_n(\mathbb{C})$ is a CPTP linear map. (Sometimes one allows trace non-increasing maps.) Choi’s theorem [7, Theorem 1] assures that there exist m operators $F_j \in M_n(\mathbb{C})$ (called Kraus operators in quantum information), $1 \leq m \leq n^2$, such that

$$\Phi(D) = \sum_{j=1}^m F_j D F_j^* \quad \forall D \in M_n(\mathbb{C}). \quad (2.19)$$

One can represent Φ pictorially as a tensor network, namely

$$\Phi(D) = \sum_{j=1}^m \begin{array}{c} \boxed{F_j^*} \\ | \\ \boxed{D} \\ | \\ \boxed{F_j} \end{array} = \sum_{j=1}^m \begin{array}{c} \boxed{D} \end{array} \quad (2.20)$$

The Fourier transform $\hat{\Phi}$ then is the following 4-tensor:

$$\hat{\Phi} = \sum_{i=1}^m \begin{array}{c} \boxed{F_i^*} \\ | \\ \boxed{\hat{\Phi}} \\ | \\ \boxed{F_i} \end{array}, \quad (2.21)$$

which is a positive operator in $M_n(\mathbb{C}) \otimes M_n(\mathbb{C})$. This gives a graphical interpretation of Choi's theorem [7, Theorem 2]: A linear map $\Phi \in \text{Hom}(M_n(\mathbb{C}), M_n(\mathbb{C}))$ is completely positive if and only if its Fourier transform $\hat{\Phi}$ is positive semidefinite.

Remark 2.2. Let Φ be a completely positive map on $M_n(\mathbb{C})$. By the spectral decomposition of $\hat{\Phi}$, we could choose Kraus operators $\{F_j\}_{j=1}^m$ in $M_n(\mathbb{C})$ such that $\text{Tr}(F_i^* F_j) = 0$, $\forall i \neq j$, where m is the rank of $\hat{\Phi}$.

A quantum channel Φ is called irreducible if $\lambda p - \Phi(p)$ is positive semidefinite for some projection $p \in M_n(\mathbb{C})$ and $\lambda > 0$ implies $p = 0$ or I . This definition has several equivalent descriptions that are discussed in Theorem 4.5.

Theorem 2.3 (The PF theorem for quantum channel [10]). *Let Φ be a quantum channel on $M_n(\mathbb{C})$. Then there exists a nonzero positive-semidefinite matrix D such that*

$$\begin{array}{c} \boxed{D} \end{array} \begin{array}{c} \boxed{\hat{\Phi}} \end{array} = r \begin{array}{c} \boxed{D} \end{array}, \quad (2.22)$$

where r is the spectral radius of Φ . Moreover, if Φ is irreducible, then D is positive definite and unique up to a scalar.

Remark 2.4. Let Φ be a quantum channel on $M_n(\mathbb{C})$. Considering the right action of $\widehat{\Phi}$, we have that there exists a nonzero positive-semidefinite matrix C such that

$$\begin{array}{c} \text{---} \\ | \\ \boxed{\widehat{\Phi}} \\ | \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ | \\ \boxed{C} \\ | \\ \text{---} \end{array} = r \begin{array}{c} \text{---} \\ | \\ \boxed{C} \\ | \\ \text{---} \end{array}, \quad (2.23)$$

where r is the spectral radius of Φ . We define the 4-tensor $\Phi_{C,D}$ as

$$\begin{array}{c} \text{---} \\ | \\ \boxed{\Phi_{C,D}} \\ | \\ \text{---} \end{array} = \begin{array}{c} \text{---} \\ | \\ \boxed{C} \\ | \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ | \\ \boxed{D} \\ | \\ \text{---} \end{array}. \quad (2.24)$$

The Fourier transform $\widehat{\Phi_{C,D}}$ is

$$\begin{array}{c} \text{---} \\ | \\ \boxed{\widehat{\Phi_{C,D}}} \\ | \\ \text{---} \end{array} = \begin{array}{c} \text{---} \\ | \\ \boxed{C} \\ | \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ | \\ \boxed{D} \\ | \\ \text{---} \end{array}, \quad (2.25)$$

which is positive semidefinite. Moreover, we have

$$\begin{array}{c} \text{---} \\ | \\ \boxed{\widehat{\Phi_{C,D}}} \\ | \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ | \\ \boxed{\widehat{\Phi}} \\ | \\ \text{---} \end{array} = \begin{array}{c} \text{---} \\ | \\ \boxed{\widehat{\Phi}} \\ | \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ | \\ \boxed{\widehat{\Phi_{C,D}}} \\ | \\ \text{---} \end{array} = r \begin{array}{c} \text{---} \\ | \\ \boxed{\widehat{\Phi_{C,D}}} \\ | \\ \text{---} \end{array}. \quad (2.26)$$

We now show Theorem 2.3 unifies the classic PF theorem. An n -dimensional vector $v \in \mathbb{C}^n$ can be regarded as a diagonal matrix D in $M_n(\mathbb{C})$; its graphical representation has the same label i :

$$v_i = D_{ii} = \begin{array}{c} | \\ i \\ \boxed{D} \\ | \\ i \end{array}. \quad (2.27)$$

A matrix A acting on the vector space could then be represented as a 4-tensor $\widehat{\Phi}_A$ acting on the diagonal matrix subalgebra of $M_n(\mathbb{C})$. Specifically,

$$Av = \sum_{i,j=1}^n A_{ij} \begin{array}{c} \text{---} \\ | \\ \boxed{E_{ji}} \\ | \\ \boxed{D} \\ | \\ \boxed{E_{ij}} \\ \text{---} \end{array} = \sum_{i,j=1}^n A_{ij} \begin{array}{c} \text{---} \\ | \\ \boxed{D} \\ | \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ | \\ \boxed{E_{ji}} \\ | \\ \boxed{E_{ij}} \\ \text{---} \end{array}. \quad (2.28)$$

We define $\widehat{\Phi}_A$ as the 4-tensor

$$\begin{array}{c} \text{---} \\ | \\ \boxed{\widehat{\Phi}_A} \\ | \\ \text{---} \end{array} = \sum_{i,j=1}^n A_{ij} \begin{array}{c} \text{---} \\ | \\ \boxed{D} \\ | \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ | \\ \boxed{E_{ji}} \\ | \\ \boxed{E_{ij}} \\ \text{---} \end{array}. \quad (2.29)$$

Note that $\widehat{\Phi}_A$ is a linear sum of n^2 orthogonal rank-one projections. So $\widehat{\Phi}_A$ is positive semidefinite if and only if A is a positive matrix, that is, $A_{ij} \geq 0$. Moreover, if A is irreducible, then $\widehat{\Phi}_A$ is also irreducible. Now take $\Phi = \widehat{\Phi}_A$ in Theorem 2.3, then we recover the classic PF theorem.

3. Quantum channels on subfactors

In this section, we provide a framework to study quantum channels on an operator algebra $\mathcal{M}$ preserving a subalgebra $\mathcal{N}$. The operator algebra $\mathcal{N}$ can be considered as conserved quantities. Such a quantum channel is an $\mathcal{N}$ - $\mathcal{N}$ -bimodule unital completely positive map (BUCP). When $\mathcal{N} \subseteq \mathcal{M}$ is an extremal subfactor with finite index, the Fourier transform is a linear map from the C^* -algebra of $\mathcal{N}$ - $\mathcal{N}$ bimodule maps on $\mathcal{M}$ into $\mathcal{M}' \cap \mathcal{M}_2$, where $\mathcal{M} \subseteq \mathcal{M}_1 \subseteq \mathcal{M}_2$ is the Jones basic construction. We will review these concepts in Section 3.1. Choi's matrix of a bounded $\mathcal{N}$ - $\mathcal{N}$ bimodule map Φ on $\mathcal{M}$ is defined as its Fourier transform $\widehat{\Phi}$. An $\mathcal{N}$ - $\mathcal{N}$ bimodule map Φ on $\mathcal{M}$ is completely positive if and only if its Fourier transform $\widehat{\Phi}$ is positive [19, Proposition 2.10], generalizing Choi's theorem [7, Theorem 2]: the equivalence between the complete positivity of a quantum channel and the positivity of its Choi matrix.

Remarkably, the Fourier transform can be implemented as a one-click rotation on 2-boxes in planar algebras, called the string Fourier transform. This provides a way to study

quantum channels as 2-boxes whose Fourier transforms are positive operators. We briefly call such 2-boxes $\mathfrak{F}$ -positive elements. The $\mathfrak{F}$ -positivity is an intrinsic property in planar algebras without referring to the underlying operator algebra $\mathcal{M}$. In this paper, we study $\mathfrak{F}$ -positivity intrinsically in planar algebras. We will explore its relation with the underlying operator algebras in the future.

3.1. Fourier transform

Let us consider the Fourier transform on subfactors. Let $\mathcal{M}$ be a II_1 factor with the normal faithful trace τ . A subfactor of type II_1 is an inclusion of type II_1 factors $\mathcal{N} \subseteq \mathcal{M}$, and its index reflects the relative size of these two factors. Jones [20] proved the remarkable result that the set of indices is

$$\left\{4 \cos^2\left(\frac{\pi}{n}\right) : n = 3, 4, 5, \dots\right\} \cup [4, +\infty).$$

Let G be a finite group. If we consider the group subfactor $\mathcal{N} \subseteq \mathcal{N} \rtimes G$, then the index is the order of G . Subfactors generalize group symmetries. We always assume that the subfactor is extremal, that is, the normalized traces on the commutant $\mathcal{N}'$ of $\mathcal{N}$ and $\mathcal{M}$ coincide on $\mathcal{N}' \cap \mathcal{M}$.

Suppose $\mathcal{N} \subseteq \mathcal{M}$ is a type II_1 subfactor with finite index δ^2 , $\delta > 0$. Let $L^2(\mathcal{M})$ be the Hilbert space obtained from the normal faithful trace τ on $\mathcal{M}$ by GNS construction. Let e_1 be the orthogonal projection from $L^2(\mathcal{M})$ onto the subspace $L^2(\mathcal{N})$, which is the Jones projection. Let $\mathcal{M}_1$ be the von Neumann algebra generated by $\mathcal{M}$ and e_1 in $B(L^2(\mathcal{M}))$, and let $E_{\mathcal{M}}$ be the conditional expectation from $\mathcal{M}_1$ onto $\mathcal{M}$. The inclusion $\mathcal{M} \subseteq \mathcal{M}_1$ is also a type II_1 subfactor with index δ^2 . Repeating the Jones basic construction [20], we have the Jones tower

$$\begin{array}{ccccccc} & & e_1 & & e_2 & & e_3 \\ \mathcal{N} & \subset & \mathcal{M} & \subset & \mathcal{M}_1 & \subset & \mathcal{M}_2 & \subset & \mathcal{M}_3 & \subset & \dots \end{array}$$

and the standard invariant

$$\begin{array}{ccccccc} \mathcal{N}' \cap \mathcal{N} & \subset & \mathcal{N}' \cap \mathcal{M} & \subset & \mathcal{N}' \cap \mathcal{M}_1 & \subset & \mathcal{N}' \cap \mathcal{M}_2 & \subset & \dots \\ & & \cup & & \cup & & \cup & & \\ & & \mathcal{M}' \cap \mathcal{M} & \subset & \mathcal{M}' \cap \mathcal{M}_1 & \subset & \mathcal{M}' \cap \mathcal{M}_2 & \subset & \dots \end{array}$$

Then $\mathcal{N}' \cap \mathcal{M}_{n-1}$ and $\mathcal{M}' \cap \mathcal{M}_n$ are finite-dimensional C^* -algebras with faithful traces, where $\mathcal{M}_{-1} = \mathcal{N}$ and $\mathcal{M}_0 = \mathcal{M}$. Moreover, $\dim_{\mathbb{C}}(\mathcal{N}' \cap \mathcal{M}_{n-1}) = \dim_{\mathbb{C}}(\mathcal{M}' \cap \mathcal{M}_n)$. In particular, when $n = 2$, there is a Fourier duality between $\mathcal{N}' \cap \mathcal{M}_1$ and $\mathcal{M}' \cap \mathcal{M}_2$. Define the following two maps:

$$\psi : \mathcal{N}' \cap \mathcal{M}_1 \rightarrow \mathcal{M}' \cap \mathcal{M}_2, \quad \psi(x) = \delta^3 E_{\mathcal{M}' \cap \mathcal{M}_2}(x e_2 e_1), \quad (3.1)$$

$$\phi : \mathcal{M}' \cap \mathcal{M}_2 \rightarrow \mathcal{N}' \cap \mathcal{M}_1, \quad \phi(x) = \delta^3 E_{\mathcal{M}_1}(x e_1 e_2), \quad (3.2)$$

where $E_{\mathcal{M}' \cap \mathcal{M}_2}$ is the conditional expectation from $\mathcal{N}' \cap \mathcal{M}_2$ onto $\mathcal{M}' \cap \mathcal{M}_2$ and $E_{\mathcal{M}_1}$ is the conditional expectation from $\mathcal{M}_2$ onto $\mathcal{M}_1$, respectively. Then ψ and ϕ are isometries (in the norm $\|\cdot\|_2$), inverse to each other; we refer to [3, Proposition 3.2] for a detailed proof. The maps ψ and ϕ are usually called the quantum Fourier transforms.

In [23], Jones introduced subfactor planar algebras to study standard invariants, which provide a pictorial tool to study subfactors. Let $\mathcal{P} = \{\mathcal{P}_{n,\pm}\}_{n \geq 0}$ be the subfactor planar algebra of the subfactor $\mathcal{N} \subseteq \mathcal{M}$, where $\mathcal{P}_{n,+} = \mathcal{N}' \cap \mathcal{M}_{n-1}$ and $\mathcal{P}_{n,-} = \mathcal{M}' \cap \mathcal{M}_n$ for any $n \geq 0$. Every element in $\mathcal{P}_{n,\pm}$ is called an n -box. Graphical representations can simplify algebraic expressions in subfactors. Let us take $\mathcal{P}_{2,+}$ as a typical example. One has the pictorial representations for element x in $\mathcal{P}_{2,+}$, the multiplication xy , the quantum Fourier transform $\mathfrak{F}(x)$, and the trace $\text{tr}(x)$ as follows:

Exchanging the gray and white regions, we obtain the pictorial representations for elements in $\mathcal{P}_{2,-}$. We will give a more detailed explanation of these graphs in Section 3.3. From equations (3.1) and (3.3), we have that for any $x \in \mathcal{N}' \cap \mathcal{M}_1$,

$$\psi(x) = \mathfrak{F}(x).$$

From equation (3.3), we see that the Fourier transform is a 90° rotation. Hence the period of $\mathfrak{F}$ is 4 which is not obvious from the algebraic expressions. Subfactor planar algebras help us understand the structures of subfactors well. Subfactor planar algebras are spherical and satisfy that $\dim_{\mathbb{C}}(\mathcal{P}_{0,\pm}) = 1$. We will consider a more general class of planar algebras in Section 3.3, namely C^* -planar algebras, which are not necessarily spherical or $\dim_{\mathbb{C}}(\mathcal{P}_{0,\pm}) = 1$, which can unify the tensor network representations of matrices and completely positive maps in Section 2.

3.2. Fourier multipliers and bimodule quantum channels

In quantum information, a linear map on a finite-dimensional von Neumann algebra is completely positive if and only if its Choi matrix is positive semidefinite. This correspondence can be generalized to infinite-dimensional quantum symmetries.

Let $\mathcal{N} \subseteq \mathcal{M}$ be an inclusion of type II_1 subfactor with index δ^2 , $\delta > 0$. A linear map $\Phi: \mathcal{M} \rightarrow \mathcal{M}$ is called an $\mathcal{N}$ - $\mathcal{N}$ bimodule map if $\Phi(y_1 x y_2) = y_1 \Phi(x) y_2$, $y_1, y_2 \in \mathcal{N}$, $x \in \mathcal{M}$. For any bounded bimodule map Φ , Φ commutes with the left and right actions of $\mathcal{N}$ on $L^2(\mathcal{M})$. Hence Φ corresponds to an element $y_\Phi \in \mathcal{N}' \cap \mathcal{M}_1$. Moreover,

$$\Phi(x) = \delta^2 E_{\mathcal{M}}(y_\Phi x e_1), \quad x \in \mathcal{M}$$

(see [19, Proposition 2.3]). The Fourier multiplier $\hat{\Phi} \in \mathcal{P}_{2,-}$ of Φ is defined to be the Fourier inverse of y_Φ

$$\hat{\Phi} = \text{[Diagram: A box labeled } y_\Phi \text{ with a vertical line passing through it, and a shaded region to the right]} . \quad (3.4)$$

It is proved in [19] that the bounded bimodule map Φ is completely positive if and only if $\hat{\Phi}$ is positive. We can recover the finite-dimensional case. For any $x \in \mathcal{P}_{1,+} = \mathcal{N}' \cap \mathcal{M}$, we have

$$\Phi(y) = \text{[Diagram: A box labeled } y \text{ followed by a box labeled } \hat{\Phi} \text{, with a vertical line passing through them, and a shaded region to the right]} . \quad (3.5)$$

If we restrict Φ on $\mathcal{P}_{1,+}$, then $\hat{\Phi}$ is Choi's matrix of Φ . Therefore, the action of a 2-box on a 1-box space can be extended to an $\mathcal{N}$ - $\mathcal{N}$ bimodule map on $\mathcal{M}$. We also study the Perron–Frobenius theorem for such maps in [19].

3.3. C^* -planar algebras

In this section, we recall planar algebras [22, 23], which contain the graphical computations in tensor networks in Section 2. Moreover, planar algebras are a more general picture language to study quantum symmetries such as quantum groups and subfactors. We emphasize that all the C^* -algebras considered in the paper are finite dimensional. We focus on the C^* -planar algebra $\mathcal{P} = \{\mathcal{P}_{n,\pm}\}_{n \geq 0}$. This algebra is graded, with each $\mathcal{P}_{n,\pm}$ being a finite-dimensional C^* -algebra consisting of shaded n -boxes with a trace $\text{tr}_{n,\pm}$. Let us introduce some planar tangles such as multiplication, convolution, the Fourier transform, etc. We are interested in 2-box space. An element $x \in \mathcal{P}_{2,+}$ is depicted as the following picture:

$$\text{[Diagram: A box labeled } x \text{ with a vertical line passing through it, and a shaded region to the right]} , \quad (3.6)$$

which is a 2-box with shading and the dollar sign \$ on the left side of the rectangle. The shading and the dollar sign represent the orientation of the box. If $x \in \mathcal{P}_{2,-}$, then x has the following graphical representation:

$$\text{[Diagram: A box labeled } x \text{ with a vertical line passing through it, and a shaded region to the right, with a dollar sign $ above the box]} . \quad (3.7)$$

We may omit the shading and the dollar sign when considering an element in $\mathcal{P}_{2,\pm}$ if there is no confusion. Multiplication of two elements $x, y \in \mathcal{P}_{2,+}$ is given by their vertical composition:

$$xy := \begin{array}{c} \text{---} \\ \boxed{y} \\ \text{---} \\ \boxed{x} \\ \text{---} \end{array} . \quad (3.8)$$

One obtains the unnormalized trace of x , by connecting the inputs with the outputs:

$$\text{tr}_{2,+}(x) := \text{tr}_{0,+} \left(\begin{array}{c} \text{---} \\ \boxed{x} \\ \text{---} \end{array} \right) . \quad (3.9)$$

Making the element x into a loop defines a map π from $\mathcal{P}_{2,+}$ into $\mathcal{P}_{0,+}$ such that $\pi(xy) = \pi(yx)$ by cyclic property, $x, y \in \mathcal{P}_{2,+}$. If a C^* -planar algebra is a subfactor

planar algebra, then $\dim_{\mathbb{C}}(\mathcal{P}_{0,\pm}) = 1$, and the trace is just $\begin{array}{c} \text{---} \\ \boxed{x} \\ \text{---} \end{array}$. It has the following property:

$$\text{tr}_{2,+}(xy) = \begin{array}{c} \text{---} \\ \boxed{y} \\ \text{---} \\ \boxed{x} \\ \text{---} \end{array} = \begin{array}{c} \text{---} \\ \boxed{x} \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ \boxed{y} \\ \text{---} \end{array} = \begin{array}{c} \text{---} \\ \boxed{x} \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ \boxed{\bar{y}} \\ \text{---} \end{array} = \begin{array}{c} \text{---} \\ \boxed{x} \\ \text{---} \\ \boxed{y} \\ \text{---} \end{array} = \text{tr}_{2,+}(yx), \quad (3.10)$$

where $\bar{y}$ denotes the 180° rotation of y . The equality of the first and second closed diagrams is because they are orientation-preserving differentially homeomorphic. By the graphical representations of 2-box in $\mathcal{P}_{2,\pm}$, the 90° rotation defines a map $\mathfrak{F}: \mathcal{P}_{2,+} \rightarrow \mathcal{P}_{2,-}$:

$$\mathfrak{F}(x) := \begin{array}{c} \text{---} \\ \boxed{x} \\ \text{---} \end{array} , \quad (3.11)$$

which is called the Fourier transform. The existence of the Fourier transform is a part of definition of C^* -planar algebras. It is clear that $\mathfrak{F}$ is a bijection. Similarly, we can also

define the Fourier transform from $\mathcal{P}_{2,-}$ into $\mathcal{P}_{2,+}$, still denoted by $\mathfrak{F}: \mathcal{P}_{2,-} \rightarrow \mathcal{P}_{2,+}$:

$$\mathfrak{F}(x) := \text{diagram of a box labeled } x \text{ with a cup and cap} . \quad (3.12)$$

From the graphical representation of $\mathfrak{F}$, we have $\mathfrak{F}^4(x) = x$. Hence the period of the Fourier transform is 4, which is the same as the classic Fourier transform. If $\mathcal{P}$ is a subfactor planar algebra, then $\mathfrak{F}$ is a unitary with respect to the L^2 -norm. For any $R \in \mathcal{P}_{2,+}$, we have

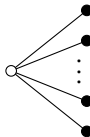
$$\begin{aligned} \|\mathfrak{F}(R)\|_2^2 &= \text{tr}_{2,+}(\mathfrak{F}(R)^* \mathfrak{F}(R)) = \text{diagram of } R \text{ and } \mathfrak{F}(R)^* \text{ in a box} = \text{diagram of } R \text{ and } R^* \text{ in a box} = \text{diagram of } R^* R \text{ in a box} \\ &= \text{tr}_{2,+}(R^* R) = \|R\|_2^2. \end{aligned}$$

The first and the second diagrams are equal because subfactor planar algebras are spherical. The second and the third diagrams are equal because they are orientation-preserving diffeomorphically. Hence $\mathfrak{F}$ is a unitary. The Fourier transform induces the convolution on $\mathcal{P}_{2,+}$:

$$x * y := \mathfrak{F}^{-1}(\mathfrak{F}(y)\mathfrak{F}(x)) = \text{diagram of } x \text{ and } y \text{ in a box} . \quad (3.13)$$

Notation 3.1. We use $\hat{x}$ to denote $\mathfrak{F}(x)$ in the subsequent sections for simplicity.

We next introduce a typical example of C^* -planar algebra. Let Γ be the bipartite graph with one white vertex and n black vertices, $n \in \mathbb{N}$, and there is exactly one edge connecting the white and black vertices:



A spin model is a bipartite graph planar algebra [21] associated to Γ , denoted by Spin . Then $\text{Spin}_{0,+} = \mathbb{C}$ and $\text{Spin}_{0,-}$ is spanned by black vertices, that is, $\text{Spin}_{0,-}$ is of

dimension n . Moreover,

$$\begin{aligned} \text{Spin}_{1,+} &= \text{span} \left\{ \begin{array}{|c|} \hline j \\ \hline \end{array} : j = 1, \dots, n \right\}, \\ \text{Spin}_{1,-} &= \text{span} \left\{ \begin{array}{|c|} \hline j \\ \hline \end{array} : j = 1, \dots, n \right\}, \\ \text{Spin}_{2,+} &= \text{span} \left\{ \begin{array}{c} \text{---} j \text{---} \\ \text{---} i \text{---} \end{array} : i, j = 1, \dots, n \right\}, \\ \text{Spin}_{2,-} &= \text{span} \left\{ \begin{array}{|c|} \hline j \\ \hline \end{array} \begin{array}{|c|} \hline i \\ \hline \end{array} : i, j = 1, \dots, n \right\}, \end{aligned}$$

and

$$\bigcirc = n^{1/2}, \quad \begin{array}{|c|} \hline j \\ \hline i \\ \hline \end{array} = \begin{cases} 1/n^{1/2} & i = j, \\ 0 & i \neq j. \end{cases}$$

We see that $\text{Spin}_{2,+}$ is the matrix algebra $M_n(\mathbb{C})$ with basis

$$E_{ij} = n^{1/2} \begin{array}{c} \text{---} j \text{---} \\ \text{---} i \text{---} \end{array}, \quad i, j = 1, \dots, n. \quad (3.14)$$

Dually, $\text{Spin}_{2,-}$ is the commutative C^* -algebra of functions on $n \times n$ points, denoted also by $M_n(\mathbb{C})$. We also see that $\text{Spin}_{1,+}$ and $\text{Spin}_{1,-}$ are n -dimensional row vector space and n -dimensional column vector space, respectively. The following is the correspondence between matrix units in spin model and those in tensor networks:

$$n^{1/2} \begin{array}{c} \text{---} j \text{---} \\ \text{---} i \text{---} \end{array} \longleftrightarrow \begin{array}{|c|} \hline j \\ \hline E_{ij} \\ \hline i \\ \hline \end{array}, \quad n^{1/2} \begin{array}{|c|} \hline j \\ \hline \end{array} \begin{array}{|c|} \hline i \\ \hline \end{array} \longleftrightarrow \begin{array}{c} j \text{---} \end{array} \begin{array}{|c|} \hline E_{ij} \\ \hline \end{array} \begin{array}{c} \text{---} i \end{array}. \quad (3.15)$$

We have that $\{\text{Spin}_{2n,+}\}_{n \geq 0}$ is an unshaded planar algebra, and a completely positive map could be realized by an element in $\text{Spin}_{4,+}$, which has a pictorial representation as equation (2.12).

We know that a matrix $A \in M_n(\mathbb{C})$ is positive if and only if the Fourier transform $\widehat{\Phi}_A$ is positive semidefinite. This means that we should interpret the matrix as pictures in $\text{Spin}_{2,-}$, where multiplication corresponds to horizontal composition and positivity is read vertically. Furthermore, a linear map $\Phi \in \text{Hom}(M_n(\mathbb{C}), M_n(\mathbb{C}))$ is completely positive if and only if the Fourier transform $\widehat{\Phi}$ is positive semidefinite. This fact inspires us to introduce the notion of $\mathfrak{F}$ -positivity for any 2-box $x \in \mathcal{P}_{2,\pm}$.

Suppose $\mathfrak{A}$ is a C^* -algebra. Recall that an element $x \in \mathfrak{A}$ is called *positive* if $x = yy^*$ for some $y \in \mathfrak{A}$ and is called *strictly positive* if x is positive and invertible, that is, there exists $\lambda > 0$ such that $x \geq \lambda$.

In C^* -planar algebras, every element $x \in \mathcal{P}_{2,\pm}$ induces a linear map Φ_x on $\mathcal{P}_{2,\pm}$, which is defined as follows:

$$\Phi_x(y) := y * \hat{x} = \begin{array}{c} \text{Diagram: Two vertical lines on the left and right. Between them are two boxes. The left box is labeled 'y' and the right box is labeled '\hat{x}'. A curved line connects the top of the 'y' box to the top of the '\hat{x}' box. Another curved line connects the bottom of the 'y' box to the bottom of the '\hat{x}' box. This represents the product y * \hat{x} in the planar algebra diagrammatic calculus.} \end{array}.$$

In abstract C^* -algebras, complete positivity is stronger than positivity for linear maps. However, in C^* -planar algebras, the complete positivity of Φ_x is equivalent to its positivity. Recall that the Jones projection

$$e_1 = \frac{1}{\delta} \begin{array}{c} \text{Diagram: Two vertical lines on the left and right. Between them are two arcs. The top arc connects the left line to the right line. The bottom arc connects the right line to the left line. This represents the Jones projection e_1 in the planar algebra diagrammatic calculus.} \end{array},$$

where δ is the value of the circle. If $\mathcal{P}$ is the subfactor planar algebra, then δ is the square root of the Jones index. By the Schur product theorem (see [29, Theorem 4.1]), the positivity of Φ_x is equivalent to the condition that $\Phi_x(e_1)$ is positive, where e_1 denotes the Jones projection. Notably, $\Phi_x(e_1) = \frac{\hat{x}}{\delta}$. Consequently, the positivity of Φ_x is synonymous with $\hat{x}$ being positive. We introduce the following definition.

Definition 3.2. Suppose $\mathcal{P}$ is a C^* -planar algebra. An element $x \in \mathcal{P}_{2,\pm}$ is called (strictly) $\mathfrak{F}$ -positive if $\hat{x}$ is a (strictly) positive element in $\mathcal{P}_{2,\mp}$.

Remark 3.3. Suppose $\mathcal{P}$ is a C^* -planar algebra. An element $x \in \mathcal{P}_{2,\pm}$ is (strictly) $\mathfrak{F}$ -positive if and only if $x = \hat{y}$ for some (strictly) positive element $y \in \mathcal{P}_{2,\mp}$.

We can interpret complete positivity of a bounded bimodule map in terms of $\mathfrak{F}$ -positivity.

Proposition 3.4. Suppose $\mathcal{N} \subseteq \mathcal{M}$ is a type II_1 subfactor with index δ^2 , $\delta > 0$. Let Φ be a bounded $\mathcal{N}$ - $\mathcal{N}$ bimodule map. Then Φ is completely positive if and only if y_Φ is $\mathfrak{F}$ -positive.

Proof. By [19, Proposition 2.10], Φ is completely positive if and only if $\hat{\Phi} = \mathfrak{F}^{-1}(y_\Phi)$ is positive. Note that $\mathfrak{F}^2$ is an anti- $*$ -isomorphism. Hence $\mathfrak{F}^{-1}(y_\Phi)$ is positive if and only if $\widehat{y_\Phi}$ is positive. Therefore, Φ is completely positive if and only if y_Φ is $\mathfrak{F}$ -positive. This completes the proof. $\blacksquare$

Remark 3.5. The positivity of an element in a C^* -algebra is different from the positivity of a matrix. If we consider $M_n(\mathbb{C})$ as a C^* -algebra, then an element A in this C^* -algebra

is positive, which means that A is a positive-semidefinite matrix. If we consider $M_n(\mathbb{C})$ as the 2-box space $\text{Spin}_{2,+}$ of the spin model, then an element $A \in \text{Spin}_{2,+}$ is $\mathfrak{F}$ -positive, which means that A is a positive matrix. Here positive matrix is equivalent to pointwise positive.

Remark 3.6. The $\mathfrak{F}$ -positivity can be regarded as the complete positivity for maps between pairs of trousers Frobenius algebras in categorical quantum mechanics, with its applications to quantum information. We refer to [39] for the generalization of complete positivity to the categorical context. Further insights and discussions on this topic can be found in, for example, [9, 17].

4. PF theory for planar algebras

In this section, we obtain the PF theorem for C^* -planar algebras. Moreover, we provide many equivalent conditions for the irreducibility, and planar algebra structures play an essential role. We also prove the quantum Collatz–Wielandt formula.

We shall recall the quantum Schur product theorem [29] in C^* -planar algebras: For two (strictly) positive elements $x, y \in \mathcal{P}_{2,\pm}$, their convolution $x * y$ is (strictly) positive. This fact has been deeply applied in the classification of subfactor planar algebras [29] and as an analytic obstruction to the unitary categorification of fusion rings [30]. Here we would use it again to show the PF theorem.

4.1. The PF theorem for C^* -planar algebras

We give a proof of the PF theorem in C^* -planar algebras.

Proposition 4.1. *Let x be an $\mathfrak{F}$ -positive element of the 2-box space $\mathcal{P}_{2,\pm}$ of a C^* -planar algebra. Then there exists a nonzero, $\mathfrak{F}$ -positive $y \in \mathcal{P}_{2,\pm}$ such that*

$$\begin{array}{c} \text{y} \\ \boxed{} \\ \text{x} \end{array} = \begin{array}{c} \text{x} \\ \boxed{} \\ \text{y} \end{array} = r \begin{array}{c} \text{y} \\ \boxed{} \end{array}, \quad (4.1)$$

where r is the spectral radius of x . Equivalently,

$$\begin{array}{c} \hat{x} \quad \hat{y} \\ \boxed{} \quad \boxed{} \end{array} = \begin{array}{c} \hat{y} \quad \hat{x} \\ \boxed{} \quad \boxed{} \end{array} = r \begin{array}{c} \hat{y} \\ \boxed{} \end{array}. \quad (4.2)$$

Proof. We first show that $r \in \sigma(x)$, the spectrum of x . Let $\rho(x) = \mathbb{C} \setminus \sigma(x)$ be the resolvent set of x . For any $\lambda \in \rho(x)$, let $R(\lambda, x) = (\lambda - x)^{-1}$ be the resolvent of x , which defines an element in $\mathcal{P}_{2,\pm}$. We have that $R(\lambda, x)$ is analytic in $\rho(x)$. For any $|\lambda| > r$, we have that

$$R(\lambda, x) = \sum_{j=0}^{\infty} \frac{x^j}{\lambda^{j+1}}. \quad (4.3)$$

We assume that $r \notin \sigma(x)$. By the fact that $R(\lambda, x)$ is analytic outside of the spectrum $\sigma(x)$, we see that $\lim_{\lambda \rightarrow r} R(\lambda, x) = R(r, x)$. By the continuity of the Fourier transform,

$$\lim_{z \rightarrow r} \widehat{R(\lambda, x)} = \widehat{R(r, x)}. \quad (4.4)$$

By the quantum Schur product theorem, we have that $\hat{x}^{(*j)} \geq 0$ for any $j \geq 1$, where $\hat{x}^{(*j)}$ is $\underbrace{\hat{x} * \cdots * \hat{x}}_j$. Therefore, for any real number $\lambda > r$,

$$\widehat{R(\lambda, x)} = \sum_{j=0}^{\infty} \frac{\hat{x}^{(*j)}}{\lambda^{j+1}} \geq 0.$$

By equation (4.4),

$$\widehat{R(r, x)} = \sum_{j=0}^{\infty} \frac{\hat{x}^{(*j)}}{r^{j+1}} \geq 0.$$

For any positive linear functional ϕ on $\mathcal{P}_{2,\pm}$,

$$\phi(\widehat{R(r, x)}) = \sum_{j=0}^{\infty} \frac{\phi(\hat{x}^{(*j)})}{r^{j+1}} < \infty.$$

Hence for any $|\lambda| = r$,

$$\left| \sum_{j=0}^{\infty} \frac{\phi(\hat{x}^{(*j)})}{\lambda^{j+1}} \right| < \infty.$$

By the uniform boundedness principle, we have that $\sum_{j=0}^{\infty} \frac{\hat{x}^{(*j)}}{\lambda^{j+1}}$ is bounded for any $|\lambda| = r$. This implies that $\{\lambda \in \mathbb{C} : |\lambda| = r\} \subset \mathbb{C} \setminus \sigma(x)$, which contradicts to the fact that r is the spectral radius of x . So $r \in \sigma(x)$.

Next we show that there exists a nonzero $\mathfrak{F}$ -positive element $y \in \mathcal{P}_{2,\pm}$ such that $xy = yx = ry$. Note that the algebraic multiplicity of r for x is finite; we see that the order of the pole r for $R(\lambda, x)$ is also finite. Let ℓ be the order of the pole r for $R(\lambda, x)$. We take the Laurent series of $R(\lambda, x)$ at $\lambda = r$ as follows:

$$R(\lambda, x) = \sum_{j=-\ell}^{\infty} x_{(j)}(\lambda - r)^j,$$

where

$$x_{(j)} = \frac{1}{2\pi i} \int_C \frac{R(\lambda, x)}{(\lambda - r)^{j+1}} d\lambda, \quad (4.5)$$

and C is a counterclockwise circle centered at r with radius small enough such that r is the only pole. For any $k, m \geq -\ell$, we have

$$\begin{aligned} x_{(k)}x_{(m)} &= \frac{1}{2\pi i} \int_{C_1} \frac{R(\lambda_1, x)}{(\lambda_1 - r)^{k+1}} d\lambda_1 \frac{1}{2\pi i} \int_{C_2} \frac{R(\lambda_2, x)}{(\lambda_2 - r)^{m+1}} d\lambda_2 \\ &= -\frac{1}{4\pi^2} \int_{C_1} \int_{C_2} \frac{R(\lambda_1, x)R(\lambda_2, x)}{(\lambda_1 - r)^{k+1}(\lambda_2 - r)^{m+1}} d\lambda_1 d\lambda_2 \\ &= -\frac{1}{4\pi^2} \int_{C_1} \int_{C_2} \frac{R(\lambda_1, x) - R(\lambda_2, x)}{(\lambda_2 - \lambda_1)(\lambda_1 - r)^{k+1}(\lambda_2 - r)^{m+1}} d\lambda_1 d\lambda_2 \\ &= \begin{cases} -x_{(k+m+1)} & k, m \geq 0, \\ 0 & k < 0, m \geq 0 \text{ or } k \geq 0, m < 0, \\ x_{(k+m+1)} & k, m < 0, \end{cases} \end{aligned}$$

where C_1, C_2 are circles centered at r with different radii such that C_1 is contained in C_2 . We add the details on the last step of the calculus when $k, m \geq 0$. By the residue theorem, we have

$$\begin{aligned} \frac{1}{2\pi i} \int_{C_2} \frac{1}{(\lambda_2 - \lambda_1)(\lambda_2 - r)^{m+1}} d\lambda_2 &= 0, \\ \frac{1}{2\pi i} \int_{C_1} \frac{1}{(\lambda_2 - \lambda_1)(\lambda_1 - r)^{k+1}} d\lambda_1 &= \frac{1}{(\lambda_2 - r)^{k+1}}. \end{aligned}$$

Then

$$\begin{aligned} x_{(k)}x_{(m)} &= \frac{1}{4\pi^2} \int_{C_1} \int_{C_2} \frac{R(\lambda_2, x)}{(\lambda_2 - \lambda_1)(\lambda_1 - r)^{k+1}(\lambda_2 - r)^{m+1}} d\lambda_1 d\lambda_2 \\ &= -\frac{1}{2\pi i} \int_{C_2} \frac{R(\lambda_2, x)}{(\lambda_2 - r)^{k+m+2}} d\lambda_2 = -x_{(k+m+1)}. \end{aligned}$$

We have

$$x_{(-\ell)}R(\lambda, x) = \sum_{j=-\ell}^{\infty} (\lambda - r)^j x_{(-\ell)}x_{(j)} = (\lambda - r)^{-1}x_{(-\ell)}x_{(-1)} = \frac{x_{(-\ell)}}{\lambda - r}.$$

Thus $(\lambda - r)x_{(-\ell)}R(\lambda, x) = x_{(-\ell)}$. By equation (4.3) and taking the coefficient of λ^{-2} , we have

$$x_{(-\ell)}x = rx_{(-\ell)}. \quad (4.6)$$

Note that

$$\widehat{x_{(-\ell)}} = \lim_{\mathbb{R} \ni \lambda \rightarrow r^+} (\lambda - r)^\ell \widehat{R(\lambda, x)} = \lim_{\mathbb{R} \ni \lambda \rightarrow r^+} (\lambda - r)^\ell \sum_{j=0}^{\infty} \frac{\widehat{x}^{(*j)}}{\lambda^{j+1}} \geq 0. \quad (4.7)$$

So $x_{(-\ell)}$ is $\mathfrak{F}$ -positive. Similarly, we have $xx_{(-\ell)} = rx_{(-\ell)}$. Let $y = x_{(-\ell)}$, then the conclusion holds. ■

Remark 4.2. In Proposition 4.1, we let

$$y_1 = \left[\begin{array}{c} \text{loop} \\ \boxed{\hat{y}} \\ \text{loop} \end{array} \right], \quad y_2 = \left[\begin{array}{c} \boxed{\hat{y}} \\ \text{loop} \end{array} \right].$$

We have that for any $\mathfrak{F}$ -positive element $x \in \mathcal{P}_{2,\pm}$, there exist nonzero positive elements $y_1 \in \mathcal{P}_{1,\pm}$ and $y_2 \in \mathcal{P}_{1,\mp}$ such that

$$y_1 * \hat{x} := \left[\begin{array}{c} \text{loop} \\ \boxed{y_1} \end{array} \right] \left[\begin{array}{c} \text{loop} \\ \boxed{\hat{x}} \end{array} \right] = r \left[\begin{array}{c} \text{loop} \\ \boxed{y_1} \end{array} \right], \quad \hat{x} * y_2 := \left[\begin{array}{c} \text{loop} \\ \boxed{\hat{x}} \end{array} \right] \left[\begin{array}{c} \text{loop} \\ \boxed{y_2} \end{array} \right] = r \left[\begin{array}{c} \text{loop} \\ \boxed{y_2} \end{array} \right],$$

where r is the spectral radius of x .

Remark 4.3. Suppose $\mathcal{P}$ is a C^* -planar algebra. For any element $x \in \mathcal{P}_{2n,+}$, it has the following graphical representation:

$$\left[\begin{array}{c} \text{2n strings} \\ \boxed{x} \end{array} \right],$$

where $\left[\begin{array}{c} \text{1 string} \end{array} \right]$ represents n strings. We call a $2n$ -box $x \in \mathcal{P}_{2n,\pm}$ is $\mathfrak{F}$ -positive if $\mathfrak{F}^n(x) \geq 0$, graphically,

$$\mathfrak{F}^n(x) := \left[\begin{array}{c} \text{one-click rotation} \\ \boxed{x} \end{array} \right] \geq 0. \quad (4.8)$$

Here $\mathfrak{F}$ is the one-click rotation and $\mathfrak{F}^n$ is the composition of $\mathfrak{F}^{n-1}$ and $\mathfrak{F}$, which is the n -click rotation. Proposition 4.1 also holds for $2n$ -box by replacing 1 string $\left[\begin{array}{c} \text{1 string} \end{array} \right]$ by n strings $\left[\begin{array}{c} \text{n strings} \end{array} \right]$:

Let x be an $\mathfrak{F}$ -positive element of the $2n$ -box space $\mathcal{P}_{2n,\pm}$ of a C^* -planar algebra. Then there exists a nonzero, $\mathfrak{F}$ -positive $y \in \mathcal{P}_{2n,\pm}$ such that

$$\begin{array}{c} \text{Diagram 1: } y \text{ on top, } x \text{ on bottom} \\ \text{Diagram 2: } x \text{ on top, } y \text{ on bottom} \end{array} = r \begin{array}{c} \text{Diagram 3: } y \text{ on top, } y \text{ on bottom} \end{array}, \quad (4.9)$$

where r is the spectral radius of x and  represents n strings. In particular, for any completely positive map Φ on $M_n(\mathbb{C})$, Φ can be realized by an $\mathfrak{F}$ -positive element, still denoted by Φ , in $\text{Spin}_{4,+}$. Then there exists another nonzero completely positive map Ψ such that $\Phi\Psi = \Psi\Phi = r\Psi$, where r is the spectral radius of Φ . Now let

$$D = \begin{array}{c} \text{Diagram: A loop with a box labeled } \hat{\Psi} \text{ on the right} \end{array}. \quad (4.10)$$

Then D is a nonzero density matrix such that

$$\begin{array}{c} \text{Diagram 1: } D \text{ on left, } \hat{\Phi} \text{ on right} \\ \text{Diagram 2: } D \text{ on right} \end{array} = \begin{array}{c} \text{Diagram 3: } D \text{ on right} \end{array}. \quad (4.11)$$

This recovers the Perron–Frobenius theorem for quantum channels.

4.2. Uniqueness of PF eigenvector

We introduce the irreducibility for $\mathfrak{F}$ -positive elements in planar algebras, which is analogous to irreducible matrices, and use it to study the uniqueness of PF eigenvector.

Definition 4.4. Suppose $\mathcal{P}$ is a C^* -planar algebra and $x \in \mathcal{P}_{2,\pm}$ is $\mathfrak{F}$ -positive. We say x is *irreducible* on $\mathcal{P}_{1,\pm}$, if for any projection $p \in \mathcal{P}_{1,\pm}$, the inequality $p * \hat{x} \leq \lambda p$, with $\lambda > 0$, ensures that $p = 0$ or I .

In the following, we present some equivalent descriptions for irreducibility of an $\mathfrak{F}$ -positive element. Moreover, we also have several easy-to-check, sufficient conditions for irreducibility.

Theorem 4.5. Suppose $\mathcal{P}$ is a C^* -planar algebra and $x \in \mathcal{P}_{2,+}$ is $\mathfrak{F}$ -positive. Let $B = \bigvee_{k=0}^{\infty} \mathcal{R}(\hat{x}^{(*k)})$, where $\mathcal{R}(\hat{x}^{(*k)})$ is the range projection of $\hat{x}^{(*k)}$. Then the following statements satisfy

$$(1) \Rightarrow (2) \Rightarrow (3) \Leftrightarrow (4) \Leftrightarrow (5) \Leftrightarrow (6) \Leftrightarrow (7) \Leftrightarrow (8) \Leftrightarrow (9).$$

(1) $B = I$ and $\dim \mathcal{P}_{0,+} = 1$.

(2) $B = I$ and $\text{tr}_{0,+} \left(\left(\begin{array}{|c|} \hline p_1 \\ \hline \end{array} \quad \begin{array}{|c|} \hline p_2 \\ \hline \end{array} \right) \right) \neq 0$ for any minimal projections p_1, p_2 in $\mathcal{P}_{1,+}$.

(3) $B(\overline{p_1} \otimes p_2) \neq 0$ for any minimal projections p_1, p_2 in $\mathcal{P}_{1,+}$, where $\overline{p_1}$ is the 180° rotation of p_1 .

(4) $\text{tr}_{2,-}(B(\overline{p_1} \otimes p_2)) \neq 0$ for any minimal projections p_1, p_2 in $\mathcal{P}_{1,+}$.

(5) $\mathcal{R}(p * B) = I$ for any nonzero projection $p \in \mathcal{P}_{1,+}$.

(6) x is irreducible on $\mathcal{P}_{1,+}$.

(7) For any nonzero projection $p \in \mathcal{P}_{1,+}$,

$$p * (e_1 + \hat{x})^{(*d-1)} > 0,$$

where $d = \dim \mathcal{P}_{1,+}$, e_1 is the Jones projection.

(8) For any positive elements $y_1, y_2 \in \mathcal{P}_{1,+}$ with $\langle y_1, y_2 \rangle = 0$, there exists $k_0 \in \mathbb{N}$, $k_0 < d$, such that

$$\langle y_1 * \hat{x}^{(*k_0)}, y_2 \rangle > 0.$$

(9) For any projection $q \in \mathcal{P}_{1,-}$, the inequality $\hat{x} * q \leq \lambda q$, with $\lambda > 0$, ensures that $q = 0$ or I .

Proof. (1) $\Rightarrow$ (2) Suppose that p_1, p_2 are minimal projections in $\mathcal{P}_{1,+}$ such that

$$\text{tr}_0 \left(\left(\begin{array}{|c|} \hline p_1 \\ \hline \end{array} \quad \begin{array}{|c|} \hline p_2 \\ \hline \end{array} \right) \right) = 0.$$

Then

$$\begin{array}{|c|} \hline p_1 \\ \hline \end{array} \neq 0, \quad \begin{array}{|c|} \hline p_2 \\ \hline \end{array} \neq 0,$$

and

$$\begin{array}{|c|} \hline p_1 \\ \hline \end{array} \quad \begin{array}{|c|} \hline p_2 \\ \hline \end{array} = 0.$$

This implies that $\boxed{p_1}, \boxed{p_2}$ are linearly independent in $\mathcal{P}_{0,+}$, which leads to a contradiction.

(2)⇒(3)⇔(4) They are not hard to check.

(4)⇒(5) Suppose that (5) does not hold. Then there exist minimal projections p_1, p_2 in $\mathcal{P}_{1,+}$ such that $p_2(p_1 * B) = 0$. Then

$$\mathrm{tr}_1(p_1(p_2 * B)) = \mathrm{tr}_2(B(\overline{p_1} \otimes p_2)) = 0,$$

which leads to a contradiction.

(5)⇒(4) Suppose that p_1, p_2 are minimal projections in $\mathcal{P}_{1,+}$ such that $\mathrm{tr}_{2,-}(B(\overline{p_1} \otimes p_2)) = 0$. Then $\mathrm{tr}_{1,+}(p_1(p_2 * B)) = 0$, that is, $p_1(p_2 * B) = 0$. We see that $\mathcal{R}(p_2 * B) \leq I - p_1$. This leads to a contradiction. Hence (4) is true.

(5)⇒(6) Suppose there exists a nonzero projection $p \in \mathcal{P}_{1,+}$ such that

$$p * \hat{x} \leq \lambda p$$

for some $\lambda > 0$. By the assumption, we see that there exists $n_0 > 0$ such that

$$\sum_{k=0}^{n_0} \hat{x}^{(*k)} \geq \lambda_1 B, \quad p * B \geq \lambda_0 I$$

for some $\lambda_0, \lambda_1 > 0$. Hence

$$\left(\sum_{j=0}^{n_0} \lambda^j \right) p \geq \sum_{j=0}^{n_0} p * \hat{x}^{(*j)} \geq \lambda_1 (p * B) \geq \lambda_0 \lambda_1 I.$$

Finally, we obtain that $p = I$ and x is irreducible on $\mathcal{P}_{1,+}$.

(6)⇒(7), (5) Suppose that x is irreducible on $\mathcal{P}_{1,+}$. Let p be a nonzero projection in $\mathcal{P}_{1,+}$ and $y = p + p * \hat{x} \in \mathcal{P}_{1,+}$. Then $\mathcal{R}(y) = p \vee \mathcal{R}(p * \hat{x})$. If $\mathcal{R}(y) = p$, then $\mathcal{R}(p * \hat{x}) = p$ and $p = I$ by the irreducibility of x on $\mathcal{P}_{1,+}$. If $\mathcal{R}(y) \neq p$, then we consider $p * (e_1 + \hat{x})^{(*k)}$ for $k \geq 2$, and its range space is increasing. Note that the dimension of $\mathcal{P}_{1,+}$ is d . We have that

$$\mathcal{R}(p * (e_1 + \hat{x})^{*(d-1)}) = \mathcal{R}(p * (e_1 + \hat{x})^{*(d)}) = I.$$

Hence we see that $p * (e_1 + \hat{x})^{*(d-1)} > 0$ and $\mathcal{R}(p * B) = I$ for any nonzero projection p in $\mathcal{P}_{1,+}$. Moreover, we have that $y * (e_1 + \hat{x})^{*(d-1)} > 0$ for any nonzero positive element y in $\mathcal{P}_{1,+}$.

(7)⇒(8) Note that for any positive elements $y_1, y_2 \in \mathcal{P}_{1,+}$ with $\langle y_1, y_2 \rangle = 0$, we have

$$\langle y_1 * (e_1 + \hat{x})^{*(d-1)}, y_2 \rangle > 0.$$

By the expansion, we see that there exists $k_0 < d$ such that $\langle y_1 * \hat{x}^{(*k_0)}, y_2 \rangle > 0$.

(8) $\Rightarrow$ (6) Suppose x is not irreducible on $\mathcal{P}_{1,+}$. Then there exists a nonzero projection $I \neq p \in \mathcal{P}_{1,+}$ such that $p * \hat{x} \leq \lambda p$ for some $\lambda > 0$. We see that for any $k \in \mathbb{N}$, $\langle p * \hat{x}^{(*k)}, I - p \rangle = 0$ which leads to a contradiction. Hence x is irreducible on $\mathcal{P}_{1,+}$.

(9) $\Leftrightarrow$ (8) Since (6) $\Leftrightarrow$ (8), we have similarly that (9) is equivalent to the following statement: For any $z_1, z_2 \in \mathcal{P}_{1,-}$, with $\langle z_1, z_2 \rangle = 0$, there exists $k_0 \in \mathbb{N}$, $k_0 < d$, such that

$$\langle z_1, \hat{x}^{(*k_0)} * z_2 \rangle > 0.$$

Note that $\langle z_1, \hat{x}^{(*k_0)} * z_2 \rangle = \langle \overline{z_1} * \hat{x}^{(*k_0)}, \overline{z_2} \rangle$. So this statement is equivalent to (8), and (9) $\Leftrightarrow$ (8). $\blacksquare$

Remark 4.6. The minimal projections in the statements (2), (3), (4) in Theorem 4.5 can be replaced by projections or positive elements in $\mathcal{P}_{1,+}$.

Remark 4.7. The statement (1) in Theorem 4.5 is stronger than the irreducibility. Let $\mathcal{P}$ be an irreducible subfactor planar algebra with a non-trivial (not identity) biprojection $q \in \mathcal{P}_{2,-}$. Take x such that $\mathcal{R}(\hat{x}) \leq q$ in Theorem 4.5, then x satisfies the statement (3) since $\mathcal{P}_{1,+} = \mathbb{C}I$, so x is irreducible. On the other hand, the statement (1) does not hold since $B \leq q < I$.

Remark 4.8. For spin model, the irreducibility in Definition 4.4 is the same as the irreducibility for matrices. Moreover, the statements (1) and (2) are equivalent to irreducibility.

Remark 4.9. Suppose that $\mathcal{P}$ is an irreducible subfactor planar algebra, then

$$B = \bigvee_{k=1}^{\infty} \mathcal{R}(\hat{x}^{(*k)})$$

is a biprojection (See [29, Theorem 4.12]).

We obtain the uniqueness result under the irreducibility.

Theorem 4.10 (Uniqueness of PF eigenvector). *Suppose $\mathcal{P}$ is a C^* -planar algebra, $x \in \mathcal{P}_{2,\pm}$ is $\mathfrak{F}$ -positive and irreducible on $\mathcal{P}_{1,\pm}$. Then there exists a unique (up to a scalar) strictly positive element $y \in \mathcal{P}_{1,\pm}$ such that*

$$\begin{array}{c} \text{Diagram: A box labeled } y \text{ is connected to a box labeled } \hat{x} \text{ by a loop. The box } \hat{x} \text{ is connected to a box labeled } y \text{ by a vertical line.} \end{array} = r \begin{array}{c} \text{Diagram: A box labeled } y \text{ connected to a vertical line.} \end{array}, \quad (4.12)$$

where r is the spectral radius of x . Moreover, if there exists a (not necessarily positive) element $y' \in \mathcal{P}_{1,\pm}$ such that $y' * \hat{x} = r y'$, then y' is a multiple of y .

Proof. From Proposition 4.1, there exists a nonzero positive element $y \in \mathcal{P}_{1,\pm}$ such that $y * \hat{x} = ry$. Thus $\mathcal{R}(y) * \hat{x} \leq \lambda \mathcal{R}(y)$ for some $\lambda > 0$. Since x is irreducible on $\mathcal{P}_{1,\pm}$, so $\mathcal{R}(y) = I$ and y is invertible. If y_1 is another strictly positive element in $\mathcal{P}_{1,\pm}$ such that y and y_1 are linearly independent and $y_1 * \hat{x} = ry_1$. We define the set $\mathcal{S} := \{t \geq 0 : y - ty_1 \geq 0\}$. Let $t = \sup_{t \in \mathcal{S}} t$ and let $y_2 = y - ty_1$. Then $y_2 \geq 0$ and $0 < \mathcal{R}(y_2) < I$. Note that $y_2 * \hat{x} = ry_2$, so $\mathcal{R}(y_2) = I$, which leads to a contradiction.

Since $\hat{x}$ is positive, we may assume that y' is self-adjoint. There exists $\lambda > 0$ such that $y' + \lambda y > 0$. Thus $y' + \lambda y$ is a multiple of y . Therefore, y' is a multiple of y . ■

Corollary 4.11. *Let $\mathcal{P}$ be a C^* -planar algebra. Suppose that $x \in \mathcal{P}_{2,\pm}$ is $\mathfrak{F}$ -positive and irreducible on $\mathcal{P}_{1,\pm}$. If $\bigvee_{j=0}^{\infty} \mathcal{R}(\hat{x}^{(*j)}) = I$ and $\dim \mathcal{P}_{1,\pm} = 1$, then there exists a unique strictly positive element $y \in \mathcal{P}_{1,\pm}$ such that $y * \hat{x} = ry$.*

Proof. It follows from Theorems 4.5 and 4.10. ■

We now consider the space of PF eigenvectors.

Notation 4.12. Suppose $\mathcal{P}$ is a C^* -planar algebra and $x \in \mathcal{P}_{2,\pm}$ is $\mathfrak{F}$ -positive. Let

$$\mathcal{E} = \{y \in \mathcal{P}_{1,\pm} : y * \hat{x} = ry\} \subseteq \mathcal{P}_{1,\pm}$$

be the eigenspace of r for x in $\mathcal{P}_{1,\pm}$, where r is the spectral radius of x . We denote by $\mathcal{E}^+$ the set of all positive elements in $\mathcal{E}$ and by $(\mathcal{E}^+)_1$ the set of all trace-one elements in $\mathcal{E}^+$. Proposition 4.1 indicates that $(\mathcal{E}^+)_1$ is a nonempty compact convex set. Let

$$\mathcal{Q} = \{\mathcal{R}(y) : y \in \mathcal{E}^+\}$$

be the set of all range projections of positive eigenvectors. For any $p, q \in \mathcal{Q}$, there exists $y_1, y_2 \in \mathcal{E}^+$ such that $\mathcal{R}(y_1) = p$ and $\mathcal{R}(y_2) = q$. So $\mathcal{R}(y_1 + y_2) = p \vee q$ and $p \vee q \in \mathcal{Q}$. Let $p_{\max}$ be the maximal projection in $\mathcal{Q}$. Then $p_{\max} = \bigvee_{p \in \mathcal{Q}} p$.

Proposition 4.13. *Suppose $\mathcal{P}$ is a C^* -planar algebra and $x \in \mathcal{P}_{2,\pm}$ is $\mathfrak{F}$ -positive. Then $y \in (\mathcal{E}^+)_1$ is an extreme point if and only if $\mathcal{R}(y)$ is a minimal projection in $\mathcal{Q}$. Moreover, if there exist $y_1, y_2 \in (\mathcal{E}^+)_1$ such that y_1 is an extreme point and $\mathcal{R}(y_1) = \mathcal{R}(y_2)$, then $y_1 = y_2$.*

Proof. Suppose y is an extremal point. If $\mathcal{R}(y)$ is not minimal, then there exists $z \in (\mathcal{E}^+)_1$ such that $\mathcal{R}(z) < \mathcal{R}(y)$. Take $\omega = y - tz \geq 0$ for some $0 < t < 1$. Then $y = (1 - t)\frac{\omega}{1-t} + tz$, which is a contradiction.

On the other hand, if $\mathcal{R}(y)$ is minimal but y is not extremal. Then y can be expressed as a convex combination of two different elements, that is, $y = \lambda y' + (1 - \lambda)y''$, $0 < \lambda < 1$, $y', y'' \in (\mathcal{E}^+)_1$. It is easy to see $\mathcal{R}(y') \leq \mathcal{R}(y)$. Thus $\mathcal{R}(y') = \mathcal{R}(y)$ from the minimality of $\mathcal{R}(y)$. We can choose $0 < t < 1$ such that $y - ty' \geq 0$ and $0 < \mathcal{R}(y - ty') < \mathcal{R}(y)$ by a continuity argument. Note that $\frac{y - ty'}{1-t} \in (\mathcal{E}^+)_1$. So we find another eigenvector with a smaller support, which leads to a contradiction. ■

At the end of this section, we present the Collatz–Wielandt formula for planar algebras.

Theorem 4.14 (Collatz–Wielandt formula). *Suppose $\mathcal{P}$ is a C^* -planar algebra, $x \in \mathcal{P}_{2,\pm}$ is $\mathfrak{F}$ -positive. Let r be the spectral radius of x . Then the following statements are true:*

- (1) *If there exists a nonzero positive element $z \in \mathcal{P}_{1,\mp}$ such that*

$$\hat{x} * z \leq rz \quad \text{and} \quad \mathcal{R}(z) \leq \overline{p_{\max}},$$

then $\hat{x} * z = rz$.

- (2) *If there exists a nonzero positive element $z \in \mathcal{P}_{1,\mp}$ such that*

$$rz \leq \hat{x} * z \quad \text{and} \quad \mathcal{R}(\hat{x} * z) \leq \overline{p_{\max}},$$

then $\hat{x} * z = rz$.

- (3) *If there exists a nonzero positive element $z \in \mathcal{P}_{1,\mp}$ such that $\hat{x} * z = \tilde{r}z$ and $\mathcal{R}(z) \leq \overline{p_{\max}}$, then $\tilde{r} = r$.*

Proof. (1) Let $y \in \mathcal{P}_{1,\pm}$ be the positive element such that $y * \hat{x} = ry$ and $\mathcal{R}(y) = p_{\max}$. Then we have

$$ry * z = (y * \hat{x}) * z = y * (\hat{x} * z) \leq ry * z.$$

The last inequality is the quantum Schur product theorem. Hence $y * (rz - \hat{x} * z) = 0$ and $p_{\max} * (rz - \hat{x} * z) = 0$. We obtain that

$$\boxed{p_{\max}} \quad \boxed{rz - \hat{x} * z} = 0.$$

Hence $\text{tr}_{1,\mp}(rz - \hat{x} * z) = 0$. By the faithful of the trace, we obtain $\hat{x} * z = rz$.

- (2) The argument is similar to (1).

- (3) Note that $\tilde{r} \in \sigma(x)$ and $\tilde{r} \leq r$. By the statement (1), we see that $\tilde{r} = r$. ■

Corollary 4.15. *Suppose $\mathcal{P}$ is a C^* -planar algebra, $x \in \mathcal{P}_{2,\pm}$ is $\mathfrak{F}$ -positive and irreducible. Let r be the spectral radius of x . Then the following statements are true:*

- (1) *If there exists a nonzero positive element $z \in \mathcal{P}_{1,\mp}$ such that $\hat{x} * z \leq rz$ or $\hat{x} * z \geq rz$, then $\hat{x} * z = rz$.*
- (2) *If there exists a nonzero positive element $z \in \mathcal{P}_{1,\mp}$ such that $\hat{x} * z = \tilde{r}z$, then $\tilde{r} = r$.*

Proof. By Theorem 4.10, there exists a unique strictly positive element $y \in \mathcal{P}_{1,\pm}$ such that $y * \hat{x} = ry$. So we have $p_{\max} = I$. Then by Theorem 4.14, we could obtain the conclusion. ■

Remark 4.16. Evans and Høegh-Krohn proved the Collatz–Wielandt formula for irreducible positive maps on finite-dimensional C^* -algebras (see [8, Theorems 2.3 and 2.4]). We give the Collatz–Wielandt formula on a weaker condition that is not necessarily irreducible for x in Theorem 4.14.

Remark 4.17. Suppose $\mathcal{P}_{1,\pm}$ is abelian and $\{p_i\}_{i=1}^n$ are orthogonal minimal projections with $\sum_{i=1}^n p_i = I$, and $x \in \mathcal{P}_{2,\pm}$ is $\mathfrak{F}$ -positive. We have another form of the Collatz–Wielandt formula: Let $z \in \mathcal{P}_{1,\mp}$ be a strictly positive element. Then

$$\min_{1 \leq i \leq n} \frac{\mathrm{tr}_{1,\mp}(\overline{p_i}[\hat{x} * z])}{\mathrm{tr}_{1,\mp}(\overline{p_i}z)} \leq r(x) \leq \max_{1 \leq i \leq n} \frac{\mathrm{tr}_{1,\mp}(\overline{p_i}[\hat{x} * z])}{\mathrm{tr}_{1,\mp}(\overline{p_i}z)}. \quad (4.13)$$

This could recover the Collatz–Wielandt formula for (not necessarily irreducible) positive matrices. Suppose Γ is a bipartite graph with simple edges. Let $\mathcal{P}(\Gamma)$ be the bipartite graph C^* -planar algebra of Γ . Then $\mathcal{P}(\Gamma)_{1,\pm}$ is abelian and inequality (4.13) holds.

5. PF theorem on ordered Banach spaces

For a positive map on an (infinite-dimensional) ordered Banach space, there exists an approximate PF eigenvector with respect to the spectral radius. The space of PF eigenvectors is at most one dimensional if the positive map has ergodicity property. We give a self-contained proof of these results. We refer to [26] for the PF theorem for positive, compact operators on ordered Banach spaces.

Definition 5.1. An *ordered Banach space* $\mathcal{X}$ is a Banach space with a closed positive cone $\mathcal{C}$ such that $\mathcal{C} \cap -\mathcal{C} = 0$ and $\mathcal{C} + \mathcal{C} \subseteq \mathcal{C}$. For any $x, y \in \mathcal{X}$, we define the order relation $x \leq y$ if $y - x \in \mathcal{C}$. Moreover, the positive cone $\mathcal{C}$ also satisfies the following properties:

- (1) If $0 \leq x \leq y$, then $\|x\| \leq c_{\mathcal{X}}\|y\|$, where $c_{\mathcal{X}}$ is a positive constant independent of x, y .
- (2) Each $x \in \mathcal{X}$ is a linear sum of four positive elements with norm bounded by $d_{\mathcal{X}}\|x\|$, where $d_{\mathcal{X}}$ is a positive constant independent of x .

Remark 5.2. Property (1) is usually called normality. Property (2) holds if only if $\mathcal{C}$ is a generating cone, that is, $\mathcal{X} = \mathcal{C} - \mathcal{C}$, and in this case $\mathcal{X}$ is a real ordered Banach space; see, for instance, [1, Theorem 2.37 (1) and (3)]. If $\mathcal{X} = \mathcal{C} - \mathcal{C} + i(\mathcal{C} - \mathcal{C})$, then $\mathcal{X}$ is a complex Banach space.

Remark 5.3. Suppose $\mathcal{M}$ is a von Neumann algebra with a normal semifinite faithful weight. Let $L^p(\mathcal{M})$ be the noncommutative spatial L^p space [18], $1 \leq p \leq \infty$. Define $\mathcal{C} := \{x \in L^p(\mathcal{M}) : x \geq 0\}$. Then $(L^p(\mathcal{M}), \mathcal{C})$ is an ordered Banach space.

Definition 5.4. A bounded linear map T between two ordered Banach spaces $(\mathcal{X}_1, \mathcal{C}_1)$ and $(\mathcal{X}_2, \mathcal{C}_2)$ is *positive* if $T\mathcal{C}_1 \subseteq \mathcal{C}_2$.

Lemma 5.5. Suppose T is a linear map from the ordered Banach space $(\mathcal{X}, \mathcal{C})$ to a Banach space. We have that T is bounded if and only if $\sup_{x \in \mathcal{C}, x \neq 0} \frac{\|Tx\|}{\|x\|} < \infty$, that is, T is bounded on the cone $\mathcal{C}$.

Proof. It could be proved by using condition (2) in Definition 5.1. ■

Lemma 5.6. Let $(\mathcal{X}, \mathcal{C})$ be an ordered Banach space. Suppose $\{x_n\}_{n=1}^\infty \subseteq \mathcal{C}$ is a sequence of positive elements such that $\sum_{n=1}^\infty x_n$ converges. Then for any sequence of complex numbers $\{w_n\}_{n=1}^\infty$ with modulus uniformly bounded by a positive number c , we have $\sum_{n=1}^\infty w_n x_n$ converges and

$$\left\| \sum_{n=1}^\infty w_n x_n \right\| \leq 4cc_{\mathcal{X}} \left\| \sum_{n=1}^\infty x_n \right\|.$$

Proof. For any $1 \leq N < M$, we have

$$\begin{aligned} \left\| \sum_{n=N}^M w_n x_n \right\| &= \left\| \sum_{n=N}^M \Re(w_n) x_n + i \sum_{n=N}^M \Im(w_n) x_n \right\| \\ &\leq \left\| \sum_{n=N}^M \Re(w_n) x_n \right\| + \left\| \sum_{n=N}^M \Im(w_n) x_n \right\| \\ &\leq 4cc_{\mathcal{X}} \left\| \sum_{n=N}^M x_n \right\|. \end{aligned}$$

Thus $\sum_{n=1}^\infty w_n x_n$ converges. Let $N = 1$, and $M \rightarrow \infty$. We have

$$\left\| \sum_{n=1}^\infty w_n x_n \right\| \leq 4cc_{\mathcal{X}} \left\| \sum_{n=1}^\infty x_n \right\|. \quad \blacksquare$$

Theorem 5.7 (Approximate existence). Let T be a positive linear map on an ordered Banach space $(\mathcal{X}, \mathcal{C})$. Then $r(T) \in \sigma(T)$ and there exists a sequence of positive elements $\{x_n\}_{n=1}^\infty \subseteq \mathcal{C}$ with $\|x_n\| = 1$ such that

$$\lim_{n \rightarrow \infty} \|Tx_n - r(T)x_n\| = 0,$$

where $r(T)$ is the spectral radius of T .

Proof. Let $r = r(T)$ and

$$R(\lambda, T) = (\lambda - T)^{-1}, \quad \lambda \in \mathbb{C} \setminus \sigma(T)$$

be the resolvent of T . Then $R(\lambda, T)$ is analytic in $\mathbb{C} \setminus \sigma(T)$. For any $|\lambda| > r$, we have that

$$R(\lambda, T) = \sum_{j=0}^{\infty} \frac{T^j}{\lambda^{j+1}}.$$

If $r \notin \sigma(T)$, then $R(r, T) = \lim_{\lambda \rightarrow r^+} R(\lambda, T)$ is bounded. For any $x \in \mathcal{C}$, we have

$$R(r, T)(x) = \lim_{\lambda \rightarrow r^+} R(\lambda, T)(x) = \sum_{j=0}^{\infty} \frac{T^j(x)}{r^{j+1}} \in \mathcal{X},$$

where the second equality is due to property (1) in Definition 5.1. By Lemma 5.6, we have $\sum_{j=0}^{\infty} \frac{T^j(x)}{w^{j+1}}$ converges for any $|w| = r$ and

$$\left\| \sum_{j=0}^{\infty} \frac{T^j(x)}{w^{j+1}} \right\| \leq \frac{4c_{\mathcal{X}}}{r} \left\| \sum_{j=0}^{\infty} \frac{T^j(x)}{r^{j+1}} \right\| \leq \frac{4c_{\mathcal{X}} \|R(r, T)\|}{r} \|x\|.$$

Then we see that $R(w, T) = \lim_{\lambda \rightarrow w} R(\lambda, T)$ is bounded on the cone, hence is bounded by Lemma 5.5. Thus $w \notin \sigma(T)$, which leads to a contradiction. Hence $r \in \sigma(T)$.

Let

$$T_n = \left(r + \frac{1}{n} - T \right)^{-1}, \quad n = 1, 2, \dots$$

Then $\{T_n\}_{n \in \mathbb{N}}$ is an unbounded sequence of positive linear maps. Thus there exists a sequence of positive elements $y_n \in \mathcal{C}$ such that $\|T_n y_n\| = 1$ and $\lim_{n \rightarrow \infty} y_n = 0$. Now take $x_n = T_n y_n$, we have $\lim_{n \rightarrow \infty} \|T x_n - r x_n\| = 0$. ■

Remark 5.8. Glück and Mironchenko proved the approximate existence of PF eigenvector for a positive map on an ordered Banach space in [15, Lemma 3.5].

We now assume that there exists an element $I \in \mathcal{C}$ such that $x \leq e_{\mathcal{X}} \|x\| I$ for any $x \in \mathcal{C}$, where $e_{\mathcal{X}}$ is a positive constant independent of x . We call $x \in \mathcal{C}$ strictly positive, written $x > 0$, if there exists $\lambda > 0$ such that $\lambda I \leq x$.

Definition 5.9. Let $(X, \mathcal{C})$ be an ordered Banach space. Let T be a positive linear map on X . We say T is *ergodic* if for any nonzero $x \in \mathcal{C}$, there exists $N > 0$ such that $\sum_{j=1}^N T^j(x)$ is strictly positive.

Remark 5.10. Let $\mathcal{X}^*$ be the dual Banach space of $\mathcal{X}$. Define the dual cone $\mathcal{C}^*$ of $\mathcal{C}$ in $\mathcal{X}^*$ by

$$\mathcal{C}^* := \{x^* \in \mathcal{X}^* : \langle x, x^* \rangle \geq 0, \forall x \in \mathcal{C}\},$$

where the inner product means the linear functional x^* acting on x . From the characterizations of strictly positive elements that can, for instance, be found in [16, Proposition 2.11

and Corollary 2.8], where they are called interior points, we know that $x \in \mathcal{C}$ is strictly positive if and only if $\langle x, x^* \rangle > 0$ for all $x^* \in \mathcal{C}^*$, $x^* \neq 0$. It follows that the definition of ergodicity property is equivalent to Sawashima's definition of irreducibility: For every couple $x \in \mathcal{C}$, $x \neq 0$, $x^* \in \mathcal{C}$, $x^* \neq 0$, there exists a positive integer $n = n(x, x^*)$ such that $\langle T^n x, x^* \rangle > 0$.

Theorem 5.11 (Uniqueness). *Let $(X, \mathcal{C})$ be an ordered Banach space and T be a positive linear map on X . If T is ergodic on X , then there exists at most one nonzero element $x \in \mathcal{C}$ up to a scalar such that $Tx = r(T)x$.*

Proof. Suppose there exists $0 \neq x \in \mathcal{C}$ such that $Tx = r(T)x$. Since T is ergodic, $\sum_{j=1}^N T(x)$ is strictly positive for some positive integer N , so $x > 0$. Suppose there is $y \in \mathcal{C}$ such that $Ty = r(T)y$. We consider the following set

$$S := \{t > 0 : x - ty > 0\}.$$

Since $x \leq e_{\mathcal{X}} \|x\| I$ and $y \leq e_{\mathcal{X}} \|y\| I$, then S is a nonempty bounded open set. Take $t_0 = \sup_{t \in S} t$, then $x - t_0 y$ is positive but is not strictly positive. Note that $T(x - t_0 y) = r(T)(x - t_0 y)$; if $x - t_0 y$ is nonzero, then it is strictly positive, which is a contradiction. Therefore, $x - t_0 y = 0$. ■

Remark 5.12. Theorem 5.11 is another way to formulate the uniqueness result in [25, Theorem 11.1 (d), p. 111].

Remark 5.13. Krein and Rutman proved the PF theorem for positive compact linear maps on ordered Banach spaces (see, e.g., [2, Theorem 1.23]). We prove the PF theorem for positive (not necessarily compact) linear maps on an ordered Banach space with two extra conditions for positive cones. They proved the uniqueness of PF eigenvector if a compact linear map T is strongly positive, which means that $T(\mathcal{C} \setminus \{0\}) \subseteq \text{Int}(\mathcal{C})$. We prove the uniqueness if a positive linear map T is ergodic. The ergodicity of T means that there exists $N > 0$ such that $\sum_{j=1}^N T^j$ is strongly positive. So the ergodic condition is weaker than the strong positivity.

6. Structures of the PF eigenspace

In Section 4, we found that the support of the PF eigenvector of an irreducible $\mathfrak{F}$ -positive element is identity. In this section, we study the eigenspace of PF eigenvectors of (non-irreducible) $\mathfrak{F}$ -positive elements.

We show that an $\mathfrak{F}$ -positive element yields a power series that defines a maximally supported PF eigenvector ζ . We obtain from this PF eigenvector the PF eigenspace $\mathcal{E}$ which has a C^* -algebraic structure. We introduce exchange algebras $\mathcal{A}$ and $\mathcal{B} = \mathcal{A}^*$ and $\mathcal{C} = \mathcal{A} \cap \mathcal{B}$ for an $\mathfrak{F}$ -positive element x . We prove that ζ commutes with $\mathcal{C}$ and

is independent of $\mathcal{C}$ and $\zeta\mathcal{C} \subseteq \mathcal{E}$. Let $\mathcal{C}_\zeta$ be the C^* -algebra $\mathcal{C}$ with respect to the $\mathfrak{F}$ -positive element $(\zeta^{1/2} \otimes \zeta^{1/2})x(\zeta^{-1/2} \otimes \zeta^{-1/2})$. We prove that the space of positive PF eigenvectors $\mathcal{E}^+ = \zeta^{1/2}\mathcal{C}_\zeta^+\zeta^{1/2}$ under a certain condition. Let $\mathcal{A}_p$ and $\mathcal{B}_p$ be the exchange algebras $\mathcal{A}$ and $\mathcal{B}$ for the $\mathfrak{F}$ -positive element $(p \otimes \bar{p})x(p \otimes \bar{p})$. We prove that $\mathcal{E}^+ = (\mathcal{A}_p\zeta)^+ = (\zeta\mathcal{B}_p)^+$ under the same condition. These characterizations of PF eigenspaces are true for CPTP maps. We elaborate on the connection between this multiplicative structure and the multiplication of the C^* -algebra of density matrices. Based on it, we explore a PF theoretical approach to quantum error correction in the next section.

6.1. The maximal-support PF eigenvector

Suppose $\mathcal{P}$ is a C^* -planar algebra and $x \in \mathcal{P}_{2,\pm}$ is $\mathfrak{F}$ -positive with the spectral radius r . We recall that the eigenvector space of x corresponding to r (the PF eigenspace) is denoted by

$$\mathcal{E} = \{y \in \mathcal{P}_{1,\pm} : y * \hat{x} = ry\} \subseteq \mathcal{P}_{1,\pm}.$$

Let k be the order of the resolvent $(z - x)^{-1}$, $z \in \mathbb{C}$, at the pole r . From equation (4.7) in the proof of Proposition 4.1, the following limit exists:

$$y := \lim_{\mathbb{R} \ni z \rightarrow r^+} (z - r)^k \sum_{j=0}^{\infty} \frac{x^j}{z^{j+1}}. \quad (6.1)$$

Moreover, $y \in \mathcal{P}_{2,\pm}$ is an $\mathfrak{F}$ -positive element such that $xy = yx = ry$. Recall that

$$\mathcal{Q} = \{\mathcal{R}(y) : y \in \mathcal{E}^+\}$$

is the set of all range projections of positive eigenvectors and $p = \bigvee_{q \in \mathcal{Q}} q$ is the maximal projection in $\mathcal{Q}$. (Here we use p instead of $p_{\max}$ in Notation 4.12 for simplicity.) We define

$$x_p := \begin{array}{c} \text{Diagram: A box labeled } x_p \text{ with four vertical lines (two on the left, two on the right). To its right is an equals sign followed by a box labeled } x \text{ with four vertical lines. Above and below the } x \text{ box are boxes labeled } p \text{ and } \bar{p} \text{ respectively, connected by lines.} \end{array} \quad (6.2)$$

We next prove that the spectral radius of x_p is r and the order k_p of the resolvent $(z - x_p)^{-1}$, $z \in \mathbb{C}$, at the pole r , is 1.

Lemma 6.1. *The following equation holds:*

$$(6.3)$$

Proof. Let $y \in \mathcal{P}_{1,\pm}$ be the PF eigenvector of x with $\mathcal{R}(y) = p$. So

$$(I - p)(y * \hat{x})(I - p) = 0,$$

which is equivalent to $(I - p)(p * \hat{x})(I - p) = 0$. This implies

$$= 0,$$

which is equivalent to $(\bar{p} \otimes (I - p))\hat{x}(\bar{p} \otimes (I - p)) = 0$. Since x is $\mathfrak{F}$ -positive and $\hat{x} \geq 0$, we have $(\bar{p} \otimes (I - p))\hat{x}^{1/2} = 0$, which has the following graphical representation by cutting the above diagram horizontally:

$$= 0.$$

Thus

This implies that equation (6.3) holds. ■

Proposition 6.2. *Suppose $\mathcal{P}$ is a C^* -planar algebra and $x \in \mathcal{P}_{2,\pm}$ is $\mathfrak{F}$ -positive and x_p is defined in equation (6.2). Then we have the following statements:*

- (1) The spectral radius of x_p is r and x_p is $\mathfrak{F}$ -positive.
 (2) Define

$$\mathcal{E}_p := \{y \in \mathcal{P}_{1,\pm} : y * \widehat{x}_p = ry\},$$

then $\mathcal{E}_p^+ = \mathcal{E}^+$.

- (3) The following limit exists:

$$y_p := \lim_{\mathbb{R} \ni z \rightarrow r^+} (z - r) \sum_{j=0}^{\infty} \frac{x_p^j}{z^{j+1}}. \quad (6.4)$$

Moreover, the element y_p is a nonzero $\mathfrak{F}$ -positive element in $\mathcal{P}_{2,\pm}$ such that $x_p y_p = y_p x_p = r y_p$.

Proof. (1) It is clear that $\widehat{x}_p$ is positive. Let $r(x_p)$ be the spectral radius of x_p and $z_p \in \mathcal{P}_{1,\pm}$ be the PF eigenvector. Then we have $r(x_p)z_p = \widehat{z}_p * \widehat{x}_p = \widehat{z}_p * \widehat{x}$. The second equality is due to equation (6.3). So $r(x_p) \leq r$. Similarly, let $z \in \mathcal{P}_{1,\pm}$ be the PF eigenvector of x . Then $\mathcal{R}(z) \leq p$ and $rz = z * \widehat{x} = z * \widehat{x}_p$. We obtain $r \leq r(x_p)$. Thus $r(x_p) = r$.

(2) It is easy to check.

(3) Let k_p be the order of the resolvent $(z - x_p)^{-1}$, $z \in \mathbb{C}$, at the pole r . We define

$$y_p := \lim_{\mathbb{R} \ni z \rightarrow r^+} (z - r)^{k_p} \sum_{j=0}^{\infty} \frac{x_p^j}{z^{j+1}}. \quad (6.5)$$

From equation (4.7) in the proof of Proposition 4.1, we see that $y_p \in \mathcal{P}_{2,\pm}$ is $\mathfrak{F}$ -positive such that $x_p y_p = y_p x_p = r y_p$. We have

$$\mathrm{tr}_{1,\pm}(p * \widehat{y}_p) = \begin{array}{c} \text{Diagram: A box labeled } p \text{ is connected to a box labeled } \widehat{y}_p \text{ by two loops.} \end{array} = \begin{array}{c} \text{Diagram: A box labeled } \widehat{y}_p \text{ is connected to itself by two loops.} \end{array} = \mathrm{tr}_{2,\pm}(\widehat{y}_p) > 0,$$

which implies $p * \widehat{y}_p \neq 0$. Note that $p * \widehat{x}_p^{*j} \leq \lambda z_p * \widehat{x}_p^{*j} = \lambda r^j z_p$ for some $\lambda > 0$, where z_p is the PF eigenvector of x_p . Now if $k_p \geq 2$, then

$$0 \leq p * \widehat{y}_p \leq \lim_{\mathbb{R} \ni z \rightarrow r^+} (z - r)^{k_p} \sum_{j=0}^{\infty} \frac{\lambda r^j z_p}{z^{j+1}} = \lambda z_p \lim_{\mathbb{R} \ni z \rightarrow r^+} (z - r)^{k_p-1} = 0,$$

which leads to a contradiction. So $k_p = 1$, and equation (6.4) is well defined. $\blacksquare$

We define

$$\zeta := p * \widehat{y}_p = \lim_{\mathbb{R} \ni z \rightarrow r^+} (z - r) \sum_{j=0}^{\infty} \frac{p * \widehat{x}_p^{*j}}{z^{j+1}} = \lim_{\mathbb{R} \ni z \rightarrow r^+} (z - r) \sum_{j=0}^{\infty} \frac{p * \widehat{x}^{*j}}{z^{j+1}}. \quad (6.6)$$

The last equality follows from equation (6.3). Then

$$\zeta * \hat{x} = \zeta * \hat{x}_p = p * \hat{y}_p * \hat{x}_p = rp * \hat{y}_p = r\zeta.$$

This implies $\zeta \in \mathcal{E}^+$ and $\mathcal{R}(\zeta) = p$. So ζ is the PF eigenvector of x with maximal support.

6.2. C^* -algebraic structures of eigenspaces

Let $x \in \mathcal{P}_{2,\pm}$ be an $\mathfrak{F}$ -positive element. We introduce three subalgebras of $\mathcal{P}_{1,\pm}$ for x , which are given by the following graphical representations:

$$\mathcal{A} := \left\{ y \in \mathcal{P}_{1,\pm} : \begin{array}{c} \boxed{\bar{y}} \\ | \\ \boxed{x} \end{array} = \begin{array}{c} \boxed{x} \\ | \\ \boxed{\bar{y}} \end{array} \right\} = \left\{ y \in \mathcal{P}_{1,\pm} : \begin{array}{c} \boxed{\hat{x}} \\ | \\ \boxed{\bar{y}} \end{array} = \begin{array}{c} \boxed{\hat{x}} \\ | \\ \boxed{y} \end{array} \right\} \subseteq \mathcal{P}_{1,\pm}. \quad (6.7)$$

$$\mathcal{B} := \left\{ y \in \mathcal{P}_{1,\pm} : \begin{array}{c} \boxed{y} \\ | \\ \boxed{x} \end{array} = \begin{array}{c} \boxed{x} \\ | \\ \boxed{y} \end{array} \right\} = \left\{ y \in \mathcal{P}_{1,\pm} : \begin{array}{c} \boxed{y} \\ | \\ \boxed{\hat{x}} \end{array} = \begin{array}{c} \boxed{\bar{y}} \\ | \\ \boxed{\hat{x}} \end{array} \right\} \subseteq \mathcal{P}_{1,\pm}. \quad (6.8)$$

We have that $\mathcal{A} = \mathcal{B}^*$ since x is $\mathfrak{F}$ -positive. The third algebra $\mathcal{C}$ is the intersection of $\mathcal{A}$ and $\mathcal{B}$, which is a C^* -algebra.

$$\mathcal{C} := \left\{ y \in \mathcal{P}_{1,\pm} : \begin{array}{c} \boxed{\bar{y}} \\ | \\ \boxed{x} \end{array} = \begin{array}{c} \boxed{x} \\ | \\ \boxed{\bar{y}} \end{array}, \begin{array}{c} \boxed{y} \\ | \\ \boxed{x} \end{array} = \begin{array}{c} \boxed{x} \\ | \\ \boxed{y} \end{array} \right\} \subseteq \mathcal{P}_{1,\pm}. \quad (6.9)$$

Equivalently, we have

$$\mathcal{C} := \left\{ y \in \mathcal{P}_{1,\pm} : \begin{array}{c} \boxed{\hat{x}} \\ | \\ \boxed{\bar{y}} \end{array} = \begin{array}{c} \boxed{\hat{x}} \\ | \\ \boxed{y} \end{array}, \begin{array}{c} \boxed{y} \\ | \\ \boxed{\hat{x}} \end{array} = \begin{array}{c} \boxed{\bar{y}} \\ | \\ \boxed{\hat{x}} \end{array} \right\} \subseteq \mathcal{P}_{1,\pm}. \quad (6.10)$$

Remark 6.3. Let $\Phi: M_n(\mathbb{C}) \rightarrow M_n(\mathbb{C})$ be a completely positive map. There are Kraus operators $F_j \in \mathcal{M}$ such that

$$\Phi(D) = \sum_{j=1}^m F_j D F_j^* \quad \forall D \in M_n(\mathbb{C}).$$

Then we have

$$\begin{aligned}\mathcal{A} &= \{F_j^* : j = 1, \dots, m\}' \cap M_n(\mathbb{C}), \\ \mathcal{B} &= \{F_j : j = 1, \dots, m\}' \cap M_n(\mathbb{C}), \\ \mathcal{C} &= \{F_j, F_j^* : j = 1, \dots, m\}' \cap M_n(\mathbb{C}),\end{aligned}$$

where $\mathcal{S}' \cap M_n(\mathbb{C}) = \{A \in M_n(\mathbb{C}) : AB = BA, B \in \mathcal{S}\}$ is the commutant of $\mathcal{S}$, and $\mathcal{S}$ is a subset of $M_n(\mathbb{C})$. We could obtain the above three equations from equation (2.21) by choosing $\{F_j\}_{j=1}^m$ as $\text{Tr}(F_i^* F_j) = 0$ if $i \neq j$. According to [31, Theorem 8.2], different choices of Kraus operators of Φ remain the commutants invariant.

The C^* -algebra $\mathcal{C}$ is useful to characterize the eigenspace $\mathcal{E}$. Let $\zeta \in \mathcal{E}^+$ be the maximal-support eigenvector of the $\mathfrak{F}$ -positive element x defined in equation (6.6). We will show that $\zeta\mathcal{C} \subseteq \mathcal{E}$ and ζ commutes with $\mathcal{C}$. Moreover, we will give a sufficient condition for $\zeta\mathcal{C}^+ = \mathcal{E}^+$.

Theorem 6.4. *Suppose $\mathcal{P}$ is a C^* -planar algebra and $x \in \mathcal{P}_{2,\pm}$ is $\mathfrak{F}$ -positive. Then the following statements hold:*

- (1) $[p_{\max}, y] = 0$ for any $y \in \mathcal{C}$. Furthermore, $[\zeta, y] = 0$ for any $y \in \mathcal{C}$.
- (2) $\zeta\mathcal{C} \subseteq \mathcal{E}$.
- (3) For any direct summand $\mathcal{D}$ of $\mathcal{C}$ such that $\mathcal{D} \cong M_n(\mathbb{C})$, we have

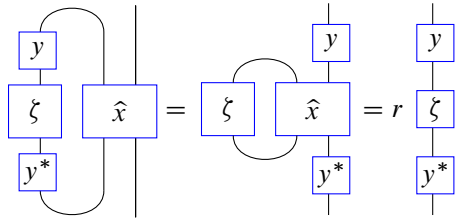
$$\{\zeta\}''\mathcal{D} = \{p_{\mathcal{D}}\zeta\}''\mathcal{D} \cong \{p_{\mathcal{D}}\zeta\}'' \otimes \mathcal{D},$$

where $p_{\mathcal{D}}$ is the union of projections in $\mathcal{D}$.

Proof. (1) For any $y \in \mathcal{C}$, we have

$$(y^*\zeta y) * \hat{x} = y^*(\zeta * \hat{x})y = ry^*\zeta y,$$

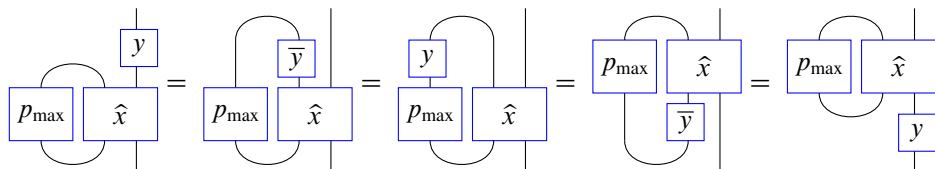
pictorially,



We see that $y^*\zeta y \in \mathcal{E}^+$. Hence $\mathcal{R}(y^*\zeta) = \mathcal{R}(y^*\zeta y) \leq p_{\max}$ and then $y^*\zeta = p_{\max}y^*\zeta$. Now we obtain that

$$y^*p_{\max} = p_{\max}y^*p_{\max} \quad \forall y \in \mathcal{C}.$$

This implies that $p_{\max}y = yp_{\max}$ for any $y \in \mathcal{C}$. Note that $y(p_{\max} * \hat{x}^{(*j)}) = (p_{\max} * \hat{x}^{(*j)})y$ for any $j \geq 0$. The following is the graph calculus for $j = 1$:

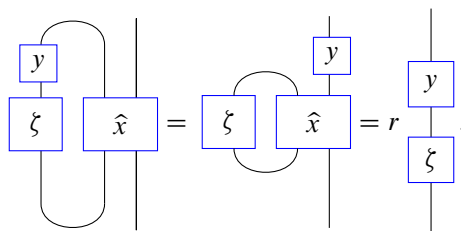


We have $\zeta y = y\zeta$.

(2) For any $y \in \mathcal{C}$,

$$(\zeta y) * \hat{x} = (\zeta * \hat{x})y = r\zeta y,$$

pictorially,



(3) Note that $\mathcal{P}_{1,\pm} \cong \bigoplus_{k=1}^m M_{n_k}(\mathbb{C})$. Suppose $\mathcal{D} \subseteq M_{n_{k_0}}(\mathbb{C})$. Then $\mathcal{D}' \cap M_{n_{k_0}}(\mathbb{C}) \cong I_n \otimes M_t(\mathbb{C}) \oplus M_{n_{k_0}-nt}(\mathbb{C})$ for some $t \in \mathbb{N}$ via unitary equivalence α . Since $\zeta \in \mathcal{D}' \cap \mathcal{P}_{1,\pm}$, we have that $p_{\mathcal{D}}\zeta$ is in $\alpha^{-1}(I_n \otimes M_t(\mathbb{C}) \oplus 0)$. Thus $\{p_{\mathcal{D}}\zeta\}''\mathcal{D} \cong \{p_{\mathcal{D}}\zeta\}'' \otimes \mathcal{D}$. Since $\zeta p_{\mathcal{D}} = p_{\mathcal{D}}\zeta$, we have $\{\zeta\}''\mathcal{D} = \{p_{\mathcal{D}}\zeta\}''\mathcal{D}$. ■

We next prove $\zeta\mathcal{C}^+ = \mathcal{E}^+$ under some conditions and begin with the following two lemmas.

Lemma 6.5. *Let $q \in \mathcal{P}_{1,\pm}$ be a projection such that $q \leq p_{\max}$. Suppose $p_{\max} \in \mathcal{C}$. Then $q \in \mathcal{C}$ if and only if $\mathcal{R}(q * \hat{x}) \leq q$ and $\mathcal{R}((p_{\max} - q) * \hat{x}) \leq p_{\max} - q$.*

Proof. Note that a projection $q \in \mathcal{C}$ if and only if

$$\begin{array}{c} \overline{q} \quad I - q \\ \hline \hat{x} \end{array} = 0 \quad \text{and} \quad \begin{array}{c} \overline{I - q} \quad q \\ \hline \hat{x} \end{array} = 0.$$

They are equivalent to

$$\begin{array}{c} \text{---} \\ | \\ \boxed{I - q} \\ | \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ | \\ \boxed{q} \\ | \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ | \\ \boxed{\hat{x}} \\ | \\ \text{---} \end{array} = 0 \quad \text{and} \quad \begin{array}{c} \text{---} \\ | \\ \boxed{q} \\ | \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ | \\ \boxed{I - q} \\ | \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ | \\ \boxed{\hat{x}} \\ | \\ \text{---} \end{array} = 0.$$

Note that

$$\begin{array}{c} \text{---} \\ | \\ \boxed{q} \\ | \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ | \\ \boxed{I - q} \\ | \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ | \\ \boxed{\hat{x}} \\ | \\ \text{---} \end{array} = \begin{array}{c} \text{---} \\ | \\ \boxed{q} \\ | \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ | \\ \boxed{I - p_{\max}} \\ | \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ | \\ \boxed{\hat{x}} \\ | \\ \text{---} \end{array} + \begin{array}{c} \text{---} \\ | \\ \boxed{q} \\ | \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ | \\ \boxed{p_{\max} - q} \\ | \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ | \\ \boxed{\hat{x}} \\ | \\ \text{---} \end{array}.$$

By the assumption, we see the lemma is true. $\blacksquare$

We say a nonzero positive element $y \in \mathcal{P}_{1,\mp}$ is a right PF eigenvector of an $\mathfrak{F}$ -positive element x if $\hat{x} * y = ry$.

Lemma 6.6. *Suppose $\mathcal{P}$ is a C^* -planar algebra and $x \in \mathcal{P}_{2,\pm}$ is $\mathfrak{F}$ -positive. Suppose that there exists a right PF eigenvector $y_1 \in \mathcal{P}_{1,\mp}$ such that $p_{\max} \leq \mathcal{R}(y_1)$. Then for any $y \in \mathcal{E}^+$ satisfying $\mathcal{R}(y)\zeta = \zeta\mathcal{R}(y)$, we have $\mathcal{R}((p_{\max} - \mathcal{R}(y)) * \hat{x}) = p_{\max} - \mathcal{R}(y)$.*

Proof. Note that

$$r\zeta = \zeta * \hat{x} = (\mathcal{R}(y)\zeta) * \hat{x} + ((p_{\max} - \mathcal{R}(y))\zeta) * \hat{x}.$$

Recall that $\mathcal{R}((\mathcal{R}(y)\zeta) * \hat{x}) = \mathcal{R}(y)$. We obtain

$$r\mathcal{R}(y)\zeta = r\mathcal{R}(y)\zeta\mathcal{R}(y) = (\mathcal{R}(y)\zeta) * \hat{x} + \mathcal{R}(y)((p_{\max} - \mathcal{R}(y))\zeta) * \hat{x}\mathcal{R}(y).$$

Note that

$$(\mathcal{R}(y)\zeta) * \hat{x} * y_1 = r(\mathcal{R}(y)\zeta) * y_1.$$

By taking the convolution of y_1 , we see that

$$(\mathcal{R}(y)((p_{\max} - \mathcal{R}(y))\zeta) * \hat{x})\mathcal{R}(y)) * y_1 = 0.$$

By the assumption that $p_{\max} \leq \overline{\mathcal{R}(y_1)}$ again, we see that $\mathcal{R}(y)((p_{\max} - \mathcal{R}(y))\zeta) * \hat{x}\mathcal{R}(y) = 0$. Thus

$$\mathcal{R}((p_{\max} - \mathcal{R}(y)) * \hat{x}) = \mathcal{R}(((p_{\max} - \mathcal{R}(y))\zeta) * \hat{x}) \leq p_{\max} - \mathcal{R}(y).$$

Note that

$$\begin{aligned} p_{\max} &= \mathcal{R}(p_{\max} * \hat{x}) \\ &= \mathcal{R}(\mathcal{R}(y) * \hat{x}) \vee \mathcal{R}((p_{\max} - \mathcal{R}(y)) * \hat{x}) \\ &= \mathcal{R}(y) \vee \mathcal{R}((p_{\max} - \mathcal{R}(y)) * \hat{x}). \end{aligned}$$

We have

$$\mathcal{R}((p_{\max} - \mathcal{R}(y)) * \hat{x}) = p_{\max} - \mathcal{R}(y).$$

This completes the proof of the proposition. $\blacksquare$

Corollary 6.7. *Suppose $\mathcal{P}$ is a C^* -planar algebra and $x \in \mathcal{P}_{2,\pm}$ is $\mathfrak{F}$ -positive. Suppose that there exists a right PF eigenvector $y_1 \in \mathcal{P}_{1,\mp}$ such that $p_{\max} = \overline{\mathcal{R}(y_1)}$. Then for any $y \in \mathcal{E}^+$, $\zeta \mathcal{R}(y) = \mathcal{R}(y) \zeta$ if and only if $\mathcal{R}(y) \in \mathcal{C}^+$.*

Proof. Note that $p_{\max} = \overline{\mathcal{R}(y_1)}$ implies $p_{\max} \in \mathcal{C}^+$. From Lemmas 6.5 and 6.6, we could prove the corollary. $\blacksquare$

Proposition 6.8. *Suppose $\mathcal{P}$ is a C^* -planar algebra and $x \in \mathcal{P}_{2,\pm}$ is $\mathfrak{F}$ -positive. Suppose that there exists a right PF eigenvector $y_1 \in \mathcal{P}_{1,\mp}$ such that $p_{\max} = \overline{\mathcal{R}(y_1)}$, and $\zeta \mathcal{R}(y) = \mathcal{R}(y) \zeta$ for any $y \in \mathcal{E}^+$. Then*

$$\mathcal{E}^+ = \zeta \mathcal{C}^+.$$

Proof. By Theorem 6.4, we have $\zeta \mathcal{C}^+ \subseteq \mathcal{E}^+$. For any extreme point $y \in (\mathcal{E}^+)_1$, we have $\mathcal{R}(y) \in \mathcal{C}^+$ from Corollary 6.7. Note that $\frac{\zeta \mathcal{R}(y)}{\text{tr}_{1,\pm}(\zeta \mathcal{R}(y))} \in (\mathcal{E}^+)_1$ and $\mathcal{R}(\zeta \mathcal{R}(y)) = \mathcal{R}(y)$. By Proposition 4.13, we have

$$\frac{\zeta \mathcal{R}(y)}{\text{tr}_{1,\pm}(\zeta \mathcal{R}(y))} = y.$$

Thus $\mathcal{E}^+ \subseteq \zeta \mathcal{C}^+$. In summary, we have that $\mathcal{E}^+ = \zeta \mathcal{C}^+$. $\blacksquare$

We need a little more effort to release the conditions in Proposition 6.8. Define

$$x_{\zeta} := \begin{array}{c} \begin{array}{cc} \boxed{\zeta^{-1/2}} & \boxed{\overline{\zeta^{-1/2}}} \\ \swarrow & \searrow \\ \boxed{x} \\ \swarrow & \searrow \\ \boxed{\zeta^{1/2}} & \boxed{\overline{\zeta^{1/2}}} \end{array} \end{array}. \quad (6.11)$$

Remark 6.9. It is clear that x_{ζ} is $\mathfrak{F}$ -positive. Moreover, the spectral radius of x_{ζ} is the same as x , that is r , and $p_{\max} * \hat{x}_{\zeta} = r p_{\max}$.

Notation 6.10. We denote the C^* -algebra $\mathcal{C}$ for x_ζ by $\mathcal{C}_\zeta$, where $\mathcal{C}$ is defined in equation (6.9). For clarity, we also write the graphical representation for $\mathcal{C}_\zeta$.

$$\mathcal{C}_\zeta := \left\{ y \in \mathcal{P}_{1,\pm} : \begin{array}{c} \overline{y} \\ \boxed{x_\zeta} \end{array} = \begin{array}{c} \boxed{x_\zeta} \\ y \end{array}, \begin{array}{c} y \\ \boxed{x_\zeta} \end{array} = \begin{array}{c} \boxed{x_\zeta} \\ y \end{array} \right\} \subseteq \mathcal{P}_{1,\pm}. \quad (6.12)$$

We denote the PF eigenspace $\mathcal{E}$ for x_ζ by $\mathcal{E}_\zeta$, where $\mathcal{E}$ is defined in Notation 4.12. Specifically,

$$\mathcal{E}_\zeta = \{y \in \mathcal{P}_{1,\pm} : y * \widehat{x}_\zeta = ry\} \subseteq \mathcal{P}_{1,\pm}.$$

It is clear that $\mathcal{E}_\zeta^+ = \zeta^{-1/2} \mathcal{E}^+ \zeta^{-1/2}$.

Theorem 6.11 (C^* -algebraic structure). *Suppose $\mathcal{P}$ is a C^* -planar algebra and $x \in \mathcal{P}_{2,\pm}$ is $\mathfrak{F}$ -positive. Suppose that there exists a right PF eigenvector $y_1 \in \mathcal{P}_{1,\mp}$ of x such that $p_{\max} \wedge \overline{\mathcal{R}(y_1)}^\perp = 0$. Then*

$$\mathcal{E}^+ = \zeta^{1/2} \mathcal{C}_\zeta^+ \zeta^{1/2}.$$

Proof. We have

$$\widehat{x}_\zeta * (\overline{\zeta^{1/2} y_1 \zeta^{1/2}}) = \overline{\zeta^{1/2} (\widehat{x}_p * y_1) \zeta^{1/2}} = \overline{\zeta^{1/2} (\widehat{x} * y_1) \zeta^{1/2}} = r(\overline{\zeta^{1/2} y_1 \zeta^{1/2}}).$$

Thus $\overline{\zeta^{1/2} y_1 \zeta^{1/2}}$ is a right PF eigenvector of x_ζ such that $p_{\max} = \overline{\mathcal{R}(\zeta^{1/2} y_1 \zeta^{1/2})}$. Take the maximal-support eigenvector of x_ζ as $p_{\max}$, we have $p_{\max} \mathcal{R}(y) = \mathcal{R}(y) = \mathcal{R}(y) p_{\max}$ for any $y \in \mathcal{E}_\zeta^+$. So x_ζ satisfies the conditions of Proposition 6.8, and we have $\mathcal{E}_\zeta^+ = p_{\max} \mathcal{C}_\zeta^+$. From $\mathcal{E}_\zeta^+ = \zeta^{-1/2} \mathcal{E}^+ \zeta^{-1/2}$, we could obtain the conclusion. ■

Remark 6.12. Let $x \in \mathcal{P}_{2,\pm}$ be an $\mathfrak{F}$ -positive element such that

$$\begin{array}{c} \boxed{\widehat{x}} \end{array} = I. \quad (6.13)$$

Then $\widehat{x}: \mathcal{P}_{1,\pm} \rightarrow \mathcal{P}_{1,\pm}$, $y \mapsto y * \widehat{x}$, defines a CPTP map. Theorem 6.11 gives the C^* -algebraic structure of the PF eigenspace of x .

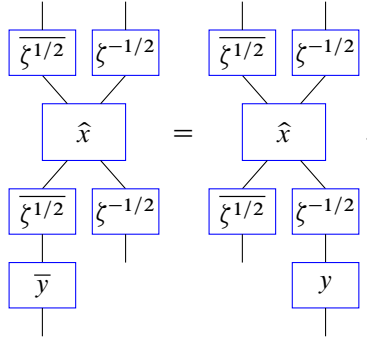
6.3. Algebraic structures of eigenspaces

In this section, we let $p = p_{\max}$ for simplicity, denote the algebras $\mathcal{A}$ and $\mathcal{B}$ for x_p by $\mathcal{A}_p$ and $\mathcal{B}_p$, where $\mathcal{A}$ and $\mathcal{B}$ are defined in equations (6.7) and (6.8), and x_p is defined in equation (6.2). We denote the PF eigenspace $\mathcal{E}$ for x_p by $\mathcal{E}_p$, where $\mathcal{E}$ is defined in Notation 4.12. It is clear that $\mathcal{E}_p^+ = \mathcal{E}^+$.

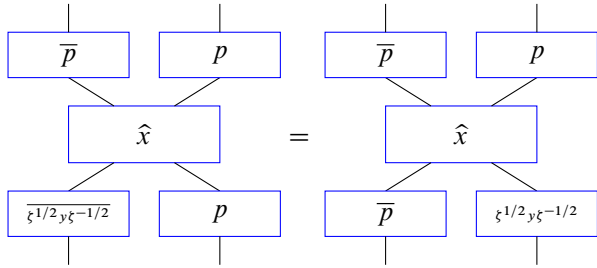
Lemma 6.13. Suppose $\mathcal{P}$ is a C^* -planar algebra and $x \in \mathcal{P}_{2,\pm}$ is $\mathfrak{F}$ -positive. Then

$$p\mathcal{C}_\zeta^+ \subseteq \zeta^{-1/2}\mathcal{A}_p\zeta^{1/2} \cap \zeta^{1/2}\mathcal{B}_p\zeta^{-1/2}.$$

Proof. For any $y \in p\mathcal{C}_\zeta^+$,



This is equivalent to



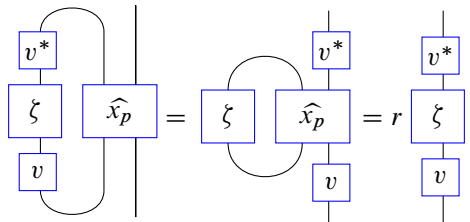
So $\zeta^{1/2}y\zeta^{-1/2} \in \mathcal{A}_p$. Analogously, we have $\zeta^{-1/2}y\zeta^{1/2} \in \mathcal{B}_p$. ■

Now we obtain the structure of the eigenvector space with respect to $\mathcal{A}_p$ and $\mathcal{B}_p$.

Theorem 6.14 (Algebraic structure I). Suppose $\mathcal{P}$ is a C^* -planar algebra and $x \in \mathcal{P}_{2,\pm}$ is $\mathfrak{F}$ -positive. Suppose that there exists a right PF eigenvector $y_1 \in \mathcal{P}_{1,\mp}$ of x such that $p_{\max} \wedge \overline{\mathcal{R}(y_1)}^\perp = 0$. Then

$$\mathcal{E}^+ = \{v\zeta v^* : v \in \mathcal{A}_p\} = \{v^*\zeta v : v \in \mathcal{B}_p\}.$$

Proof. Suppose that $v \in \mathcal{A}_p$. Then



So $v\zeta v^* \in \mathcal{E}_p^+ = \mathcal{E}^+$.

By Theorem 6.11, for any $y \in \mathcal{E}^+$, there exists $w \in \mathcal{C}_\zeta^+$ such that $y = \zeta^{1/2} w \zeta^{1/2}$. By Lemma 6.13, we see that there exists $v \in \mathcal{A}_p$ such that

$$pw^{1/2} = \zeta^{-1/2} v \zeta^{1/2}.$$

Hence

$$y = \zeta^{1/2} (\zeta^{-1/2} v \zeta^{1/2}) (\zeta^{-1/2} v \zeta^{1/2})^* \zeta^{1/2} = v \zeta v^*.$$

This completes the proof of the theorem. $\blacksquare$

We obtain another representation of $\mathcal{E}^+$ with a little effort.

Theorem 6.15 (Algebraic structure II). *Suppose $\mathcal{P}$ is a C^* -planar algebra and $x \in \mathcal{P}_{2,\pm}$ is $\mathfrak{F}$ -positive. Suppose that there exists a right PF eigenvector $y_1 \in \mathcal{P}_{1,\mp}$ of x such that $p_{\max} \wedge \overline{\mathcal{R}(y_1)}^\perp = 0$. Then*

$$\mathcal{E}^+ = (\mathcal{A}_p \zeta)^+ = (\zeta \mathcal{B}_p)^+.$$

Proof. For any $v \in \mathcal{A}_p$, we have $\mathcal{R}(v\zeta) = \mathcal{R}(v\zeta^{1/2}) = \mathcal{R}(v\zeta v^*) \leq p$. The last inequality is due to $v\zeta v^* \in \mathcal{E}^+$ in Theorem 6.14. So $p\mathcal{A}_p\zeta = \mathcal{A}_p\zeta$. By Theorem 6.11 and Lemma 6.13, we know that $\mathcal{E}^+ = \zeta^{1/2} \mathcal{C}_\zeta^+ \zeta^{1/2} \subseteq p\mathcal{A}_p\zeta = \mathcal{A}_p\zeta$. On the other hand, $\mathcal{A}_p\zeta \subseteq \mathcal{E}$, so $(\mathcal{A}_p\zeta)^+ \subseteq \mathcal{E}^+$. Thus $\mathcal{E}^+ = (\mathcal{A}_p\zeta)^+$. Similarly, $\mathcal{E}^+ = (\zeta \mathcal{B}_p)^+$. $\blacksquare$

Remark 6.16. Let $x \in \mathcal{P}_{2,\pm}$ be an $\mathfrak{F}$ -positive element such that $\hat{x}: \mathcal{P}_{1,\pm} \rightarrow \mathcal{P}_{1,\pm}$, $y \mapsto y * \hat{x}$, defines a CPTP map. Theorems 6.14 and 6.15 give the algebraic structures of the PF eigenspace of x .

Remark 6.17. Let $x \in \mathcal{P}_{2,\pm}$ be an $\mathfrak{F}$ -positive element such that

$$\begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} \boxed{\hat{x}} \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} = I. \quad (6.14)$$

Then $\hat{x}: \mathcal{P}_{1,\pm} \rightarrow \mathcal{P}_{1,\pm}$, $y \mapsto y * \hat{x}$, defines a unital completely positive map. In [8], Choi and Effros proved that the PF eigenspace $\mathcal{E}$ is a C^* -algebra with respect to the multiplication $\circ$ defined as follows:

$$y_1 \circ y_2 = \lim_{k \rightarrow \infty} \frac{1}{k+1} \sum_{j=0}^k \frac{(y_1 y_2) * \hat{x}^{*j}}{r^{j+1}}$$

for any $y_1, y_2 \in \mathcal{E}$. Note that a CPTP map is the adjoint of a unital completely positive map. Here we are able to consider completely positive maps besides trace-preserving case and describe the C^* -algebraic structures and algebraic structures of their PF eigenspaces.

7. Quantum error correction

Quantum error correction provides a method to protect quantum information. Developing robust quantum error correction is necessary for the realization of scalable quantum computing. The first quantum error correction scheme for a single qubit state was proposed by Shor [41]. Inspired by the PF theorem, we encode logical qubits by mixed states, and we prove Theorem 7.1 for quantum error corrections of mixed states. Using the multiplicative structure of the PF eigenspace, these mixed states in the PF eigenspace could be regarded as the tensor product of pure states and a fixed ancillary mixed state. In Section 7.2, we contrast this approach with the Knill–Laflamme Theorem 7.4, that characterizes quantum error corrections of pure states.

7.1. Quantum error correction

A quantum channel Φ on $M_{2^n}(\mathbb{C})$ is a CPTP linear map. (Sometimes one allows trace non-increasing maps.)

We recall some basic notions of error correction (e.g., [31]). Let $\Phi_{\mathcal{E}}$ be a quantum channel with Kraus operators $\{E_i\}$ representing errors. Given a reference state $\rho \in M_n(\mathbb{C})$ with $\mathcal{R}(\rho) = p$ and $\mathcal{S}$ a subspace of $pM_{2^n}(\mathbb{C})p$. Error correction means there exists a CPTP map $\Phi_{\mathcal{R}}$ such that

$$\Phi_{\mathcal{R}}\Phi_{\mathcal{E}}(\sigma) = \sigma \quad \forall \sigma \in \mathcal{S}.$$

The map $\Phi_{\mathcal{R}}$ is known as the recovery map of $\Phi_{\mathcal{E}}$ for the subspace $\mathcal{S}$. A well-known class of recovery maps is the Petz recovery map [33–35], defined as

$$\Phi_{\mathcal{R}}^{\rho}(\sigma) = \rho^{1/2} \Phi_{\mathcal{E}}^*(\Phi_{\mathcal{E}}(\rho)^{-1/2} \sigma \Phi_{\mathcal{E}}(\rho)^{-1/2}) \rho^{1/2}, \quad (7.1)$$

where $\Phi_{\mathcal{E}}^*$ is the adjoint operator of $\Phi_{\mathcal{E}}$. The recovery space of $\Phi_{\mathcal{R}}^{\rho}$ is defined by the invariability of the relative entropy under the action of $\Phi_{\mathcal{E}}$ (see, e.g., [28]):

$$S(\sigma \parallel \rho) = S(\Phi_{\mathcal{E}}(\sigma) \parallel \Phi_{\mathcal{E}}(\rho)) \quad \forall \sigma \in \mathcal{S}.$$

In this section, we apply the structure of PF eigenspace to study quantum error correction. Precisely, we prove the following theorem.

Theorem 7.1 (PF to QEC). *Suppose $\Phi_{\mathcal{E}}: M_{2^n}(\mathbb{C}) \rightarrow M_{2^n}(\mathbb{C})$ is a CPTP map with Kraus operators $\{E_i\}_{i \in \mathcal{I}}$ representing errors. For any density matrix $\rho \in M_{2^n}(\mathbb{C})$ with $\mathcal{R}(\rho) = p$, there exists a CPTP recovery map $\Phi_{\mathcal{R}}$ such that for any*

$$D \in \mathcal{D} = \{\rho^{1/2} E_j^* \Phi_{\mathcal{E}}(\rho)^{-1/2} E_i p : i, j \in \mathcal{I}' \cap pM_{2^n}(\mathbb{C})p,$$

we have $\rho' D = D \rho'$ and

$$\Phi_{\mathcal{R}}^{\rho} \Phi_{\mathcal{E}}(\rho' D) = \rho' D, \quad (7.2)$$

where ρ' is the Perron–Frobenius eigenvector of $\Phi_{\mathcal{R}}^{\rho} \Phi_{\mathcal{E}}$.

Proof. Let $\Phi_{\mathcal{E}}^{\rho}$ be the recovery map defined in equation (7.1). We consider the CPTP map $\Phi_{\mathcal{R}}^{\rho} \Phi_{\mathcal{E}}$ acting on the subalgebra $pM_{2^n}(\mathbb{C})p$. It is clear that $\rho^{1/2} E_j^* \Phi_{\mathcal{E}}(\rho)^{-1/2} E_i p$, $i, j \in \mathcal{I}$, are Kraus operators of $\Phi_{\mathcal{R}}^{\rho} \Phi_{\mathcal{E}}$. Next we let $\Phi_{\mathcal{R}}^{\rho} \Phi_{\mathcal{E}}$ be the Fourier transform of $\Phi_{\mathcal{R}}^{\rho} \Phi_{\mathcal{E}}$ defined in equation (2.21). By Remark 6.3, the subalgebra $\mathcal{D}$ is equal to the C^* -algebra $\mathcal{C}$ in equation (6.10) associated to $\Phi_{\mathcal{R}}^{\rho} \Phi_{\mathcal{E}}$.

Let ρ' be the Perron–Frobenius eigenvector of $\Phi_{\mathcal{R}}^{\rho} \Phi_{\mathcal{E}}$, defined in equation (6.6). Precisely,

$$\rho' = \lim_{\mathbb{R} \ni z \rightarrow r^+} (z - 1) \sum_{j=0}^{\infty} \frac{(\Phi_{\mathcal{R}}^{\rho} \Phi_{\mathcal{E}})^j(p)}{z^{j+1}}.$$

By Theorem 6.4, we have $\rho' D = D \rho'$. Moreover,

$$\Phi_{\mathcal{R}}^{\rho} \Phi_{\mathcal{E}}(\rho' D) = \Phi_{\mathcal{R}}^{\rho} \Phi_{\mathcal{E}}(\rho') D = \rho' D.$$

We obtain the conclusion. ■

Remark 7.2. Let $\Phi_{\mathcal{R}}^{\rho'}$ be the recovery map with the reference state ρ' . Note that ρ' and ρ are faithful states on $pM_{2^n}(\mathbb{C})p$. Since $\Phi_{\mathcal{R}}^{\rho} \Phi_{\mathcal{E}}(\rho') = \rho'$, then by [34, Theorem 3], $\Phi_{\mathcal{R}}^{\rho'} = \Phi_{\mathcal{R}}^{\rho}$.

Remark 7.3. We can also see Theorem 7.1 from the fact that the adjoint map $(\Phi_{\mathcal{R}}^{\rho} \Phi_{\mathcal{E}})^{\dagger}$ in the Heisenberg picture on the support of ρ is a unital completely positive map symmetric to the KMS inner product of ρ . Then the fixed point is the multiplicative domain which is the commutant of the Kraus operators. Hence it is a finite-dimensional C^* -algebra. By taking adjoint, one can obtain the fixed point (error correcting space) of $\Phi_{\mathcal{R}}^{\rho} \Phi_{\mathcal{E}}$.

7.2. Comparison with Knill–Laflamme theory

Here we relate our Theorem 7.1 to the well-known result of Knill and Laflamme (KL).

Theorem 7.4 (Knill–Laflamme [24]). *Suppose a noisy environment is a CPTP map $\Phi_{\mathcal{E}}$ with Kraus operators $\{E_i\}$. For a projection $p \in M_{2^n}(\mathbb{C})$, there is a CPTP recovery map $\Phi_{\mathcal{R}}$ such that*

$$\Phi_{\mathcal{R}} \Phi_{\mathcal{E}}(D) = D \quad \forall D \in pM_{2^n}(\mathbb{C})p$$

if and only if

$$p E_i^* E_j p = \lambda_{i,j} p, \lambda_{i,j} \in \mathbb{C} \quad \forall i, j.$$

In our framework, the KL condition is equivalent to

$$\mathcal{D} = \{\rho^{1/2} E_j^* \Phi_{\mathcal{E}}(\rho)^{-1/2} E_i p : i, j \in \mathcal{I}\}' \cap pM_{2^n}(\mathbb{C})p = pM_{2^n}(\mathbb{C})p.$$

We now show how this comes. Suppose that $\mathcal{D} = pM_{2^n}(\mathbb{C})p$, then

$$\rho^{1/2} E_j^* \Phi_{\mathcal{E}}(\rho)^{-1/2} E_i p = \lambda_{i,j} p.$$

We have

$$\begin{aligned} \sum_{j \in \mathcal{I}} p E_k^* \Phi_{\mathcal{E}}(\rho)^{-1/2} E_j \rho E_j^* \Phi_{\mathcal{E}}(\rho)^{-1/2} E_i p &= p E_k^* \Phi_{\mathcal{E}}(\rho)^{-1/2} \Phi_{\mathcal{E}}(\rho) \Phi_{\mathcal{E}}(\rho)^{-1/2} E_i p \\ &= p E_k^* E_i p. \end{aligned}$$

Note that the left side of the equation equals $\sum_{j \in \mathcal{I}} \lambda_{i,j} \overline{\lambda_{k,j}} p$. Hence the KL condition holds. On the contrary, suppose that $p E_i^* E_j p = \lambda_{i,j} p$. Then

$$\begin{aligned} \rho^{1/2} E_j^* \Phi_{\mathcal{E}}(\rho)^{-1/2} E_i p &= \rho^{1/2} E_j^* \left(\sum_{k \in \mathcal{I}} E_k \rho E_k^* \right)^{-1/2} E_i p \\ &= \rho^{1/2} E_j^* \sum_{k \in \mathcal{I}} (E_k \rho E_k^*)^{-1/2} E_i p \\ &= \rho^{1/2} E_j^* \sum_{k \in \mathcal{I}} (\lambda_{k,k}^{-3/2} E_k \rho^{-1/2} E_k^*) E_i p \\ &= \delta_{i,j} \lambda_{i,j}^{1/2} p. \end{aligned}$$

In the above calculations, we assume that $p E_i^* E_j p = \delta_{i,j} \lambda_{i,j} p$. We have that

$$(E_k \rho E_k^*)(E_j \rho E_j^*) = (E_j \rho E_j^*)(E_k \rho E_k^*) = 0, \quad j \neq k.$$

Hence

$$\left(\sum_{k \in \mathcal{I}} E_k \rho E_k^* \right)^{-1/2} = \sum_{k \in \mathcal{I}} (E_k \rho E_k^*)^{-1/2}.$$

The second equality holds. Note that $(E_k \rho^{1/2} E_k^*)^2 = \lambda_{k,k} E_k \rho E_k^*$, hence

$$(E_k \rho E_k^*)^{1/2} = \lambda_{k,k}^{-1/2} E_k \rho^{1/2} E_k^*.$$

Moreover,

$$(E_k \rho^{1/2} E_k^*)(E_k \rho^{-1/2} E_k^*) = \lambda_{k,k} E_k p E_k^*.$$

Note that $(E_k p E_k^*)^2 = \lambda_{k,k} E_k p E_k^*$. Hence $E_k p E_k^* / \lambda_{k,k}$ is the range projection of $E_k p E_k^*$. Then

$$(E_k \rho^{1/2} E_k^*)^{-1} = \lambda_{k,k}^{-2} E_k \rho^{-1/2} E_k^*.$$

Therefore,

$$(E_k \rho E_k^*)^{-1/2} = \lambda_{k,k}^{1/2} (E_k \rho^{1/2} E_k^*)^{-1} = \lambda_{k,k}^{-3/2} E_k \rho^{-1/2} E_k^*.$$

The third equality holds. We obtain that $\mathcal{D} = \{\delta_{i,j} \lambda_{i,j}^{1/2} p : i, j \in \mathcal{I}\}' \cap p M_{2^n}(\mathbb{C}) p = p M_{2^n}(\mathbb{C}) p$.

The KL condition is a strong constraint to construct p . In this case, the correctable information is $pM_{2^n}(\mathbb{C})p$. In Theorem 7.1, there is no constraint on p , and the recovery map always exists. The correctable information is

$$\mathcal{D} = \{\rho^{1/2} E_j^* \Phi_\varepsilon(\rho)^{-1/2} E_i p : i, j \in \mathcal{I}\}' \cap pM_{2^n}(\mathbb{C})p.$$

In principle, p induces a good QECC if $\mathcal{D}$ is large. The PF theorem suggests that we study QECCs whose logical qubits are mixed states.

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