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A Kakeya maximal estimate for regulus strips

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Abstract. We prove Kakeya-type estimates for regulus strips. As a result, we obtain another epsilon improvement over the Kakeya conjecture in $\mathbb{R}^3$, by showing that the regulus strips in the SL_2 example are essentially disjoint. We also establish an L^p inequality regarding a Nikodym-type maximal function in the first Heisenberg group.

1. Introduction

Recall that a Kakeya set in $\mathbb{R}^n$ is a compact set $K \subset \mathbb{R}^n$ that contains a unit line segment pointing in every direction. The three-dimensional Kakeya conjecture asserts that every Kakeya set in $\mathbb{R}^3$ has Hausdorff dimension 3. After standard discretization arguments, it is a consequence of the following statement.

Conjecture 1.1. *For all $\varepsilon > 0$, there exist $c_\varepsilon > 0$ and $M = M(\varepsilon) > 0$ such that the following is true for all $\delta \in (0,1)$. Let L be a family of directional δ-separated lines and let $E \subset [0,1]^3$ be a union of δ-balls. Suppose $|\ell \cap E| \geq \lambda$ for all $\ell \in L$. Then*

$$|E| \geq c_\varepsilon \delta^\varepsilon \lambda^M (\delta^2 \# L). \tag{1.1}$$

When $\lambda \sim 1$, (1.1) can be rephrased as follows: suppose $\mathcal{T}$ is a family of δ-tubes pointing at δ-separated directions. Then $|\bigcup_{\mathcal{T}}| \gtrapprox \delta^2 \# \mathcal{T}$ ($A \lessapprox B$ means $A \leq C_\varepsilon \delta^{-\varepsilon} B$ for all $\varepsilon > 0$). In other words, the δ-tubes in $\mathcal{T}$ are essentially disjoint.

The best known result regarding Conjecture 1.1 was obtained by Katz–Zahl [4], who showed that a three-dimensional Kakeya set must have Hausdorff dimensions greater than $5/2 + \varepsilon_0$ for some absolute $\varepsilon_0 > 0$. In the same paper, an enemy scenario of the Kakeya conjecture, called the "SL_2 example", was discovered. To describe this example, we briefly go through the definition of a regulus strip.

Let $\delta \in (0,1)$. A regulus strip is the intersection of the δ-neighborhood of a regulus (a quadratic surface in $\mathbb{R}^3$ doubly-ruled by lines) and the $\delta^{1/2}$-neighborhood of a line contained within that regulus. In other words, a regulus strip is a $(\delta \times \delta^{1/2} \times 1)$-plank twisting at a constant speed. Note that a single regulus strip contains $\sim \delta^{-1/2}$ many δ-tubes.

The SL_2 example, roughly speaking, contains around $\delta^{-3/2}$ well-spaced regulus strips. Hence, it also contains approximately δ^{-2} many δ-tubes. Thus, if a Kakeya phenomenon

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is expected for the SL_2 example, its measure should be $\gtrapprox 1$. It was shown in [4] that the SL_2 example has a measure strictly bigger than $\delta^{1/2}$, which is a key ingredient of their $(5/2+\varepsilon_0)$-bound for the Kakeya conjecture in $\mathbb{R}^3$.

The SL_2 example motivates the study of the SL_2-Kakeya set in [1,3]. An SL_2-Kakeya set is a Kakeya set whose unit line segments come from the SL_2-lines

$$\mathcal{L}_{SL_2} := \{\ell_{(a,b,c,d)} : ad - bc = 1\}.$$

In particular, a certain point-line duality regarding SL_2-lines was observed in [1], and the authors there used tools from Fourier analysis developed in [2] to study SL_2-Kakeya sets. We remark that both the SL_2 example and the SL_2-Kakeya set are useful models of "plany" Kakeya sets. For further discussion, we refer to [4] and [3].

In this paper, we apply similar tools from Fourier analysis to study regulus strips. Our result can be viewed as a Kakeya maximal estimate for regulus strips. As a consequence, we show that the measure of the SL_2 example is $\gtrapprox 1$, which results in an epsilon improvement over the epsilon improvement of the Kakeya conjecture in $\mathbb{R}^3$ obtained in [4].

To give the formal definition of regulus strips, let us first recall a standard parameterization of non-horizontal lines in $\mathbb{R}^3$.

Definition 1.2. Given a line $\ell_{(a,b,c,d)}$: $(a,b,0)+\mathrm{span}(c,d,1)$ in $\mathbb{R}^3$, we define $\breve{\ell} := (a,b,c,d)$ to be a point in the parameter space $\mathbb{R}^4$. Generally, if $S = \bigcup \ell \cap [0,1]^3$ is a union of line segments in $\mathbb{R}^3$, define $\breve{S} = \bigcup \breve{\ell}$ as a set in $\mathbb{R}^4$.

For each $\ell \in \mathcal{L}_{SL_2}$ and each $p = (p_1, p_2, p_3) \in \ell$, direct calculation shows that all the lines $\ell' \in \mathcal{L}_{SL_2}$ intersecting p are trapped in a plane P_p containing ℓ, whose normal direction is $(p_2, -p_1, 1)$ (see (1.3) in [3]). Let $\ell(p) = p + \mathrm{span}(p_1, p_2, 0)$ be the intersection of P_p and the horizontal plane at height p_3. This defines a regulus $R(\ell)$ as

$$R(\ell) := \bigcup_{p\in\ell} \ell(p). \tag{1.2}$$

The regulus $R(\ell)$ is a double-ruled surface. The family of lines $\{\ell(p) : p \in \ell\}$ forms one ruling of $R(\ell)$. Note that $\{\ell' : \breve{\ell}' \in \mathrm{span}(\breve{\ell})\}$ forms another ruling of $R(\ell)$ (just need to check that such an ℓ' intersects $\ell(p)$ for all $p \in \ell$).

Definition 1.3 (Regulus strip). Let $\ell \in \mathcal{L}_{SL_2}$. Define a *regulus strip*

$$S = S(\ell) := N_\delta(R(\ell)) \cap N_{\delta^{1/2}}(\ell) \cap [0,1]^3,$$

which is a twisted plank of dimensions $\delta \times \delta^{1/2} \times 1$.

Since $\{\ell' : \breve{\ell}' \in \mathrm{span}(\breve{\ell})\}$ forms a ruling of $R(\ell)$, $\breve{S}(\ell)$ is the δ-neighborhood of the $\delta^{1/2}$ line segment $\{(1+u)\breve{\ell} : |u| \le \delta^{1/2}\}$ in the parameter space $\mathbb{R}^4$. In other words, $\breve{S}(\ell)$ is a δ-tube of length $\delta^{1/2}$.

After suitable affine transformations, we only need to focus on the following family of regulus strips:

$$\bar{\mathcal{S}} := \{S(\ell_{(a,b,c,d)}) : a, d \sim 1, ad - bc = 1\}. \tag{1.3}$$

The next definition captures a non-concentration property of the regulus strips.

Definition 1.4. Let $\mathcal{S} \subset \bar{\mathcal{S}}$ be a family of regulus strips and let $W = \bigcup_{S \in \mathcal{S}} \breve{S} \subset \mathbb{R}^4$ be a union of δ-tubes of length $\delta^{1/2}$. We say $\mathcal{S}$ obeys the *two-dimensional ball condition* if for any $r \in [\delta, 1]$ and any four-dimensional r-ball B_r,

$$|W \cap B_r|\delta^{-4} \leq 100(r\delta^{-1})^2. \tag{1.4}$$

Note that when W is a general union of δ-balls, (1.4) is the non-concentration inequality that forces W to obey a standard two-dimensional ball condition. Moreover, it is also the minimal requirement for a family of regulus strips $\mathcal{S}$ to be essentially disjoint, that is, $|\bigcup_{\mathcal{S}}| \gtrapprox (\delta^{3/2}\#\mathcal{S})$.

To see this, suppose $\mathcal{S}$ is a family of regulus strips that fails (1.4). Then there exists an absolute $\varepsilon > 0$ and an r-ball B_r such that $|W \cap B_r|\delta^{-4} \geq \delta^{-\varepsilon}(r\delta^{-1})^2$ for some absolute $\varepsilon > 0$. Assume $\mathcal{S} = \mathcal{S}(B_r) := \{S \in \mathcal{S} : \breve{S} \cap B_r \neq \varnothing\}$. We will show

$$\Big|\bigcup_{\mathcal{S}}\Big| \lesssim \delta^{\varepsilon}(\delta^{3/2}\#\mathcal{S}).$$

If $r \geq \delta^{1/2}$, then $\mathcal{S} \subset \{S \in \bar{\mathcal{S}} : \breve{S} \subset 2B_r\}$. This shows that $\bigcup_{\mathcal{S}}$ is contained in an r-tube, so $|\bigcup_{\mathcal{S}}| \lesssim r^2$. Note that

$$\#\mathcal{S} = \#\mathcal{S}(B_r) \gtrsim \delta^{-\varepsilon}(r\delta^{-1})^2\delta^{1/2}.$$

Thus, we get

$$(\delta^{3/2}\#\mathcal{S}) \gtrsim \delta^{-\varepsilon}r^2 \gtrsim \delta^{-\varepsilon}\Big|\bigcup_{\mathcal{S}}\Big|.$$

If $r \leq \delta^{1/2}$, then $\bigcup_{\mathcal{S}}$ is contained in $N_{2r}(S)$, where $S \in \mathcal{S}$ is any regulus strip. This implies $|\bigcup_{\mathcal{S}}| \lesssim r\delta^{1/2}$. Note that $\#\mathcal{S} = \#\mathcal{S}(B_r) \gtrsim \delta^{-\varepsilon}r\delta^{-1}$. Thus, we have

$$(\delta^{3/2}\#\mathcal{S}) \gtrsim \delta^{-\varepsilon}r\delta^{1/2} \gtrsim \delta^{-\varepsilon}\Big|\bigcup_{\mathcal{S}}\Big|.$$

Now we state the main result of the paper.

Theorem 1.5. *Suppose $\mathcal{S} \subset \bar{\mathcal{S}}$ is a collection of regulus strips satisfying the two-dimensional ball condition* (1.4). *Let $Y : \mathcal{S} \to [0,1]^3$ be a shading such that $Y(S) \subset S \cap [0,1]^3$ for any $S \in \mathcal{S}$. Let $\lambda \in (0,1]$. Suppose $|Y(S)| \geq \lambda|S|$ for all $S \in \mathcal{S}$. Then for any $\varepsilon > 0$,*

$$\Big|\bigcup_{S \in \mathcal{S}} Y(S)\Big| \geq c_\varepsilon \delta^{\varepsilon}\lambda^6 \sum_{S \in \mathcal{S}} |S|.$$

The SL_2 example discovered in [4] is a family of regulus strips $\mathcal{S}$ with $\#\mathcal{S} \approx \delta^{-3/2}$ so that in the parameter space, each $\delta^{1/2}$-ball in $N_{\delta^{1/2}}Z(ad - bc = 1)$ contains ≈ 1 many $\breve{S}$. Note that, up to a factor ≈ 1, the SL_2 example obeys the two-dimensional ball condition (1.4). Therefore, as a direct corollary of Theorem 1.5, we have the following.

Corollary 1.6. *The* SL_2 *example has measure* $\gtrapprox 1$.

As a result, there is another epsilon improvement over the epsilon improvement of the three-dimensional Kakeya conjecture in [4].

Finally, let us discuss an application of Theorem 1.5 in the first Heisenberg group. One feature of the SL_2 lines $\mathcal{L}_{\mathrm{SL}_2}$ is that they coincide with the horizontal lines in the first Heisenberg group (see [1]). In particular, for each $\ell \in \mathcal{L}_{\mathrm{SL}_2}$, the regulus strip $S(\ell)$ is essentially the $\delta^{1/2}$-neighborhood of the horizontal line ℓ under the Korányi metric. See [5].

Therefore, Theorem 1.5 also gives a Nikodym maximal estimate for horizontal lines in the first Heisenberg group.

Definition 1.7. Let T be the δ-neighborhood of an arbitrary horizontal unit line segment contained in the unit ball, under the Korányi metric in the first Heisenberg group $\mathbb{H}$. For any function $f : \mathbb{H} \to \mathbb{C}$, define a Nikodym maximal function as

$$Mf(z) = \sup_{z \in T} \frac{1}{|T|} \int_T |f(z')|\, dz' \tag{1.5}$$

By a standard duality, Theorem 1.5 has the following corollary.

Corollary 1.8. *Let M be the Nikodym maximal operator defined in (1.5). Then for any $\varepsilon > 0$, and when $p \geq 6$,*

$$\|Mf\|_p \leq C_\varepsilon \delta^{-\varepsilon} \|f\|_p.$$

Proof. It suffices to consider $f = \mathbf{1}_E$, where E is a union of disjoint Heisenberg δ-balls contained in the unit ball. For each point $z \in \mathbb{H}$, let $z = (x, y, t)$ be its standard coordinate. Using the foliation of $\mathbb{H}$ by vertical planes through the vertical axis and by the triangle, it suffices to show that

$$\int |M^* \mathbf{1}_E(0, y, t)|^6 \, dy\, dt \lessapprox |E|. \tag{1.6}$$

Here M^* is the maximal operator restricted to horizontal lines that are transverse to the plane $\{x = 0\}$. Note that (1.6) boils down to the weak-type estimate

$$\lambda^6 |F_\lambda| \lessapprox |E|, \tag{1.7}$$

where

$$F_\lambda = \{(y, t) : |M^* \mathbf{1}_E(0, y, t)| \geq \lambda\}.$$

Note that, after discretization, F_λ can be viewed as a union of horizontal $(\delta \times \delta^2)$-rectangles. For each such rectangle, choose a regulus strip S of dimensions $1 \times \delta \times \delta^2$ (the δ-neighborhood of a horizontal line) such that $|S \cap E| \geq \lambda |S|$. Denote by $Y(S) = S \cap E$. It is straightforward to check that the family of regulus strips generated by the set of $(\delta \times \delta^2)$-rectangles from F_λ obeys the two-dimensional ball condition. By Theorem 1.5, we have

$$|E| \geq \Big|\bigcup_S Y(S)\Big| \gtrapprox \lambda^6 \sum_S |S| \sim \lambda^6 |F_\lambda|.$$

This shows (1.7) and hence the corollary. ∎

Notations. We write $A \lesssim B$ if $A \le CB$ for some absolute constant C, and write $A \lessapprox B$ if $A \le C_\varepsilon \delta^{-\varepsilon} B$ for any $\varepsilon > 0$. We use $A = \Omega(B)$ if $B \lesssim A$ and $A = \Omega^*(B)$ if $B \lessapprox A$. If $\mathscr{E}$ is a family of sets in $\mathbb{R}^n$, we use $\bigcup_{\mathscr{E}}$ to denote $\bigcup_{E \in \mathscr{E}} E$. For two sets $A, B \in \mathbb{R}^n$, let $A + B = \{a + b : a \in A, b \in B\}$. For two finite sets E and F, we say E is an $\Omega(c)$-refinement of F if $E \subset F$ and $\#E \gtrsim c\#F$; we say E is a $\Omega^*(c)$-refinement of F if $E \subset F$ and $\#E \gtrapprox c\#F$.

2. Proof of Theorem 1.5

Let $\mathscr{L}$ be the collection of lines $\ell_{(a,b,c,d)}$ satisfying $a - d = 0$. It was observed in [1] that a generic line in $\mathscr{L}_{\mathrm{SL}_2}$ can be parameterized by lines in $\mathscr{L}$ as well. This observation will simplify our calculation later.

Lemma 2.1 (SL_2 is linear). *Lines in $\mathscr{L}_{SL_2}$ satisfying $ad \neq 0$ can be identified as lines in $\mathscr{L}$.*

Proof. For any non-horizontal line $\ell_{(a,b,c,d)}$, let $\ell_{(a,b,c,d)}(t) = (a, b, 0) + t(c, d, 1)$ be its intersection with the horizontal plane at height t. Thus, each line $\ell \in \mathscr{L}_{\mathrm{SL}_2}$ with $ad \neq 0$ has the parameterization

$$\ell(t) = (a + ct,\, b + (1 + bc)t/a,\, t).$$

Choose t_0 and t_1 so that the second entries of $\ell(t_0)$ and $\ell(t_1)$ are 0 and 1, respectively. Hence

$$t_0 = -\frac{ab}{1 + bc} \quad \text{and} \quad t_1 = \frac{a - ab}{1 + bc}.$$

Moreover, we have

$$\ell(t_0) = \Big(\frac{a}{1 + bc}, 0, \frac{-ab}{1 + bc}\Big) \quad \text{and} \quad \ell(t_1) = \Big(\frac{a + ac}{1 + bc}, 1, \frac{a - ab}{1 + bc}\Big).$$

This gives a directional vector

$$v = \ell(t_1) - \ell(t_0) = \Big(\frac{ac}{1 + bc}, 1, \frac{a}{1 + bc}\Big).$$

Thus, if swapping the second and the third coordinates of $\ell(t_0)$, we have

$$\ell(t_0) = \Big(\frac{a}{1 + bc}, \frac{-ab}{1 + bc}, 0\Big) \quad \text{and} \quad v = \Big(\frac{ac}{1 + bc}, \frac{a}{1 + bc}, 1\Big). \tag{2.1}$$

This suggests that the line ℓ can be re-parameterized as $\ell = \ell_{(a',b',c',d')}$ obeying $a' = d'$ ($a' = d' = a/(1 + bc)$ in (2.1)). ∎

Now we can identify a line $\ell_{(a,b,c,a)} \in \mathscr{L}$ as a point $(a, b, c) \in \mathbb{R}^3$, and define

$$\ell_{(a,b,c)} := \ell_{(a,b,c,a)}.$$

This gives a point-line duality between lines in $\mathscr{L}$ and points in $\mathbb{R}^3$. As a direct consequence of Lemma 2.1, any regulus strip in $\bar{\mathcal{S}}$ can be re-parameterized as follows.

Corollary 2.2. *Let $\bar{\mathcal{S}}$ be the family of regulus strips defined in (1.3). After choosing an appropriate parameterization, each $S(\ell) \in \bar{\mathcal{S}}$ can be expressed as*

$$S(\ell) = \{(a + ct, b + at, t) + u(1, -t, 0),\ t \in [0, 1],\ |u| \lesssim \delta^{1/2}\} + B^3_\delta,$$

where $\ell = \ell_{(a,b,c)} \in \mathcal{L}$. In particular, if we let

$$R(\ell_{(a,b,c)}) := \{(a + ct, b + at, t) + u(1, -t, 0),\ t \in [0, 1],\ u \in \mathbb{R}\} \tag{2.2}$$

be a regulus, then $S(\ell) = N_\delta(R(\ell)) \cap N_{\delta^{1/2}}(\ell)$.

Proof. Suppose ℓ is parameterized as $\ell_{(A,B,C,D)}$ with $AD - BC = 1$. By (1.2), we can write the regulus $R(\ell)$ explicitly as

$$\begin{aligned} R(\ell) &= \{(A + Cs, B + Ds, s) + r(A + Cs, B + Ds, 0) : s, r \in \mathbb{R}\} \\ &= \{(\lambda(A + Cs), \lambda(B + Ds), s) : s, \lambda \in \mathbb{R}\}. \end{aligned}$$

As in Lemma 2.1, consider the coordinate swap $\Phi(X, Y, Z) = (X, Z, Y)$. The new height variable is $t = Y$. Let

$$a = \frac{1}{D}, \quad b = -\frac{B}{D} \quad \text{and} \quad c = \frac{C}{D}.$$

Note that a point of the regulus $R(\ell)$ is $(\lambda(A + Cs), \lambda(B + Ds), s)$. After the coordinate swap, its new height is $t = \lambda(B + Ds)$. Solving for s, we get

$$s = \frac{t - \lambda B}{\lambda D} = -\frac{B}{D} + \frac{t}{\lambda D} = b + \frac{a}{\lambda} t.$$

As a result, the first coordinate of $R(\ell)$ becomes

$$\lambda(A + Cs) = \lambda A + \lambda C \,\frac{t - \lambda B}{\lambda D} = \lambda \left(A - \frac{CB}{D}\right) + \frac{C}{D} t = a\lambda + ct,$$

since $AD - BC = 1$. Therefore, after the coordinate swap, the regulus $\Phi(R(\ell))$ can be parameterized as

$$\left\{\left(a\lambda + ct,\ b + \frac{a}{\lambda} t,\ t\right) : t, \lambda \in \mathbb{R}\right\}.$$

Put $u = a(\lambda - 1)$, so $\lambda = 1 + u/a$. Consequently,

$$\left(a\lambda + ct,\ b + \frac{a}{\lambda} t,\ t\right) = (a + ct, b + at, t) + u(1, -t, 0) + \left(0, \frac{u^2}{a + u} t, 0\right).$$

Note that the assumptions $|u| \lesssim \delta^{1/2}$ and $a \sim 1$ imply that

$$\frac{u^2}{a + u} t = O(\delta) \quad \text{for } t \in [0, 1].$$

This completes the proof of the corollary. ∎

In order to prove Theorem 1.5 by induction, we need to consider a wider class of regulus strips: (δ,ρ)-regulus strips.

Definition 2.3. For $\rho\in[\delta,\delta^{1/2}]$, a (δ,ρ)-*regulus strip* $S_\rho(\ell)$ is defined as $S_\rho(\ell):=S(\ell)\cap N_\rho(\ell)$ for some regulus strip $S(\ell)\in\bar{\mathcal{S}}$. In other words, if $\ell=\ell_{(a,b,c)}\in\mathcal{L}$, then $S_\rho(\ell)$ is a regulus strip of dimensions $\delta\times\rho\times 1$ in $\mathbb{R}^3$ that can be expressed as

$$\{(a+ct,b+at,t)+u(1,-t,0),\ t\in[0,1],\ |u|\le\rho\}+B^3_\delta. \tag{2.3}$$

That is, $S_\rho(\ell)=N_\delta(R(\ell))\cap N_\rho(\ell)$. Note that when $\rho=\delta$, a (δ,ρ)-regulus strip is a δ-tube; when $\rho=\delta^{1/2}$, a (δ,ρ)-regulus strip is a regulus strip.

Given a family of (δ,ρ)-regulus strips $\mathcal{S}$, by (2.3), each $S_\rho=S_\rho(\ell_{(a,b,c)})\in\mathcal{S}$ can be parameterized as a point $(a,b,c)\in\mathbb{R}^3$. This correspondence gives a set of points

$$E(\mathcal{S})=\{x\in\mathbb{R}^3:\exists\ell_x \text{ such that } S_\rho(\ell_x)\in\mathcal{S}\} \tag{2.4}$$

so that there is a bijection $E(\mathcal{S})\longleftrightarrow\mathcal{S}$, given by $x\longleftrightarrow S_\rho(\ell_x)$.

Definition 2.4. We say a set of (δ,ρ)-regulus strips $\mathcal{S}$ is δ-*separated* if the parameter set $E(\mathcal{S})$ is a set of δ-separated points in $\mathbb{R}^3$.

For each $t\in[0,1]$, consider the vectors $e_j=e_j(t)$ defined as

$$e_1=(1,0,t),\quad e_2=(t,1,0),\quad\text{and}\quad e_3=e_1\wedge e_2=(-t,t^2,1). \tag{2.5}$$

Define π_t as follows:

$$\pi_t \text{ is the orthogonal projection onto } e_3^\perp(t). \tag{2.6}$$

For a family of (δ,ρ)-regulus strips $\mathcal{S}$, let

$$X(\mathcal{S})=\bigcup_{\mathcal{S}},$$

and let

$$X_t(\mathcal{S})=X(\mathcal{S})\cap\{z_3=t\}$$

be the slice of $X(\mathcal{S})$ at height t.

The next lemma is about a duality for regulus strips. Before stating the lemma, we first recalled the point-line duality found in [1].

Let $\ell_x\in\mathcal{L}$ be a line, where $x=(a,b,c)\in\mathbb{R}^3$ is a point in the parameter space. Then for any physical point $p=(p_1,p_2,t)\in\mathbb{R}^3$, there exists a light ray $\bar{\ell}(p)$ with direction $e_3(t)$ in the parameter space such that the following is true: $p\in\ell_x$ if and only if $x\in\bar{\ell}(p)$. In other words, $p\in\ell_x$ if and only if $\bar{\ell}(p)$ is the line $x+\operatorname{span}(e_3(t))$. Thus, each $\ell_x\in\mathcal{L}$ corresponds to a one-dimensional family of light rays $\{x+\operatorname{span}(e_3(t)):t\in[0,1]\}$ intersecting x. Consequently, given a family of lines $L=\{\ell_x:x\in E\}\subset\mathcal{L}$ for some parameter set $E\subset\mathbb{R}^3$, the horizontal slice $L\cap\{z_3=t\}$ is the projection $\pi_t(E)$, up to a linear transformation depending on t. In [1], this duality was used to convert the SL_2-Kakeya problem to a restricted projection problem.

Note that each line ℓ_x corresponds to only one single point x in the parameter space. The situation for regulus strips here is a bit different. For each regulus strip $S(\ell_x)$, the horizontal slice $S(\ell_x) \cap \{z_3 = t\}$ corresponds to a $(\delta \times \delta^{1/2})$-tube in the parameter space, that depends not only on x, but also on the parameter t. In the next lemma, we make the following crucial observation: the $(\delta \times \delta^{1/2})$-tube corresponding to the horizontal slice $S(\ell_x) \cap \{z_3 = t\}$ is indeed contained in a "conical wave packet" containing x with direction $e_3(t)$. As a consequence, $|X_t(\mathcal{S})|$ is essentially the same as the area of the union of certain conical wave packets determined by the set $E(\mathcal{S})$ and the projection π_t. This observation allows us to use the tools developed in [2] to study regulus strips.

Lemma 2.5 (Regulus strips vs. wave packets). *Let $\mathcal{S}$ be a set of δ-separated (δ, ρ)-regulus strips and let $E(\mathcal{S})$ be its corresponding parameter set. For each $x \in E(\mathcal{S})$ and each $t \in [0, 1]$, there exists a $(1 \times \rho \times \delta)$-plank*

$$T_x = T_x(t) := x + \{y_1 e_3 + y_2 (e_3/|e_3|)' + y_3 e_3 \wedge (e_3/|e_3|)' : |y_1| \le 1, |y_2| \le \rho, |y_3| \le \delta\}$$

so that

$$|X_t(\mathcal{S})| \sim \Big|\pi_t\Big(\bigcup_{x \in E(\mathcal{S})} T_x(t)\Big)\Big| \sim \Big|\bigcup_{x \in E(\mathcal{S})} T_x(t)\Big|.$$

Remark 2.6. Note that $X_t(\mathcal{S})$ is a union of $(\delta \times \rho)$-tubes. Lemma 2.5 implies that the number of distinct $(\delta \times \rho)$-tubes in $X_t(\mathcal{S})$ is about the same as the number of distinct $(1 \times \rho \times \delta)$-planks in $\bigcup_{x \in E(\mathcal{S})} T_x(t)$.

Proof of Lemma 2.5. Denote by $\xi(t)$ the normalized vector

$$\xi(t) = \frac{e_3}{|e_3|} = \frac{1}{(1 + t^2 + t^4)^{1/2}} (-t, t^2, 1),$$

so that

$$\xi'(t) = \frac{1}{(1 + t^2 + t^4)^{3/2}} (t^4 - 1, 2t + t^3, -t - 2t^3). \tag{2.7}$$

For $x = (a, b, c) \in E(\mathcal{S})$, denote by $P_x = \{(a + ct, b + at) + u(1, -t), |u| \le \rho\}$, which is a horizontal line segment of length ρ. Then $N_\delta(P_x)$ is the intersection of the regulus strip $S_\rho(\ell_x) \in \mathcal{S}$ with the horizontal plane $\{z_3 = t\}$. Notice that P_x can be expressed as

$$\big\{\big((x + u(0, -t, t^{-1})) \cdot e_1,\ (x + u(0, -t, t^{-1})) \cdot e_2\big),\ |u| \le \rho\big\}.$$

Since $e_1 \wedge e_2 \sim 1$, we have $|N_\delta(P_x')| \sim |N_\delta(P_x)| \sim \rho\delta$, where

$$P_x' = \pi_t(\{x + u(0, -t, t^{-1}),\ |u| \le \rho\}). \tag{2.8}$$

Thus, $P_x \cap P_y \ne \varnothing$ if and only if $P_x' \cap P_y' \ne \varnothing$. In other words, $S(\ell_x) \cap S(\ell_y) \cap \{z_3 = t\} \ne \varnothing$ if and only if $P_x' \cap P_y' \ne \varnothing$.

The preimage of $N_\delta(P_x')$ regarding π_t is an infinite strip whose cross-section is a $(\delta \times \rho)$-rectangle. Denote its intersection with the unit ball by

$$T_x := N_\delta(\{P_x' + k\xi, |k| \le 1\}),$$

which is a $(1 \times \rho \times \delta)$-plank whose longest direction is e_3. To prove the lemma, it suffices to show that the second-longest direction of T_x is parallel to $\xi' = (e_3/|e_3|)'$.

Indeed, let the direction of the second-longest side of T_x be a vector $v(t)$ so that it is orthogonal to $\xi(t)$. By (2.8), P'_x is a line segment with direction $(0, -t, t^{-1})$. Thus, there exists a number α such that $v(t)$ can be parameterized as $v(t) = (0, -t, t^{-1}) + \alpha\xi(t)$. To find α, we solve the equation $\langle (0, -t, t^{-1}) + \alpha\xi, \xi\rangle = 0$ and obtain

$$\alpha = \frac{t^3 - t^{-1}}{1 + t^2 + t^4},$$

which implies

$$v(t) = \frac{1}{1 + t^2 + t^4}(1 - t^4, -2t - t^3, 2t^3 + t). \tag{2.9}$$

By (2.7), this means that v is indeed parallel to ξ' (we will explain this coincidence later in Section 3). Hence the second-longest direction of T_x is $\xi' = (e_3/|e_3|)'$. ∎

Suppose that $S(\ell)$ is a (δ, ρ)-regulus strip. Align with the notation in Definition 1.2, by (2.3), $\breve{S}(\ell)$ is a δ-tube of length ρ in $\mathbb{R}^4$ that can be parameterized as

$$\breve{S}(\ell) = \{\breve{\ell} + u(1, 0, 0, -1) : |u| \le \rho\} + B^4_\delta. \tag{2.10}$$

As a result, we can extend Definition 1.4 to a family of (δ, ρ)-regulus strips naturally as the following.

Definition 2.7. Let $\mathcal{S}$ be a set of (δ, ρ)-regulus strips and let $W = \bigcup_{S \in \mathcal{S}} \breve{S} \subset \mathbb{R}^4$ be a union of δ-tubes of length ρ (recall (2.10)). We say $\mathcal{S}$ obeys the *two-dimensional ball condition* if, for any $r \in [\delta, 1]$ and any four-dimensional r-ball B_r,

$$|W \cap B_r|\delta^{-4} \le 100(r\delta^{-1})^2. \tag{2.11}$$

Lemma 2.8. *Let $\mathcal{S}$ be a set of distinct (δ, ρ)-regulus strips with the two-dimensional ball condition (2.11), and let $\mathcal{S}'$ be a set of (δ', ρ')-regulus strips with $\delta \le \delta'$ and $\rho \le \rho'$ such that each $S' \in \mathcal{S}'$ contains some $S \in \mathcal{S}$. Then there exists $\mathcal{S}'_1 \subset \mathcal{S}'$, with $(\#\mathcal{S}'_1) \ge |\log\delta|^{-1}(\delta/\delta')(\rho/\rho')(\#\mathcal{S}')$, such that $\mathcal{S}'_1$ is a set of (δ', ρ')-regulus strips with the two-dimensional ball condition (2.11).*

Proof. Let $X \subset \mathbb{R}^4$ be the finite set such that for all $x \in X$, there exists an $S \in \mathcal{S}$ such $S = S(\ell)$ with $\breve{\ell} = x$. In other words, X is the parameter set in $\mathbb{R}^4$ of the (δ, ρ)-regulus strips $\mathcal{S}$. Notice that the statement "$\mathcal{S}$ satisfies the two-dimensional ball condition (2.11)" is equivalent to the following:

$$\#(X \cap B_r) \le \begin{cases} r\delta^{-1}, & \text{if } \delta \le r \le \rho, \\ r^2\delta^{-1}\rho^{-1}, & \text{if } \rho \le r \le 1. \end{cases}$$

Let X' be a random sample of X with probability $|\log\delta|^{-1}(\delta/\delta')(\rho/\rho')$. Then with high probability, we have

$$\#X' \ge |\log\delta|^{-1}(\delta/\delta')(\rho/\rho')\#X$$

and

$$\#(X' \cap B_r) \leq \begin{cases} r(\delta')^{-1}(\rho'/\rho), & \text{if } \delta \leq r \leq \rho, \\ r^2(\delta\rho)^{-1}, & \text{if } \rho \leq r \leq 1. \end{cases}$$

Let $\mathcal{S}'$ be the set of (δ', ρ')-regulus strips such that for any $S' \in \mathcal{S}'$, $S' = S'(\ell)$ for some $\breve{\ell} \in X'$. Such $\mathcal{S}'$ exists since each $S' \in \mathcal{S}'$ contains some $S \in \mathcal{S}$. Then $\mathcal{S}'$ satisfies the two-dimensional ball condition (2.7). ■

Recall that in theorem 1.5, we need to consider a shading $Y(S)$ for each regulus strip $S \in \mathcal{S}$. The definition of shading can be extended to (δ, ρ)-regulus strips naturally.

Definition 2.9. Suppose that $\mathcal{S}$ is a family of (δ, ρ)-regulus strips. Define a *shading* $Y : \mathcal{S} \to [0, 1]^3$ as a map that $Y(S) \subset S \cap [0, 1]^3$ for any $S \in \mathcal{S}$.

Definition 2.10. Let $(\mathcal{S}, Y)$ be a family of (δ, ρ)-regulus strips and shading. If for all $S \in \mathcal{S}$, $|Y(S)| \geq \lambda|S|$, then we say that $(\mathcal{S}, Y)$ is a set of regulus strips with a λ-*dense* shading.

Definition 2.11. Let $(\mathcal{S}, Y)$ and $(\mathcal{S}', Y')$ be two families of (δ, ρ)-regulus strips and shading. If $\mathcal{S}' \subset \mathcal{S}$, $Y'(S) \subset Y(S)$ for all $S \in \mathcal{S}'$, and $\sum_{S \in \mathcal{S}'} |Y'(S)| \gtrapprox \sum_{S \in \mathcal{S}} |Y(S)|$, then we say $(\mathcal{S}', Y')$ is a *refinement* of $(\mathcal{S}, Y)$.

Let $\mathcal{S}$ be a family of (δ, ρ)-regulus strips. Note that if $x \in S \cap S'$ for two (δ, ρ)-regulus strips $S, S' \in \mathcal{S}$, then $S \cap S'$ contains a horizontal $(\delta \times \rho \times \delta)$-tube which contains x (in fact, $S \cap S'$ contains a $(\delta \times \rho \times \rho)$-slab that contains x). This is a consequence of (2.3). As a result, we may think of $\bigcup_{\mathcal{S}}$ as a union of $(\delta \times \rho \times \delta)$-tubes.

Definition 2.12. Let $(\mathcal{S}, Y)$ be a family of (δ, ρ)-regulus strips and shading. Suppose that $Y(S)$ is a union of horizontal $(\delta \times \rho \times \delta)$-tubes for all $S \in \mathcal{S}$. We say Y is *regular* if the following is true.

(1) $Y(S) = S \cap (\bigcup_{S' \in \mathcal{S}} Y(S'))$ for any $S \in \mathcal{S}$.
(2) For all $S \in \mathcal{S}$, $\#\{P \subset S : Y(S) \cap P \neq \varnothing,\ P$ is a horizontal $(\delta \times \rho \times \delta)$-tube$\}$ are the same up to a constant multiple.

Lemma 2.13. *Let $(\mathcal{S}, Y)$ be a set of (δ, ρ)-regulus strips and shading. Suppose that $Y(S)$ is a union of horizontal $(\delta \times \rho \times \delta)$-tubes for all $S \in \mathcal{S}$. Then there exists a refinement $(\mathcal{S}', Y')$ of $(\mathcal{S}, Y)$ so that $(\mathcal{S}', Y')$ is a family of (δ, ρ)-regulus strips with regular shading.*

Proof. If $Y(S) \subsetneq S \cap (\bigcup_{S \in \mathcal{S}} Y(S))$, update $Y(S) = S \cap (\bigcup_{S \in \mathcal{S}} Y(S))$. Apply dyadic pigeonholing on the set $\mathcal{S}$, we can find a refinement $(\mathcal{S}', Y)$ of $(\mathcal{S}, Y)$ so that for all $S \in \mathcal{S}'$, $\#\{P \subset S : Y(S) \cap P \neq \varnothing,\ P$ is a horizontal $(\delta \times \rho \times \delta)$-tube$\}$ are the same up to a constant multiple. Let $Y' = Y$. The family $(\mathcal{S}', Y')$ is what we need. ■

The next proposition is the key input in the proof of Theorem 1.5. It is built on the "high-low plus decoupling method" developed in [2]. Our proof partially relies on the argument in [2].

Proposition 2.14. *Suppose that $(\mathcal{S}, Y)$ is a set of δ-separated (δ, ρ)-regulus strips with a regular, λ-dense shading and suppose $\delta^{1/2} \geq \rho \geq \delta$. Then, for any $\varepsilon > 0$, one of the following must be true*:

(a) *There exist an* $\alpha \geq 1$ *and a set of* $(\alpha\delta^{1-\varepsilon^2}, \rho_o)$*-regulus strips* $(\mathcal{S}', Y')$*, with an* $\Omega(\lambda)$*-dense shading, so that* $\#\mathcal{S}' \gtrsim \#\mathcal{S}$ *and*

$$\bigcup_{S' \in \mathcal{S}'} Y'(S') \subset \bigcup_{S \in \mathcal{S}} Y(S),$$

where $\rho_o = \max\{\rho, \alpha\delta^{1-\varepsilon^2}\}$*. Moreover, any* $S' \in \mathcal{S}'$ *contains some* $S \in \mathcal{S}$*.*

(b) *We have the following Kakeya maximal estimate for* $(\mathcal{S}, Y)$*:*

$$\Big|\bigcup_{S \in \mathcal{S}} Y(S)\Big| \geq c_\varepsilon \delta^{\varepsilon/2}\lambda^6 \sum_{S \in \mathcal{S}} |S|.$$

Proof. Since Y is regular, it is a union of horizontal $(\delta \times \rho \times \delta)$-tubes. Let $\Sigma = [0,1] \cap \delta\mathbb{Z}$, and for each $t \in \Sigma$, define

$$\mathcal{S}_t := \{S \in \mathcal{S} : Y(S) \cap \{z_3 = t\} \neq \varnothing\}.$$

By Lemma 2.5, there are a set $E(\mathcal{S}_t) \subset \mathbb{R}^3$ and a family of $(1 \times \rho \times \delta)$-planks $\{T_x : x \in E(\mathcal{S}_t)\}$ so that

$$|X_t(\mathcal{S}_t)| \sim \Big|\bigcup_{x \in E(\mathcal{S}_t)} T_x(t)\Big|.$$

Therefore, to estimate $|\bigcup_{S \in \mathcal{S}} Y(S)|$, it suffices to estimate the number of distinct $(1 \times \rho \times \delta)$-planks in $\bigcup_{x \in E(\mathcal{S}_t)} T_x(t)$ for all $t \in \Sigma$.

Let $\mathcal{T}_t$ be the collection of distinct $(1 \times \rho \times \delta)$-planks in $\bigcup_{x \in E(\mathcal{S}_t)} T_x(t)$ (two planks T_1 and T_2 are considered the same if $T_1 \subset 2T_2$ and $T_2 \subset 2T_1$). For each $T \in \mathcal{T}_t$, let $\mathbf{1}_T^*$ be a bump function adapted to T so that $\widehat{\mathbf{1}}_T^*$ is supported in the dual $(1 \times \rho^{-1} \times \delta^{-1})$-plank associated with T and containing the origin. Denote by $\omega = \omega(t)$ this dual plank, so the shortest direction of ω is e_3, and the second shortest direction of ω is $(e_3/|e_3|)'$ (recall (2.5)). Note that $\omega(t)$ is tangent to the conical surface

$$\Gamma = \{r \cdot e_3(s) \wedge (e_3(s)/|e_3(s)|)' : |r| \leq \delta^{-1}, s \in [0,1]\}$$

along the line $\mathrm{span}\{e_3(t) \wedge (e_3(t)/|e_3(t)|\}$.

Recall from (2.4) that $E(\mathcal{S}) \subset \mathbb{R}^3$ is the corresponding parameter set of $\mathcal{S}$. Since Y is a regular shading, $\#\{P \subset S : Y(P) \neq \varnothing\}$ are the same up to a constant for all $S \in \mathcal{S}$. Denote by μ this uniform number. Since $Y(S)$ is λ-dense, we have

$$\mu \sim \#\{P \subset S : Y(P) \neq \varnothing\} \gtrsim \lambda\delta^{-1}, \qquad \forall S \in \mathcal{S}. \tag{2.12}$$

It follows that any $x \in E(\mathcal{S})$ belongs to $\gtrsim \mu$ copies of $E(\mathcal{S}_t)$. Thus, if we let

$$f_t = \sum_{T \in \mathcal{T}_t} \mathbf{1}_T^*, \tag{2.13}$$

then

$$\mu\delta^3 \lesssim \int_{B^3(x,\delta)} \sum_{t \in \Sigma} f_t, \quad \text{for all } x \in E(\mathcal{S}).$$

Let $\varphi \geq 0$ be so that $\widehat{\varphi}$ is a bump for $B^3_{\delta^{-1+\varepsilon^2}}$ and let δ_0 be the Dirac delta measure concentrated at the origin. Thus, by partitioning, for all f_t,

$$\widehat{f_t} = \widehat{f_t}\,\widehat{\varphi} + \widehat{f_t}(1-\widehat{\varphi}),$$

and by the triangle inequality,

$$\begin{aligned}\mu\delta^3 \lesssim \int_{B^3(x,\delta)} \sum_{t\in\Sigma} f_t \leq \int_{B^3(x,\delta)} \Big|\sum_{t\in\Sigma} f_t * \varphi\Big| + \int_{B^3(x,\delta)} \Big|\sum_{t\in\Sigma} f_t * (\delta_0 - \varphi)\Big| \\ = \mathrm{I}(x) + \mathrm{II}(x).\end{aligned}$$

We consider two separate cases:

(i) $2|\mathrm{I}(x)| \geq |\mathrm{I}(x) + \mathrm{II}(x)|$ for $\geq \#E(\mathcal{S})/2$ points x.

(ii) $2|\mathrm{I}(x)| \geq |\mathrm{I}(x) + \mathrm{II}(x)|$ for $\leq \#E(\mathcal{S})/2$ points x.

Suppose that $2|\mathrm{I}(x)| \geq |\mathrm{I}(x) + |\mathrm{II}(x)|$ for $\geq \#E(\mathcal{S})/2$ points x. Denote this set of points by E_1. Then for any $x \in E_1$,

$$\mu\delta^3 \lesssim \int_{B^3(x,\delta)} \sum_{t\in\Sigma} f_t \lesssim \int_{B^3(x,\delta)} \Big|\Big(\sum_{t\in\Sigma} f_t\Big) * \varphi\Big|.$$

For each $k \in \mathbb{N}$, consider the interval

$$A_k = \{y \in \mathbb{R} : \delta^{1-\varepsilon^2} 2^{k-2} \leq |y| \leq \delta^{1-\varepsilon^2} 2^{k-1}\},$$

with $A_{-2} := \varnothing$. Recall the definition of π_t at (2.6). For each k and each $t \in \Sigma$, let

$$\mathcal{T}_t(k) = \{T \in \mathcal{T}_t : \mathrm{dist}(\pi_t(T) - \pi_t(x)) \in A_k\}$$

be the family of $(1 \times \rho \times \delta)$-planks that is $\sim \delta^{1-\varepsilon^2} 2^k$ away from x. The partition $\mathcal{T}_t = \bigcup_k \mathcal{T}_t(k)$ gives

$$\mu\delta^3 \lesssim \int_{B^3(x,\delta)} \Big|\Big(\sum_{t\in\Sigma} f_t\Big) * \varphi\Big| \leq \sum_k \int_{B^3(x,\delta)} \Big|\Big(\sum_{t\in\Sigma}\sum_{T\in\mathcal{T}_t(k)} \mathbf{1}_T^*\Big) * \varphi\Big|.$$

Since

$$\varphi(x) \lesssim_\varepsilon \delta^{3\varepsilon^2-3}(1 + |\delta^{\varepsilon^2-1}x|)^{-4\varepsilon^{-3}},$$

the support of $\mathbf{1}_T^* * \varphi$ is essentially a $(\delta^{1-\varepsilon^2} \times \rho \times 1)$-plank when $\rho \geq \delta^{1-\varepsilon^2}$, and is essentially a $(\delta^{1-\varepsilon^2} \times \delta^{1-\varepsilon^2} \times 1)$-tube when $\rho \leq \delta^{1-\varepsilon^2}$. Let $\beta = \delta^{1-\varepsilon^2}\rho^{-1}$ if $\rho \leq \delta^{1-\varepsilon^2}$, and $\beta = 1$ if $\rho \geq \delta^{1-\varepsilon^2}$. Thus

$$\mu\delta^{-\varepsilon^2}\beta \lesssim \sum_k \Big(\sum_{t\in\Sigma} \#\mathcal{T}_t(k)\Big) 2^{-4k\varepsilon^{-3}}.$$

By pigeonholing, there exists a k such that

$$\mu\delta^{-\varepsilon^2}\beta\, 2^{2k} \lesssim \mu\delta^{-\varepsilon^2}\beta\, 2^{4k\varepsilon^{-3}}/k^2 \lesssim \sum_{t\in\Sigma} \#\mathcal{T}_t(k).$$

Let $\alpha = 2^k$ and let $\rho_o = \max\{\rho, \alpha\delta^{1-\varepsilon^2}\}$, so that $\alpha\beta \geq (\rho_o/\rho)$. For each $t \in \Sigma$, let $T'_x(t)$ be the $(1 \times \rho_o \times \alpha\delta^{1-\varepsilon^2})$-plank containing x, with directions given by e_3, $(e_3/|e_3|)'$ and $e_3 \wedge (e_3/|e_3|)'$, as in the statement of Lemma 2.5. Then, the above estimate implies

$$\sum_{t\in\Sigma} \#\{T \in \mathcal{T}_t : T \subset T'_x\} \gtrsim \mu\delta^{-\varepsilon^2}\beta 2^{2k} = \mu\delta^{-\varepsilon^2}\beta\alpha^2 \geq \mu\alpha\delta^{-\varepsilon^2}(\rho_o/\rho). \tag{2.14}$$

By reversing the argument in Lemma 2.5, we know that $T'_x(t)$ corresponds to an $(\alpha\delta^{1-\varepsilon^2} \times \rho_o)$-tube on the horizontal plane $\{z_3 = t\}$ in the physical space, which, recalling (2.2) and (2.3), is the intersection between the plane $\{z_3 = t\}$ and the $(\alpha\delta^{1-\varepsilon^2}, \rho_o)$-regulus strip

$$S'_x := N_{\alpha\delta^{1-\varepsilon^2}}(R(\ell_x)) \cap N_{\rho_o}(\ell_x).$$

Recall that $\mathcal{T}_t$ is the collection of distinct $(1 \times \rho \times \delta)$-planks in $\bigcup_{x\in E(\mathcal{S}_t)} T_x(t)$. Thus, each $(1 \times \rho \times \delta)$-plank in $\mathcal{T}_t$ corresponds to a horizontal $(\delta \times \rho \times \delta)$-tube $P_t(S)$ at height t so that $P_t(S) \subset Y(S)$ for some $S \in \mathcal{S}$. Therefore, by (2.14), the $(\alpha\delta^{1-\varepsilon^2}, \rho_o)$-regulus strip S'_x contains $\gtrsim \mu\alpha\delta^{-\varepsilon^2}(\rho_o/\rho)$ distinct horizontal $(\delta \times \rho \times \delta)$-tubes P with $Y(S) \cap P \neq \varnothing$ for some $S \in \mathcal{S}$. Let $Y'(S_x)$ be the union of these $(\delta \times \rho \times \delta)$-tubes, which defines a shading for S'_x. Via (2.12), we have

$$|Y(S'_x)| \gtrsim \mu\alpha\delta^{-\varepsilon^2}(\rho_o/\rho)\cdot\delta|S| \gtrsim \lambda(\alpha\delta^{-\varepsilon^2}(\rho_o/\rho)|S|) \gtrsim \lambda|S'_x|.$$

Denote by $\mathcal{S}' = \{S'_x : x \in E_1\}$. We conclude case (a).

Suppose on the other hand that $2|\mathrm{I}(x)| \geq |\mathrm{I}(x) + \mathrm{II}(x)|$ for $\leq \#E(\mathcal{S})/2$ points x. Then $2|\mathrm{II}(x)| \geq |\mathrm{I}(x) + \mathrm{II}(x)|$ for $\geq \#E(\mathcal{S})/2$ points x. Denote this set of points by E_2. Then for any $x \in E_2$,

$$\mu\delta^3 \lesssim \int_{B^3(x,\delta)} \Big|\Big(\sum_{t\in\Sigma} f_t\Big) * (\delta_0 - \varphi)\Big|.$$

By Minkowski's inequality and Hölder's inequality,

$$\mu^6(\#E_2)\delta^3 \lesssim \int \Big|\sum_{t\in\Sigma} f_t * (\delta_0 - \varphi)\Big|^6.$$

We are going to make further frequency decomposition for the function $\widehat{f}_t(1-\widehat{\varphi})$, similar to that of Definition 3 in [2]. When $\rho = \delta$, the situation considered here is identical to that in [2], after rescaling. Recall that $\widehat{f}_t(1-\widehat{\varphi})$ is supported in $\omega(t) \setminus B^3_{\delta^{-1+\varepsilon^2}}$, a punctured $(1 \times \rho^{-1} \times \delta^{-1})$-plank. In particular, when $\rho \geq \delta^{1-\varepsilon^2}$, $\omega(t) \setminus B^3_{\delta^{-1+\varepsilon^2}}$ is a union of two $(1 \times \rho^{-1} \times \delta^{-1})$-planks, ω^+ and ω^-, each of which is $\delta^{-1+\varepsilon^2}$-away from the origin. Both ω^+ and ω^- have e_3 as their shortest direction and $(e_3/|e_3|)'$ as their second shortest direction. For simplicity, denote by $v_1 = e_3$, $v_2 = (e_3/|e_3|)'$, and $v_3 = e_3 \wedge (e_3/|e_3|)'$.

Similar to Definition 3 in [2], we define

$$P_{t,\mathrm{high}} = \Big\{\sum_{j=1}^{3} \xi_j v_j : |\xi_1| \leq 1, \delta^{\varepsilon^2-1} \leq |\xi_2| \leq \delta^{-1}, |\xi_3| \leq \delta^{-1}\Big\},$$

and for a dyadic number $\nu \in [\delta^{-1/2}, \delta^{\varepsilon^2-1}]$, define $P_{t,\nu}$, with the convention $\nu/2 = 0$ when $\nu/2 < \delta^{-1/2}$, as (see Figure 1 in [2])

$$P_{t,\nu} = \Big\{ \sum_{j=1}^{3} \xi_j v_j : |\xi_1| \le 1, \nu/2 \le |\xi_2| \le \nu, \delta^{\varepsilon^2-1} \le |\xi_3| \le \delta^{-1} \Big\}.$$

Hence

$$\omega(t) \setminus B^3_{\delta^{-1+\varepsilon^2}} \subset P_{t,\mathrm{high}} \cup \Big(\bigcup_{\nu} P_{t,\nu} \Big).$$

Let $\{\varphi_{\mathrm{high}}, \{\varphi_\nu\}\}$ be bump functions associated to $\{P_{t,\mathrm{high}}, \{P_{t,\nu}\}_\nu\}$ obeying

$$\widehat{\varphi}_{\mathrm{high}} + \sum_{\nu} \widehat{\varphi}_\nu = 1$$

on $\omega(t) \setminus B^3_{\delta^{-1+\varepsilon^2}}$. Thus, by the triangle inequality,

$$\begin{aligned} \mu^6 (\#E_2) \delta^3 &\lesssim \int \Big| \sum_{t\in\Sigma} f_t * (\delta_0 - \varphi) * \varphi_{\mathrm{high}} \Big|^6 + \sum_{\nu} \int \Big| \sum_{t\in\Sigma} f_t * (\delta_0 - \varphi) * \varphi_\nu \Big|^6 \\ &:= \mathrm{I} + \mathrm{II}. \end{aligned} \tag{2.15}$$

Suppose $\mu^6 (\#E_2) \delta^3 \lesssim \mathrm{II}$. Apply the first inequality of equation (47) in [2] to the δ^{-1}-dilate of each ν-term in II so that (with the following choice of parameters: $t = 1$, $\lambda = \nu\delta$, $\Theta = \Sigma$, $\#\mathbb{T} = \sum_{t\in\Sigma} \#\mathcal{T}_t$)

$$\int \Big| \sum_{t\in\Sigma} f_t * (\delta_0 - \varphi) * \varphi_\nu \Big|^6 \lesssim \delta^3 \delta^{-O(\varepsilon^2)} (\#\Sigma)^2 \nu^2 (\nu\delta)^{-1} \sum_{t\in\Sigma} \#\mathcal{T}_t (\delta^{-2}\rho). \tag{2.16}$$

The expression $(\#\mathbb{T})\delta^{-1}$ in [2] is replaced by $\sum_{t\in\Sigma} \#\mathcal{T}_t (\delta^{-2}\rho)$ here for the following reason: in our situation, after a δ^{-1}-dilate, $\mathcal{T}_t$ (recall (2.13)) becomes a set of $(\delta^{-1} \times \rho\delta^{-1} \times 1)$-planks, each of which has volume $\delta^{-2}\rho$. Whereas in [2], their corresponding $\mathcal{T}_t$ is a set of $(\delta^{-1} \times 1 \times 1)$-tubes, each of which has volume δ^{-1} (see also the first inequality in the line above equation (47) in [2]). Hence, regarding the last factor in (2.16), we have $\delta^{-2}\rho$ instead of δ^{-1}.

Note that $\sum_{t\in\Sigma} f_t * (\delta_0 - \varphi) * \varphi_\nu = 0$ when $\nu \gtrsim \rho^{-1}$. Recall $\mu \gtrsim \lambda\delta^{-1}$ and $\#\Sigma \sim \delta^{-1}$. Sum up all $\nu \lesssim \rho^{-1}$ in (2.16) so that

$$\lambda^6 \delta^{-3} (\#E_2) \lesssim \mu^6 \delta^3 (\#E_2) \lesssim \sum_{\nu} \delta^{-O(\varepsilon^2)} \delta^{-2} \sum_{t\in\Sigma} \#\mathcal{T}_t.$$

Since $\#\nu \lessapprox 1$, since $2(\#E_2) \ge \#E(\mathcal{S})$, and since $\#E(\mathcal{S}) = \#\mathcal{S}$, we thus get

$$\sum_{t\in\Sigma} \#\mathcal{T}_t \gtrsim_\varepsilon \delta^{O(\varepsilon^2)} \lambda^6 \delta^{-1} (\#\mathcal{S}). \tag{2.17}$$

Reversing the argument in Lemma 2.5, we know that the left-hand side of (2.17) counts the number of distinct $(\delta \times \rho \times \delta)$-tubes in $\bigcup_{S\in\mathcal{S}} Y(S)$. Thus, we have

$$\Big| \bigcup_{S\in\mathcal{S}} Y(S) \Big| \gtrsim \rho\delta^2 \sum_{t\in\Sigma} (\#\mathcal{T}_t) \gtrsim_\varepsilon \delta^{O(\varepsilon^2)} \lambda^6 \rho\delta (\#\mathcal{S}) = \delta^{O(\varepsilon^2)} \lambda^6 \sum_{S\in\mathcal{S}} |S|. \tag{2.18}$$

Suppose $\mu^6|E_2|\delta^3 \lesssim I$ in (2.15). Then $\rho \le \delta^{1-\varepsilon^2}$, since otherwise we would have $f_t * (\delta_0 - \varphi) * \varphi_{\text{high}} = 0$. It was shown in equation (35) of [2] that (take $\Theta = \Sigma$)

$$\int \Big|\sum_{t\in\Sigma} f_t * (\delta_0 - \varphi) * \varphi_{\text{high}}\Big|^6 \lesssim \delta^{-O(\varepsilon^2)}(\#\Sigma)^4 \sum_{t\in\Sigma}\int \big|f_t * (\delta_0 - \varphi) * \varphi_{\text{high}}\big|^6$$

$$\lesssim \delta^{-O(\varepsilon^2)}(\#\Sigma)^4 \sum_{t\in\Sigma}\Big\|\sum_{T\in\mathcal{T}_t}\mathbf{1}_T^*\Big\|_6^6 \lesssim \delta^{-O(\varepsilon^2)}(\#\Sigma)^4 \sum_{t\in\Sigma}\big((\delta\rho)\#\mathcal{T}_t\big).$$

Since $\rho \le \delta^{1-\varepsilon^2}$, similar to the calculation in the first case, we have

$$\Big|\bigcup_{S\in\mathcal{S}} Y(S)\Big| \gtrsim_\varepsilon \delta^{O(\varepsilon^2)}\lambda^6\rho\delta(\#\mathcal{S}) = \delta^{O(\varepsilon^2)}\lambda^6\sum_{S\in\mathcal{S}}|S|. \tag{2.19}$$

The estimates (2.18) and (2.19) give case (b). ∎

Finally, we use Proposition 2.14 to prove Theorem 1.5. Via Corollary 2.2, Theorem 1.5 is a consequence of the following theorem by taking $\rho = \delta^{1/2}$.

Theorem 2.15. *Let $\delta \le \rho \le \delta^{1/2}$. Suppose that $(\mathcal{S}, Y)$ is a set of δ-separated (δ,ρ)-regulus strips with a λ-dense shading, and that $\mathcal{S}$ obeys the two-dimensional ball condition* (2.11). *Then for any $\varepsilon > 0$,*

$$\Big|\bigcup_{S\in\mathcal{S}} Y(S)\Big| \ge c_\varepsilon\delta^\varepsilon\lambda^6\sum_{S\in\mathcal{S}}|S|. \tag{2.20}$$

Proof. If $\lambda \le \delta$, then

$$|Y(S)| \sim \lambda\delta\rho \ge c_\varepsilon\delta^\varepsilon\lambda^6\sum_{S\in\mathcal{S}}|S|$$

since $\#\mathcal{S} \lesssim \delta^{-3/2}$, so there is nothing to prove. Let us assume $\lambda \ge \delta$.

We first reduce (2.20) to the case when $Y(S)$ is a union of horizontal $(\delta\times\rho\times\delta)$-tubes for all $S\in\mathcal{S}$. Let

$$E = \bigcup_{S\in\mathcal{S}} Y(S).$$

Recall that $S\cap S'$ is a union of horizontal $(\delta\times\rho\times\delta)$-tubes for any two (δ,ρ)-regulus strips $S, S'\in\mathcal{S}$. Let $\mathcal{P}$ be the family of horizontal $(\delta\times\rho\times\delta)$-tubes that for any $P\in\mathcal{P}$, $P\subset S$ for some $S\in\mathcal{S}$. Partition $\mathcal{P}$ into a disjoint union of $\mathcal{P}_k$ such that for all $P\in\mathcal{P}_k$, $|P\cap E| \sim 2^{-k}|P|$.

For each $S\in\mathcal{S}$, let

$$Y_0'(S) = S\cap E\cap\Big(\bigcup_{2^{-k}\ge\delta^2}\bigcup\mathcal{P}_k\Big).$$

Since $\lambda \ge \delta$, $|Y_0'(S)| \ge |Y(S)|/2$. By pigeonholing, there exist a k and a new shading Y_0 so that $(\mathcal{S}, Y_0)$ is a refinement of $(\mathcal{S}, Y)$, where $Y_0(S) = S\cap E\bigcap\bigcup_{\mathcal{P}_k}$ for each $S\in\mathcal{S}$. By dyadic pigeonholing, there is a refinement $(\mathcal{S}_0, Y_0)$ of $(\mathcal{S}, Y_0)$ such that $(\mathcal{S}_0, Y_0)$ has $\Omega^*(\lambda)$-dense shading.

For each $S \in \mathcal{S}_0$, consider a new shading $\tilde{Y}_0(S) = \mathcal{S} \cap \bigcup_{\mathcal{P}_k}$ by horizontal $(\delta \times \rho \times \delta)$-tubes. Since $|P \cap E| \sim 2^{-k}|P|$ for all $P \in \mathcal{P}_k$, $\tilde{Y}_0$ is $\Omega^*(2^k\lambda)$-dense. Now $(\mathcal{S}_0, \tilde{Y}_0)$ is a set of δ-separated (δ, ρ)-regulus strips that $\tilde{Y}_0(S)$ is a union of horizontal $(\delta \times \rho \times \delta)$-tubes for all $S \in \mathcal{S}_0$, and $\tilde{Y}_0$ is $\Omega^*(2^k\lambda)$-dense. Apply (2.20) so that

$$\Big|\bigcup_{S\in\mathcal{S}} Y(S)\Big| \gtrsim 2^{-k}\Big|\bigcup_{S\in\mathcal{S}_0} \tilde{Y}_0(S)\Big| \gtrapprox c_\varepsilon \delta^\varepsilon 2^{5k}\lambda^6 \sum_{S\in\mathcal{S}_0} |S|.$$

Since $k \geq 1$, this proves Theorem 2.15, assuming that it is true when $Y(S)$ is a union of horizontal $(\delta \times \rho \times \delta)$-tubes for all $S \in \mathcal{S}$.

From now on, let us assume $Y(S)$ is a union of horizontal $(\delta \times \rho \times \delta)$-tubes for all $S \in \mathcal{S}$. We will prove it by induction on the pair (δ, ρ). Apply Lemma 2.13 to $(\mathcal{S}, Y)$ to get a refinement $(\mathcal{S}_1, Y_1)$ of $(\mathcal{S}, Y)$ so that $(\mathcal{S}_1, Y_1)$ is a set of (δ, ρ)-regulus strips with a regular, $\Omega^*(\lambda)$-dense shading.

Apply Proposition 2.14 to $(\mathcal{S}_1, Y_1)$, so there are two cases.

If case (b) holds, then

$$\Big|\bigcup_{S\in\mathcal{S}} Y(S)\Big| \geq \Big|\bigcup_{S\in\mathcal{S}_1} Y_1(S)\Big| \gtrapprox c_\varepsilon \delta^{\varepsilon/2}\lambda^6 \sum_{S\in\mathcal{S}_1} |S| \geq c_\varepsilon \delta^\varepsilon \lambda^6 \sum_{S\in\mathcal{S}} |S|.$$

This concludes the proof of Theorem 2.15 in this case.

Now let us assume case (a) holds. Recall $\rho_o = \max\{\rho, \alpha\delta^{1-\varepsilon^2}\}$ for some $\alpha \geq 1$. In this case, we obtain a set of $(\alpha\delta^{1-\varepsilon^2}, \rho_o)$-regulus strips $(\mathcal{S}_2, Y_2)$ with an $\Omega(\lambda)$-dense shading so that the following is true:

(1) $\#\mathcal{S}_2 \gtrapprox \#\mathcal{S}_1$,

(2) each $S_2 \in \mathcal{S}_2$ contains some $S_1 \in \mathcal{S}_1$,

(3) we have

$$\bigcup_{S\in\mathcal{S}_2} Y_2(S) \subset \bigcup_{S\in\mathcal{S}_1} Y_1(S). \tag{2.21}$$

Apply Lemma 2.8 with $(\mathcal{S}, \mathcal{S}') = (\mathcal{S}_1, \mathcal{S}_2)$ to find an $\Omega^*(\alpha\delta^{-\varepsilon^2}(\rho_o/\rho))$-refinement $\mathcal{S}_2'$ of $\mathcal{S}_2$ so that $\mathcal{S}_2'$ is a family of $(\alpha\delta^{1-\varepsilon^2}, \rho_o)$-regulus strips that satisfies the two-dimensional ball condition (2.11). Apply Theorem 2.15 at scale $(\alpha\delta^{1-\varepsilon^2}, \rho_o)$ to $(\mathcal{S}_2', Y_2)$ so that

$$\Big|\bigcup_{S'\in\mathcal{S}_2'} Y_2(S')\Big| \geq c_\varepsilon \lambda^6 \alpha^\varepsilon \delta^{\varepsilon(1-\varepsilon^2)} \sum_{S'\in\mathcal{S}_2'} |S'| \gtrapprox \delta^{-\varepsilon^3} c_\varepsilon \delta^\varepsilon \lambda^6 \sum_{S\in\mathcal{S}_1} |S|,$$

which, together with (2.21), yields

$$\Big|\bigcup_{S\in\mathcal{S}} Y(S)\Big| \geq \Big|\bigcup_{S\in\mathcal{S}_1} Y_1(S)\Big| \geq c_\varepsilon \delta^\varepsilon \lambda^6 \sum_{S\in\mathcal{S}} |S|.$$

This concludes the proof of Theorem 2.15. ∎

3. An explanation for Lemma 2.5

In the proof of Lemma 2.5, we find that v is (surprisingly) parallel to ξ' (see below (2.9)). Here we attempt to explain this coincidence in a more general setting.

Let $\gamma\colon [0,1] \to \mathbb{R}^3$ be a spatial curve. The special case $\gamma = (-t, t^2, 1)$ corresponds to the SL_2 lines studied in the previous section (see (2.5)). Denote by $\gamma(t) = \gamma = (\gamma_1, \gamma_2, \gamma_3)$. Note that $\gamma^\perp$ is spanned by the vectors

$$e_1 = (\gamma_3, 0, -\gamma_1) \quad \text{and} \quad e_2 = (\gamma_2, -\gamma_1, 0).$$

Let E be a set in the parameter space. For $x = (x_1, x_2, x_3) \in E$, introduce an affine-modified projection

$$\pi(x) = (x_1\gamma_3 - x_3\gamma_1,\ x_1\gamma_2 - x_2\gamma_1) \tag{3.1}$$

and a family of curves $\mathscr{C} = \{C_x : x \in E\}$ in the physical space, where

$$C_x = (\pi(x), t) = (x_1\gamma_3 - x_3\gamma_1,\ x_1\gamma_2 - x_2\gamma_1,\ t), \quad t \in [0,1].$$

For a point z in the physical space, denote $\mathscr{C}(z) = \{C \in \mathscr{C} : z \in C\}$. Then the claim $C_x \in \mathscr{C}(z)$ gives the following equations:

$$\begin{cases} x_1\gamma_3(z_3) - x_3\gamma_1(z_3) = z_1, \\ x_1\gamma_2(z_3) - x_2\gamma_1(z_3) = z_2. \end{cases}$$

Hence the curves in $\mathscr{C}(z)$ can be parameterized as

$$\begin{aligned} \{s \cdot (\gamma_3 - \gamma_3(z_3)\gamma_1(z_3)^{-1}\gamma_1,\ \gamma_2 - \gamma_2(z_3)\gamma_1(z_3)^{-1}\gamma_1,\ 0) \\ + (z_1\gamma_1(z_3)^{-1}\gamma_1,\ z_2\gamma_1(z_3)^{-1}\gamma_1,\ t),\ t \in [0,1]\}_{s\in\mathbb{R}}, \end{aligned}$$

which implies that the tangent vector of curves in $\mathscr{C}(z)$ at the point z all lie in a plane, spanned by the two vectors

$$\begin{aligned} &(\gamma_3'(z_3) - \gamma_3(z_3)\gamma_1(z_3)^{-1}\gamma_1'(z_3),\ \gamma_2'(z_3) - \gamma_2(z_3)\gamma_1(z_3)^{-1}\gamma_1'(z_3),\ 0) \\ \text{and}\quad &(z_1\gamma_1(z_3)^{-1}\gamma_1'(z_3),\ z_2\gamma_1(z_3)^{-1}\gamma_1'(z_3),\ 1). \end{aligned} \tag{3.2}$$

The intersection between the plane spanned by (3.2) and the horizontal plane $\{t = z_3\}$ is important to us. It is a line that intersects z with a normal vector $n(z_3)$, where the normal vector n is defined as (here γ_1, γ_2 and γ_3 all depend on t)

$$n(t) = n = \gamma_1^{-1}(-\gamma_1\gamma_2' + \gamma_2\gamma_1',\ \gamma_1\gamma_3' - \gamma_3\gamma_1',\ 0). \tag{3.3}$$

Remark 3.1. From (3.3), we can define a "regulus strip" S_x of dimensions $\delta \times \delta^{1/2} \times 1$ associated to a curve C_x (cf. (1.5) in [3]) as

$$S_x = \{z + un(z_3)^\perp : z = (z_1, z_2, z_3) \in C_x,\ |u| \le \delta^{1/2}\} + B_\delta^3.$$

It is possible that using the approach developed in [3], this observation will lead to an alternative proof of the restricted projection theorem in [2] that avoids Fourier analysis.

The vector n introduced in (3.3) corresponds to the vector $(1, -t, 0)$ in (2.3) (see also below (2.7)) when $\gamma = (-t, t^2, 1)$. Let $v_1 = v_1(t)$ be the directional vector orthogonal to n (inside the horizontal plane):

$$v_1 = (\gamma_1\gamma_3' - \gamma_3\gamma_1', \gamma_1\gamma_2' - \gamma_2\gamma_1').$$

Observe that $v_1 = \pi(\gamma')$, where π is the projection introduced in (3.1). This means that the vector $(0, -t, t^{-1})$ in (2.8) corresponds γ' when $\gamma = (-t, t^2, 1)$. Since the vector e_3 defined in (2.5) corresponds to γ, the claim that v is parallel to $\xi' = (e_3/|e_3|)'$ (see below (2.9)) is equivalent to the claim that $\gamma, \gamma', (\gamma/|\gamma|)'$ are coplanar. On the other hand, we have the following.

Lemma 3.2. *The three vectors γ, γ' and $(\gamma/|\gamma|)'$ are coplanar.*

Proof. Just need to notice that $(\gamma/|\gamma|)' = \gamma'/|\gamma| - \gamma(\gamma \cdot \gamma')/|\gamma|^3$. ■

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