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The space of foliations on projective spaces in positive characteristic

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Abstract. This work explores the space of foliations on projective spaces over algebraically closed fields of positive characteristic, with a particular focus on the codimension one case. It describes how the irreducible components of these spaces vary with the characteristic of the base field in very low degrees, and establishes an arbitrary characteristic version of Calvo-Andrade's stability of generic logarithmic 1-forms under deformation.

1. Introduction

1.1. Context

The study of the space of holomorphic foliations on projective spaces has been a subject of significant interest since the pioneering work of Jouanolou. Inspired by classical contributions from Jacobi, Darboux, and others, Jouanolou described codimension one foliations of degree one on projective spaces in his book [11]. In particular, his results imply that the space of codimension one holomorphic foliations of degree one on projective spaces, of dimension at least three, has exactly two irreducible components.

More recently, in [12], it was noted that work by Medeiros [8] provides an analogous description for degree one holomorphic foliations of arbitrary codimension on projective spaces.

Cerveau and Lins Neto [3] proved that the space of codimension one holomorphic foliations of degree two on projective spaces of dimension at least three has exactly six irreducible components, and described precisely each one of them. The analogous problem for degree two holomorphic foliations of arbitrary codimension was recently studied by Corrêa and Muniz [4]. However, a complete description of the irreducible components of the space of degree two holomorphic foliations of codimension q on $\mathbb{P}^n_{\mathbb{C}}$ remains an open question for any q strictly between 1 and $n-1$. The situation for degree three foliations is even less understood. While a partial classification exists for codimension one foliations, see [7], the problem remains unexplored for higher codimensions.

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1.2. Purpose and discussion of the main results

This work aims to initiate the study of the space of foliations on projective spaces over algebraically closed fields of positive characteristic, with a focus on the codimension one case. Our initial motivation was to understand how the geometry of these spaces vary with the characteristic of the base field, though our perspective evolved throughout the study and we were led to an analog of Calvo-Andrade's stability of generic logarithmic 1-forms under deformation in arbitrary characteristic, see Theorem A below.

For codimension one foliations of degree zero and one, we provide a complete description of all irreducible components in arbitrary characteristic. For degree zero foliations, there is always a single irreducible component, see Theorem 3.5, which also describes degree zero foliations of arbitrary codimension. Nevertheless, the behavior in characteristic two is different from other characteristics. Specifically, when the characteristic is different from two, the space of degree zero foliations of codimension one can be identified with the Grassmannian of lines in $(\mathbb{P}^n_{\mathsf{k}})^\vee$, whereas in characteristic two, it corresponds to the projective space $\mathbb{P} H^0(\mathbb{P}^n_{\mathsf{k}}, \Omega^1_{\mathbb{P}^n_{\mathsf{k}}}(2))$.

For degree one foliations, the number of irreducible components depends on the characteristic. In characteristic two, there is a single irreducible component, whereas for higher characteristics, there are exactly two. Moreover, the description is uniform when the characteristic is at least 5, see Theorem 4.8.

A key technical tool in our analysis of foliations of degree zero and one is Medeiros' description of locally decomposable integrable differential forms with homogeneous coefficients of degree one, see [8]. Although Medeiros originally formulated his result in the complex setting, the result remains valid in arbitrary characteristic.

For degree two foliations, our results are less definitive. We are able to show that each of the six irreducible components that exist in characteristic zero specialize to irreducible components in characteristic p, when p is at least 5. We achieve this by combining Theorem B in [15] with the main result of this paper, namely:

Theorem A. *Let $r \geq 2$ and $1 \leq d_1 \leq \cdots \leq d_r$ be positive integers. Let k be an algebraically closed field of characteristic $p > 0$. Assume that one of the following conditions holds*:

(1) $r = 2$ *and none of the integers* d_1, d_2, *and* $d_1 + d_2$ *is divisible by* p; *or*

(2) $r = 2$ *and both* d_1 *and* d_2 *are divisible by* p; *or*

(3) $r > 2$ *and the set*

$$S = \{i \in \{1, \ldots, r\} \mid p \text{ divides } d_i\}$$

has cardinality different from $r - 1$ *and* $r - 2$.

Then the set $\mathsf{Log}_{(d_1,\ldots,d_r)}(\mathbb{P}^n_{\mathsf{k}})$ *is an irreducible component of* $\mathsf{Fol}^1_d(\mathbb{P}^n_{\mathsf{k}})$ (*with its reduced scheme structure*) *for* $n \geq 3$ *and* $d = \sum_{i=1}^r d_i - 2$.

For the definition of the scheme $\mathsf{Fol}^1_d(\mathbb{P}^n_{\mathsf{k}})$, we refer to Section 2.4, while the set

$$\mathsf{Log}_{(d_1,\ldots,d_r)}(\mathbb{P}^n_{\mathsf{k}})$$

is defined in Section 4.4.

Over the complex numbers, Theorem A was established by Calvo-Andrade [2], with a proof based on complex-geometric and topological arguments. For an alternative approach using algebraic methods, which actually proves that $\mathsf{Fol}^1_d(\mathbb{P}^n_{\mathbb{C}})$ is generically reduced along $\mathsf{Log}_{(d_1,\dots,d_r)}(\mathbb{P}^n_{\mathbb{C}})$, see [5].

Theorem A offers yet another proof of Calvo-Andrade's result, this time from a more arithmetic perspective, see Corollary 6.3.

2. Basic definitions

2.1. Foliations and distributions

Let X be a smooth algebraic variety defined over an arbitrary field k. A foliation $\mathcal{F}$ on X consists of a pair $(T_{\mathcal{F}}, N^*_{\mathcal{F}})$ of coherent subsheaves of T_X and Ω^1_X such that $T_{\mathcal{F}}$ is the annihilator of $N^*_{\mathcal{F}}$, $N^*_{\mathcal{F}}$ is the annihilator of $T_{\mathcal{F}}$, and $T_{\mathcal{F}}$ is closed under the Lie bracket. The sheaf $T_{\mathcal{F}}$ is the tangent sheaf of $\mathcal{F}$ and the sheaf $N^*_{\mathcal{F}}$ is the conormal sheaf of $\mathcal{F}$. The rank of $T_{\mathcal{F}}$ is the dimension of $\mathcal{F}$, while the rank of $N^*_{\mathcal{F}}$ is the codimension of $\mathcal{F}$. Since $T_{\mathcal{F}}$ and $N^*_{\mathcal{F}}$ are annihilators of each other, we have that $\dim \mathcal{F} + \operatorname{codim} \mathcal{F} = \dim X$.

Dropping the closedness under Lie brackets, one obtains the definition of a distribution. To wit, a distribution $\mathcal{D}$ on X consists of a pair $(T_{\mathcal{D}}, N^*_{\mathcal{D}})$ of subsheaves of T_X and Ω^1_X such that $T_{\mathcal{D}}$ is the annihilator of $N^*_{\mathcal{D}}$ and vice-versa.

2.2. Differential forms

If $\mathcal{D}$ is a distribution of codimension q, taking the determinant of the inclusion $N^*_{\mathcal{D}} \to \Omega^1_X$ and tensoring the result by $N = \det(N^*_{\mathcal{D}})^*$ yields a q-differential form with coefficients in N, that is,

$$\omega \in H^0(X, \Omega^q_X \otimes N),$$

satisfying the following conditions:

(1) the zero locus of ω has codimension at least two;

(2) at any closed point x outside the zero locus of ω, and in any local trivialization of N, the germ of the differential q-form ω at x can be written as the product of q germs of 1-forms, $\omega_1, \dots, \omega_q$;

(3) moreover, if $\mathcal{D}$ is not only a distribution but also a foliation then the germs of 1-forms of the previous item are such that

$$d\omega_j \wedge \omega_1 \wedge \cdots \wedge \omega_q = 0$$

for any $j \in \{1, \dots, q\}$.

The twisted q-form ω is not unique, but is unique up to multiplication by elements of $H^0(X, \mathcal{O}^*_X)$, that is, nowhere-vanishing regular functions.

We define saturated q-forms as those satisfying condition (1); locally decomposable q-forms as those satisfying condition (2); and integrable q-forms as those satisfying conditions (2) and (3). In Medeiros' original terminology, locally decomposable forms were referred to as locally decomposable off the singular set (LDS) forms. We adopt the shorter term locally decomposable for the sake of brevity.

Conversely, given a nonzero twisted q-form $\omega \in H^0(X, \Omega^q_X \otimes N)$, the kernel of the natural morphism induced by contraction with ω,

$$\omega^\flat : T_X \to \Omega^{q-1}_X \otimes N,$$

is a saturated subsheaf of T_X. Consequently, $(\ker \omega^\flat, \operatorname{Ann}(\ker \omega^\flat)) = (T_{\mathcal{D}}, N^*_{\mathcal{D}})$ for some codimension q distribution $\mathcal{D}$.

2.3. Plücker's equations

As it was noted by Medeiros in [8], see also the Appendix B of [15], a twisted q-form $\omega \in H^0(X, \Omega^q_X \otimes N)$ is locally decomposable if and only if the morphism

$$\begin{aligned} \bigwedge^{q-1} T_X &\longrightarrow \Omega^{q+1}_X \otimes N^{\otimes 2}, \\ v &\longmapsto i_v \omega \wedge \omega, \end{aligned}$$

is identically zero. Notice that this condition can be rephrased as the classical Plücker conditions

$$i_v \omega \wedge \omega = 0 \quad \text{for any } v \in \bigwedge^{q-1} T_X.$$

Similarly, a q-form ω is integrable if and only if it is locally decomposable and the morphism

$$\begin{aligned} \bigwedge^{q-1} T_X &\longrightarrow \Omega^{q+2}_X \otimes N^{\otimes 2} \\ v &\longmapsto i_v \omega \wedge d\omega \end{aligned}$$

is identically zero. Here and above, when $q = 1$, we adopt the convention that $\wedge^0 T_X = \mathcal{O}_X$ with interior product with elements of $\Omega^\bullet_X$ given by the usual multiplication. Hence, for $q = 1$, the integrability condition is the usual $\omega \wedge d\omega = 0$ as a section of $\Omega^3_X \otimes N^{\otimes 2}$.

Remark 2.1. The formulation of Plücker's condition above is independent of the characteristic. However, note that in characteristic zero, the local decomposability of a 2-form is equivalent to the condition $\omega \wedge \omega = 0$. This equivalence fails in characteristic two. For instance, the 2-form $\omega = dx_0 \wedge dx_1 + dx_2 \wedge dx_3$ is not decomposable in any characteristic, yet $\omega \wedge \omega = 2dx_0 \wedge dx_1 \wedge dx_2 \wedge dx_3$, making $\omega \wedge \omega = 0$ equal to zero in characteristic two.

2.4. The schemes $\mathsf{Dist}^q_d(\mathbb{P}^n_{\mathsf{k}})$ and $\mathsf{Fol}^q_d(\mathbb{P}^n_{\mathsf{k}})$

Let k be an arbitrary algebraically closed field. Let $\mathcal{D}$ be a codimension q distribution on the projective space $\mathbb{P}^n_{\mathsf{k}}$, $n \geq q + 1$. Let $\omega \in H^0(\mathbb{P}^n_{\mathsf{k}}, \Omega^q_{\mathbb{P}^n_{\mathsf{k}}} \otimes N)$ be a twisted q-form with zero set of codimension at least two whose kernel equals $T_{\mathcal{D}}$. The degree of $\mathcal{D}$ is, by definition, the degree of the zero locus of $i^*\omega \in H^0(\mathbb{P}^q_{\mathsf{k}}, \Omega^q_{\mathbb{P}^q_{\mathsf{k}}} \otimes i^*N)$, where $i : \mathbb{P}^q_{\mathsf{k}} \to \mathbb{P}^n_{\mathsf{k}}$ is a general linear embedding. Hence, if $d = \deg(\mathcal{D})$, then

$$N = \mathcal{O}_{\mathbb{P}^n_{\mathsf{k}}}(d + q + 1).$$

It follows from Euler's sequence that we can identify $H^0(\mathbb{P}^n_{\mathsf{k}}, \Omega^q_{\mathbb{P}^n_{\mathsf{k}}}(d+q+1))$ with the vector space of homogeneous polynomial q-forms

$$\sum_{\substack{I=(i_1,\dots,i_q)\\ 0\le i_1<\cdots<i_q\le n}} a_I(x_0,\dots,x_n)\, dx_I$$

which are annihilated by the radial vector field $R=\sum_{i=0}^n x_i\,\partial/\partial x_i$, and have coefficients $a_0,\dots,a_n$ of degree $d+1$. We will say that a homogeneous polynomial q-form on $\mathbb{A}^{n+1}_{\mathsf{k}}$ representing an element of $H^0(\mathbb{P}^n_{\mathsf{k}}, \Omega^q_{\mathbb{P}^n_{\mathsf{k}}}(d+q+1))$ is a projective q-form of degree $d+q+1$.

The space of codimension q foliations of degree d on the projective space $\mathbb{P}^n_{\mathsf{k}}$ is, by definition, the locally closed subscheme $\mathsf{Fol}^q_d(\mathbb{P}^n_{\mathsf{k}})$ of $\mathbb{P}H^0(\mathbb{P}^n_{\mathsf{k}}, \Omega^q_{\mathbb{P}^n_{\mathsf{k}}}(d+q+1))$ defined by the conditions

$$[\omega]\in \mathsf{Fol}^q_d(\mathbb{P}^n_{\mathsf{k}}) \quad \text{if and only if} \quad \begin{cases} \operatorname{codim}\operatorname{sing}(\omega)\ge 2, \\ i_v\,\omega\wedge\omega=0 \quad \text{for every } v\in \bigwedge^{q-1}\mathbb{A}^{n+1}, \text{ and} \\ i_v\,\omega\wedge d\omega=0 \quad \text{for every } v\in \bigwedge^{q-1}\mathbb{A}^{n+1}. \end{cases}$$

Likewise, the space of codimension q distributions of degree d on the projective space $\mathbb{P}^n_{\mathsf{k}}$ is, by definition, the locally closed subscheme $\mathsf{Dist}^q_d(\mathbb{P}^n_{\mathsf{k}})$ of $\mathbb{P}H^0(\mathbb{P}^n_{\mathsf{k}}, \Omega^q_{\mathbb{P}^n_{\mathsf{k}}}(d+q+1))$ defined by the conditions

$$[\omega]\in \mathsf{Dist}^q_d(\mathbb{P}^n_{\mathsf{k}}) \quad \text{if and only if} \quad \begin{cases} \operatorname{codim}\operatorname{sing}(\omega)\ge 2, \text{ and} \\ i_v\,\omega\wedge\omega=0 \text{ for every } v\in \bigwedge^{q-1}\mathbb{A}^{n+1}. \end{cases}$$

3. Foliations of degree zero

3.1. Euler's relation

If ω is a homogeneous q-form of degree e on $\mathbb{A}^{n+1}_{\mathsf{k}}$, meaning that the coefficients of ω have degree $e-q$, then by Euler's relation, we have

$$i_R\, d\omega + d\, i_R\,\omega = e\cdot\omega. \tag{3.1}$$

Note that in our conventions both x_i and dx_i are assigned degree one.

As observed in Proposition 1.4 in Chapter 1 of [11], when the characteristic of k does not divide $d+q+1$, there is a bijection of the set $H^0(\mathbb{P}^n, \Omega^q_{\mathbb{P}^n}(d+q+1))$ of projective q-forms of degree $d+q+1$ and the set of closed homogeneous $(q+1)$-forms of degree $d+q+1$. This bijection is given by the maps

$$\omega\mapsto d\omega \quad\text{and}\quad \beta\mapsto (d+q+1)^{-1} i_R\,\beta.$$

Furthermore, when $q=1$ and still under the same assumption on the characteristic of k, the set of integrable projective 1-forms of degree $d+2$ is in bijection with the set of closed homogeneous 2-forms β of degree $d+2$ satisfying $\beta\wedge\beta=0$.

For later use, we record a consequence of Euler's formula.

Lemma 3.1. *Let $\omega \in H^0(\mathbb{P}^n_{\mathsf{k}}, \Omega^q_{\mathbb{P}^n_{\mathsf{k}}}(e))$ be a nonzero projective q-form of degree e on $\mathbb{A}^{n+1}_{\mathsf{k}}$. If ω is closed, then the characteristic of k divides e. Moreover, if $e < p(q+1)$, then ω is exact.*

Proof. Since ω is projective, Euler's relation (3.1) reads

$$i_R\, d\omega = e \cdot \omega.$$

If ω is closed, then the left-hand side vanishes and, consequently, the characteristic of k divides e as claimed.

If ω is closed, then Theorem 1.3.4 in [1] implies that

$$\omega = d\beta + \sum_{\substack{I=(i_1,\dots,i_q)\\ i_1<\cdots<i_q}} a_I^p (x_{i_1}^{p-1} dx_{i_1}) \wedge (x_{i_2}^{p-1} dx_{i_2}) \wedge \cdots \wedge (x_{i_q}^{p-1} dx_{i_q})$$

for a certain homogeneous $(q-1)$-form β and some homogeneous polynomials $a_I \in \mathsf{k}[x_0, \dots, x_n]$. The homogeneity of the a_I's follows from that of ω and β, together with the identity $(b_1 + \cdots + b_k)^p = b_1^p + \cdots + b_k^p$. If ω is projective, closed and not exact, then the sum is non-zero, so some a_I is non-zero. The projectiveness condition then forces $\deg(a_I) \geq 1$, and the corresponding summand has degree $p\deg(a_I) + qp \geq p(q+1)$. Hence $e \geq p(q+1)$, as wanted. ■

3.2. Foliations defined by closed projective polynomial 1-forms

Lemma 3.1 implies that there are no closed projective 1-forms of degree e when p does not divide e. In sharp contrast, we have the following proposition.

Proposition 3.2. *Let k be an algebraically closed field of characteristic $p > 0$ and let $n \geq 3$ and $e \geq 1$ be integers. Consider the following subscheme of $\mathbb{P}H^0(\mathbb{P}^n_{\mathsf{k}}, \Omega^1_{\mathbb{P}^n_{\mathsf{k}}}(pe))$:*

$$\mathsf{Cl}_{pe-2}(\mathbb{P}^n_{\mathsf{k}}) := \mathbb{P}\big(\{\omega \in H^0(\mathbb{P}^n_{\mathsf{k}}, \Omega^1_{\mathbb{P}^n_{\mathsf{k}}}(pe)) \mid d\omega = 0 \text{ and } \operatorname{codim} \operatorname{sing}(\omega) \geq 2\}\big)$$

Then $\mathsf{Cl}_{pe-2}(\mathbb{P}^n_{\mathsf{k}})$ is non-empty and is an irreducible component of $\mathsf{Fol}^1_{pe-2}(\mathbb{P}^n_{\mathsf{k}})$. Moreover, $\mathsf{Fol}^1_{pe-2}(\mathbb{P}^n_{\mathsf{k}})$ is generically reduced at a general point of $\mathsf{Cl}_{pe-2}(\mathbb{P}^n_{\mathsf{k}})$.

Proof. Let $F \in \mathsf{k}[x_0, \dots, x_n]$ be the homogeneous polynomial

$$F = \sum_{i=1}^{n} x_{i-1} x_i^{pe-1}$$

borrowed from Section 2 of [16]. The differential dF of F defines a closed (hence integrable) polynomial projective 1-form on $\mathbb{A}^{n+1}_{\mathsf{k}}$ showing that $\mathsf{Cl}_{pe-2}(\mathbb{P}^n_{\mathsf{k}})$ is non-empty. To establish the proposition we will compute the Zariski tangent space of $\mathsf{Fol}^1_{pe-2}(\mathbb{P}^n_{\mathsf{k}})$ at the foliation $\mathcal{F}$ determined by dF.

Collecting coefficients, we can write

$$dF = x_1^{pe-1} dx_0 + \sum_{j=1}^{n-1} (x_{j+1}^{pe-1} - x_{j-1} x_j^{pe-2})\, dx_j - x_{n-1} x_n^{pe-2}\, dx_n.$$

At a zero of dF, the vanishing of the dx_0-coefficient forces $x_1 = 0$. Assuming inductively $x_1 = \cdots = x_j = 0$ for some $1 \leq j \leq n-1$, the dx_j-coefficient reduces to x_{j+1}^{pe-1}, forcing $x_{j+1} = 0$. Thus dF vanishes only along the line $\{x_1 = \cdots = x_n = 0\}$ through the origin of $\mathbb{A}_{\mathsf{k}}^{n+1}$, and the foliation $\mathcal{F}$ determined by dF has singular set of codimension n. In particular, $\mathcal{F}$ is a closed point of $\mathsf{Cl}_{pe-2}(\mathbb{P}_{\mathsf{k}}^n)$, which is thus non-empty.

The Zariski tangent space of $\mathsf{Fol}_{pe-2}^1(\mathbb{P}_{\mathsf{k}}^n)$ at $\mathcal{F}$ can be identified with the vector space

$$\frac{\{\eta \in H^0(\mathbb{P}_{\mathsf{k}}^n, \Omega_{\mathbb{P}_{\mathsf{k}}^n}^1(pe)) \mid dF \wedge d\eta = 0\}}{\mathsf{k} \cdot dF}.$$

If η is a polynomial projective 1-form of degree pe satisfying $dF \wedge d\eta = 0$, then since $\operatorname{codim} \operatorname{sing}(dF) = n \geq 3$, de Rham–Saito's lemma [18] implies that $d\eta = dF \wedge \beta$ for some polynomial form β. Comparing degrees, we deduce that $d\eta = 0$. It follows that the Zariski tangent space of $\mathsf{Fol}_{pe-2}^1(\mathbb{P}_{\mathsf{k}}^n)$ at $\mathcal{F}$ coincides with the Zariski tangent space of $\mathsf{Cl}_{pe-2}(\mathbb{P}_{\mathsf{k}}^n)$ at $\mathcal{F}$. Since $\mathsf{Cl}_{pe-2}(\mathbb{P}_{\mathsf{k}}^n)$ is defined by linear equations on the coefficients of ω and is contained in the Zariski closure of $\mathsf{Fol}_{pe-2}^1(\mathbb{P}_{\mathsf{k}}^n)$, we conclude that $\mathsf{Cl}_{pe-2}(\mathbb{P}_{\mathsf{k}}^n)$ is an irreducible component of $\mathsf{Fol}_{pe-2}^1(\mathbb{P}_{\mathsf{k}}^n)$ and that $\mathsf{Fol}_{pe-2}^1(\mathbb{P}_{\mathsf{k}}^n)$ is generically reduced at $\mathcal{F}$. ■

3.3. Locally decomposable differential forms with linear coefficients

Below, we recall Medeiros' description of linear locally decomposable and integrable differential forms.

Theorem 3.3. *Let $n \geq 3$ and $1 \leq q \leq n-1$ be integers, and let ω be a homogeneous locally decomposable q-form on $\mathbb{A}_{\mathsf{k}}^{n+1}$. If the coefficients of ω are linear, then there exists a linear change of coordinates such that*

(a) *$\omega = \alpha \wedge dx_1 \wedge \cdots \wedge dx_{q-1}$ for some linear 1-form α (when $q = 1$, this reads simply $\omega = \alpha$); or*

(b) *ω can be written, for suitable linear polynomials A_i, as*

$$\sum_{i=0}^{q} A_i \, dx_0 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_q.$$

Moreover, if ω is integrable, then in Case (a), $\alpha = df$ for some quadratic polynomial $f \in \mathsf{k}[x_0, \ldots, x_n]$ and, in Case (b), the polynomials A_i belong to $\mathsf{k}[x_0, \ldots, x_q]$.

The original statement was formulated over the field $\mathbb{C}$. However, as we proceed to verify, the result remains valid over any algebraically closed field k. The key modifications to Medeiros' argument are presented in the following result.

Lemma 3.4. *Let ω be a linear locally decomposable and integrable q-form on $\mathbb{A}_{\mathsf{k}}^{n+1}$. If ω is non-closed, then there is a linear change of coordinates such that*

$$\omega = \sum_{i=0}^{q} A_i \, dx_0 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_q,$$

where A_i are linear forms in $\mathsf{k}[x_0, \ldots, x_q]$ for all i.

Proof. Since ω is integrable, it is locally decomposable by definition and we can write

$$\omega = \omega_1 \wedge \cdots \wedge \omega_q,$$

with $\omega_1, \ldots, \omega_q$ suitable rational 1-forms. Moreover, the integrability of ω implies that

$$d\omega_j = \sum_{\ell=1}^{q} \theta_{j\ell} \wedge \omega_\ell$$

for suitable rational 1-forms $\theta_{j\ell}$. Therefore

$$d\omega = \sum_{j=1}^{q} (-1)^{j-1}\, \omega_1 \wedge \cdots \wedge d\omega_j \wedge \cdots \wedge \omega_q = \theta \wedge \omega$$

for some rational 1-form θ. Hence $d\omega$ is a decomposable $(q+1)$-form. Moreover, after a suitable linear change of coordinates, decomposable $(q+1)$-forms with coefficients of degree zero can be written as $dx_0 \wedge \cdots \wedge dx_q$.

The integrability condition $i_v\,\omega \wedge d\omega = 0$, for any $(q-1)$-vector v implies that

$$\omega = \sum_{i=0}^{q} A_i\, dx_0 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_q,$$

where A_i are linear homogeneous polynomials in $\mathsf{k}[x_0, \ldots, x_n]$. Furthermore, since $d\omega = dx_0 \wedge \cdots \wedge dx_q$ by assumption, computing the exterior derivative of ω in the above form implies that the linear homogeneous polynomials A_i actually belong to $\mathsf{k}[x_0, \ldots, x_q]$ as wanted. ■

Proof of Theorem 3.3. Medeiros' argument remains valid over an arbitrary algebraically closed field, except for his use of Proposition 1.3.1 in [8], which relies on analytic methods. However, this can be replaced by Lemma 3.4 above. ■

3.4. Foliations of degree zero

We now have all the necessary ingredients to state and prove the classification of degree zero foliations in arbitrary characteristic.

Theorem 3.5. *Let k be an algebraically closed field. Let $n \geq 3$ and $n-1 \geq q \geq 1$ be integers. If the characteristic of k is different from 2, or if $q > 1$, then $\mathsf{Fol}_0^q(\mathbb{P}^n_{\mathsf{k}})$ is isomorphic to the Grassmannian of $(q+1)$-codimensional linear subspaces of $\mathbb{P}^n_{\mathsf{k}}$, and foliations of degree 0 are defined by linear projections $\mathbb{P}^n_{\mathsf{k}} \dashrightarrow \mathbb{P}^q_{\mathsf{k}}$. If instead the characteristic of k is 2 and $q = 1$, then*

$$\mathsf{Fol}_0^1(\mathbb{P}^n_{\mathsf{k}}) = \mathbb{P}H^0(\mathbb{P}^n_{\mathsf{k}}, \Omega^1_{\mathbb{P}^n_{\mathsf{k}}}(2)) = \mathsf{Cl}_0(\mathbb{P}^n_{\mathsf{k}}).$$

In all cases, $\mathsf{Fol}_0^q(\mathbb{P}^n_{\mathsf{k}})$ (with its reduced scheme structure) is a smooth and irreducible algebraic variety.

Proof. Let $\mathcal{F}$ be a foliation of degree zero, and let $\omega \in H^0(\mathbb{P}^n_{\mathsf{k}}, \Omega^q_{\mathbb{P}^n_{\mathsf{k}}}(q+1))$ be a projective q-form of degree $q+1$ defining it. We analyze the two cases predicted by Theorem 3.3.

If ω falls in Case (a) of Theorem 3.3, i.e., if $\omega = df \wedge dx_1 \wedge \cdots \wedge dx_{q-1}$, we can and will assume that $f \in \mathsf{k}[x_q, \ldots, x_n]$. Then, expanding the projectiveness condition $i_R\omega = 0$, we obtain:

(1) $i_R df = 0$; and

(2) $i_R(dx_1 \wedge \cdots \wedge dx_{q-1}) = 0$.

Since f is quadratic, the first condition implies that the characteristic of k is 2. The second condition forces $q = 1$, that is, $\omega = df$.

If instead ω fits in Case (b) of Theorem 3.3, that is,

$$\sum_{i=0}^{q} A_i(x_0, \ldots, x_q)\, dx_0 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_q,$$

then the projectiveness condition $i_R\omega = 0$, together with the exactness of the Koszul complex, implies that ω is a scalar multiple of $i_R(dx_0 \wedge \cdots \wedge dx_q)$. Consequently, the foliation $\mathcal{F}$ is defined by the kernel of the differential of the linear projection

$$\begin{aligned} \mathbb{P}^n_{\mathsf{k}} &\dashrightarrow \mathbb{P}^q_{\mathsf{k}}, \\ (x_0 : \ldots : x_n) &\mapsto (x_0 : \ldots : x_q). \end{aligned}$$

As such, $\mathcal{F}$ is completely characterized by the indeterminacy locus of this linear projection, which is the linear subspace $\{x_0 = \cdots = x_q = 0\}$ of codimension $q+1$. This proves the result when $q \neq 1$ or the characteristic of k is different from 2. It remains to treat the case $q = 1$ and $\mathrm{char}(\mathsf{k}) = 2$.

We first show that $\mathbb{P}H^0(\mathbb{P}^n_{\mathsf{k}}, \Omega^1_{\mathbb{P}^n_{\mathsf{k}}}(2)) = \mathsf{Cl}_0(\mathbb{P}^n_{\mathsf{k}})$, i.e., that every projective 1-form of degree 2 is closed in characteristic 2. Write $\omega = \sum_i a_i\, dx_i$, with $a_i = \sum_j A_{ij} x_j$ linear. The projectivity condition $i_R\omega = \sum_i x_i a_i = 0$ gives $A_{ii} = 0$ for all i, and $A_{ij} + A_{ji} = 0$ for $i \neq j$. In characteristic 2, the latter reads $A_{ij} = A_{ji}$, so

$$d\omega = \sum_{i<j} (A_{ij} + A_{ji})\, dx_i \wedge dx_j = 0.$$

Thus

$$\mathsf{Fol}^1_0(\mathbb{P}^n_{\mathsf{k}}) \subseteq \mathbb{P}H^0(\mathbb{P}^n_{\mathsf{k}}, \Omega^1_{\mathbb{P}^n_{\mathsf{k}}}(2)) = \mathsf{Cl}_0(\mathbb{P}^n_{\mathsf{k}}) \subseteq \mathsf{Fol}^1_0(\mathbb{P}^n_{\mathsf{k}}),$$

and all three coincide.

Moreover, every $\omega \in \mathsf{Fol}^1_0(\mathbb{P}^n_{\mathsf{k}})$ is exact: in Case (a), the argument above gives $\omega = df$; in Case (b), ω is a scalar multiple of $i_R(dx_0 \wedge dx_1) = x_0\, dx_1 - x_1\, dx_0$, which in characteristic 2 equals $x_0\, dx_1 + x_1\, dx_0 = d(x_0 x_1)$. This completes the proof. ∎

Remark 3.6. Our proof does not explicitly address the scheme structure of $\mathsf{Fol}^q_0(\mathbb{P}^n_{\mathsf{k}})$. However, the proof of Proposition 3.2 shows that in the exceptional case where $q = 1$ and the characteristic of k is 2, the scheme $\mathsf{Fol}^1_0(\mathbb{P}^n_{\mathsf{k}})$ is generically reduced. In all other cases, we believe that $\mathsf{Fol}^q_0(\mathbb{P}^n_{\mathsf{k}})$ is reduced everywhere. An attentive reader will notice that when $q = 1$ and the characteristic is different from 2, this follows from the proof of Proposition 4.6 below. Moreover, it is very likely that the results and arguments of [6] can be adapted to prove the reducedness of $\mathsf{Fol}_0^q(\mathbb{P}^n_{\mathsf{k}})$ when $q > 1$.

4. Foliations of degree one

In this section, we focus on foliations of degree one on projective varieties over fields of arbitrary characteristic. Unlike Section 3, where we examined foliations of arbitrary codimension, here we restrict our attention to codimension one foliations.

After reviewing key properties of codimension one foliations in positive characteristic, following [14], we introduce two families of irreducible components of the space of foliations: linear pull-back components and logarithmic components with only two distinct residues. These families contain special members of degree one.

We conclude the section with a detailed description of codimension one foliations of degree one in positive characteristic, building on the previous results.

4.1. The p-curvature and its degeneracy divisor for codimension one foliations

The main reference for the definitions and results discussed in this subsection is [14]. Let X be a smooth projective variety defined over an algebraically closed field k of characteristic $p > 0$. Denote by $\mathrm{Frob}\colon X \to X$ the absolute Frobenius morphism. If $\mathcal{F}$ is a foliation on X, its p-curvature is the morphism of $\mathcal{O}_X$-modules given by raising vector fields to theirs p-th powers:

$$\begin{aligned} \psi_{\mathcal{F}} : \mathrm{Frob}^* T_{\mathcal{F}} &\longrightarrow N_{\mathcal{F}}, \\ \sum f_i \otimes v_i &\mapsto \sum f_i v_i^p \mod T_{\mathcal{F}}. \end{aligned}$$

A foliation $\mathcal{F}$ is called p-closed if $\psi_{\mathcal{F}}$ is identically zero. When $\psi_{\mathcal{F}}$ is generically surjective, meaning that the image of $\psi_{\mathcal{F}}$ has rank equal to the codimension of $\mathcal{F}$, we say that $\mathcal{F}$ is p-dense.

When a foliation $\mathcal{F}$ has codimension one, then it is either p-closed or p-dense. In the p-dense case, there exist a uniquely determined effective divisor $\Delta_{\mathcal{F}}$ and a uniquely determined codimension two subfoliation $\mathcal{V}(\mathcal{F}) \subset \mathcal{F}$ such that

(1) away from a subset of codimension at least two, the image of $\psi_{\mathcal{F}}$ coincides with $N_{\mathcal{F}} \otimes \mathcal{O}_X(-\Delta_{\mathcal{F}})$; and
(2) the kernel of the $\psi_{\mathcal{F}}$ is equal to $\mathrm{Frob}^* T_{\mathcal{V}(\mathcal{F})}$.

When $X = \mathbb{P}^n_{\mathsf{k}}$, the degrees of $\mathcal{F}$, $\mathcal{V}(\mathcal{F})$, and $\Delta_{\mathcal{F}}$ satisfy the relation

$$\deg(\Delta_{\mathcal{F}}) = p(\deg(\mathcal{F}) - \deg(\mathcal{V}(\mathcal{F})) - 1) + \deg(\mathcal{F}) + 2, \tag{4.1}$$

which is a specialization of Proposition 4.12 in [14].

Example 4.1. Every foliation on $\mathbb{P}^n_{\mathsf{k}}$ of degree zero is p-closed. Indeed, the codimension one case follows from equation (4.1) and in higher codimension this follows from Theorem 3.5.

4.2. Linear pull-backs

Let $\mathrm{Lin}_d(\mathbb{P}^n_{\mathsf{k}}) \subset \mathrm{Fol}^1_d(\mathbb{P}^n_{\mathsf{k}})$ denote the reduced subscheme whose closed points correspond to foliations on $\mathbb{P}^n_{\mathsf{k}}$ that arise as pull-backs of degree d foliations on $\mathbb{P}^2_{\mathsf{k}}$ under a linear projection $\pi\colon \mathbb{P}^n_{\mathsf{k}} \dashrightarrow \mathbb{P}^2_{\mathsf{k}}$. It is well known that for every $d \geq 0$ and $n \geq 3$, $\mathrm{Lin}_d(\mathbb{P}^n_{\mathbb{C}})$ is an irreducible component of $\mathrm{Fol}^1_d(\mathbb{P}^n_{\mathbb{C}})$. Observe that $\mathrm{Lin}_0(\mathbb{P}^n_{\mathsf{k}})$ corresponds to foliations on $\mathbb{P}^n_{\mathsf{k}}$ that are defined by a pencil of hyperplanes, hence have singular set of codimension two. As a corollary of Theorem 3.5, we obtain that $\mathrm{Lin}_0(\mathbb{P}^n_{\mathsf{k}})$ is an irreducible component of $\mathrm{Fol}^1_0(\mathbb{P}^n_{\mathsf{k}})$ if and only if the characteristic of k is different from two. Indeed, when the characteristic of k is two, a general element of $\mathrm{Cl}_0(\mathbb{P}^n_{\mathsf{k}})$ has empty singular set when n is odd or a unique singularity when n is even. In both cases, $\mathrm{Lin}_0(\mathbb{P}^n_{\mathsf{k}})$ cannot be equal to $\mathrm{Cl}_0(\mathbb{P}^n_{\mathsf{k}})$.

Proposition 4.2. *Let $d > 0$. If k has characteristic $p > 0$, then $\mathrm{Lin}_d(\mathbb{P}^n_{\mathsf{k}})$ is an irreducible component of $\mathrm{Fol}^1_d(\mathbb{P}^n_{\mathsf{k}})$ if and only if $d < p$.*

Proof. Let $\mathcal{G}$ be a general foliation of $\mathbb{P}^2_{\mathsf{k}}$ of degree $d \geq 1$. Consider the linear projection $\pi\colon \mathbb{P}^n_{\mathsf{k}} \dashrightarrow \mathbb{P}^2_{\mathsf{k}}$ given by

$$\pi(x_0 : \ldots : x_n) = (x_0 : x_1 : x_2),$$

and let $\mathcal{F} = \pi^*\mathcal{G}$.

According to [13], see also Theorem 5.10 in [14], the degeneracy divisor $\Delta_{\mathcal{G}}$ is a reduced divisor. Moreover, equation (4.1) implies it has degree $d + 2 + p(d-1)$. Due to the explicit form of π, $\Delta_{\mathcal{F}} = \pi^*\Delta_{\mathcal{G}}$ and $\Delta_{\mathcal{F}}$ is also reduced and of degree $d + 2 + p(d-1)$.

If $\Sigma \subset \mathrm{Fol}^1_d(\mathbb{P}^n_{\mathsf{k}})$ is an irreducible component containing $\mathcal{F}$, then by Section 6 of [14], for a general foliation $\mathcal{F}'$ in Σ, the kernel of the p-curvature of $\mathcal{F}'$ is a codimension two foliation $\mathcal{V}(\mathcal{F}')$ of degree zero. Theorem 3.5 implies that in suitable homogeneous coordinates, $\mathcal{V}(\mathcal{F})'$ is defined by the linear projection $(x_0 : \ldots : x_n) \mapsto (x_0 : x_1 : x_2)$. Consequently, $\mathcal{F}'$ is defined by a homogeneous polynomial 1-form ω' that can be written as

$$\omega' = a\,dx_0 + b\,dx_1 + c\,dx_2,$$

where a, b and c are homogeneous polynomials in $\mathsf{k}[x_0, x_1, x_2, \ldots, x_n]$ of degree $d + 1$. The integrability condition $\omega' \wedge d\omega' = 0$ implies that

$$\begin{aligned}(adb - bda) \wedge dx_0 \wedge dx_1 \wedge dx_2 &= (adc - cda) \wedge dx_0 \wedge dx_1 \wedge dx_2 \\ &= (cdb - bdc) \wedge dx_0 \wedge dx_1 \wedge dx_2 = 0.\end{aligned}$$

Therefore we deduce from the integrability condition that

$$\frac{\partial}{\partial x_i}\left(\frac{a}{b}\right) = \frac{\partial}{\partial x_i}\left(\frac{a}{c}\right) = \frac{\partial}{\partial x_i}\left(\frac{b}{c}\right) = 0,$$

for every $i \in \{3, \ldots, n\}$. Consequently, we obtain the existence of a polynomial

$$f \in \mathsf{k}[x_0, x_1, x_2, \ldots, x_n]$$

such that $a, b, c \in f \cdot \mathsf{k}[x_0, x_1, x_2, x_3^p, \dots, x_n^p]$. Since ω' has singularities of codimension at least two, $f \in \mathsf{k}$.

Thus, for every multi-index $I = (i_3, \dots, i_n) \in \mathbb{N}^{n-2}$ such that $d + 2 > p \cdot |I|$, there exists a projective 1-form β_I of degree $d + 2 - p \cdot |I|$ on $\mathbb{A}^3_{\mathsf{k}}$ with variables x_0, x_1, x_2 such that

$$\omega' = \sum_{I \in \mathbb{N}^{n-2}} (x_3^{i_3} \cdots x_n^{i_n})^p \beta_I .$$

Since any projective 1-form has degree at least two, we deduce that ω' is a linear pull-back when $d + 2 - p < 2$. If instead $d + 2 - p \geq 2$, then the above expression for ω' provides a method to construct deformations of ω that are not linear pull-backs, since ω' as above satisfies the Frobenius integrability condition $\omega' \wedge d\omega' = 0$. ∎

For later use, we isolate part of the argument used in the proof of Proposition 4.2 in the form of a lemma.

Lemma 4.3. *Let $\mathcal{F}$ be a codimension one foliation on $\mathbb{P}^n_{\mathsf{k}}$, where k is an algebraically closed field of characteristic $p > 0$. If $\mathcal{F}$ contains a codimension q foliation of degree zero, $2 \leq q \leq n-1$, and $\deg(\mathcal{F}) < p$, then there exists a linear projection $\pi\colon \mathbb{P}^n_{\mathsf{k}} \dashrightarrow \mathbb{P}^q_{\mathsf{k}}$ and a foliation $\mathcal{G}$ on $\mathbb{P}^q_{\mathsf{k}}$ such that $\mathcal{F} = \pi^*\mathcal{G}$.*

Proof. The argument follows the proof of Proposition 4.2 with $\mathbb{P}^q_{\mathsf{k}}$ in place of $\mathbb{P}^2_{\mathsf{k}}$. Since, by assumption, $\mathcal{F}$ contains a degree zero foliation of codimension q, Theorem 3.5 provides coordinates in which

$$\omega' = \sum_{i=0}^{q-1} a_i \, dx_i, \quad \text{with } a_i \in \mathsf{k}[x_0, \dots, x_n].$$

The integrability condition forces $a_i \in \mathsf{k}[x_0, \dots, x_{q-1}, x_q^p, \dots, x_n^p]$. Since $\deg(\mathcal{F}) < p$, any term involving x_j^p (for $j \geq q$) would contribute degree at least $p > d + 1$ to a_i, which is impossible. Hence $a_i \in \mathsf{k}[x_0, \dots, x_{q-1}]$ and ω' is the pull-back of a form on $\mathbb{P}^q_{\mathsf{k}}$. ∎

As a consequence of Proposition 4.2 we get the following proposition.

Proposition 4.4. *Let k be an algebraically closed field of characteristic $p > 2$. Let $\mathcal{F}$ be a codimension one foliation of degree one in $\mathbb{P}^n_{\mathsf{k}}$ and suppose that $\mathcal{F}$ is not p-closed. Then, $\mathcal{F}$ is a linear pull-back of a degree one foliation on $\mathbb{P}^2_{\mathsf{k}}$.*

Proof. Since $\mathcal{F}$ is not p-closed, the degeneracy divisor of its p-curvature, $\Delta_{\mathcal{F}}$, has degree $-p\deg(\mathcal{V}(\mathcal{F})) + 3$ according to equation (4.1).

If $\deg(\mathcal{V}(\mathcal{F})) = 0$, then, arguing as in the proof of Proposition 4.2, we obtain that $\mathcal{F}$ is a linear pull-back of a degree one foliation on $\mathbb{P}^2_{\mathsf{k}}$.

If $\deg(\mathcal{V}(\mathcal{F})) \geq 1$, then, since $\Delta_{\mathcal{F}}$ is an effective divisor, it follows that $p = 3$ and that $\Delta_{\mathcal{F}}$ is the trivial divisor. By Propositions 4.7 and 4.14 in [14], any projective 1-form $\omega \in H^0(\mathbb{P}^n_{\mathsf{k}}, \Omega^1_{\mathbb{P}^n_{\mathsf{k}}}(3))$ defining $\mathcal{F}$ is closed. Lemma 3.1 implies that $\omega = df$ for some polynomial $f \in \mathsf{k}[x_0, \dots, x_n]$. Therefore, $\mathcal{F}$ is p-closed, contradicting our assumptions. Thus, we conclude that $\mathcal{F}$ is necessarily a linear pull-back of a degree one foliation on $\mathbb{P}^2_{\mathsf{k}}$. ∎

4.3. Numerically linear pull-backs

Let $\mathsf{NLin}_d(\mathbb{P}^n_{\mathsf{k}}) \subset \mathsf{Fol}^1_d(\mathbb{P}^n_{\mathsf{k}})$ denote the reduced subscheme whose closed points correspond to codimension one foliations $\mathcal{F}$ on $\mathbb{P}^n_{\mathsf{k}}$ of degree d such that the sheaf $T_{\mathcal{F}}$ contains $\mathcal{O}_{\mathbb{P}^n_{\mathsf{k}}}(+1)^{\oplus n-2}$.

Over the complex numbers, we have $\mathsf{NLin}_d(\mathbb{P}^n_{\mathbb{C}}) = \mathsf{Lin}_d(\mathbb{P}^n_{\mathbb{C}})$. However, in positive characteristic, the proof of Proposition 4.2 highlights the distinction between these two spaces, leading to the more precise statement below.

Proposition 4.5. *Let $d > 0$. If k has characteristic $p > 0$, then $\mathsf{NLin}_d(\mathbb{P}^n_{\mathsf{k}})$ is an irreducible component of $\mathsf{Fol}^1_d(\mathbb{P}^n_{\mathsf{k}})$. Moreover, $\mathsf{NLin}_d(\mathbb{P}^n_{\mathsf{k}}) = \mathsf{Lin}_d(\mathbb{P}^n_{\mathsf{k}})$ if and only if $d < p$.*

We point out that one can deduce that $\mathsf{NLin}_d(\mathbb{P}^n_{\mathsf{k}})$ is an irreducible component of $\mathsf{Fol}^1_d(\mathbb{P}^n_{\mathsf{k}})$ using Theorem B and Corollary 4.14 in [15], as an alternative to the argument involving the p-th powers of vector fields. We opt for the latter method because it leverages specific features of positive characteristic, in contrast to the former method.

4.4. Logarithmic components I

Let $(d_1, \dots, d_r)$ be an ordered partition of $d + 2$, i.e., $1 \leq d_1 \leq d_2 \leq \cdots \leq d_r$ and $\sum_{i=1}^r d_i = d + 2$. Set Λ as the k-vector space

$$\Lambda = \Big\{(\lambda_1, \dots, \lambda_r) \in \mathsf{k}^r \;\Big|\; \sum_{i=1}^r d_i \lambda_i = 0\Big\}.$$

Observe that $\Lambda = \mathsf{k}^r$ if all integers d_i are divisible by p, otherwise $\dim \Lambda = r - 1$.

Consider the rational map

$$\Phi_{n,d_1,\dots,d_r} : \mathbb{P}(\Lambda) \times \Big(\prod_{i=1}^r \mathbb{P} H^0\big(\mathbb{P}^n_{\mathsf{k}}, \mathcal{O}_{\mathbb{P}^n_{\mathsf{k}}}(d_i)\big)\Big) \dashrightarrow \mathbb{P} H^0\big(\mathbb{P}^n_{\mathsf{k}}, \Omega^1_{\mathbb{P}^n_{\mathsf{k}}}(d+2)\big),$$

$$([\lambda_1 : \dots : \lambda_r], [f_1], \dots, [f_r]) \longmapsto \Big(\prod_{i=1}^r f_i\Big)\Big(\sum_{i=1}^r \lambda_i \frac{df_i}{f_i}\Big).$$

Define $\mathsf{Log}_{(d_1,\dots,d_r)}(\mathbb{P}^n_{\mathsf{k}})$ as the Zariski closure of the image of $\Phi_{n,d_1,\dots,d_r}$ in the open subset of $\mathbb{P} H^0(\mathbb{P}^n_{\mathsf{k}}, \Omega^1_{\mathbb{P}^n_{\mathsf{k}}}(d+2))$ formed by projective 1-forms with singular set of codimension at least 2. Note that $\mathsf{Log}_{(d_1,\dots,d_r)}(\mathbb{P}^n_{\mathsf{k}})$ is empty if all but one of the d_i's are divisible by p since, in this case, the image of $\Phi_{n,d_1,\dots,d_r}$ will be contained in the closed subset formed by 1-forms with codimension one zeros. In every other case, $\mathsf{Log}_{(d_1,\dots,d_r)}(\mathbb{P}^n_{\mathsf{k}})$ is non-empty.

Let us first study the case $r = 2$. We adapt the proof of Theorem 1 in [6].

Proposition 4.6. *Let d_1 and d_2 be two positive integers and let k be an algebraically closed field of characteristic $p \geq 0$. If $n \geq 3$ and p does not divide any of the integers d_1, d_2, and $d_1 + d_2$, then $\mathsf{Log}_{(d_1,d_2)}(\mathbb{P}^n_{\mathsf{k}})$ is an irreducible component of $\mathsf{Fol}^1_{d_1+d_2-2}(\mathbb{P}^n_{\mathsf{k}})$. Moreover, $\mathsf{Fol}^1_{d_1+d_2-2}(\mathbb{P}^n_{\mathsf{k}})$ is generically reduced, i.e., reduced at the general point of $\mathsf{Log}_{(d_1,d_2)}(\mathbb{P}^n_{\mathsf{k}})$.*

Proof. Let $f_1, f_2 \in \mathsf{k}[x_0, \dots, x_n]$ be homogeneous polynomials of degrees d_1, d_2, defining smooth hypersurfaces H_1 and H_2 that intersect transversely. Let $\mathcal{F} \in \mathsf{Log}_{(d_1,d_2)}(\mathbb{P}^n_{\mathsf{k}})$ be the foliation of degree $d = d_1 + d_2 - 2$ determined by $\omega = i_R df_1 \wedge df_2$. The proof of the proposition will follow the same lines of the proof of Proposition 3.2. More specifically, it will consist in determining the tangent space of $\mathsf{Fol}^1_d(\mathbb{P}^n_{\mathsf{k}})$ at $\mathcal{F}$. For that, it suffices to solve the equation

$$d\omega \wedge d\eta = 0$$

for η projective 1-form of degree $d + 2 = d_1 + d_2$, since p does not divide $d + 2$ by assumption, and because of that, we can go back and forth from this equation to $\omega \wedge d\alpha + \alpha \wedge d\omega$ using Euler's relation, see Section 3.1.

First observe that the projectivization Z of the zero set of $df_1 \wedge df_2$ does not intersect the hypersurfaces H_1 and H_2. Since they are ample divisors, it follows that $\dim Z = 0$. As we are assuming $n \geq 3$, we have that the codimension of the zero set of $df_1 \wedge df_2$ is at least three. Therefore, we are in place to apply de Rham–Saito's lemma and write

$$d\eta = \alpha \wedge df_1 + \beta \wedge df_2$$

for homogeneous 1-forms α of degree $d + 2 - d_1 = d_2$ and β of degree $d + 2 - d_2 = d_1$. Differentiating this expression and taking the wedge product with df_1, we deduce that $d\beta \wedge df_1 \wedge df_2 = 0$. Similarly, $d\alpha \wedge df_1 \wedge df_2 = 0$.

We apply De Rham–Saito's lemma again to write

$$d\beta = \beta' \wedge df_1 + \beta'' \wedge df_2,$$

with β' of degree $d_1 - d_1 = 0$ and β'' of degree $d_1 - d_2 \leq 0$. It follows that $d\beta = 0$. Therefore, there exist polynomials $a_1, \dots, a_n, b \in \mathsf{k}[x_0, \dots, x_n]$ such that

$$\beta = db + \sum_{i=0}^{n} a_i(x_0, \dots, x_n)^p x_i^{p-1} dx_i.$$

Since the degree of β is not divisible by p, the polynomials a_i are equal to zero and we deduce that β is exact, i.e., $\beta = db$.

If $d_1 = d_2$, then the argument above shows that α is also exact. We deduce that

$$d\eta = da \wedge df_1 + db \wedge df_2$$

for homogeneous polynomials a of degree d_2 and b of degree d_1. This shows that the Zariski tangent space of $\mathsf{Fol}^1_d(\mathbb{P}^n_{\mathsf{k}})$ coincides with the image of the differential of Φ_{n,d_1,d_2} at the point corresponding to $\mathcal{F}$. If instead $d_1 < d_2$ then a variant of the argument above shows that $\alpha = da + c\,df_1$ for some polynomials $a, c \in \mathsf{k}[x_0, \dots, x_n]$, cf. the proof of Proposition 2.1 in [6]. Anyway, the conclusion is the same: $d\eta = da \wedge df_1 + db \wedge df_2$ as above.

Since $\mathsf{Log}_{(d_1,d_2)}(\mathbb{P}^n_{\mathsf{k}})$ is irreducible and its Zariski tangent space at $\mathcal{F}$ coincides with that of $\mathsf{Fol}^1_d(\mathbb{P}^n_{\mathsf{k}})$, we conclude as in the proof of Proposition 3.2 that $\mathsf{Log}_{(d_1,d_2)}(\mathbb{P}^n_{\mathsf{k}})$ is an irreducible component of $\mathsf{Fol}^1_d(\mathbb{P}^n_{\mathsf{k}})$ and that $\mathsf{Fol}^1_d(\mathbb{P}^n_{\mathsf{k}})$ is generically reduced at the general point of $\mathsf{Log}_{(d_1,d_2)}(\mathbb{P}^n_{\mathsf{k}})$. ∎

Remark 4.7. Let k be an algebraically closed field of characteristic $p > 0$. If d_1, d_2 are not divisible by p and $d_1 + d_2 = 0 \mod p$, then $\mathsf{Log}_{(d_1,d_2)}(\mathbb{P}^n_{\mathsf{k}})$ is not an irreducible component of $\mathsf{Fol}^1_d(\mathbb{P}^n_{\mathsf{k}})$ since it is contained in the irreducible component $\mathsf{Cl}_{d_1+d_2-2}(\mathbb{P}^n_{\mathsf{k}})$.

We will show in Theorem 6.2 that the subvarieties $\mathsf{Log}_{(d_1,d_2)}(\mathbb{P}^n_{\mathsf{k}})$ when both d_1 and d_2 are divisible by the characteristic of k are irreducible components of $\mathsf{Fol}^1_{d_1+d_2-2}(\mathbb{P}^n_{\mathsf{k}})$.

4.5. Foliations of degree one

With the necessary tools in place, we can now state and prove the classification of codimension one foliations of degree one in arbitrary characteristic.

Theorem 4.8. *Let k be an algebraically closed field of characteristic $p \geq 0$.*

(1) *If $p = 2$, then $\mathsf{Fol}^1_1(\mathbb{P}^n_{\mathsf{k}})$ has exactly one irreducible component*:

$$\mathsf{Lin}_1(\mathbb{P}^n_{\mathsf{k}}).$$

(2) *If $p = 3$, then $\mathsf{Fol}^1_1(\mathbb{P}^n_{\mathsf{k}})$ has exactly two irreducible components*:

$$\mathsf{Lin}_1(\mathbb{P}^n_{\mathsf{k}}) \quad \textit{and} \quad \mathsf{Cl}_1(\mathbb{P}^n_{\mathsf{k}}).$$

(3) *If $p \notin \{2, 3\}$, then $\mathsf{Fol}^1_1(\mathbb{P}^n_{\mathsf{k}})$ has exactly two irreducible components*:

$$\mathsf{Lin}_1(\mathbb{P}^n_{\mathsf{k}}) \quad \textit{and} \quad \mathsf{Log}_{(1,2)}(\mathbb{P}^n_{\mathsf{k}}).$$

Proof. As a preliminary remark, let us note that $\mathsf{Lin}_1(\mathbb{P}^n_{\mathsf{k}})$ is an irreducible component of $\mathsf{Fol}^1_1(\mathbb{P}^n_{\mathsf{k}})$ for every $p \geq 0$. For $p > 0$, this follows from Proposition 4.2. For $p = 0$, it follows by semi-continuity: since $\mathsf{Fol}^1_1(\mathbb{P}^n_{\mathsf{k}})$ and $\mathsf{Lin}_1(\mathbb{P}^n_{\mathsf{k}})$ are defined over $\mathbb{Z}$, upper semi-continuity of the Zariski tangent space dimension implies that its value at a general point of $\mathsf{Lin}_1(\mathbb{P}^n_{\mathsf{k}})$ in characteristic zero is at most its value in characteristic $p \gg 0$, which equals $\dim \mathsf{Lin}_1(\mathbb{P}^n_{\mathsf{k}}) = \dim \mathsf{Fol}^1_1(\mathbb{P}^n_{\mathsf{k}})$ by Proposition 4.2. When $p = 3$, Proposition 3.2 provides an additional irreducible component $\mathsf{Cl}_1(\mathbb{P}^n_{\mathsf{k}})$. For $p > 3$, Proposition 4.6 also provides an extra irreducible component $\mathsf{Log}_{(1,2)}(\mathbb{P}^n_{\mathsf{k}})$.

Let ω be a projective 1-form defining a degree one foliation $\mathcal{F}$ and consider the 2-form $d\omega$. Since $d\omega$ is closed, it is integrable. Therefore, we are in position to apply Theorem 3.3 to obtain homogeneous coordinates such that either:

(a) $d\omega = df \wedge dx_0$ for some quadratic function $f \in \mathsf{k}[x_0, \dots, x_n]$, or

(b) $d\omega = \sum_{i,j \in \{0,1,2\}} A_{ij}\, dx_i \wedge dx_j$ for some linear polynomials $A_{ij} \in \mathsf{k}[x_0, x_1, x_2]$.

If $p \neq 3$ and $d\omega$ is as in Case (b), then multiplying its contraction with the radial vector field by $1/3$ recovers ω, showing that it depends only on the variables x_0, x_1, x_2. Thus, it defines a foliation in $\mathsf{Lin}_1(\mathbb{P}^n_{\mathsf{k}})$.

If $p \notin \{2, 3\}$ and $d\omega$ is as in Case (a), then the explicit form of $d\omega$ shows that ω defines a foliation in $\mathsf{Log}_{(1,2)}(\mathbb{P}^n_{\mathsf{k}})$. This completes the proof of item (3).

Now consider the case $p = 2$. If $d\omega$ is as in Case (a), then

$$\omega = 3\omega = i_R(d\omega) = 2f\,dx_0 + x_0\,df = x_0\,df.$$

This implies that ω is either identically zero or that ω has a zero locus of codimension one. In either case, ω does not define a codimension one foliation of degree one. Thus, in characteristic two, the only irreducible component of $\mathsf{Fol}_1^1(\mathbb{P}^n_{\mathsf{k}})$ is $\mathsf{Lin}_1(\mathbb{P}^n_{\mathsf{k}})$.

For $p = 3$, we already know from Propositions 4.2 and 3.2 that $\mathsf{Lin}_1(\mathbb{P}^n_{\mathsf{k}})$ and $\mathsf{Cl}_1(\mathbb{P}^n_{\mathsf{k}})$ are irreducible components of $\mathsf{Fol}_1^1(\mathbb{P}^n_{\mathsf{k}})$. If $d\omega$ is as in Case (a), then

$$0 = 3\omega = i_R d\omega = 2f\,dx_0 - x_0\,df = -(f\,dx_0 + x_0\,df) = -d(fx_0).$$

This implies that f is a nonzero multiple of x_0^2, so $d\omega = 0$, implying that ω defines a foliation in $\mathsf{Cl}_1(\mathbb{P}^n_{\mathsf{k}})$ in this case.

Finally, we analyze the possibility that $d\omega$ is as in Case (b). Since $p = 3$, we have

$$i_R\,d\omega = 3\omega = 0.$$

As the coefficients of $d\omega$ have degree one, we may write

$$d\omega = \alpha \cdot (i_R\,dx_0 \wedge dx_1 \wedge dx_2)$$

for some $\alpha \in \mathsf{k}$. If $\alpha = 0$, then ω defines a foliation in $\mathsf{Cl}_1(\mathbb{P}^n_{\mathsf{k}})$. Otherwise, if $\alpha \neq 0$, consider the vector field

$$v = \frac{\alpha}{2}\,x_0\,\frac{\partial}{\partial x_0}.$$

Then, a straightforward computation shows that

$$d\omega = d(i_v\,i_R\,dx_0 \wedge dx_1 \wedge dx_2).$$

Therefore, we can write

$$\omega = i_v\,i_R\,dx_0 \wedge dx_1 \wedge dx_2 + \theta,$$

where θ is a closed polynomial 1-form with coefficients of degree two annihilated by the radial vector field R. By Lemma 3.1, we conclude that

$$\theta = df.$$

The integrability condition $\omega \wedge d\omega = 0$ implies

$$0 = df \wedge d\omega = \alpha \cdot df \wedge (i_R\,dx_0 \wedge dx_1 \wedge dx_2) = \alpha \cdot i_R(df \wedge dx_0 \wedge dx_1 \wedge dx_2).$$

Thus $f \in \mathsf{k}[x_0, x_1, x_2, x_3^3, \dots, x_n^3]$, and consequently,

$$\theta = df \in \bigoplus_{i=0}^{2} \mathsf{k}[x_0, x_1, x_2] \cdot dx_i.$$

It follows that ω defines a foliation in $\mathsf{Lin}_1(\mathbb{P}^n_{\mathsf{k}})$ when $p = 3$ and $d\omega \neq 0$. This completes the proof. ∎

5. Closed logarithmic 1-forms and their residues

5.1. Logarithmic 1-forms

Let X be a smooth algebraic variety defined over an algebraically closed field k. Let

$$D = \sum_{i=1}^{\ell} H_i$$

be a reduced and effective divisor on X, that is supported on irreducible hypersurfaces $H_1, \dots, H_\ell$. Following [19], we denote by $\Omega^1_X(\log D)$ the sheaf of rational 1-forms ω on X such that both ω and $d\omega$ have simple poles contained in the support of D. More precisely, for a point $x \in X$, if $h \in \mathcal{O}_{X,x}$ is a local equation for D, then a germ of rational 1-form ω belongs to $\Omega^1_X(\log D)_x$ if and only if both $h\omega$ and $hd\omega$ are germs of regular differential forms.

If $\omega \in \Omega^1_X(\log D)_x$ is a germ of logarithmic 1-form with poles along D, then there exist $g \in \mathcal{O}_{X,x}$, $\eta \in \Omega^1_{X,x}$, and $f \in \mathcal{O}_{X,x}$ such that $g|_{H_i}$ is not identically zero for any i, and

$$g\omega = f\frac{dh}{h} + \eta.$$

Indeed, if $z_1, \dots, z_n \in \mathcal{O}_{X,x}$ form a local system of parameters such that z_1 is not identically zero on any H_j, then we can take $g = \partial h/\partial z_1$, see Section 1 of [19].

5.2. Residues

Let $n_i\colon H_i^n \to H_i$ be the normalization of H_i. The (local) residue of ω along H_i is defined as the rational function $n_i^*(f/g)$. This local residue is well defined, independent of the choices above, and induces a morphism

$$\operatorname{Res} : \Omega^1_X(\log D) \longrightarrow \bigoplus_{i=1}^{\ell} \mathcal{M}_{H_i^n},$$

where $\mathcal{M}_{H_i^n}$ is the sheaf of rational functions on H_i^n, the normalization of H_i. The kernel of Res is precisely Ω^1_X.

We follow the terminology of Tag 0CBN in [20]: a normal crossing divisor is a divisor that, locally in the étale topology, is supported on a union of hypersurfaces defined by a subset of a local system of parameters. A strict normal crossing divisor, or a simple normal crossing divisor, is one that locally, in the Zariski topology, is supported on a union of hypersurfaces defined by a subset of a local system of parameters.

Proposition 5.1. *Under the assumptions above, if there exists a closed subset $S \subset X$ such that* $\operatorname{codim} S \geq 3$ *and $D|_{X-S}$ is a normal crossing divisor on $X - S$, then the image of* Res *is contained in*

$$\bigoplus_{i=1}^{\ell} \mathcal{O}_{H_i^n},$$

meaning that the residues of 1-forms in $\Omega^1_X(\log D)$ are regular functions on the normalization of D.

Proof. If $D|_{X-S}$ is a simple normal crossing divisor, then for every $x \in X - S$ there exists a local system of parameters $z_1, \dots, z_n$ such that

$$\Omega^1_X(\log D)_x = \sum_{i=1}^{r} \mathcal{O}_{X,x} \frac{dz_i}{z_i} + \Omega^1_{X,x},$$

as can be verified from Theorem 2.9 in [19] and Section 10 of [9]. Consequently, the residues of $\omega|_{X-S}$ are regular functions on the normalization of $D\big|_{X-S}$ by Theorem 2.9 in [19]. Since $S \cap H_i$ has codimension at least two in H_i, the same holds for $n_i^{-1}(S)$ in H_i^n, as the normalization is a finite morphism. Therefore, the residues of ω are regular functions on $D^n = \coprod H_i^n$, the normalization of D. If $D|_{X-S}$ is normal crossing, but not simple normal crossing, then it suffices to observe that the residue morphism commutes with étale morphisms. The regularity of the residues then follows from simple normal crossing case. ∎

A divisor D satisfying the assumptions of Proposition 5.1 is called a normal crossing divisor in codimension two.

Corollary 5.2. *Let*

$$D = \sum_{i=1}^{\ell} H_i$$

be a reduced and effective divisor on $\mathbb{P}^n_{\mathsf{k}}$ for some algebraically closed field k. If D is a normal crossing divisor in codimension two and $\omega \in H^0(\mathbb{P}^n_{\mathsf{k}}, \Omega^1_{\mathbb{P}^n_{\mathsf{k}}}(\log D))$ is a logarithmic 1-form with poles on D, then there exist constants $\lambda_1, \dots, \lambda_\ell \in \mathsf{k}$ such that, in homogeneous coordinates,

$$\omega = \sum_{i=1}^{\ell} \lambda_i \frac{dh_i}{h_i},$$

where h_i are irreducible and reduced homogeneous polynomials defining H_i.

Proof. The residues of ω are regular functions on proper algebraic varieties, hence they are constants $\lambda_1, \dots, \lambda_\ell$. Applying the residue theorem to the restriction of ω to a general line implies that

$$\sum_{i=1}^{\ell} \lambda_i \deg(H_i) = 0.$$

Consequently, the homogeneous rational form

$$\eta = \sum_{i=1}^{\ell} \lambda_i \frac{dh_i}{h_i}$$

is annihilated by the radial vector field, hence it defines a section of $H^0(\mathbb{P}^n_{\mathsf{k}}, \Omega^1_{\mathbb{P}^n_{\mathsf{k}}}(\log D))$. Observe that the difference $\omega - \eta$ is a logarithmic 1-form with vanishing residues, and thus belongs to $H^0(\mathbb{P}^n_{\mathsf{k}}, \Omega^1_{\mathbb{P}^n_{\mathsf{k}}})$. Since $\mathbb{P}^n_{\mathsf{k}}$ has no global 1-forms, we conclude that $\omega = \eta$, as claimed. ∎

5.3. A characterization of normal crossing in codimension two

For later use, we record a characterization of divisors that are normal crossing in codimension two.

Lemma 5.3. *Let X be an affine smooth variety defined over an algebraically closed field k. Let $h \in \mathcal{O}_X$ be a regular function, and let D be the divisor defined by h. Assume that D is reduced. The following conditions are equivalent*:

(1) *The divisor D is normal crossing in codimension two.*

(2) *The subvariety defined by the ideal $I(D)$, which is generated by h, $v(h)$ for every $v \in H^0(X, T_X)$, and*

$$\det\begin{pmatrix} v_1^2(h) & v_2(v_1(h)) \\ v_1(v_2(h)) & v_2^2(h) \end{pmatrix} \quad \textit{for every } v_1, v_2 \in H^0(X, T_X)$$

has codimension at least three.

Proof. We prove the equivalence by considering both directions.

First, suppose that D is normal crossing in codimension two. Then, there exists a closed subset $S \subset X$ of codimension at least three such that $D|_{X-S}$ is a normal crossing divisor. We may assume that S also contains all loci where three or more irreducible components of D meet, since these have codimension at least three in X. Consequently, for every $x \in D - S$, the germ of D at x is either smooth or, in a suitable étale neighborhood, equal to the sum of exactly two smooth irreducible divisors, say H_1 and H_2, intersecting transversely. In the first case, if we take $v \in T_{X,x}$ with value at x not contained in the kernel of dh then $v(h)$ is a unity. In the second case, we take $v_1, v_2 \in T_{X,x}$ such that

(1) the value of v_1 at x is transverse to $T_{H_1}(x) := T_{H_1}/\mathfrak{m}_x T_{H_1}$ and it is contained in $T_{H_2}(x) := T_{H_2}/\mathfrak{m}_x T_{H_2}$; and

(2) the value of v_2 at x is transverse to $T_{H_2}(x)$ and it is contained in $T_{H_1}(x)$.

A simple computation shows that the determinant

$$\det\begin{pmatrix} v_1^2(h) & v_2(v_1(h)) \\ v_1(v_2(h)) & v_2^2(h) \end{pmatrix}$$

is a unit in $\mathcal{O}_{X,x}$. Consequently, $I(D)_x$ contains a unit for every $x \in X - S$. Thus, the variety defined by $I(D)$ has codimension at least 3 as wanted.

To establish the converse direction, we will prove the contrapositive. Suppose that D is reduced and not normal crossing in codimension two. We will show that $I(D)$ defines a subvariety of codimension at most two in X.

Let S be an irreducible component of the singular locus of D that has codimension one in D. Since we assume that D is not normal crossing in codimension two, such a component exists. Consider the local ring $\mathcal{O}_{X,S}$ of regular functions on X along S. As X is regular and S has codimension two in X, it follows that $\mathcal{O}_{X,S}$ is a regular local ring of dimension two. Let $x, y \in \mathcal{O}_{X,S}$ be minimal generators of the maximal ideal $\mathfrak{m}_{X,S}$ of $\mathcal{O}_{X,S}$. Since S is contained in the singular locus of D, h is an element of $\mathfrak{m}_{X,S}^2$. Writing $h = ax^2 + bxy + cy^2$ for some functions $a, b, c \in \mathcal{O}_{X,S}$, we claim that the quadratic form

$ax^2 + bxy + cy^2$ degenerates along S, i.e., $b^2 - 4ac = 0$ along S. Indeed, D is normal crossing at the generic point s of S if and only if the initial form of h in $\mathfrak{m}^2_{X,s}/\mathfrak{m}^3_{X,s}$ is a product of two nonproportional linear forms over the residue field, which holds if and only if $b^2 - 4ac \neq 0$ at s. Since D fails to be normal crossing at s by assumption, we conclude that $b^2 - 4ac = 0$ along S, meaning S is contained in the hypersurface $\{b^2 - 4ac = 0\}$. On the other hand, for any sections $v_1, v_2 \in T_X$, the determinant

$$\det \begin{pmatrix} v_1^2(h) & v_2(v_1(h)) \\ v_1(v_2(h)) & v_2^2(h) \end{pmatrix}$$

has the form $s + (b^2 - 4ac)t$, where $s \in \mathfrak{m}_{X,S}$ and $t \in \mathcal{O}_{X,S}$. It follows that $I(D)\mathcal{O}_{X,S}$ is contained in the ideal $\mathfrak{m}_{X,S}$, and thus S is contained in the subvariety defined by $I(D)$. Hence, this subvariety has codimension at most two as desired. ∎

Corollary 5.4. *Let X be a smooth projective variety and let T be an algebraic variety, both defined over an algebraically closed field k. If Δ is a divisor on $\mathscr{X} = X \times T$, then the set*

$$\{t \in T \mid \Delta_t = \Delta|_{X \times \{t\}} \text{ is a reduced normal crossing divisor in codimension two}\}$$

is an open subset of T.

Proof. Since the normal crossing in codimension two condition and the codimension of the subvariety defined by $I(D_t)$ are both local on X, we may cover X by finitely many affine open subvarieties and apply Lemma 5.3 to each.

For each $t \in T$, let h_t be a local equation of Δ_t and let $I(D_t)$ be the ideal generated by h_t, $v(h_t)$ and

$$\det \begin{pmatrix} v_1^2(h_t) & v_2(v_1(h_t)) \\ v_1(v_2(h_t)) & v_2^2(h_t) \end{pmatrix} \quad \text{for every } v_1, v_2 \in T_X.$$

By Lemma 5.3, D_t is normal crossing in codimension two if and only if the subvariety V_t of X defined by $I(D_t)$ has codimension at least three. The result follows from the upper semi-continuity of the function $t \in T \mapsto \dim V_t$, see, e.g., Corollary 13.1.5 in [10]. ∎

6. Logarithmic components II

We return to the study of the logarithmic components of the space of foliations on projective spaces. Unlike the proof of Proposition 4.6, where we adapted arguments from characteristic zero to positive characteristic, here we leverage specific properties of foliations over fields of positive characteristic. In particular, we explore fundamental properties of the Cartier transform of foliations, which we recall below.

6.1. The Cartier transform of a logarithmic foliation

Let X be a smooth projective variety defined over an algebraically closed field k of characteristic $p > 0$. Let $\mathcal{F}$ be a p-dense codimension one foliation defined on X. By

Proposition 4.4 in [14], $\mathcal{F}$ is defined by a closed rational 1-form ω. The Cartier transform $\mathscr{C}(\mathcal{F})$ of $\mathcal{F}$, as introduced in Definition 4.17 in Section 4.5 of [14], is the distribution given by the kernel of 1-form obtained by applying the Cartier operator on ω.

In the particular case that ω is of the form

$$\omega = \sum_{i=1}^{r} \lambda_i \frac{df_i}{f_i},$$

the result of applying the Cartier operator $\mathbf{C}$ to it is rather easy to compute:

$$\mathbf{C}(\omega) = \sum_{i=1}^{r} \lambda_i^{1/p} \frac{df_i}{f_i}. \tag{6.1}$$

Lemma 6.1. *Let $\mathcal{F}$ a codimension one foliation on $\mathbb{P}^n_{\mathsf{k}}$ defined by a closed logarithmic 1-form*

$$\omega = \sum_{i=1}^{r} \lambda_i \frac{df_i}{f_i},$$

where $\lambda_1, \dots, \lambda_r \in \mathsf{k}$ are constants. Assume that the following conditions hold:

(1) *for any i, there exists a j such that the ratio λ_i/λ_j does not belong to $\mathbb{F}_p$*;

(2) *the polar divisor of ω is a simple normal crossing divisor.*

Then

$$\deg(\mathscr{V}(\mathcal{F})) = \deg(\mathcal{F}) - 1 \quad \textit{and} \quad \deg(\Delta_{\mathcal{F}}) = \deg(\mathcal{F}) + 2.$$

Proof. By Proposition 4.19 in [14], the tangent sheaf of $\mathscr{V}(\mathcal{F})$ coincides with the saturation of $T_{\mathcal{F}} \cap T_{\mathscr{C}(\mathcal{F})}$ inside T_X. Under our assumptions, the distribution $\mathscr{V}(\mathcal{F})$ is defined by the rational 2-form

$$\theta = \omega \wedge \mathbf{C}(\omega) = \sum_{0 \le i < j \le r} (\lambda_i \lambda_j^{1/p} - \lambda_j \lambda_i^{1/p}) \frac{df_i}{f_i} \wedge \frac{df_j}{f_j}.$$

Since the determinant of the normal bundle of $\mathscr{V}(\mathcal{F})$ satisfies

$$\det(N_{\mathscr{V}(\mathcal{F})}) = \mathcal{O}_{\mathbb{P}^n_{\mathsf{k}}}((\theta)_\infty - (\theta)_0) = \mathcal{O}_{\mathbb{P}^n_{\mathsf{k}}}(\deg(\mathscr{V}(\mathcal{F})) + 3,)$$

it suffices to verify that $(\theta)_\infty = (\omega)_\infty$ and that θ does not have codimension one zeros.

The equality

$$(\theta)_\infty = (\omega)_\infty$$

follows from the assumption that for any i, there exists a j such that $\lambda_i/\lambda_j \notin \mathbb{F}_p$, which ensures that

$$\lambda_i \lambda_j^{1/p} - \lambda_j \lambda_i^{1/p} \neq 0.$$

To show that θ has no codimension one zeros, assume for contradiction that its zero locus contains a hypersurface Z. The hypersurface Z must intersect $(\theta)_\infty = (\omega)_\infty$, as both are ample divisors. However, because $(\theta)_\infty$ is a simple normal crossing divisor, a local computation shows that θ remains non-zero in a neighborhood of its polar divisor, contradicting the existence of Z. This completes the proof. ∎

6.2. Proof of Theorem A

For the reader's convenience, we recall the statement of Theorem A.

Theorem 6.2. *Let $r \geq 2$ and $1 \leq d_1 \leq \cdots \leq d_r$ be positive integers, and let k be an algebraically closed field of characteristic $p > 0$. Assume that one of the following conditions holds:*

(1) *$r = 2$ and none of the integers d_1, d_2, and $d_1 + d_2$ is divisible by p; or*

(2) *$r = 2$ and both d_1 and d_2 are divisible by p; or*

(3) *$r > 2$ and the set of integers $i \in \{1, \ldots, r\}$ such that p divides d_i has cardinality different from $r - 1$ and $r - 2$.*

Then $\mathsf{Log}_{(d_1,\ldots,d_r)}(\mathbb{P}^n_{\mathsf{k}})$ is an irreducible component of $\mathsf{Fol}^1_d(\mathbb{P}^n_{\mathsf{k}})$ (with its reduced scheme structure) for $n \geq 3$ and $d = \sum_{i=1}^r d_i - 2$.

Proof. Case (1) is the content of Proposition 4.6. We therefore assume that condition (2) or (3) holds. Our assumptions ensure the existence of $\lambda_1, \ldots, \lambda_r \in \mathsf{k}$ such that

$$\sum_{i=1}^r d_i \lambda_i = 0,$$

with the additional property that $\lambda_i/\lambda_j \notin \mathbb{F}_p$ for every $i \neq j$.

Let $f_i \in H^0(\mathbb{P}^n_{\mathsf{k}}, \mathcal{O}_{\mathbb{P}^n_{\mathsf{k}}}(d_i))$ be homogeneous polynomials defining smooth hypersurfaces H_i such that the divisor $D = \sum_{i=1}^r H_i$ is a simple normal crossing divisor. Let $\omega \in H^0(\mathbb{P}^n_{\mathsf{k}}, \Omega^1_{\mathbb{P}^n_{\mathsf{k}}}(\log D))$ be the logarithmic 1-form defined in homogeneous coordinates by

$$\omega = \sum_{i=1}^r \lambda_i \frac{df_i}{f_i},$$

and let $\mathcal{F}$ be the foliation defined by ω. Let Σ be any irreducible component of $\mathsf{Fol}^1_d(\mathbb{P}^n_{\mathsf{k}})$ containing $\mathcal{F}$. We will show that $\Sigma = \mathsf{Log}_{(d_1,\ldots,d_r)}(\mathbb{P}^n_{\mathsf{k}})$.

By Lemma 6.1, the polar divisor D of ω coincides with the degeneracy divisor of the p-curvature $\Delta_{\mathcal{F}}$. Furthermore, by Lemma 6.3 in [14], there exists an open subset $U \subset \Sigma$, containing $\mathcal{F}$, such that for any closed point in U corresponding to a foliation $\mathcal{F}'$, the degeneracy divisor $\Delta_{\mathcal{F}'}$ is reduced. Possibly after further restricting U, every $\mathcal{F}' \in U$ is p-dense (this is an open condition). By Proposition 4.4 in [14], $\mathcal{F}'$ is defined by a closed rational 1-form ω' whose polar locus is $\Delta_{\mathcal{F}'}$. Since $\Delta_{\mathcal{F}'}$ is reduced and normal crossing in codimension two (Corollary 5.4), ω' is logarithmic with poles along $\Delta_{\mathcal{F}'}$. By Corollary 5.2, this ensures that each $\mathcal{F}' \in U$ belongs to some $\mathsf{Log}_{(d'_1,\ldots,d'_{r'})}(\mathbb{P}^n_{\mathsf{k}})$, where r' and $d'_1 \leq \cdots \leq d_{r'}$ satisfy $\sum_{i=1}^{r'} d'_i = d + 2$.

To conclude the proof, we must show that $r' = r$ and $(d'_1, \ldots, d'_r) = (d_1, \ldots, d_r)$. On one hand, the polar divisor $\Delta_{\mathcal{F}'}$ lives in the fixed linear system $|\mathcal{O}_{\mathbb{P}^n_{\mathsf{k}}}(d+2)|$, and the universal family of effective divisors over a linear system is flat (all fibers are Cartier divisors with the same Hilbert polynomial). Since the number of irreducible components of the fibers of a flat proper family is upper semi-continuous, and $\Delta_{\mathcal{F}}$ has r irreducible components, we conclude that $r' \leq r$ for a general $\mathcal{F}' \in U$. On the other hand, consider an inclusion $\varphi\colon \mathbb{P}^1_{\mathsf{k}} \to \mathbb{P}^n_{\mathsf{k}}$ of a general line. The number r equals the number of distinct residues of $\varphi^*\omega$. Since $\mathcal{F}' \in U$ is defined by a deformation $\omega' \in H^0(\mathbb{P}^n_{\mathsf{k}}, \Omega^1_{\mathbb{P}^n_{\mathsf{k}}}(\log \Delta_{\mathcal{F}'}))$ of ω, the residues of $\varphi^*\omega'$ deform those of $\varphi^*\omega$, so $r' \geq r$. Therefore $r' = r$.

Finally, each degree d_j corresponds to the set of points in $\mathbb{P}^1_{\mathsf{k}}$ where $\varphi^*\omega$ has the same residue λ_j. Since deformations can only decrease the values of each d_j, but their sum remains fixed at $d + 2$, it follows that $(d'_1, \dots, d'_r) = (d_1, \dots, d_r)$, as claimed. ■

As a corollary, we obtain a new proof of Calvo-Andrade's theorem on the stability of generic logarithmic 1-forms, this time from an arithmetic perspective.

Corollary 6.3. *Let $r \geq 2$ and $1 \leq d_1 \leq \cdots \leq d_r$ be positive integers, let $n \geq 3$, and set $d = \sum_{i=1}^r d_i - 2$. Then $\mathsf{Log}_{(d_1,\dots,d_r)}(\mathbb{P}^n_{\mathbb{C}})$ is an irreducible component of $\mathsf{Fol}^1_d(\mathbb{P}^n_{\mathbb{C}})$.*

Proof. Note that both $\mathsf{Fol}^1_d(\mathbb{P}^n_{\mathsf{k}})$ and $\mathsf{Log}_{(d_1,\dots,d_r)}(\mathbb{P}^n_{\mathsf{k}})$ are defined over $\mathbb{Z}$. Suppose, for a contradiction, that $\mathsf{Log}_{(d_1,\dots,d_r)}(\mathbb{P}^n_{\mathbb{C}})$ is strictly contained in some irreducible component $W_{\mathbb{C}} \subset \mathsf{Fol}^1_d(\mathbb{P}^n_{\mathbb{C}})$. Then there exist a finitely generated $\mathbb{Z}$-subalgebra $A \subset \mathbb{C}$ and an integral closed subscheme $W_A \subset \mathsf{Fol}^1_d(\mathbb{P}^n_A)$, with geometrically irreducible fibers over $\mathrm{Spec}(A)$, whose base change to $\mathbb{C}$ recovers $W_{\mathbb{C}}$ and which strictly contains $\mathsf{Log}_{(d_1,\dots,d_r)}(\mathbb{P}^n_A)$. Since A has characteristic zero, the morphism $\mathrm{Spec}(A) \to \mathrm{Spec}(\mathbb{Z})$ is dominant, so its image contains $\mathrm{Spec}(\mathbb{Z}[1/N])$ for some integer $N \geq 1$. For every prime $p \nmid N$ and every closed point of $\mathrm{Spec}(A)$ above p, the corresponding geometric fiber of W_A is an irreducible closed subscheme of $\mathsf{Fol}^1_d(\mathbb{P}^n_{\overline{\mathbb{F}}_p})$ strictly containing $\mathsf{Log}_{(d_1,\dots,d_r)}(\mathbb{P}^n_{\overline{\mathbb{F}}_p})$. For every p larger than $\max(d_1, \dots, d_r, d_1 + d_2)$, the hypotheses of Theorem 6.2 hold (case (1) for $r = 2$; case (3) for $r > 2$, since the set S is empty), and therefore $\mathsf{Log}_{(d_1,\dots,d_r)}(\mathbb{P}^n_{\overline{\mathbb{F}}_p})$ is already an irreducible component of $\mathsf{Fol}^1_d(\mathbb{P}^n_{\overline{\mathbb{F}}_p})$. This is a contradiction, and Calvo-Andrade's result follows. ■

7. Foliations of degree two

7.1. Foliations defined by Lie algebras of vector fields

We identify $\mathfrak{pgl}_{n+1}(\mathsf{k})$, the quotient of $\mathrm{End}(\mathbb{A}^{n+1}_{\mathsf{k}})$ by multiples of the identity, with the Lie algebra of regular vector fields on $\mathbb{P}^n_{\mathsf{k}}$, i.e., $H^0(\mathbb{P}^n_{\mathsf{k}}, T_{\mathbb{P}^n_{\mathsf{k}}})$.

If the characteristic of k does not divide $n + 1$, then we can concretely, in homogeneous coordinates, identify $\mathfrak{pgl}_{n+1}(\mathsf{k})$ with the Lie algebra of vector fields on $\mathbb{A}^{n+1}_{\mathsf{k}}$ with linear coefficients and zero divergence. In general, we will think of $\mathfrak{pgl}_{n+1}(\mathsf{k})$ as the quotient of the Lie algebra of linear vector fields on $\mathbb{A}^{n+1}_{\mathsf{k}}$ with linear coefficients modulo multiples of the radial vector field.

Given a Lie subalgebra $\mathfrak{h} \subset \mathfrak{pgl}_{n+1}(\mathsf{k})$, we consider the subsheaf of $T_{\mathbb{P}^n_{\mathsf{k}}}$ defined by the image of the evaluation morphism:

$$\mathrm{ev} : \mathfrak{h} \otimes \mathcal{O}_{\mathbb{P}^n} \longrightarrow T_{\mathbb{P}^n_{\mathsf{k}}}.$$

Since, by assumption, $\mathfrak{h}$ is a Lie subalgebra of $\mathfrak{pgl}_{n+1}(\mathsf{k})$, the image of ev is an involutive subsheaf of $T_{\mathbb{P}^n}$. Its saturation then defines the tangent sheaf of a foliation, which we denote by $\mathcal{F}(\mathfrak{h})$.

Before stating our next result, we briefly recall the terminology concerning Kupka and non-Kupka singularities that will be used below. Let $\mathcal{F}$ be a codimension q foliation on a smooth variety X, and let

$$\omega \in H^0(X, \Omega_X^q \otimes \mathcal{N})$$

be a twisted q-form defining $\mathcal{F}$. A point $x \in \operatorname{sing}(\mathcal{F})$ is called a Kupka singularity if there exist an open neighborhood U of x, a nowhere vanishing local section $\tau \in H^0(U, \mathcal{N})$, and a local q-form $\eta \in H^0(U, \Omega_X^q)$ such that

$$\omega_{|U} = \eta \otimes \tau$$

and

$$\eta(x) = 0 \quad \text{and} \quad d\eta(x) \neq 0.$$

This definition is independent of the choice of local trivialization of $\mathcal{N}$. The set of non-Kupka singularities of $\mathcal{F}$ is the complement of the Kupka set inside $\operatorname{sing}(\mathcal{F})$.

Theorem 7.1. *Let $\mathfrak{h} \subset \mathfrak{pgl}_{n+1}(\mathsf{k})$ be such that* ev *is injective and has saturated image, i.e., $T_{\mathcal{F}(\mathfrak{h})} \simeq \mathfrak{h} \otimes \mathcal{O}_{\mathbb{P}^n_{\mathsf{k}}}$. Let q denote the codimension of $\mathcal{F}(\mathfrak{h})$. If the set of non-Kupka singularities of $\mathcal{F}(\mathfrak{h})$ has codimension at least three, then any irreducible component Σ of $\mathsf{Fol}^q_d(\mathbb{P}^n_{\mathsf{k}})$ containing $\mathcal{F}(\mathfrak{h})$ is such that a general closed point of Σ corresponds to a foliation defined by a Lie subalgebra of $\mathfrak{pgl}_{n+1}(\mathsf{k})$ of dimension* $\dim \mathfrak{h}$.

Proof. We draw on results from [15], which are proved in a characteristic-free setting.

By Corollary 4.11 in [15] (the restatement of Theorem B in loc. cit.), the set of points in $\mathsf{Fol}^q_d(\mathbb{P}^n_{\mathsf{k}})$ corresponding to foliations with locally free tangent sheaf and non-Kupka singular locus of codimension at least three is open. Since $\mathcal{F}(\mathfrak{h})$ belongs to this set by hypothesis, every foliation in a sufficiently small open neighborhood $U \subset \Sigma$ of $\mathcal{F}(\mathfrak{h})$ also has locally free tangent sheaf and non-Kupka singular locus of codimension at least three.

By Corollary 4.14 in [15], the isomorphism class of the tangent sheaf is locally constant on Σ. Since $T_{\mathcal{F}(\mathfrak{h})} \simeq \mathfrak{h} \otimes \mathcal{O}_{\mathbb{P}^n_{\mathsf{k}}}$ by hypothesis, the tangent sheaf of every foliation $\mathcal{F}' \in U$ is isomorphic to $\mathfrak{h} \otimes \mathcal{O}_{\mathbb{P}^n_{\mathsf{k}}}$, and in particular is defined by a Lie subalgebra of $\mathfrak{pgl}_{n+1}(\mathsf{k})$ of dimension $\dim \mathfrak{h}$. ∎

Example 7.2. Let k be an algebraically closed field of characteristic $p > 0$. Let $\mathfrak{h}$ be the subalgebra of $\mathfrak{pgl}_4(\mathsf{k})$ generated by the classes of the linear vector fields

$$v_s = -x_1 \frac{\partial}{\partial x_1} - 2x_2 \frac{\partial}{\partial x_2} - 3x_3 \frac{\partial}{\partial x_3} \quad \text{and} \quad v_n = x_0 \frac{\partial}{\partial x_1} + x_1 \frac{\partial}{\partial x_2} + x_2 \frac{\partial}{\partial x_3}.$$

Note that $[v_s, v_n] = v_n$.

The corresponding foliation on $\mathbb{P}^3_{\mathsf{k}}$ is defined by the projective 1-form

$$\omega = i_R i_{v_s} i_{v_n} dx_0 \wedge \cdots \wedge dx_3,$$

where R is the radial vector field. The set of non-Kupka singularities of $\mathcal{F}(\mathfrak{h})$ is, by definition, the zero locus of $d\omega$. An explicit computation shows that

$$\begin{aligned} \omega = {} & (-x_0 x_2 x_3 + 2x_1^2 x_3 - x_1 x_2^2)\, dx_0 + (-3x_0 x_1 x_3 + 2x_0 x_2^2)\, dx_1 \\ & + (3x_0^2 x_3 - x_0 x_1 x_2)\, dx_2 + (-2x_0^2 x_2 + x_0 x_1^2)\, dx_3 \end{aligned}$$

and that

$$\begin{aligned} d\omega = {} & (7x_1x_3 - 3x_2^2)\, dx_0 \wedge dx_1 + (-7x_0x_3 - x_1x_2)\, dx_0 \wedge dx_2 \\ & + (3x_0x_2 + x_1^2)\, dx_0 \wedge dx_3 + (5x_0x_2)\, dx_1 \wedge dx_2 \\ & + (-5x_0x_1)\, dx_1 \wedge dx_3 + (5x_0^2)\, dx_2 \wedge dx_3. \end{aligned}$$

The zero locus of $d\omega$ is then equal to

(1) the line $\{x_0 = x_1 = 0\}$ when $p = 3$; and
(2) the two points $(1:0:0:0)$ and $(0:0:0:1)$ when $p = 5$; and
(3) the point $(0:0:0:1)$ when $p \notin \{3, 5\}$.

Our next result shows that the so-called exceptional component $\mathsf{Exc}(\mathbb{P}^3_{\mathbb{C}})$ of $\mathsf{Fol}^1_2(\mathbb{P}^3_{\mathbb{C}})$ has natural analogs in all characteristics $p \neq 3$.

Proposition 7.3. *Let* k *be an algebraically closed field of positive characteristic* $p \neq 3$. *The closure of the orbit through the point in* $\mathsf{Fol}^1_2(\mathbb{P}^3_{\mathsf{k}})$ *determined by Example 7.2 under the action of* $\mathrm{Aut}(\mathbb{P}^3_{\mathsf{k}})$ *is an irreducible component* $\mathsf{Exc}(\mathbb{P}^3_{\mathsf{k}})$ *of* $\mathsf{Fol}^1_2(\mathbb{P}^3_{\mathsf{k}})$.

Proof. Notation as in Example 7.2. Let $\mathfrak{h}$ be the non-abelian Lie algebra of $\mathfrak{pgl}_4(\mathsf{k})$ generated by the classes of v_s and v_n. If $p \neq 3$, then the induced foliation $\mathcal{F}(\mathfrak{h})$ satisfies the assumptions of Theorem 7.1. Therefore, if $\Sigma \subset \mathsf{Fol}^1_2(\mathbb{P}^3_{\mathsf{k}})$ is an irreducible component containing $\mathcal{F}(\mathfrak{h})$, there exists an open subset $U \subset \Sigma$ containing $\mathcal{F}(\mathfrak{h})$ such that every foliation in U is defined by a non-abelian subalgebra $\mathfrak{h}'$ of $\mathfrak{pgl}_4(\mathsf{k})$. The image of the Lie-bracket morphism $\wedge^2\mathfrak{h}' \to \mathfrak{h}'$ is generated by a nilpotent element v_n' of $\mathfrak{pgl}_4(\mathsf{k})$. Perhaps after passing to a smaller open subset of U, we may assume that this nilpotent element corresponds to a 4×4 matrix with one-dimensional kernel. Jordan normal form implies that it is conjugated to v_n and hence we can assume that it is equal to v_n. Notice also that v_n corresponds to a regular nilpotent element of $\mathfrak{gl}_4(\mathsf{k})$ (the Lie algebra of 4×4 matrices over k) and, as such, has 4-dimensional centralizer and the dimension of the corresponding nilpotent orbit is 12. Hence, Σ is birationally equivalent to a $\mathbb{P}^2$-bundle over the nilpotent orbit through v_n of the adjoint action on $\mathfrak{pgl}_4(\mathsf{k})$. In particular, Σ has dimension 13 and is equal to the closure of the PGL(4) orbit through $\mathcal{F}(\mathfrak{h})$. ∎

Corollary 7.4. *Let* k *be an algebraically closed field of positive characteristic* $p \neq 3$, *and let* $n \geq 4$ *be an integer. The closure of the orbit through the point in* $\mathsf{Fol}^1_2(\mathbb{P}^n_{\mathsf{k}})$ *determined by a dominant linear pull-back* $\mathbb{P}^n_{\mathsf{k}} \dashrightarrow \mathbb{P}^3_{\mathsf{k}}$ *of Example 7.2 under the action of* $\mathrm{Aut}(\mathbb{P}^n_{\mathsf{k}})$ *is an irreducible component* $\mathsf{Exc}(\mathbb{P}^n_{\mathsf{k}})$ *of* $\mathsf{Fol}^1_2(\mathbb{P}^n_{\mathsf{k}})$.

Proof. As verified in Example 7.2, the non-Kupka singularities of the foliation $\mathcal{F}(\mathfrak{h})$ have codimension three. The same property holds true for any pull-back of $\mathcal{F}(\mathfrak{h})$ under a dominant linear projection $\mathbb{P}^n_{\mathsf{k}} \dashrightarrow \mathbb{P}^3_{\mathsf{k}}$. By applying Theorem B and Corollary 4.14 in [15], we deduce that any foliation sufficiently close to $\mathcal{F}(\mathfrak{h})$ has tangent sheaf isomorphic to $\mathcal{O}_{\mathbb{P}^n_{\mathsf{k}}}(+1)^{\oplus n-3} \oplus \mathcal{O}_{\mathbb{P}^n_{\mathsf{k}}}^{\oplus 2}$, and therefore contains a subfoliation of degree zero and codimension three. Lemma 4.3 then implies that this foliation must be a linear pull-back of foliation on $\mathbb{P}^3$. Proposition 7.3 allows us to conclude. ∎

Over the complex numbers, there are exactly two irreducible components of $\mathsf{Fol}_2^1(\mathbb{P}^3_{\mathbb{C}})$ whose general element corresponds to a foliation with trivial tangent bundle, namely, $\mathsf{Exc}(\mathbb{P}^3_{\mathbb{C}})$ and $\mathsf{Log}_{(1,1,1,1)}(\mathbb{P}^3_{\mathbb{C}})$. When the field k is arbitrary, Theorem A ensures that $\mathsf{Log}_{(1,1,1,1)}(\mathbb{P}^3_{\mathsf{k}})$ is an irreducible component of $\mathsf{Fol}_2^1(\mathbb{P}^3_{\mathsf{k}})$. If the characteristic of k is different from 3, then Proposition 7.3 shows that $\mathsf{Exc}(\mathbb{P}^3_{\mathsf{k}})$ is also an irreducible component. Both $\mathsf{Exc}(\mathbb{P}^3_{\mathsf{k}})$ and $\mathsf{Log}_{(1,1,1,1)}(\mathbb{P}^3_{\mathsf{k}})$ are defined over $\mathbb{Z}$. Since the integrability condition is cut out by polynomial equations with integer coefficients, we have that $\mathsf{Fol}_2^1(\mathbb{P}^3_{\mathsf{k}})$ is a scheme over $\mathrm{Spec}(\mathbb{Z})$, and the function associating to each prime p the number of irreducible components of $\mathsf{Fol}_2^1(\mathbb{P}^3_{\mathsf{k}})$ whose general element has trivial tangent bundle is constructible on $\mathrm{Spec}(\mathbb{Z})$, see Tag 055B in [20]. Since its value at the generic point equals two, it equals two for all but finitely many primes p, showing that the above list is complete for $p \gg 0$. However, the situation in small characteristic remains an open question. Similarly, it would be interesting to describe the irreducible components of the variety/scheme of two-dimensional Lie subalgebras of $\mathfrak{pgl}_{n+1}(\mathsf{k})$, akin to the results established for $\mathsf{k} = \mathbb{C}$ in Theorem C in [15]. We do not expect an identical classification in positive characteristic, as the variety of commuting pairs in $\mathfrak{pgl}_{n+1}(\mathsf{k})$ fails to be irreducible when the characteristic of k divides $n + 1$, see [17].

7.2. Synthesis

Our last result summarizes our current understanding of the irreducible components of $\mathsf{Fol}_2^1(\mathbb{P}^n_{\mathsf{k}})$ for algebraically closed fields of positive characteristic. It is a straightforward combination of the results previously established throughout this paper.

Proposition 7.5. *Let k be an algebraically closed field of characteristic $p > 0$. The following subvarieties are irreducible components of the support of the scheme $\mathsf{Fol}_2^1(\mathbb{P}^n_{\mathsf{k}})$:*

(1) *When $p = 2$,*

$$\mathsf{Cl}_2(\mathbb{P}^n_{\mathsf{k}}), \quad \mathsf{Log}_{(2,2)}(\mathbb{P}^n_{\mathsf{k}}), \quad \mathsf{Log}_{(1,1,1,1)}(\mathbb{P}^n_{\mathsf{k}}), \quad \mathsf{NLin}_2(\mathbb{P}^n_{\mathsf{k}}), \quad \mathsf{Exc}(\mathbb{P}^n_{\mathsf{k}}).$$

(2) *When $p = 3$,*

$$\mathsf{Log}_{(2,2)}(\mathbb{P}^n_{\mathsf{k}}), \quad \mathsf{Log}_{(1,1,2)}(\mathbb{P}^n_{\mathsf{k}}), \quad \mathsf{Log}_{(1,1,1,1)}(\mathbb{P}^n_{\mathsf{k}}), \quad \mathsf{Lin}_2(\mathbb{P}^n_{\mathsf{k}}).$$

(3) *When $p \geq 5$,*

$$\mathsf{Log}_{(2,2)}(\mathbb{P}^n_{\mathsf{k}}),\ \mathsf{Log}_{(1,3)}(\mathbb{P}^n_{\mathsf{k}}),\ \mathsf{Log}_{(1,1,2)}(\mathbb{P}^n_{\mathsf{k}}),\ \mathsf{Log}_{(1,1,1,1)}(\mathbb{P}^n_{\mathsf{k}}),\ \mathsf{Lin}_2(\mathbb{P}^n_{\mathsf{k}}),\ \mathsf{Exc}(\mathbb{P}^n_{\mathsf{k}}).$$

Moreover, if $p \gg 0$, the above list is complete.

Proof. In all cases, $\mathsf{NLin}_2(\mathbb{P}^n_{\mathsf{k}})$ is an irreducible component by Proposition 4.5. When $p \geq 3$, the inequality $d = 2 < p$ gives $\mathsf{NLin}_2(\mathbb{P}^n_{\mathsf{k}}) = \mathsf{Lin}_2(\mathbb{P}^n_{\mathsf{k}})$.

The logarithmic components in each list are irreducible components by Theorem A.

When $p \neq 3$, the component $\mathsf{Exc}(\mathbb{P}^n_{\mathsf{k}})$ is irreducible by Proposition 7.3 for $n = 3$ and by Corollary 7.4 for $n \geq 4$. When $p = 3$, the foliation of Example 7.2 has non-Kupka singular locus of codimension two (a line), so Theorem 7.1 does not apply. Whether $\mathsf{Exc}(\mathbb{P}^n_{\mathsf{k}})$ remains an irreducible component of $\mathsf{Fol}_2^1(\mathbb{P}^n_{\mathsf{k}})$ in characteristic three is an open question.

When $p = 2$, the component $\mathsf{Cl}_2(\mathbb{P}^n_{\mathsf{k}})$ is irreducible by Proposition 3.2.

For completeness, when $p \gg 0$: since the integrability condition is cut out by polynomial equations with integer coefficients, $\mathsf{Fol}^1_2(\mathbb{P}^n_{\mathsf{k}})$ is a scheme over $\mathrm{Spec}(\mathbb{Z})$, and each component in the list is defined over $\mathbb{Z}$. The function associating to each prime p the number of irreducible components of $\mathsf{Fol}^1_2(\mathbb{P}^n_{\mathsf{k}})$ is constructible on $\mathrm{Spec}(\mathbb{Z})$, Tag 055B in [20], hence constant for all but finitely many primes. Since its value at the generic point equals six by [3], the list is complete for $p \gg 0$. ∎

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