

## Foreword

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In 1926, three independent articles by Wentzel [33], Brillouin [3], and Kramers [21] paved the way for a method devoted to the construction of asymptotic expansions of wave functions in quantum mechanics, in powers of Planck’s constant (see also [1]). Adopted and further developed by the mathematical community, this approach became known as the *WKB method* (or *méthode BKW* in French—note the order of the letters and the nationality of the authors). The letter J is sometimes added, giving rise to the WKBJ method, in reference to Jeffreys [19], a terminology apparently more common in the physics community, even though Jeffreys’ contribution is the only strictly mathematical paper among these founding articles.

Originally introduced in the context of the time-independent Schrödinger equation, the WKB method has since undergone numerous extensions, including generalizations to evolution equations and to a wide variety of other models. In the context of hyperbolic equations, it is also known as *geometric optics*; see, for instance, [26]. The aim of this volume is to celebrate the anniversary of these three seminal works marking the birth of the WKB method, and to offer a (necessarily partial) overview of some of its developments along various directions. The volume brings together both survey articles and more focused recent contributions.

The contribution of San Vũ Ngọc opens the volume with a historical perspective on the method, its evolution, and its influences. In particular, the reader will be introduced to the Maslov index and will gain a first insight into the Bohr–Sommerfeld quantization rules. As emphasized by San Vũ Ngọc, WKB methods are “at the origin of the spectacular development of microlocal and semiclassical analysis,” and one of the objectives of this volume is to illustrate the diversity of phenomena that the WKB approach makes accessible.

Several contributions focus on the role of WKB constructions in the construction of semiclassical Fourier integral operators (FIO). Jean-Marc Bouclet explains how the Isozaki–Kitada parametrix, some FIOs that describe the correction terms required in long-range scattering, as compared with short-range scattering, can be understood in terms of WKB states. Within the broader family of Fourier integral operators, those approximating propagators play a central role in semiclassical analysis. This theme is further developed in the contribution of Melissa Tacy, where the parametrix of the propagator of a semiclassical Schrödinger operator is revisited from a WKB perspective.

Schrödinger operators naturally constitute a recurring theme of the volume. Setsuro Fujiie analyzes the resonances of a one-dimensional Schrödinger operator with a matrix-valued potential exhibiting a crossing. He addresses the obstruction to the construction of WKB states induced by the presence of the crossing point and shows how this difficulty can be overcome. A one-dimensional setting is also considered by Luc Hillairet and Jeremy L. Marzuola, who study a Schrödinger operator on a segment with a potential admitting a singularity of the form  $x^\alpha$ ,  $0 < \alpha < 1$ , at the origin. Using WKB expansions, they relate the Cauchy data of eigenfunctions of this operator at the endpoints of the segment and establish Bohr–Sommerfeld quantization rules in this singular framework.

The scope of WKB methods, however, extends well beyond Schrödinger equations. Semyon Dyatlov, Henry Zeng, and Maciej Zworski consider a model arising from the analysis of bilayer graphene. Their work elucidates the WKB structure of special states—so-called symmetry-protected states—which appear when the band structure exhibits a flat band due to symmetries of the underlying operator.

While this excursion from quantum mechanics to solid-state physics remains, for the most part, within a linear setting, WKB analysis also plays a crucial role in nonlinear problems. In particular, integrable equations provide an important field of application, where refined nonlinear spectral analysis is central. This perspective is illustrated in the contribution of Joanne Dong, Peter Miller, and Giorgio Young. The authors consider singularly perturbed first-order systems which somehow extend the eikonal equation, in the case of turning points, where the direct WKB method fails. A prototypical example arises from the semiclassical limit of the defocusing one-dimensional cubic Schrödinger equation. Through suitable transformations, the authors establish Bohr–Sommerfeld quantization rules and obtain a uniform asymptotic description near the turning point, in the spirit of Maslov’s approach in the linear case, as presented in the contribution of San Vũ Ngọc.

In the field of nonlinear equations, WKB analysis has also proved to be a powerful tool in the study of ill-posedness phenomena in Sobolev spaces. In the case of positive regularity, the key mechanism is the emergence of rapid oscillations. This phenomenon was first identified for the nonlinear wave equation in [22] (see also [25]), subsequently strengthened in [23], and later transposed to the nonlinear Schrödinger equation in [4]. By contrast, in the regime of negative regularity, the dominant mechanism is the appearance of nonoscillatory modes. For the nonlinear Schrödinger equation, WKB analysis has made it possible to sharpen earlier ill-posedness results [7, 20], leading to the demonstration of an infinite loss of regularity in [6]. Some details and various extensions may be found in e.g. [5].

Finally, even if it does not appear explicitly in the contributions to this volume, it is worth mentioning that the WKB method brought success in the fine study of the tunneling effect for Schrödinger operators with some symmetric electric potentials wells (say, with nondegenerate minima). Combined with the famous semiclassical Agmon decay estimates, the WKB method (with complex phase) can indeed be used to describe the eigenfunctions in the classically forbidden region, where the eigenfunctions attached with the low-lying eigenvalues, are exponentially small. These aspects have been the object

of the seminal series of works by Simon [29–32] and by Helffer and Sjöstrand [12–18], see also the Bourbaki exposé [28] or the lectures notes [8, 10]. The interest in these questions has been renewed with the mathematical study of superconductivity, where magnetic Schrödinger operators and their eigenfunctions play a central role; see [9, 27]. In these magnetic contexts, a lack of WKB approximations has been explicitly underlined in [11, Section 6] and partially remedied in [2]. Establishing WKB-like approximation in magnetic situations is still an active area of research aiming to describe tunnelling in magnetic fields. This often leads to understanding these WKB Ansätze at a microlocal level, thanks to the famous Fourier–Bros–Iagolnitzer transform, as in the seminal paper [24].

We hope that this volume will convey to the reader both the breadth and the vitality of WKB methods, from their historical origins to their contemporary developments in linear and nonlinear theory. While necessarily selective, the contributions gathered here aim to reflect the richness of the ideas and techniques that have emerged from the WKB perspective over the past century. We wish the reader an engaging and stimulating journey through this collection of works, and warmly thank their authors for their contributions.

## References

- [1] A. Aspect and J. Villain, [The birth of wave mechanics \(1923–1926\)](#). *Comptes Rendus Physique* **18** (2017), no. 9–10, 583–585
- [2] V. Bonnaillie-Noël, F. Hérau, and N. Raymond, [Magnetic WKB constructions](#). *Arch. Ration. Mech. Anal.* **221** (2016), no. 2, 817–891 Zbl 1338.35379 MR 3488538
- [3] L. Brillouin, La mécanique ondulatoire de Schrödinger; une méthode générale de résolution par approximations successives. *C. R. hebd. Séanc. Acad. Sci. Paris* **183** (1926), 24–26 Zbl 52.0967.05
- [4] R. Carles, [Geometric optics and instability for semi-classical Schrödinger equations](#). *Arch. Ration. Mech. Anal.* **183** (2007), no. 3, 525–553 Zbl 1134.35098 MR 2278414
- [5] R. Carles, [Semi-classical analysis for nonlinear Schrödinger equations—WKB analysis, focal points, coherent states](#). World Scientific, Singapore, 2021 Zbl 1448.35461 MR 4274579
- [6] R. Carles, E. Dumas, and C. Sparber, [Geometric optics and instability for NLS and Davey–Stewartson models](#). *J. Eur. Math. Soc. (JEMS)* **14** (2012), no. 6, 1885–1921 Zbl 1273.35248 MR 2984591
- [7] M. Christ, J. Colliander, and T. Tao, [Asymptotics, frequency modulation, and low regularity ill-posedness for canonical defocusing equations](#). *Amer. J. Math.* **125** (2003), no. 6, 1235–1293 Zbl 1048.35101 MR 2018661
- [8] M. Dimassi and J. Sjöstrand, [Spectral asymptotics in the semi-classical limit](#). London Math. Soc. Lecture Note Ser. 268, Cambridge University Press, Cambridge, 1999 Zbl 0926.35002 MR 1735654
- [9] S. Fournais and B. Helffer, [Spectral methods in surface superconductivity](#). Prog. Nonlinear Differ. Equ. Appl. 77, Birkhäuser, Boston, MA, 2010 Zbl 1256.35001 MR 2662319
- [10] B. Helffer, [Semi-classical analysis for the Schrödinger operator and applications](#). Lecture Notes in Math. 1336, Springer, Berlin, 1988 Zbl 0647.35002 MR 0960278

- [11] B. Helffer, Introduction to semi-classical methods for the Schrödinger operator with magnetic field. In *Aspects théoriques et appliqués de quelques EDP issues de la géométrie ou de la physique*, pp. 49–117, Sémin. Congr. 17, Société Mathématique de France, Paris, 2009 Zbl [1225.81063](#) MR [2792093](#)
- [12] B. Helffer and J. Sjöstrand, [Multiple wells in the semiclassical limit. I.](#) *Comm. Partial Differential Equations* **9** (1984), no. 4, 337–408 Zbl [0546.35053](#) MR [0740094](#)
- [13] B. Helffer and J. Sjöstrand, [Multiple wells in the semiclassical limit. III. Interaction through nonresonant wells.](#) *Math. Nachr.* **124** (1985), 263–313 Zbl [0597.35023](#) MR [0827902](#)
- [14] B. Helffer and J. Sjöstrand, Puits multiples en limite semi-classique. II. Interaction moléculaire. Symétries. Perturbation. *Ann. Inst. H. Poincaré Phys. Théor.* **42** (1985), no. 2, 127–212 Zbl [0595.35031](#) MR [0798695](#)
- [15] B. Helffer and J. Sjöstrand, [Puits multiples en mécanique semi-classique. IV. Étude du complexe de Witten.](#) *Comm. Partial Differential Equations* **10** (1985), no. 3, 245–340 Zbl [0597.35024](#) MR [0780068](#)
- [16] B. Helffer and J. Sjöstrand, Puits multiples en mécanique semi-classique. V. Étude des minipuits. In *Current topics in partial differential equations*, pp. 133–186, Kinokuniya, Tokyo, 1986 Zbl [0628.35024](#) MR [1112146](#)
- [17] B. Helffer and J. Sjöstrand, Effet tunnel pour l'équation de Schrödinger avec champ magnétique. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)* **14** (1987), no. 4, 625–657 Zbl [0699.35205](#) MR [0963493](#)
- [18] B. Helffer and J. Sjöstrand, Puits multiples en mécanique semi-classique. VI. Cas des puits sous-variétés. *Ann. Inst. H. Poincaré Phys. Théor.* **46** (1987), no. 4, 353–372 Zbl [0648.35027](#) MR [0912154](#)
- [19] H. Jeffreys, [On certain approximate solutions of linear differential equations of the second order.](#) *Proc. London Math. Soc. (2)* **23** (1924), no. 6, 428–436 Zbl [50.0301.02](#) MR [1575202](#)
- [20] C. E. Kenig, G. Ponce, and L. Vega, [On the ill-posedness of some canonical dispersive equations.](#) *Duke Math. J.* **106** (2001), no. 3, 617–633 Zbl [1034.35145](#) MR [1813239](#)
- [21] H. A. Kramers, [Wellenmechanik und halbzahlige Quantisierung.](#) *Zeitschrift für Physik* **39** (1926), no. 10-11, 828–840 Zbl [52.0969.04](#)
- [22] G. Lebeau, Non linear optic and supercritical wave equation. *Bull. Soc. Roy. Sci. Liège* **70** (2001), no. 4-6, 267–306 Zbl [1034.35137](#) MR [1904059](#)
- [23] G. Lebeau, [Perte de régularité pour les équations d'ondes sur-critiques.](#) *Bull. Soc. Math. France* **133** (2005), no. 1, 145–157 Zbl [1071.35020](#) MR [2145023](#)
- [24] A. Martinez and V. Sordoni, [Microlocal WKB expansions.](#) *J. Funct. Anal.* **168** (1999), no. 2, 380–402 Zbl [0941.35136](#) MR [1719237](#)
- [25] G. Métivier, Exemples d'instabilités pour des équations d'ondes non linéaires (d'après G. Lebeau). In *Séminaire Bourbaki. Volume 2002/2003. Exposés 909–923*, pp. 63–75, Astérisque 294, Société Mathématique de France, Paris, 2004 Zbl [1215.35106](#) MR [2111639](#)
- [26] J. Rauch, [Hyperbolic partial differential equations and geometric optics.](#) Grad. Stud. Math. 133, American Mathematical Society, Providence, RI, 2012 Zbl [1252.35004](#) MR [2918544](#)
- [27] N. Raymond, [Bound states of the magnetic Schrödinger operator.](#) EMS Tracts Math. 27, European Mathematical Society (EMS), Zürich, 2017 Zbl [1370.35004](#) MR [3616340](#)
- [28] D. Robert, Analyse semi-classique de l'effet tunnel. In *Séminaire Bourbaki. Volume 1985–1986. Exposés 665*, pp. 257–281, Astérisque 145–146, Société Mathématique de France, Paris, 1987 Zbl [0632.35014](#) MR [0880037](#)

- [29] B. Simon, Semiclassical analysis of low lying eigenvalues. I. Nondegenerate minima: Asymptotic expansions. *Ann. Inst. H. Poincaré Sect. A (N.S.)* **38** (1983), no. 3, 295–308  
Zbl [0526.35027](#) MR [0708966](#)
- [30] B. Simon, Semiclassical analysis of low lying eigenvalues. II. Tunneling. *Ann. of Math. (2)* **120** (1984), no. 1, 89–118 Zbl [0626.35070](#) MR [0750717](#)
- [31] B. Simon, Semiclassical analysis of low lying eigenvalues. III. Width of the ground state band in strongly coupled solids. *Ann. Physics* **158** (1984), no. 2, 415–420 Zbl [0596.35028](#)  
MR [0772619](#)
- [32] B. Simon, Semiclassical analysis of low lying eigenvalues. IV. The flea on the elephant. *J. Funct. Anal.* **63** (1985), no. 1, 123–136 Zbl [0652.35090](#) MR [0795520](#)
- [33] G. Wentzel, Eine Verallgemeinerung der Quantenbedingungen für die Zwecke der Wellenmechanik. *Zeitschrift für Physik* **38** (1926), no. 6-7, 518–529 Zbl [52.0969.03](#)

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