

WKB structure in a scalar model of flat bands

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Abstract. We consider a family of periodic scalar operators for which one can define *flat bands* in the sense of Floquet–Bloch theory. One puzzling question originating in recent physics literature is a quantisation rule for the values of parameters at which these flat bands occur. We present a general theorem about the structure of solutions to the corresponding equation and a heuristic argument explaining their WKB structure in a specific case. That structure also explains the quantisation condition – both the WKB structure and that rule are confirmed by numerical experiments. Finally, we consider a simplified model in which separation of variables allows the use of complex WKB methods.

1. Introduction

Topologically nontrivial flat bands (in the sense of Bloch–Floquet theory) are of interest in condensed matter physics, as their presence has remarkable experimental consequences, see [6, 26] for theoretical references and [7] for experimental results in the context of twisted bilayer graphene.

This paper is concerned with flat bands and the structure of solutions to $Pu = 0$ for a class of periodic operators P on $\mathbb{R}^2$. They are of the form

$$P(\alpha, k) = (2D_{\bar{z}} + k)^2 - \alpha^2 V(z), \quad 2D_{\bar{z}} = \frac{1}{i}(\partial_{x_1} + i\partial_{x_2}), \quad (1.1)$$

where $x = (x_1, x_2)$, $\alpha, k \in \mathbb{C}$, and V is a real analytic, Λ -periodic potential satisfying additional symmetry assumptions, see (2.5). The study of solutions to $P(\alpha, k)u = 0$ is equivalent to the study of eigenfunctions of the system

$$P(\alpha, k)u_0 = 0 \Leftrightarrow (D_S(\alpha) + k)u = 0, \quad \alpha \neq 0, \quad (1.2)$$

$$D_S(\alpha) := \begin{pmatrix} 2D_{\bar{z}} & \alpha V(z) \\ \alpha & 2D_{\bar{z}} \end{pmatrix}, \quad u = \begin{pmatrix} \alpha^{-1}(2D_{\bar{z}} + k)u_0 \\ -u_0 \end{pmatrix}.$$

Flat bands (see (1.5) below) occur at α if

$$\text{Spec}_{L^2(\mathbb{C}/\Lambda)} D_S(\alpha) = \mathbb{C}. \quad (1.3)$$

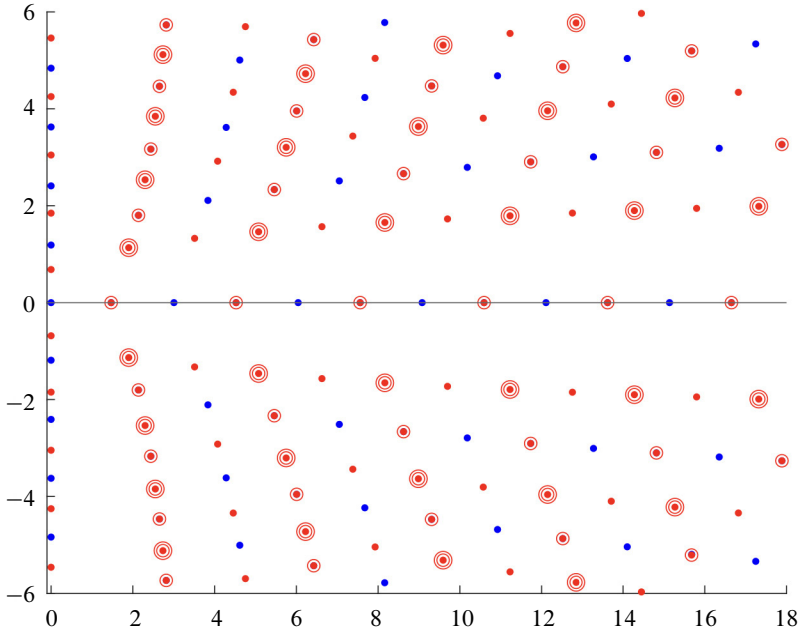


Figure 1. The sets $\mathcal{A}$ (red) and $\mathcal{B}$ (blue) of Theorem 1.1 for the scalar model based on the Bistritzer–MacDonald potential (2.3). The multiple circles indicate multiplicity (in the sense of Theorem 1.1, $m = 1, 2, 3$ all occur). A movie showing the evolution of the bands $k \mapsto E_j(\alpha, k)$, $j = 1, 2$ (over the real axis in k) as α varies can be viewed at https://math.berkeley.edu/~zworski/bands_scalar.mp4 (visited on 31 August 2026). It illustrates the behaviour of bands at real elements of $\mathcal{A} \cup \mathcal{B}$.

When V satisfies certain symmetries, see (2.5) below, we have nontrivial solutions to

$$D_S(\alpha)v(\alpha) = 0, \quad (1.4)$$

for all α . They are called (*symmetry*) *protected states*, as their existence is due to symmetries of the operator. We are interested in the distribution of α 's for which (1.3) holds (see Figure 1) and in the closely related WKB structure (as $\alpha \rightarrow \infty$) of the protected states.

Because the properties of $D_S(\alpha)$ in (1.2) are described using a scalar operator (1.1), we refer to this model as a *scalar model* of magic angles/flat bands. It was introduced in [11] with the hope of having a simpler setting for seeing the distribution of magic angles. That model is derived from the chiral model of twisted bilayer graphene (TBG) [2, 4, 26], see Section 2.1.

The self-adjoint Hamiltonian associated with $D_S(\alpha)$ is given by

$$H(\alpha) := \begin{pmatrix} 0 & D_S(\alpha) \\ D_S(\alpha)^* & 0 \end{pmatrix},$$

and the bands are the eigenvalues of $e^{-i\langle z, k \rangle} H(\alpha) e^{i\langle z, k \rangle}: H^1(\mathbb{C}/\Lambda; \mathbb{C}^4) \rightarrow L^2(\mathbb{C}/\Lambda, \mathbb{C}^4)$, $z, k \in \mathbb{C}$, considered as functions of k (they are $\pm$ the singular values of $D_S(\alpha) + k$):

$$\cdots \leq -E_j(\alpha, k) \leq \cdots \leq -E_1(\alpha, k) \leq E_1(\alpha, k) \leq \cdots \leq E_j(\alpha, k) \leq \cdots. \quad (1.5)$$

(For the general theory of Bloch–Floquet band spectrum, see [25, §5.3], and for an introduction in this context [28, §§3.1 and 3.2].) A flat band occurs at α such that $E_1(\alpha, k) \equiv 0$, and that is equivalent to (1.3).

Local solutions to $D_S(\alpha)u = 0$ (or, equivalently, solutions to $P(\alpha, 0)u_0 = 0$) are given by local holomorphic sections of a rank-2 holomorphic (topologically trivial) vector bundle over $\mathbb{C}/\Lambda$. (This is valid in greater generality, but we concentrate on the case in which we know flat bands occur.) We denote it by $\mathcal{E}(\alpha)$, and it is defined in Section 3.6. The first theorem describes the connection between the structure of this vector bundle and the behaviour of bands.

Theorem 1.1. *Suppose that $E_j(\alpha, k)$ are defined in (1.5) for $D_S(\alpha)$ with V satisfying (2.5). There exist discrete sets $\mathcal{A}, \mathcal{B} \subset \mathbb{C}$ such that we have the following trichotomy:*

$$\begin{aligned} \alpha \notin \mathcal{A} \cup \mathcal{B} &\Leftrightarrow \mathcal{E}(\alpha) \text{ is indecomposable} \Leftrightarrow E_{\pm 1}(\alpha, k) \simeq \pm c|k|^2; \\ \alpha \in \mathcal{B} &\Leftrightarrow \mathcal{E}(\alpha) = L_0 \oplus L_1, L_j \text{ trivial} \Leftrightarrow E_{\pm j}(\alpha, k) \simeq \pm c_j|k|, j = 1, 2, E_3(\alpha, k) > 0; \\ \alpha \in \mathcal{A} &\Leftrightarrow \mathcal{E}(\alpha) = L \oplus L^*, \deg L = m \Leftrightarrow E_j(\alpha, k) \equiv 0, |j| \leq m, E_{m+1}(\alpha, k) > 0, \end{aligned}$$

where L_j and L are line bundles.

We refer to m in the third statement of the theorem as the *multiplicity* of a magic $\alpha \in \mathcal{A}$. Although the model we consider does not seem to be physically relevant, Li–Yang [19] proved that the same trichotomy appears in the case of twisted multiple layers of graphene, see [27] for a mathematical presentation and references to the physics literature. The fact that double Dirac points (that is, points k at which the bands $k \mapsto E_j(\alpha, k)$ in (1.5) meet conically, as for $k = 0$ in the case of $\alpha \in \mathcal{B}$ in Theorem 1.1) can occur at a discrete set of angles was not present in the physics literature. The computations in [19] suggest that Dirac points should occur when the angle of twisting between two bilayer graphene wafers is close to 0.44° .

1.1. Quantisation condition for real magic α 's

One of the most striking observations made in [26] was a quantisation rule for real magic alphas in the chiral model (the definition is the same as (1.3) but with operator (2.1)) with the Bistritzer–MacDonald potential (2.2): if $\alpha_1 < \alpha_2 < \cdots < \alpha_j < \cdots$ is the sequence of positive α 's for which (2.6) holds, then

$$\alpha_{j+1} - \alpha_j = \gamma + o(1), \quad j \rightarrow +\infty, \quad \gamma \approx \frac{3}{2}. \quad (1.6)$$

The more accurate computations made in [2] suggest that $\gamma \approx 1.515$. In the scalar model (2.4) with $V(z) = U_{\text{BM}}(z)U_{\text{BM}}(-z)$, where U_{BM} is given by (2.2), we numerically

observe the following rule for real elements of $\mathcal{A}$: real magic α 's are double (in the sense of Theorem 1.1), $\alpha_{2j-1} = \alpha_{2j}$, $j = 1, 2, \dots$,

$$\alpha_{j+1} - \alpha_{j-1} = 2\gamma + o(1), \quad j \rightarrow +\infty, \quad (1.7)$$

where γ is the same as in (1.6). This “doubling” phenomenon in the passage from chiral to scalar models occurs for other potentials as well.

In this paper, we discuss the connection between the WKB structure of solutions to (1.4) and the distribution of α 's in the scalar model. In Section 4.1, we present the mechanism for zero creation in protected states $v(\alpha)$ for the Bistritzer–MacDonald potential. This is related to the distribution of magic alphas, see Section 3.3. In Section 4.2, we discuss the potential

$$U(z) := i\overline{U_{\text{BM}}(z)}^2, \quad V(z) := U(z)U(-z), \quad (1.8)$$

for which the structure of equation (1.1) simplifies significantly and for which we have (1.6) with $\gamma = 1/4$. This is presented in the next theorem, though we stress that we do *not* provide a full proof in this paper. At this point, we are only able to provide a heuristic WKB argument supported by numerical evidence. Hence the next theorem is only a theorem in the “physical” sense.

Theorem 1.2. *The potential V is given in (1.8) satisfies (2.5) and hence defines a set $\mathcal{A}$ of Theorem 1.1. If $\mathcal{A} \cap \mathbb{R}_+ = \{\alpha_j\}_{j=1}^\infty$, $\alpha_{j+1} \geq \alpha_j$, then each $\alpha_{2j-1} = \alpha_{2j}$, $j = 1, 2, \dots$ (that is, each real α has multiplicity 2), and*

$$\alpha_k - \alpha_{k-2} = \frac{1}{4} + o(1), \quad k \rightarrow \infty. \quad (1.9)$$

Numerically computed α_j 's for the scalar model based on (1.8) are

k	α_k	$\alpha_k - \alpha_{k-2}$
1	0.1395	
3	0.3803	0.2407
5	0.6281	0.2478
7	0.8772	0.2490
9	1.1267	0.2494
11	1.3764	0.2496
13	1.6262	0.2497
15	1.8760	0.2498
17	2.1259	0.2498
19	2.3758	0.2498

This shows that the heuristic argument for (1.9) is confirmed numerically. The magic alphas for the scalar model and the potential V in (1.8) are shown in Figure 2. They have

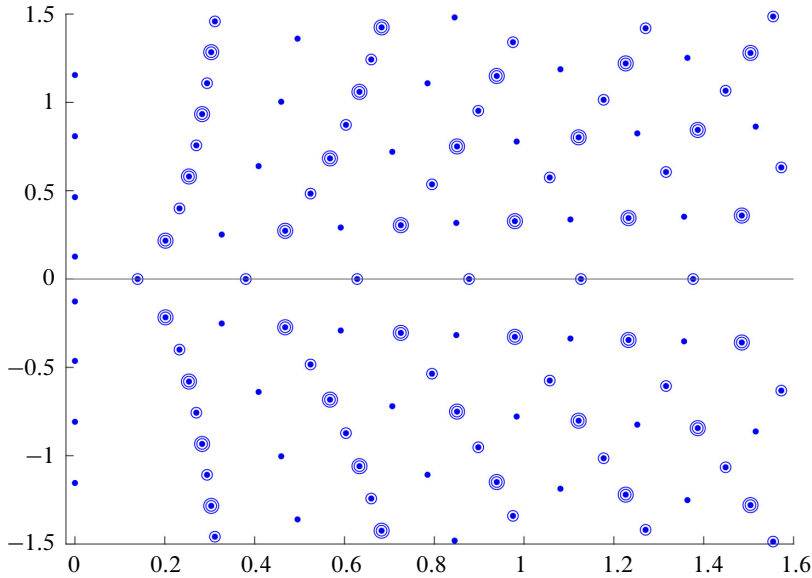


Figure 2. The sets $\mathcal{A}$ for the scalar model with the potential V given by (1.8). The multiple circles indicate multiplicity (in the sense of Theorem 1.1, $m = 1, 2, 3$ all occur). Except for different scaling, the structure of the set is the same as for $\mathcal{A}$ shown in red in Figure 1.

the same structure (on a different scale) as the magic alphas for the Bistritzer–MacDonald potential shows (in red) in Figure 1. We also remark that for the chiral model with the potential U in (2.1), we have an analogue of (1.6) and (1.7) with $\gamma = 1/8$: the spacing is halved and the multiplicities are simple.

1.2. Nonvanishing potentials

We can study properties of solutions to $P(\alpha, k)u = 0$ for more general potentials. In that case, we do not expect to have α 's for which (1.3) holds, but we could ask for which α 's and k 's we have a nontrivial kernel of $P(\alpha, k)$. One case to consider is that of

$$V = W^2 \neq 0, \quad (1.10)$$

as in that case the characteristic variety (that is, the set of $(x, \xi) \in T^*\mathbb{C}/\Lambda$ such that $q(x, \xi) := (2\bar{\xi})^2 - V(z) = 0$, $\xi = (\xi_1 - i\xi_2)/2$, $z = x_1 + ix_2$) is a union of two disjoint tori. Those tori are not Lagrangian unless $\text{Im } \partial_z W \equiv 0$.

Assuming (1.10) and that

$$W_0 := |\mathbb{C}/\Lambda|^{-1} \int_{\mathbb{C}/\Lambda} W(z) dm(z) \neq 0,$$

we have the following *naïve* WKB argument for the existence of nontrivial kernel of $P(\alpha, k)$: we construct two global phase functions by solving

$$2\partial_{\bar{z}}\varphi + k = \pm\alpha W(z), \quad \varphi(z) = \pm(\langle\alpha W_0, z\rangle + \alpha\psi) - \langle k, z\rangle, \quad \psi(z + \gamma) = \psi(z).$$

(Here $\langle k, z\rangle = \operatorname{Re}(k\bar{z})$.) A *naïve* quantisation condition is given by

$$\forall \gamma \in \Lambda, \quad \varphi(z + \gamma) \equiv \varphi(z) \bmod 2\pi\mathbb{Z} \Leftrightarrow \alpha \in W_0^{-1}(\Lambda^* \pm k). \quad (1.11)$$

It is not clear under what assumptions this gives approximate values of k and α for which we have a nontrivial solution to $P(\alpha, k)u = 0$. It certainly does not hold in general, see Figure 3. We remark that for non-self-adjoint operators for which the characteristic variety is a *single* torus, close in some sense to a Lagrangian torus, Melin and Sjöstrand [20] obtained Bohr–Sommerfeld quantisation rules for the spectrum. Their methods do not seem to be applicable to our case.

In view of these complications, we consider a yet simpler model in which one can use separation of variables and complex WKB. For that, we take $\Lambda := \mathbb{Z}^2$ and $W(x, y) = W(x)$, $(x, y) \in \mathbb{R}^2$. Taking the Fourier modes in y , we obtain a family of operators depending on $\zeta = 2\pi mi + k$, where $m \in \mathbb{Z}$ is a Fourier mode in y and k is in the fundamental domain of the dual lattice $2\pi(\mathbb{Z} + i\mathbb{Z})$:

$$P_\zeta(\alpha) := (D_x + \zeta)^2 - \alpha^2 W(x)^2. \quad (1.12)$$

We assume that $W(x)$ is an entire function satisfying $W(x + 1) = W(x)$. In that case, methods of complex WKB can be applied, but they normally require precise properties of a given $W(x)$, typically a trigonometric polynomial, see, for instance, the work of Galtsev–Shafarevich [12] and references given there. For a brief account of complex WKB method and additional references, we suggest [23, Chapter 7]. Here we present a general result which gives a condition under which the quantisation rule (1.11) holds.

Theorem 1.3. *Suppose that $W(x)$ is an entire function satisfying $W(x + 1) = W(x)$. Put $W_0 := \int_0^1 W(x) dx$ and assume that $W_0 \neq 0$. If there exists a closed smooth non-self-intersecting curve $\gamma: [0, 1] \rightarrow \mathbb{C}/\mathbb{Z}$, $\gamma'(t) \neq 0$ such that*

$$\operatorname{Im} \gamma'(t) W_0^{-1} W(\gamma(t)) = 0, \quad W(\gamma(t)) \neq 0, \quad t \in [0, 1], \quad (1.13)$$

then for $|\alpha| \gg 1$ and $|\operatorname{Re} \zeta| \leq \pi$,

$$\ker((D_x + \zeta)^2 - \alpha^2 W(x)^2) \neq \{0\} \Leftrightarrow \exists n \in \mathbb{Z} \text{ such that } \alpha = W_0^{-1}(2\pi n \pm \zeta) + \mathcal{O}(|\alpha|^{-1}).$$

The restriction on $|\operatorname{Re} \zeta|$ is not essential, as for $\ell \in \mathbb{Z}$, $P_{\zeta+2\pi\ell} = e^{-2\pi i \ell x} P_\zeta e^{2\pi i \ell x}$. The proof shares many features with standard complex WKB methods, but we are asking different questions and allow for a very large class of potentials. A more precise version is given in Theorem 5.10. An illustration of the result is given in Figure 3. One reason for presenting this case, much simpler than that considered in the heuristic Theorem 1.2, is to

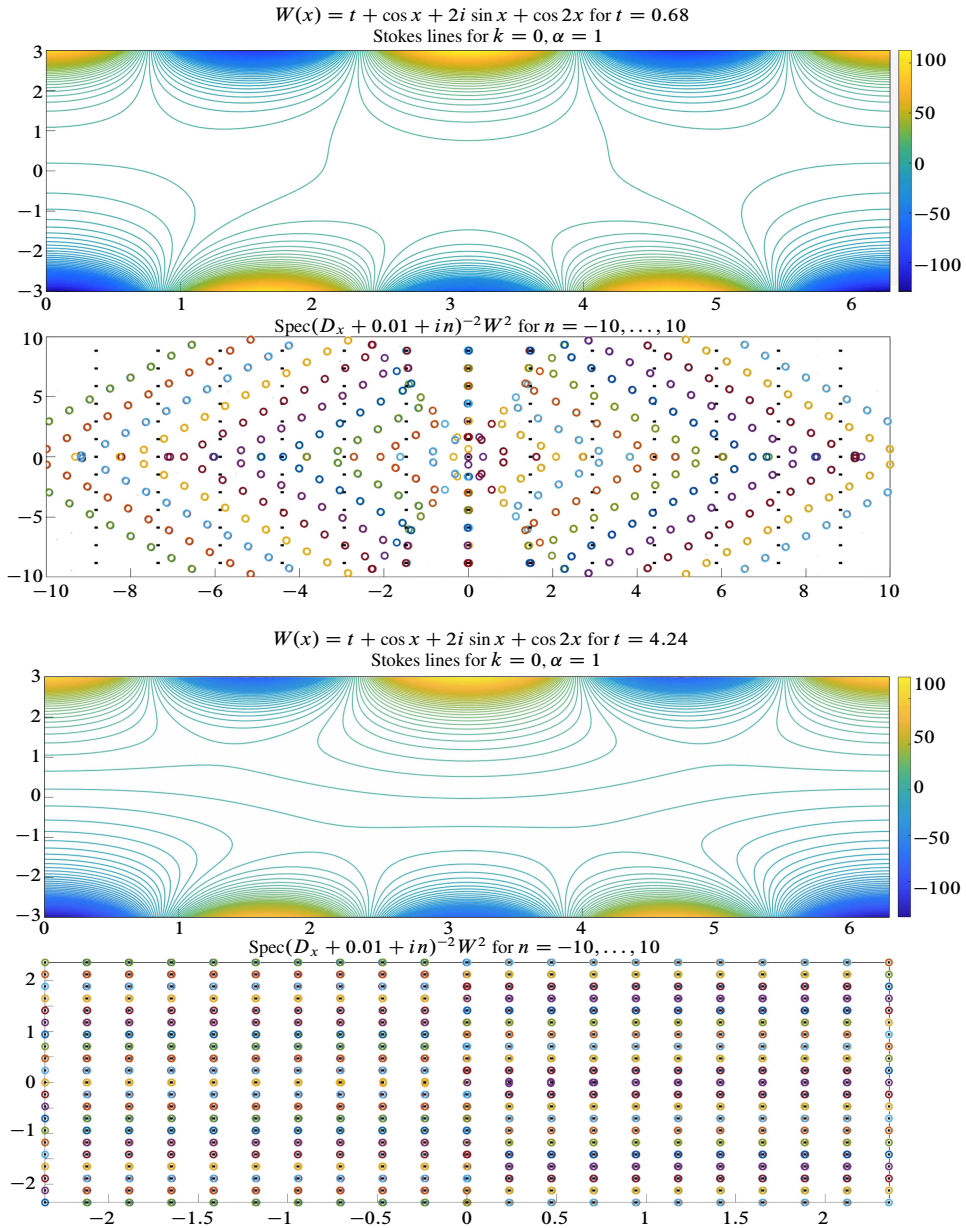


Figure 3. The bottom panels show sets of α 's (circles; coloured depending on the momentum variable of y) for which $(D_x + iD_y)^2 - \alpha^2 W(x)$, $(x, y) \in \mathbb{R}^2/2\pi\mathbb{Z}^2$ has a nontrivial kernel. The top panels show curves on which $\text{Im}(W(y(t))y'(t)) = 0$ (Stokes lines). The first W satisfies (5.4) but does not have Stokes loops (1.13); on the second W has Stokes loops. The quantisation rule (1.11) (shown by black dots) does not apply on the top and is very accurate on the bottom. For an animated version, see <https://math.berkeley.edu/~zworski/stokes.mp4> (visited on 31 August 2026).

indicate the potential complications for operators involving the potentials U_{BM} and $i\overline{U_{\text{BM}}}^2$ for which flat bands are present.

We call such γ 's satisfying (1.13) *Stokes loops*. A condition on W which guarantees their existence is given in Proposition 5.8: $\int_0^1 W(x) dx = W_0 \neq 0$, and

$$W(z) \neq 0, \quad |\operatorname{Im}(W_0^{-1}W(z))| \leq \operatorname{Re}(W_0^{-1}W(z)) \quad \text{for } |\operatorname{Im} z| \leq 1. \quad (1.14)$$

For instance, if $f(z)$ is entire and $f(z+1) = f(z)$ and $\int_0^1 f(z) dz = 0$, then $W(z) = M + f(z)$ will satisfy (1.14) for $M \geq \sqrt{2} \max_{|\operatorname{Im} z| \leq 1} |f(z)|$.

The simplest case in which the relevance of Stokes loops becomes apparent is that of $\zeta = 0$ and $\alpha \in \mathbb{R}$. A heuristic argument for this goes as follows. We first consider the case of $W(x) > 0$, $x \in \mathbb{R}$. In that case, standard WKB analysis gives two solutions,

$$u_{\pm}(x, \alpha) = \Psi(x, \alpha)^{-1/2} e^{\pm i\alpha \int_0^x \Psi(y, \alpha) dy},$$

$$\Psi(x, h) \simeq W(x) + \sum_{j=1}^{\infty} \alpha^{-j} \Psi_j(x).$$

The asymptotic condition for having a periodic solution is then

$$\alpha \int_0^1 \Psi(y, \alpha) dy = 2\pi n, \quad n \in \mathbb{Z},$$

which to the leading order is the quantisation condition in Theorem 1.3. Now suppose that for a holomorphic 1-periodic W , we have a curve satisfying (1.13) and, to simplify the notation, assume that $W_0 > 0$. Then on γ the holomorphic equation

$$(D_z^2 - \alpha^2 W(z))u(z) = 0$$

becomes

$$\left(D_t^2 - \alpha^2 (\gamma'(t)W(\gamma(t)))^2 - \frac{1}{2}\gamma'''(t) + \frac{1}{4}\gamma''(t)^2 \right) [e^{-\gamma'(t)/2} u(\gamma(t))] = 0.$$

(We note that $e^{-\gamma'(t)/2}$ is periodic.) The only difference with the real case is the presence of lower-order terms ($\mathcal{O}(1)$ corresponding to $\mathcal{O}(h^2)$ if $h := 1/\alpha$ is the semiclassical parameter) and this suggests that the same condition for periodicity under $t \mapsto t+1$, that is, the quantization condition in Theorem 1.3, holds. Since holomorphy then gives $u(z+1) = u(z)$, we obtain a periodic solution for $x \in \mathbb{R}$. To make this argument rigorous and to handle the case of general ζ , we proceed by a careful semiclassical analysis of a monodromy operator of solutions to a system equivalent to (1.12).

Notation. We write $f = \mathcal{O}_{\rho}(g)_H$ for $\|f\|_H \leq C_{\rho}g$, that is, the constant depends on ρ . When the context is clear, either ρ or H may be dropped. We denote by $\mathcal{O}(U)$ holomorphic functions on an open set $U \subset C^d$. If $z_0 \in U$ where U is a topological space, we write $\operatorname{neigh}_U(z_0)$ for an open neighbourhood of z_0 in U .

2. Review of the scalar model

In this section, we review the properties of the scalar model, starting with the chiral model from which it is derived.

2.1. Chiral model

The scalar model is derived from the chiral model of twisted bilayer graphene (TBG) [2, 4, 26]:

$$\begin{aligned} Q(\alpha, k) &:= D(\alpha) + k, \quad D(\alpha) := \begin{pmatrix} 2D_{\bar{z}} & \alpha U(z) \\ \alpha U(-z) & 2D_{\bar{z}} \end{pmatrix}, \quad \Omega = \mathbb{C}, \\ 2D_{\bar{z}} &= \frac{1}{i}(\partial_{x_1} + i\partial_{x_2}), \quad z = x_1 + ix_2 \in \mathbb{C}, \end{aligned} \quad (2.1)$$

where U satisfies

$$\begin{aligned} U(z + \gamma) &= e^{i\langle \gamma, K \rangle} U(z), \quad U(\omega z) = \omega U(z), \quad \overline{U(\bar{z})} = -U(-z), \quad \omega = e^{2\pi i/3}, \\ \gamma \in \Lambda &:= \omega\mathbb{Z} \oplus \mathbb{Z}, \quad K = \frac{4}{3}\pi, \quad \langle z, w \rangle := \operatorname{Re}(z\bar{w}). \end{aligned} \quad (2.2)$$

The constant K is determined up to sign by

$$\omega K \equiv K \not\equiv 0 \pmod{\Lambda^*}, \quad \Lambda^* := \frac{4\pi i}{\sqrt{3}}\Lambda,$$

that is, K is a high symmetry point with respect to the dual lattice. Although the chiral model is not considered accurate for large α (a dimensionless constant proportional to the reciprocal of the angle of twisting), it remains popular in physics and its mathematical properties remain puzzling, see [28].

The principal example of U is given by the Bistritzer–MacDonald potential

$$U_{\text{BM}}(z) = -\frac{4}{3}\pi i \sum_{\ell=0}^2 \omega^\ell e^{i\langle z, \omega^\ell K \rangle}. \quad (2.3)$$

It is, up to a real multiplicative constant, the unique potential satisfying (2.2) and having the lowest Fourier modes.

The scalar model is defined as follows:

$$P(\alpha, k) := (2D_{\bar{z}} + k)^2 - \alpha^2 V(z), \quad V(z) := U(z)U(-z). \quad (2.4)$$

It is derived from the chiral model by neglecting lower-order terms, as $\alpha \rightarrow \infty$, in the principally scalar reduction:

$$\begin{aligned} (D(-\alpha) + k)(D(\alpha) + k) &= P(\alpha, k)I_{\mathbb{C}^2} + \alpha R_k(\alpha), \\ R_k(\alpha) &:= \begin{pmatrix} k\alpha^{-1}4D_{\bar{z}} + k^2\alpha^{-1} & 2D_{\bar{z}}U(z) \\ -(2D_{\bar{z}}U)(-z) & k\alpha^{-1}4D_{\bar{z}} + k^2\alpha^{-1} \end{pmatrix}. \end{aligned}$$

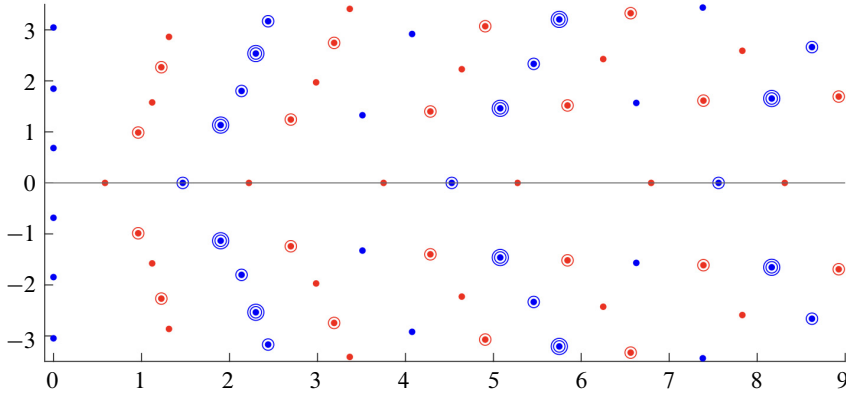


Figure 4. The sets $\mathcal{A}$ of magic α 's for the chiral model (2.1) (in red) and the scalar model (2.4) (in blue) with U given by (2.3). Higher multiplicities are indicated by additional circles: in the scalar model, the real α 's have multiplicity 2. When we interpolate between the chiral model and the scalar model, the multiplicity two real α 's split and travel in opposite directions to become magic α 's for the chiral model: see <https://math.berkeley.edu/~zworski/Spec.mp4> (visited on 31 August 2026), where the magic α 's are plotted as a function of $t \in [0, 1]$ which linearly interpolates between the chiral and the scalar models.

We refer to [13, §2.2] for the discussion of this reduction and a semiclassical explanation why $\alpha R_k(\alpha)$ is a lower-order term.

The potential V in (2.4) inherits symmetries from (2.2). We can formulate them independently as

$$V(z) = V(-z), \quad V(z + \gamma) = V(z), \quad V(\omega z) = \omega^2 V(z), \quad \overline{V(\bar{z})} = V(z). \quad (2.5)$$

Just as for the chiral model of TBG, a flat band at zero for a given α means that

$$\alpha \in \mathcal{A} \Leftrightarrow \forall k \in \mathbb{C}, \quad \ker_{H^2(\mathbb{C}/\Lambda; \mathbb{C})} P(\alpha, k) \neq \{0\}. \quad (2.6)$$

We denote the set of α 's for which (2.6) holds by $\mathcal{A}$, the same notation used in the chiral model [2]. It follows from [11] that $\mathcal{A}$ is a discrete subset of $\mathbb{C}$ and that

$$\mathcal{A} = -\mathcal{A} = \bar{\mathcal{A}}.$$

It is shown, together with the magic α 's for the chiral model (2.1), both with U given by (2.3), in Figure 4.

We define multiplicity as follows: if for $\alpha \in \Omega$, there is $k_0 \in \mathbb{C}$ such that $\ker P(\alpha, k_0) = \{0\}$, then we define

$$m(\alpha, k) := \frac{1}{2\pi i} \operatorname{tr} \oint_{\partial D} P(\alpha, \zeta)^{-1} \partial_{\bar{\zeta}} P(\alpha, \zeta) d\zeta, \quad (2.7)$$

where the integral is over the positively oriented boundary of a disc D which contains k as the only possible pole of $\zeta \mapsto P(\alpha, \zeta)$. Otherwise, we put $m(\alpha, k) = \infty$ for all $k \in \mathbb{C}$.

2.2. Symmetries

To simplify notation, we write $P(k) := P(\alpha, k)$, where $P(\alpha, k)$ was defined in (1.1) with V satisfying (2.5).

We define the following symmetries:

$$\begin{aligned}\Omega u(z) &:= u(\omega z), \quad \mathcal{L}_\gamma u(z) = u(z + \gamma), \quad \mathcal{E}_1 u(z) = u(-z), \\ \mathcal{A}_1 u(z) &= \overline{u(\bar{z})}, \quad \mathcal{R}_1 u(z) = u(\bar{z}),\end{aligned}$$

with the following commutation properties with $P(k)$: (here $\gamma \in \Lambda$ defined in (2.2))

$$\begin{aligned}\mathcal{E}_1 P(k) &= P(-k) \mathcal{E}_1, \quad \mathcal{L}_\gamma P(k) = P(k) \mathcal{L}_\gamma, \quad P(k) \Omega = \omega \Omega P(\omega k), \\ \mathcal{A}_1 P(k) &= P(-\bar{k}) \mathcal{A}_1, \quad \mathcal{R}_1 P(\alpha, k) = P(\bar{\alpha}, \bar{k})^* \mathcal{R}_1.\end{aligned}$$

It is useful to combine $\mathcal{E}_1$ and Ω :

$$\mathcal{C}_1 := \mathcal{E}_1 \Omega = \Omega \mathcal{E}_1, \quad P(k) \mathcal{C}_1 = -\tau \mathcal{C}_1 P(\tau k), \quad \mathcal{C}_1 u(z) = u(\tau z), \quad \tau := e^{-\pi i/3}.$$

(Nothing is lost here since $\mathcal{C}_1^3 = \mathcal{E}_1$ and $\mathcal{E}_1 \mathcal{C}_1 = \Omega$.)

We also note that for $\mathbf{a} \in \Lambda^*$,

$$P(k + \mathbf{a}) = e^{-i\langle z, \mathbf{a} \rangle} P(k) e^{i\langle z, \mathbf{a} \rangle}, \quad u(z) \mapsto e^{i\langle z, \mathbf{a} \rangle} u(z) \text{ is unitary on } L^2(\mathbb{C}/\Lambda). \quad (2.8)$$

Using these symmetries, definition of the multiplicity (2.7) and [11, Theorem 1], we obtain

$$m(\alpha, k) = \begin{cases} \infty, & \alpha \in \mathcal{A}, \\ 2 \mathbb{1}_{\Lambda^*}(k), & \alpha \notin \mathcal{A}. \end{cases} \quad (2.9)$$

(See Lemma 3.1 below for a direct proof.) In particular,

$$\forall \alpha \in \mathbb{C}, \quad \ker P(\alpha, 0) \neq \{0\}.$$

We now relate $P(k)$ to operator (1.2):

$$D_S = D_S(\alpha) = \begin{pmatrix} 2D_{\bar{z}} & \alpha V(z) \\ \alpha & 2D_{\bar{z}} \end{pmatrix}: H^1(\mathbb{C}/\Lambda; \mathbb{C}^2) \rightarrow L^2(\mathbb{C}/\Lambda; \mathbb{C}^2).$$

We have

$$(D_S + k) \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix} \Leftrightarrow \begin{cases} \alpha u_1 = (2D_{\bar{z}} + k)u_2 - g, \\ P(k)u_2 = \alpha f + (2D_{\bar{z}} + k)g. \end{cases}$$

Hence, with $\pi_1((u, v)^t) = u$, $\pi_2((u, v)^t) = v$, $\iota f := (f, 0)^t$,

$$\begin{aligned}P(k)^{-1} f &= \alpha^{-1} \pi_2(D_S + k)^{-1} \iota f, \quad \partial_k P(k) P(k)^{-1} f = 2\pi_1(D_S + k)^{-1} \iota f, \\ (D_S + k)^{-1} &= \begin{pmatrix} \frac{1}{2} \partial_k P(k) P(k)^{-1} & \frac{1}{2} \alpha^{-1} \partial_k P(k) P(k)^{-1} \partial_k P(k) - \alpha^{-1} \\ \alpha P(k)^{-1} & \frac{1}{2} P(k)^{-1} \partial_k P(k) \end{pmatrix}. \end{aligned} \quad (2.10)$$

For $\alpha \notin \mathcal{A}$, (2.10) gives (with integrals being over a positively oriented circle around 0)

$$\begin{aligned}\Pi(\alpha) &:= \frac{1}{2\pi i} \oint_0 (D_S(\alpha) + \zeta)^{-1} d\zeta = \frac{1}{2} \begin{pmatrix} a(\alpha) & b(\alpha) \\ c(\alpha) & d(\alpha) \end{pmatrix}, \\ a(\alpha) &:= \frac{1}{2\pi i} \oint \partial_k P(k) P(k)^{-1} dk, \quad d(\alpha) := \frac{1}{2\pi i} \oint P(k)^{-1} \partial_k P(k) dk.\end{aligned}\tag{2.11}$$

Hence

$$\operatorname{tr} \Pi(\alpha) = \frac{1}{2} (\operatorname{tr} a(\alpha) + \operatorname{tr} d(\alpha)) = \operatorname{tr} a(\alpha) = m(\alpha, 0).\tag{2.12}$$

(The equality of the traces of $a(\alpha)$ and $d(\alpha)$ follows from the Gohberg–Sigal theory, see, for instance, [10, Theorem C.11].) We note that we have analogous formula for $m(\alpha, k)$ by taking an integral around k in the definition of $\Pi(\alpha)$.

From (2.7), we conclude that

$$(D_S(\alpha) + k)^{-1} = \frac{\Pi(\alpha)}{k} - \frac{D_S \Pi(\alpha)}{k^2} + A(k),$$

and $k \mapsto A(k)$ is holomorphic near $k = 0$.

Remark. The definition of the projection $\Pi(\alpha)$ in (2.11) makes sense only for $\alpha \notin \mathcal{A}$.¹ To see how $\Pi(\alpha)$ extends to a *meromorphic family of projections*, one can proceed as follows. Suppose that $\alpha_0 \in \mathcal{A}$. Since $D(\alpha_0)$ is a Fredholm operator of index 0, [25, Proposition 2.14] produces an invertible Grushin operator for $D(\alpha_0)$,

$$\mathcal{P}(\alpha_0, 0) := \begin{pmatrix} D(\alpha_0) & R_- \\ R_+ & 0 \end{pmatrix}: H^1 \oplus \mathbb{C}^m \rightarrow L^2 \oplus \mathbb{C}^m,$$

where m is the dimension of the kernel of $D(\alpha_0)$. If $\mathcal{P}(\alpha, k)$ is defined by replacing $D(\alpha_0)$ by $D(\alpha) + k$, then $\mathcal{P}(\alpha, k)^{-1}$ exists for $|\alpha - \alpha_0| \ll 1$ and $|k| \ll 1$, and is holomorphic in k and α . Using the standard notation of [25, §2.6], for $0 < |\alpha - \alpha_0| \ll 1$ and $0 < |k| \ll 1$,

$$(D(\alpha) + k)^{-1} = E(\alpha, k) - E_+(\alpha, k) E_{-+}(\alpha, k)^{-1} E_-(\alpha, k).$$

Hence definition (2.11) gives

$$\Pi(\alpha) = -\frac{1}{2\pi i} \oint_{|k|=\varepsilon} E_+(\alpha, k) E_{-+}(\alpha, k)^{-1} E_-(\alpha, k) dk.$$

This allows a meromorphic continuation of $\Pi(\alpha)$ as follows: we can write

$$E_{-+}(\alpha, k)^{-1} = g(\alpha, k)^{-1} F(\alpha, k),$$

¹We owe this remark to Zhongkai Tao.

where $(\alpha, k) \rightarrow F(\alpha, k)$ is holomorphic and $g(\alpha, k) := \det E_{-+}(\alpha, k)$ (F is the adjugate matrix of E_{-+}). We know that

$$g(\alpha_0, k) \equiv 0, \quad g(\alpha, 0) = 0, \quad g(\alpha, k) \neq 0, \quad 0 < |\alpha - \alpha_0| \ll 1, \quad 0 < |k| \ll 1.$$

This and the fact that for $\alpha \neq \alpha_0$, $\text{tr } \Pi(\alpha) = 2$ imply that

$$g(\alpha, k) = (\alpha - \alpha_0)^p k^2 h(\alpha, k), \quad h(0, 0) \neq 0.$$

(That $k \mapsto g(\alpha, k)$ has to vanish exactly to order two in view of $\text{tr } \Pi(\alpha) = 2$ follows, for instance, from [24, Proposition 4.1].) Since $E_{-+}(\alpha_0, k) \equiv 0_m$, we conclude that $p \geq m$. The possibility that $p > m$ comes from the possibility of algebraic multiplicities (observed in the case of the chiral model, see [5, §10.1]).

We record the following symmetries of D_S : the translation symmetry $\mathcal{L}_\gamma D_S = D_S \mathcal{L}_\gamma$, and

$$\mathcal{C}_2 \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} := \begin{pmatrix} \bar{\tau} u_1(\tau \bullet) \\ u_2(\tau \bullet) \end{pmatrix}, \quad \mathcal{C}_2(D_S(\alpha) + k) = \tau(D_S(\alpha) + \bar{\tau}k) \mathcal{C}_2, \quad \mathcal{C}_2 \Pi = \Pi \mathcal{C}_2, \quad (2.13)$$

$$\mathcal{R}_2 \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} := \begin{pmatrix} u_2(\bar{\bullet}) \\ u_1(\bar{\bullet}) \end{pmatrix}, \quad \mathcal{R}_2(D_S(\alpha) + k) = (D_S(\bar{\alpha}) + \bar{k})^* \mathcal{R}_2,$$

$$\mathcal{A}_2 \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} := \begin{pmatrix} -\bar{u}_2 \\ \bar{u}_1 \end{pmatrix}, \quad \mathcal{A}_2(D_S(\alpha) + k) = -(D_S(\alpha) - k)^* \mathcal{A}_2.$$

We note that

$$\mathcal{R}_2 \mathcal{C}_2 = \bar{\tau} \mathcal{C}_2^{-1} \mathcal{R}_2, \quad \mathcal{A}_2 \mathcal{C}_2 = \tau \mathcal{C}_2 \mathcal{A}_2.$$

A self-adjoint operator associated with D_S is given by

$$H_S(\alpha, k) := \begin{pmatrix} 0 & D_S(\alpha) + k \\ (D_S(\alpha) + k)^* & 0 \end{pmatrix}: H^1(\mathbb{C}/\Lambda; \mathbb{C}^4) \rightarrow L^2(\mathbb{C}/\Lambda; \mathbb{C}^4).$$

It enjoys the standard chiral symmetry

$$\mathcal{W} H(\alpha, k) \mathcal{W} = -H(\alpha, k), \quad \mathcal{W} \begin{pmatrix} \mathbf{u} \\ \mathbf{v} \end{pmatrix} := \begin{pmatrix} -\mathbf{u} \\ \mathbf{v} \end{pmatrix}, \quad \mathbf{u}, \mathbf{v} \in L^2(\mathbb{C}/\Lambda; \mathbb{C}^2).$$

The symmetries of $P(k)$ give us also

$$\mathcal{C}_4 H(\alpha, k) = H(\alpha, \bar{\tau}k) \mathcal{C}_4, \quad \mathcal{C}_4 \begin{pmatrix} \mathbf{u} \\ \mathbf{v} \end{pmatrix} := \begin{pmatrix} \mathcal{C}_2 \mathbf{u} \\ \tau \mathcal{C}_2 \mathbf{v} \end{pmatrix}, \quad \mathcal{C}_4 \mathcal{W} = \mathcal{W} \mathcal{C}_4. \quad (2.14)$$

Finally,

$$\mathcal{R}_4 H(\alpha, k) = H(\bar{\alpha}, \bar{k}) \mathcal{R}_4, \quad \mathcal{R}_4 \begin{pmatrix} \mathbf{u} \\ \mathbf{v} \end{pmatrix} := \begin{pmatrix} \mathcal{R}_2 \mathbf{v} \\ \mathcal{R}_2 \mathbf{u} \end{pmatrix}, \quad \mathcal{C}_4 \mathcal{R}_4 = \mathcal{R}_4 \mathcal{C}_4^{-1}, \quad (2.15)$$

$$\mathcal{A}_4 H(\alpha, k) = -H(\alpha, -k) \mathcal{A}_4, \quad \mathcal{A}_4 \begin{pmatrix} \mathbf{u} \\ \mathbf{v} \end{pmatrix} := \begin{pmatrix} \mathcal{A}_2 \mathbf{v} \\ \mathcal{A}_2 \mathbf{u} \end{pmatrix}, \quad \mathcal{C}_4 \mathcal{A}_4 = \mathcal{A}_4 \mathcal{C}_2. \quad (2.16)$$

We also define for $\ell \in \mathbb{Z}/6\mathbb{Z}$ (dropping $\mathbb{C}$ when $n = 1$),

$$L_\ell^2(\mathbb{C}/\Lambda, \mathbb{C}^n) := \{u \in L^2(\mathbb{C}/\Lambda, \mathbb{C}^n) : \mathcal{C}_n u = \tau^\ell u\}, \quad n = 1, 2, 4,$$

noting that

$$\begin{aligned} L_\ell^2(\mathbb{C}/\Lambda; \mathbb{C}^2) &= L_{\ell+1}^2(\mathbb{C}/\Lambda) \oplus L_\ell^2(\mathbb{C}/\Lambda), \\ L_\ell^2(\mathbb{C}/\Lambda; \mathbb{C}^4) &= L_\ell^2(\mathbb{C}/\Lambda; \mathbb{C}^2) \oplus L_{\ell-1}^2(\mathbb{C}/\Lambda; \mathbb{C}^2), \\ \mathcal{R}_1: L_\ell^2(\mathbb{C}/\Lambda) &\rightarrow L_{-\ell}^2(\mathbb{C}/\Lambda), \quad \mathcal{R}_2: L_\ell^2(\mathbb{C}/\Lambda; \mathbb{C}^2) \rightarrow L_{-\ell-1}^2(\mathbb{C}/\Lambda; \mathbb{C}^2), \\ \mathcal{R}_4, \mathcal{A}_4: L_\ell^2(\mathbb{C}/\Lambda; \mathbb{C}^4) &\rightarrow L_{-\ell}^2(\mathbb{C}/\Lambda; \mathbb{C}^2). \end{aligned}$$

We then have

$$\begin{aligned} D_S(\alpha): (H^1 \cap L_\ell^2)(\mathbb{C}/\Lambda; \mathbb{C}^2) &\rightarrow L_{\ell+1}^2(\mathbb{C}/\Lambda; \mathbb{C}^2), \\ H_\ell(\alpha) := H(\alpha) := H(\alpha, 0): (H^1 \cap L_\ell^2)(\mathbb{C}/\Lambda; \mathbb{C}^4) &\rightarrow L_\ell^2(\mathbb{C}/\Lambda; \mathbb{C}^4). \end{aligned} \quad (2.17)$$

3. Protected states and magic alphas in the scalar model

This section is devoted to the proof of Theorem 1.1. We first investigate the family of protected states, that is, nontrivial elements of the kernel of $D_S(\alpha)$ which exist for all α . In the process, we define the discrete sets $\mathcal{A}$ and $\mathcal{B}$ which appear in Theorem 1.1. We then discuss the presence of Dirac points for $\alpha \in \mathcal{B}$ and the relation with holomorphic vector bundles.

3.1. Family of protected states

For completeness, we start with a direct proof of (2.9).

Lemma 3.1. *There exists a discrete set $\mathcal{A} \subset \mathbb{C}$ such that for $\alpha \notin \mathcal{A}$, $k \mapsto (D_S(\alpha) + k)^{-1}$ is meromorphic on $\mathbb{C}$ and, in the notation of (2.7),*

$$m(\alpha, k) = 2 \mathbb{1}_{\Lambda^*}(k).$$

For $\alpha \in \mathcal{A}$, $\text{Spec } D_S(\alpha) = \mathbb{C}$.

Proof. In the proof, we use (2.12) (see the comment below that formula) and, when $k \mapsto (D_S(\alpha) + k)^{-1}$ is meromorphic, write

$$m(\alpha, k) = \frac{1}{2\pi i} \oint_k (D_S(\alpha) + \zeta)^{-1} d\zeta. \quad (3.1)$$

We then observe that (2.8) and (2.13) show

$$m(\alpha, k + a), \quad a \in \Lambda^*, \quad m(\alpha, \omega k) = m(\alpha, k). \quad (3.2)$$

For any fixed $k_0 \notin \Lambda^*$, $m(0, k_0) = 0$, and hence (using analytic Fredholm theory for $\alpha \mapsto D_S(\alpha) + k_0$, see [10, §C.3]), $m(\alpha, k_0) = 0$ except on a discrete set of α 's which we denote by $\mathcal{A}_{k_0}$. We claim that

$$\forall \alpha \in \mathbb{C} \setminus \mathcal{A}_{k_0}, k \in \mathbb{C}, \quad m(\alpha, k) = 2 \mathbb{1}_{\Lambda^*}(k). \quad (3.3)$$

To start, we note that $\mathbb{C} \setminus \mathcal{A}_{k_0}$ is connected and for all $\alpha \in \mathbb{C} \setminus \mathcal{A}_{k_0}$, $m(\alpha, k) = 0$ except for a discrete set of k 's (again using analytic Fredholm theory). Let $Z \subset \mathbb{C} \setminus \mathcal{A}_{k_0}$ be the set of α with $m(\alpha, k) = 2 \mathbb{1}_{\Lambda^*}(k)$ for all k . The set Z is closed in $\mathbb{C} \setminus \mathcal{A}_{k_0}$ (this follows from (3.1)) and nonempty since $0 \in Z$.

Suppose $\alpha_1 \in Z$ and $k_1 \in \mathbb{C}$. If $k_1 \notin \Lambda^*$, then $m(\alpha_1, k_1) = 0$ and for $(\alpha, k) \in U \times V$,

$$U = \text{neigh}_{\mathbb{C}}(\alpha_1), \quad V = \text{neigh}_{\mathbb{C}}(k_1), \quad m(\alpha, k) = 0$$

(invertibility of $D_S(\alpha) + k: H^1(\mathbb{C}/\Lambda; \mathbb{C}^2) \rightarrow L^2(\mathbb{C}/\Lambda; \mathbb{C}^2)$ is an open condition).

If $k_1 \in \Lambda^*$, then, since $\alpha_1 \in Z$, $m(\alpha_1, k) = 0$ for $k \in D(k_1, 4\delta) \setminus D(k_1, \delta/2)$, $\delta \ll 1$. Again, since invertibility is an open condition, there exists $U = \text{neigh}_{\mathbb{C}}(\alpha_1)$ such that for $\alpha \in U$ and $k \in D(k_1, 3\delta) \setminus D(k_1, \delta)$, $m(\alpha, k) = 0$. It follows from (3.1) and continuity of $U \times \partial D(k_1, 2\delta) \ni (\alpha, k) \mapsto (D_S(\alpha) + k)^{-1}$ that for $\alpha \in U$,

$$\begin{aligned} \sum_{k \in D(k_1, \delta)} m(\alpha, k) &= \frac{1}{2\pi i} \text{tr} \int_{\partial D(k_1, 2\delta)} (D_S(\alpha) + \zeta)^{-1} d\zeta \\ &= \frac{1}{2\pi i} \text{tr} \int_{\partial D(k_1, 2\delta)} (D_S(\alpha_1) + \zeta)^{-1} d\zeta = m(\alpha_1, k_1) = 2. \end{aligned} \quad (3.4)$$

Now, if $m(\alpha, k) \neq 0$ for $k \in D(k_1, \delta) \setminus \{k_1\}$, then, using (3.2) and the fact that $\omega k_1 \equiv k_1 \pmod{\Lambda^*}$,

$$m(\alpha, k_1 + \omega(k - k_1)) = m(\alpha, \omega k_1 + \omega(k - k_1)) = m(\alpha, \omega k) = m(\alpha, k) \geq 1,$$

and consequently $\sum_{k \in D(k_1, \delta)} m(\alpha, k) \geq 3$, contradicting (3.4). This means that for $\alpha \in U$ and $k \in D(k_1, \delta)$, $m(\alpha, k) = 2 \mathbb{1}_{\Lambda^*}(k)$.

We conclude that every (α_1, k_1) , $\alpha_1 \in Z$ and $k_1 \in \mathbb{C}$, has a neighbourhood in which $m(\alpha, k) = 2 \mathbb{1}_{\Lambda^*}(k)$. In view of periodicity in (3.2), to obtain validity for all k , we only need to check it for $k \in \mathbb{C}/\Lambda^*$. Since that set is compact, the local result implies that every $\alpha_1 \in Z$ has a neighbourhood in which the equality in (3.3) holds. That means that Z is open in $\mathbb{C} \setminus \mathcal{A}_{k_0}$ and as it is closed, it is equal to $\mathbb{C} \setminus \mathcal{A}_{k_0}$. This proves (3.3) and hence the first part of the lemma with $\mathcal{A} = \mathcal{A}_{k_0}$.

Finally, we note that $\mathcal{A}_{k_0}$ is independent of $k_0 \notin \Lambda^*$. In fact,

$$\mathbb{C} \setminus \mathcal{A}_{k_0} \subset \{\alpha \in \mathbb{C} : \forall k \in \mathbb{C}, m(\alpha, k) = 2 \mathbb{1}_{\Lambda^*}(k)\} \subset \mathbb{C} \setminus \mathcal{A}_{k_1},$$

where the second inclusion follows from the fact that we can take $k = k_1$, so that $m(\alpha, k_1) = 0$. Hence we can drop k_0 in $\mathcal{A}_{k_0}$. Since by definition, $D(\alpha) + k_0$ is not invertible for $\alpha \in \mathcal{A}$ and any $k_0 \notin \Lambda^*$, it follows that $\text{Spec } D_S(\alpha) = \mathbb{C}$, completing the proof. ■

We have the following result proved using methods similar to those in [2–4] but with some new twists (no pun intended).

Proposition 3.2. *Let $\mathcal{A}$ be the set of magic α 's defined in (2.6). There exists a unique discrete set $\mathcal{B}$, with $\mathcal{B} \cap \mathcal{A} = \emptyset$, and such that for $\alpha \neq 0$,*

$$\dim \ker P(\alpha, 0) = \dim \ker D_S(\alpha) = \begin{cases} 1, & \alpha \notin \mathcal{A} \cup \mathcal{B}, \\ 2, & \alpha \in \mathcal{B}. \end{cases} \quad (3.5)$$

Moreover, there exists a holomorphic family unique up to multiplication by a holomorphic function

$$\mathbb{C} \ni \alpha \mapsto u(\alpha) \in \ker P(\alpha, 0) \setminus \{0\}, \quad u(\alpha) \in L^2_0(\mathbb{C}/\Lambda; \mathbb{C}). \quad (3.6)$$

Remarks. (1) The proposition and (2.10) show that for $\Pi(\alpha)$ defined in (2.11), we have

$$D_S(\alpha)\Pi(\alpha) \neq 0, \quad \alpha \notin \mathcal{A} \cup \mathcal{B},$$

that is, $D_S(\alpha)$ has a Jordan block structure at its eigenvalues (which are given by Λ^*). The way that $D_S(\alpha)$ was defined, we do not have (3.5) at $\alpha = 0$. Instead,

$$\dim \ker P(0, 0) = 1 \quad \text{and} \quad \dim \ker D_S(0) = 2,$$

and from the point of view of Theorem 1.2, we should include 0 in $\mathcal{B}$. To have an agreement between the system and the scalar operator, we could have considered

$$\tilde{D}_S(\alpha) := \begin{pmatrix} 2D_{\bar{z}} & \alpha^2 V(z) \\ 1 & 2D_{\bar{z}} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \alpha \end{pmatrix}^{-1} D_S(\alpha) \begin{pmatrix} 1 & 0 \\ 0 & \alpha \end{pmatrix}. \quad (3.7)$$

To maintain a closer analogy to the chiral model (2.1), we keep α as a linear coupling constant.

(2) As we will see, the set $\mathcal{B}$ has an interesting interpretation in terms of band structure: for $\alpha \in \mathcal{A}$, the operator $H(\alpha)$ has a flat band. For $\alpha \notin \mathcal{A} \cup \mathcal{B}$, the bands touch tangentially at 0, while for $\alpha \in \mathcal{B}$, the bands exhibit a Dirac point at 0. This will be explained in Section 3.2.

(3) For $\alpha \notin \mathcal{A} \cup \mathcal{B}$, the unique solution to $P(\alpha, 0)u = 0$ satisfies symmetries inherited from the equation

$$u(z) = u(-z), \quad u(\omega z) = u(z), \quad \overline{u(\bar{z})} = u(z), \quad u(z + \gamma) = u(z), \quad \gamma \in \Lambda. \quad (3.8)$$

We note that this also implies that (since $e^{\pi i/3} = -\omega^2$)

$$u(e^{\pi i/3} z) = u(z), \quad u(-\bar{z}) = \overline{u(z)}.$$

Analytic continuation shows that these properties remain true for the family $\mathbb{C} \ni \alpha \mapsto u(\alpha)$.

Proof of Proposition 3.2. We use $\tilde{D}_S(\alpha)$ defined in (3.7) and the corresponding self-adjoint family

$$\tilde{H}(\alpha) := \begin{pmatrix} 0 & \tilde{D}_S(\alpha) \\ \tilde{D}_S(\alpha)^* & 0 \end{pmatrix}.$$

Away from 0, these are equivalent to $D_S(\alpha)$ and $H(\alpha)$ using (3.7). Since $\mathbb{C}_2$ commutes with the conjugation in (3.7), we still have the mapping properties (2.17).

We have

$$\tilde{H}(0) = \begin{pmatrix} & 2D_{\bar{z}} & 0 \\ & 1 & 2D_{\bar{z}} \\ 2D_z & & 1 \\ 0 & 2D_z & \end{pmatrix}, \quad \ker \tilde{H}(0) = \{\mathbf{e}_1, \mathbf{e}_4\}, \quad \mathbf{e}_1 \in L_{-1}^2, \quad \mathbf{e}_4 \in L_1^2,$$

so that in the notation of (2.17),

$$\dim \ker \tilde{H}_\ell(0) = \begin{cases} 1, & \ell = \pm 1, \\ 0, & \text{otherwise.} \end{cases} \quad (3.9)$$

In view of (2.14), $\mathcal{W} \tilde{H}_\ell(\alpha) \mathcal{W} = -\tilde{H}_\ell(\alpha)$, and hence the spectrum of $\tilde{H}_\ell(\alpha)$ is symmetric about 0. At $\alpha = 0$, $0 \in \text{Spec}(\tilde{H}_\ell(0))$ only for $\ell = \pm 1$, and then it is a simple eigenvalue. Hence, the multiplicity of 0 as an eigenvalue of $\tilde{H}_\ell(\alpha)$, $\ell = \pm 1$ has to be odd, while that of $\tilde{H}_\ell(\alpha)$, $\ell \notin \{\pm 1\}$ has to be even. In particular,

$$\dim \ker \tilde{H}_{\pm 1}(\alpha) \geq 1.$$

In view of (2.17),

$$\begin{aligned} \ker \tilde{H}_\ell(\alpha) &= \ker \tilde{D}_S(\alpha)^*|_{L_\ell^2} \oplus \{0\} + \{0\} \oplus \ker \tilde{D}_S(\alpha)|_{L_{\ell-1}^2}, \\ \tilde{D}_S(\alpha)|_{L_{\ell-1}^2} : L_{\ell-1}^2 &\rightarrow L_\ell^2, \quad \tilde{D}_S(\alpha)^*|_{L_\ell^2} = (\tilde{D}_S(\alpha)|_{L_{\ell-1}^2})^* = L_\ell^2 \rightarrow L_{\ell-1}^2. \end{aligned}$$

Since it is true at $\alpha = 0$, the continuity of (simple) eigenfunctions shows that for $|\alpha| \ll 1$,

$$\ker \tilde{H}_1(\alpha) = \{0\} \oplus \ker \tilde{D}_S(\alpha)|_{L_0^2}$$

and

$$\ker \tilde{H}_{-1}(\alpha) = \mathcal{A}_4 \ker \tilde{H}_1(\alpha) = \ker \tilde{D}_S(\alpha)^*|_{L_{-1}^2} \oplus \{0\}.$$

This also shows that

$$\dim \ker \tilde{D}_S(\alpha) = 1, \quad |\alpha| \ll 1,$$

as $\dim \ker \tilde{D}_S(\alpha)$ is upper semicontinuous (see, for instance, [15, Theorem 2.5.3]). Since $\text{rank } \Pi(\alpha) = 2$ (this follows from (2.12)), we have $\tilde{D}_S(\alpha)\Pi(\alpha) \neq 0$ for $|\alpha| \ll 1$, that is,

$$\ker \tilde{D}_S(\alpha) = \tilde{D}_S(\alpha)\Pi(\alpha)L^2.$$

In view of the mapping properties in (2.17), we conclude that $\text{ran } \Pi(\alpha) = \mathbb{C}u(\alpha) \oplus \mathbb{C}v(\alpha)$, where

$$u(\alpha) \in \ker \tilde{D}_S(\alpha) \subset L_0^2(\mathbb{C}/\Lambda; \mathbb{C}^2), \quad v(\alpha) \in L_{-1}^2(\mathbb{C}/\Lambda; \mathbb{C}^2), \quad |\alpha| \ll 1.$$

In particular, if $\pi_\ell: L^2 \rightarrow L_\ell^2$ are the orthogonal projections, we have for $|\alpha| \ll 1$,

$$(\pi_0 + \pi_{-1})\Pi(\alpha) = \Pi(\alpha). \quad (3.10)$$

We now note that $\mathbb{C} \setminus \mathcal{A} \ni \alpha \mapsto \Pi(\alpha)$ is a holomorphic family of projections (this follows from applying $\partial_{\bar{\alpha}}$ to the definition of $\Pi(\alpha)$ in (2.11)). We then see that (3.10) holds for $\alpha \in \mathbb{C} \setminus \mathcal{A}$.

Since $\mathbb{C} \setminus \mathcal{A} \ni \alpha \mapsto \text{tr } \pi_0 \Pi(\alpha) \pi_0$ is holomorphic and hence identically equal to 1, and since $\pi_0 \Pi(\alpha) \pi_0 = \pi_0 \Pi(\alpha)$ (the projections π_ℓ commute with $\Pi(\alpha)$ thanks to (2.13)), we conclude that

$$\dim \pi_0 \Pi(\alpha) L^2 = 1, \quad \alpha \in \mathbb{C} \setminus \mathcal{A}. \quad (3.11)$$

Hence, if (3.5) does not hold, then

$$\ker \tilde{D}_S(\alpha)|_{L_{-1}^2} \neq \{0\},$$

that is, there exists a non-zero $v \in L_{-1}^2(\mathbb{C}/\Lambda)$ such that $P(\alpha, 0)v = 0$ (we then have $[2D_{\bar{z}}v, v]^t \in \ker \tilde{D}_S|_{L_{-1}^2(\mathbb{C}/\Lambda; \mathbb{C}^2)}$). Since

$$D_{\bar{z}}^2: L_{-1}^2(\mathbb{C}/\Lambda) \rightarrow L_1^2(\mathbb{C}/\Lambda)$$

is invertible, and $u \mapsto Vu$ takes $L_{-1}^2 \rightarrow L_1^2$, we have

$$P(\alpha, 0)|_{L_{-1}^2} = (2D_{\bar{z}})^2|_{L_{-1}^2} (I - \alpha^2 K), \quad K := [(2D_{\bar{z}})^2|_{L_{-1}^2}]^{-1} V: L_{-1}^2 \rightarrow L_{-1}^2.$$

The operator K is compact and hence $I - \alpha^2 K$ is invertible outside of a discrete set of α 's. We define the set $\mathcal{B}$ as follows:

$$\alpha \in \mathcal{B} \Leftrightarrow \ker P(\alpha, 0)|_{L_{-1}^2} \neq \{0\} \text{ and } \alpha \notin \mathcal{A}. \quad (3.12)$$

We note that for $\alpha \in \mathcal{B}$, $\dim \ker \tilde{D}_S(\alpha) = 2$. In fact, if there existed three independent solutions $\tilde{D}_S(\alpha)u_j = 0$, for any $z_0 \in \mathbb{C}/\Lambda$ we could find $\gamma \in \mathbb{C}^3 \setminus \{0\}$ such that $\sum_{j=1}^3 \gamma_j u_j(z_0) = 0$ (as $u_j(z_0) \in \mathbb{C}^2$ these are two scalar equations). Since $\sum_{j=1}^3 \gamma_j u_j \neq 0$ (from the independence of u_j 's), the now standard argument (see [26] and [28, §6]) shows that $\alpha \in \mathcal{A}$.

To see (3.6), we modify the proof of [4, Proposition 2.3]. For that, we fix α and define a new operator

$$\tilde{H}(\zeta) := \begin{pmatrix} 0 & \tilde{D}_S(\alpha + \zeta) \\ \tilde{D}_S(\alpha + \bar{\zeta})^* & 0 \end{pmatrix},$$

which has the property that $\tilde{H}(\zeta) = \tilde{H}(\bar{\zeta})^*$. We can restrict this operator to $L_1^2(\mathbb{C}/\Lambda; \mathbb{C}^4)$ and, as we already know that $\ker \tilde{D}_S(\alpha + \zeta)|_{L_0^2}$ is nontrivial, $\dim \ker \tilde{H}_1(\zeta) \geq 1$. Rellich's theorem [16, Chapter VII, Theorem 3.9] then shows that there exists a holomorphic family $\text{neigh}_{\mathbb{C}}(0) \ni \zeta \rightarrow \tilde{u}(\zeta) \in \ker \tilde{H}_1(\zeta) \setminus \{0\}$ (it corresponds to the 0 eigenfunctions of $\tilde{H}_1(\zeta)$). If $\alpha + \zeta \notin \mathcal{A}$, then $\tilde{u}(\zeta) = (0, \tilde{\mathbf{u}}(\zeta))$, $\ker \tilde{D}_S(\alpha + \zeta)|_{L_0^2} = \mathbb{C}\tilde{\mathbf{u}}(\zeta)$. This construction is local near any $\alpha \in \mathbb{C}$, and by partition of unity, we can find a smooth family $\alpha \mapsto \mathbf{u}_1(\alpha) \in \ker \tilde{D}_S(\alpha)|_{L_0^2}$.

To modify this construction to obtain a holomorphic family, we first note that

$$0 = \partial_{\bar{\alpha}}(\tilde{D}_S(\alpha)\mathbf{u}_1(\alpha)) = \tilde{D}_S(\alpha)(\partial_{\bar{\alpha}}\mathbf{u}_1(\alpha)) \Rightarrow \partial_{\bar{\alpha}}\mathbf{u}_1(\alpha) \in \ker \tilde{D}_S(\alpha)|_{L_0^2}.$$

Using the fact that for $\alpha \notin \mathcal{A}$ this kernel is one dimensional, we see that

$$\partial_{\bar{\alpha}}\mathbf{u}_1(\alpha) = f(\alpha)\mathbf{u}_1(\alpha), \quad f(\alpha) := \frac{\langle \partial_{\bar{\alpha}}\mathbf{u}_1(\alpha), \mathbf{u}_1(\alpha) \rangle}{\|\mathbf{u}_1(\alpha)\|^2}, \quad \alpha \notin \mathcal{A}. \quad (3.13)$$

The formula for $f(\alpha)$ defines a smooth function $f \in C^\infty(\mathbb{C})$ and that means that the first formula in (3.13) holds everywhere. The equation $\partial_{\bar{\alpha}}F(\alpha) = f(\alpha)$ (see, for instance, [14, Theorem 4.4.6] applied with $P = \partial_{\bar{\alpha}}$ and $X = \mathbb{C}$) can be solved with $F \in C^\infty(\mathbb{C})$. This shows that $\mathbf{u}(\alpha) := \exp(-F(\alpha))\mathbf{u}_1(\alpha)$ is holomorphic. Since $\mathbf{u}(\alpha) = [2D_{\bar{z}}u(\alpha), u(\alpha)]$, we obtain (3.6). ■

Remark. For $\alpha \in \mathbb{R} \setminus \{0\}$, having a kernel of $D_S(\alpha)|_{L_{-1}^2}$ is a natural possibility for another reason than the one presented before (3.12). It implies that $\ker H_0(\alpha) \neq 0$. Since $\ker H_\ell(0) = 0$, $\ell \neq \pm 1$, the chiral symmetry shows that $\dim \ker H_\ell(\alpha) \equiv 0 \pmod{2}$ and if $\dim \ker H(\alpha) > 2$, then there exists a unique (in view of (3.11)) $\ell \neq \pm 1$ such that $\dim \ker H_\ell(\alpha) = 2$. (For $\alpha \in \mathbb{R} \setminus \mathcal{A}$, we have $\dim \ker H(\alpha) \leq 4$, as otherwise $\mathcal{R}_4 H(\alpha) = H(\alpha)\mathcal{R}_4$ would show that $\dim \ker D_S(\alpha) > 2$, contradicting (2.12).) Since $\mathcal{R}_4: \ker H_\ell(\alpha) \rightarrow H_{-\ell}(\alpha)$, that means that $\ell \equiv -\ell \pmod{6}$, that is, $\ell = 0$ or $\ell = 3$. The case $\ell = 0$ corresponds to a nontrivial kernel of $D_S(\alpha)|_{L_{-1}^2}$ (see (2.17)). The case $\ell = 3$ is excluded by the assumption that $\alpha \notin \mathcal{A}$. In fact, we would then have a nontrivial kernel of $D_S(\alpha)|_{L_2^2}$. But that would mean that we would have v with $D_S(\alpha)v = 0$, $[\bar{\tau}v_1(\tau z), v_2(\tau z)]^t = [\tau^2 v_1(z), \tau^2 v_2(z)]^t$, that is, $v_1(\tau z) = -v_1(z)$ and $v_2(\tau z) = \tau^2 v_2(z)$. In particular, $v(z) = 0$ and the theta function argument (see [26] and [28, §6]) shows that $\alpha \in \mathcal{A}$ (see Proposition 3.3).

3.2. Dirac points for a discrete set of α 's

Suppose that $\alpha \in \mathcal{B}$ defined in (3.12). Then $0 \in \text{Spec } D_S(\alpha)$ has geometric (and algebraic) multiplicity 2 and there exist

$$\mathbf{u} \in \ker D_S(\alpha)|_{L_0^2}, \quad \mathbf{v} \in \ker D_S(\alpha)|_{L_{-1}^2}, \quad \langle \mathbf{u}, \mathbf{v} \rangle = 0, \quad \|\mathbf{u}\| = \|\mathbf{v}\| = 1. \quad (3.14)$$

We consider the Wronskian of $\mathbf{u}$ and $\mathbf{v}$,

$$\text{Wr}(\mathbf{u}, \mathbf{v}) = u_1 v_2 - u_2 v_1, \quad (3.15)$$

and note that it is non-zero. Indeed, note $u_1(0) = 0$ and $v_2(0) = 0$ by their symmetries. If $\text{Wr}(\mathbf{u}, \mathbf{v}) = 0$, then either $\mathbf{u}$ or $\mathbf{v}$ would have a zero at 0 and then $\alpha \in \mathcal{A}$, see Proposition 3.3. We can then set up a Grushin problem for $H(\alpha, 0)$ which remains invertible when we replace $H(\alpha, 0)$ by $H(\alpha, k)$ for small k :

$$\mathcal{H}(\alpha, k) = \begin{pmatrix} H(\alpha, k) & R_- \\ R_+ & 0 \end{pmatrix}: H^1(\mathbb{C}/\Lambda; \mathbb{C}^4) \times \mathbb{C}^4 \rightarrow L^2(\mathbb{C}/\Lambda; \mathbb{C}^4) \times \mathbb{C}^4, \quad (3.16)$$

$$R_+ := R_-^*, \quad R_- u_- = u_{-,1} \begin{pmatrix} 0 \\ \mathbf{u} \end{pmatrix} + u_{-,2} \begin{pmatrix} 0 \\ \mathbf{v} \end{pmatrix} + u_{-,3} \begin{pmatrix} 0 \\ \mathcal{A}_2 \mathbf{u} \end{pmatrix} + u_{-,3} \begin{pmatrix} 0 \\ \mathcal{A}_2 \mathbf{v} \end{pmatrix}.$$

Then

$$\mathcal{H}(\alpha, k)^{-1} = \begin{pmatrix} E(\alpha, k) & E_+(\alpha, k) \\ E_-(\alpha, k) & E_{-+}(\alpha, k) \end{pmatrix}, \quad E_{\pm}(\alpha, 0) = R_{\pm},$$

where (see [25, Proposition 2.12])

$$E_{-+}(\alpha, k) = -R_+ \begin{pmatrix} 0 & k \\ \bar{k} & 0 \end{pmatrix} R_- + \mathcal{O}(|k|^2) = \begin{pmatrix} 0 & \bar{k} A^* \\ k A & 0 \end{pmatrix} + \mathcal{O}(|k|^2),$$

$$A := \frac{\sqrt{3}}{2} W \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad (3.17)$$

where $W = W(\mathbf{u}, \mathbf{v})$. This follows from the following calculations:

$$\langle \mathbf{u}, \mathcal{A}_2 \mathbf{u} \rangle = \int_{\mathbb{C}/\Lambda} u_1 \overline{(-\bar{u}_2)} + u_2 \overline{(\bar{u}_1)} dm(z) = 0, \quad \langle \mathbf{v}, \mathcal{A}_2 \mathbf{v} \rangle = 0,$$

$$\langle \mathbf{u}, \mathcal{A}_2 \mathbf{v} \rangle = \int_{\mathbb{C}/\Lambda} u_1 \overline{(-\bar{v}_2)} + u_2 \overline{(\bar{v}_1)} dm(z) = \frac{\sqrt{3}}{2} \text{Wr}(\mathbf{v}, \mathbf{u}) = -\langle \mathbf{v}, \mathcal{A}_2 \mathbf{u} \rangle,$$

and

$$\begin{aligned} \begin{pmatrix} 0 & k \\ \bar{k} & 0 \end{pmatrix} R_- u_- &= \begin{pmatrix} 0 & k \\ \bar{k} & 0 \end{pmatrix} \left(u_{-,1} \begin{pmatrix} 0 \\ \mathbf{u} \end{pmatrix} + u_{-,2} \begin{pmatrix} 0 \\ \mathbf{v} \end{pmatrix} + u_{-,3} \begin{pmatrix} 0 \\ \mathcal{A}_2 \mathbf{u} \end{pmatrix} + u_{-,3} \begin{pmatrix} 0 \\ \mathcal{A}_2 \mathbf{v} \end{pmatrix} \right) \\ &= u_{-,1} \begin{pmatrix} k \mathbf{u} \\ 0 \end{pmatrix} + u_{-,2} \begin{pmatrix} k \mathbf{v} \\ 0 \end{pmatrix} + u_{-,3} \begin{pmatrix} 0 \\ \bar{k} \mathcal{A}_2 \mathbf{u} \end{pmatrix} + u_{-,3} \begin{pmatrix} 0 \\ \bar{k} \mathcal{A}_2 \mathbf{v} \end{pmatrix}. \end{aligned}$$

To study the eigenvalues near 0 of $H(\alpha, k)$ for $|k| \ll 1$, we observe that the Grushin problem (3.16) is well posed when $H(\alpha, k)$ is replaced by $H(\alpha, k) - \lambda$, $|\lambda| \ll 1$. Using [25, Proposition 2.12], we then see that small eigenvalues of $H(\alpha, k)$ are given by the zeros of $\lambda \rightarrow \det(E_{-+}^\lambda(\alpha, k))$, where, in the notation of (3.17),

$$\begin{aligned} E_{-+}^\lambda(\alpha, k) &= E_{-+}(\alpha, k) - \lambda E_-(\alpha, k) E_+(\alpha, k) + \mathcal{O}(\lambda^2) \\ &= \begin{pmatrix} 0 & \bar{k} A^* \\ k A & 0 \end{pmatrix} - \lambda + \mathcal{O}(\lambda^2 + |k|^2). \end{aligned}$$

The eigenvalues of the first term on the right-hand side are double and given by $\pm(\sqrt{3}/2) \times |k||W|$ and hence (with the labelling convention of (1.5))

$$E_{\pm j}(\alpha, k) = \pm \frac{\sqrt{3}}{2} |\text{Wr}(\mathbf{u}, \mathbf{v})| |k| + \mathcal{O}(|k|^2), \quad \alpha \in \mathcal{B}, \quad j = 1, 2, \quad (3.18)$$

while all the other eigenvalues are non-zero. As in [26], the Wronskian is interpreted as the Fermi velocity, that is, the slope of the Dirac cone. However, it occurs only for a discrete set of α at which the cone is double.

When $\alpha \notin \mathcal{B} \cup \mathcal{A}$, then $\ker D_S(\alpha) = \mathbb{C}\mathbf{u}$, and setting up the Grushin problem using $(0, \mathbf{u})^t$ and $(A_2 \mathbf{u}, 0)^t$, shows that

$$E_{\pm 1}(\alpha, k) = \mathcal{O}(|k|^2), \quad |E_{\pm 2}(\alpha, k)| > 0, \quad \alpha \notin \mathcal{B} \cup \mathcal{A}.$$

3.3. Zeros of $z \mapsto \mathbf{u}(\alpha, z)$

We start with some general considerations and first prove the following fact. It is an adaptation to the scalar model of the central argument of [26].

Proposition 3.3. *In the notation of (3.6), let $\mathbf{u}(\alpha) = [2D_{\bar{z}}u(\alpha), u(\alpha)]^t$. Then*

$$\alpha \in \mathcal{A} \Leftrightarrow \exists z_0 \in \mathbb{C}/\Lambda, \quad \mathbf{u}(z_0) = 0.$$

Proof. The proof of $\Leftarrow$ is the same as the proof for the chiral model in [28, §6], but we present a slightly modified version for completeness. It is based on the properties of the resolvent (see [25, §10.2.3]):

$$R(k) := (2D_{\bar{z}} + k)^{-1}: L^2(\mathbb{C}/\Lambda) \rightarrow L^2(\mathbb{C}/\Lambda), \quad k \notin \Lambda^*,$$

$$R(k)f(z) = \int_{\mathbb{C}/\Gamma} G_k(z-w)f(w) dm(w), \quad (2D_{\bar{z}} + k)G_k(z) = \delta_0(z).$$

The function G_k is periodic and $z-w$ is defined modulo Λ . (We can describe G_k explicitly using theta functions, but we do not need that.) If $\mathbf{u}(\alpha) \in \ker D_S(\alpha)$ is the protected state from Proposition 3.2, then for $k \notin \Lambda^*$,

$$(D_S(\alpha) + k)(G_k(z - z_0)\mathbf{u}(\alpha, z)) = \delta_{z_0}(z)\mathbf{u}(z_0).$$

Hence if $\mathbf{u}$ has a zero at z_0 , we obtain an eigenfunction for every k .

Now suppose that $\alpha \in \mathcal{A}$ and $\mathbf{u}$ does *not* vanish. Since $\text{Spec } D_S(\alpha) = \mathbb{C}$, for every k (or equivalently some $k \notin \Lambda^*$) there exists $\mathbf{v} \neq 0$ such that $(D_S(\alpha) + k)\mathbf{v} = 0$. Then the Wronskian (3.15) is seen to satisfy $2D_{\bar{z}} \text{Wr} = -k \text{Wr}$. The general solution of this equation is given by

$$\text{Wr}(z, \bar{z}) = e^{i(k\bar{z} + \bar{k}z)/2} w(z), \quad w \in \mathcal{O}(\mathbb{C}).$$

Since Wr is periodic and $z \mapsto e^{i(k\bar{z} + \bar{k}z)/2}$ is a bounded function, w has to be constant, and for $k \notin \Lambda^*$ that constant has to vanish.

The vanishing of $\mathbf{W}r$ means that $\mathbf{u}$ and $\mathbf{v}$ are parallel and hence, since $\mathbf{u}$ is assumed not to vanish, there exists $F \in C^\infty(\Omega)$ such that $\mathbf{v} = F\mathbf{u}$. Applying $2D_{\bar{z}}$ to both sides shows that $2D_{\bar{z}}F = -kF$. Hence for some $f \in C^\infty(\mathbb{C})$,

$$\mathbf{v}(z, \bar{z}) = e^{-i(k\bar{z} + \bar{k}z)/2} f(z) \mathbf{u}(z, \bar{z}), \quad \partial_{\bar{z}} f = 0. \quad (3.19)$$

Since $\mathbf{v}$ and $\mathbf{u}$ are periodic, and $\mathbb{C} \ni z \mapsto e^{-i(k\bar{z} + \bar{k}z)/2}$ is bounded (but not periodic for $k \notin \Lambda^*$), we conclude that f is bounded and hence constant. But then $\mathbf{v}$ is not periodic, which provides a contradiction. ■

We have the following refinement of Lemma 3.1. To formulate it, we define the notion of multiplicity.

Definition 3.4. For a solution $\mathbf{u} \in \ker D_S(\alpha)$, we define the multiplicity of a zero at z_0 as

$$M_{\mathbf{u}}(z_0) := \max\{m : [\partial_z^{m-1} \mathbf{u}](z_0) = 0\}, \quad (3.20)$$

with the convention that $M_{\mathbf{u}}(z_0) = 0$ if $\mathbf{u}(z_0) \neq 0$.

Since $\mathbf{u} = [2D_{\bar{z}}u, u]^t$, $P(\alpha)u = 0$, and $m = M_{\mathbf{u}}(z_0)$ is equivalent to

$$u(z) = (z - z_0)^m v(z), \quad v(z_0) \neq 0, \quad v \in C^\infty(\text{neigh}_{\mathbb{C}}(z_0)).$$

We note here that $\mathbf{u}(z_0) = 0$ and $D_S(\alpha)\mathbf{u}(z) = 0$ imply that $\partial_{\bar{z}}^k \mathbf{u}(z_0) = 0$ for all k . That means $\partial_{\bar{z}}^k u(z_0) = 0$ for all k and hence we can factor out a positive power of $(z - z_0)$. For an example of $\mathbf{u}(\alpha)$ with $m(\alpha) = 1$, see Figure 5.

Proposition 3.5. In the notation of (3.6), let $\mathbf{u}(\alpha) = [2D_{\bar{z}}u(\alpha), u(\alpha)]^t$. For $\alpha \in \mathcal{A}$,

$$\dim \ker D_S(\alpha) = \sum_{z \in \mathbb{C}/\Lambda} M_{\mathbf{u}(\alpha)}(z) =: m(\alpha). \quad (3.21)$$

This means that for all $k \in \mathbb{C}/\Lambda^*$,

$$\dim \ker H(\alpha, k) = 2m(\alpha),$$

which is the number of flat bands at 0.

Proof. We prove that for $\alpha \in \mathcal{A}$, $\dim \ker D_S(\alpha) \geq m(\alpha)$. For that, suppose first that $\mathbf{u}(\alpha)$ has distinct zeros $z_0, \dots, z_{m-1}$, each of multiplicity one. We can assume that $m \geq 2$, as otherwise there is nothing to prove. We then choose

$$w_j \notin \{z_j\}_{j=0}^{m-1}, \quad \sum_{j=0}^{m-1} w_j \equiv \sum_{j=0}^{m-1} z_j,$$

and put

$$\mathbf{u}_j(z, \bar{z}) = \frac{\theta(z - w_0)\theta(z - w_j)}{\theta(z - z_0)\theta(z - z_j)} \mathbf{u}(z, \bar{z}).$$

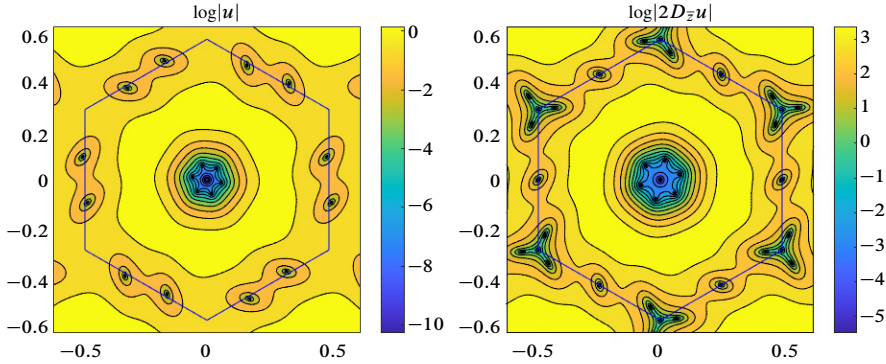


Figure 5. When multiplicity of the band is one, the only possible zero of $\mathbf{u}$ is at 0 and that is illustrated in this figure. For potential (2.2) in the scalar model with $V(z) = U_{\text{BM}}(z)U_{\text{BM}}(-z)$, this occurs only for complex α 's.

The factor in front of $\mathbf{u}$ is a meromorphic function with simple poles at z_0 and z_j (see [4, §3.1] for the review of properties of θ in a related context), but the properties of the zeros show that $\mathbf{u}_j \in \ker D_S(\alpha)$. The functions $\mathbf{u}_j$ are linearly independent, as

$$c_0 + \sum_{j=1}^{n-1} c_j \frac{\theta(z-w_0)\theta(z-w_j)}{\theta(z-z_0)\theta(z-z_j)} \equiv 0 \Rightarrow c_0 = c_1 = \cdots = c_{m-1} = 0.$$

(Put $z = z_j$, $j > 0$, to see that $c_j = 0$ for $j > 0$.) The case of higher multiplicities is dealt with similarly but using meromorphic functions with poles of higher order. This proves that $\dim \ker D_S(\alpha) \geq m(\alpha)$.

To prove the opposite inequality, suppose that $\mathbf{u}, \mathbf{v}_j$, $1 \leq j \leq M-1$, span $\ker D_S(\alpha)$, $M = \dim \ker D_S(\alpha)$. Then Wronskians (3.15), $\text{Wr}(\mathbf{u}, \mathbf{v}_j) = 0$, as $\mathbf{u}$ has to vanish somewhere by Proposition 3.3 and the Wronskian is constant. But as in (3.19),

$$\mathbf{v}_j(z, \bar{z}) = f_j(z)\mathbf{u}(z, \bar{z}), \quad f_j(z), 1 \leq j \leq m-1, \text{ meromorphic}, \quad (3.22)$$

$f_0 \equiv 1$ and $\{f_j\}_{j=0}^{M-1}$ are linearly independent. The span of f_j 's is contained in the space of meromorphic functions² $L(D)$, where D is the divisor defined by the zeros of $\mathbf{u}$ included according to their multiplicities defined in (3.20). Since $\dim L(D) = \deg D = m(\alpha)$ (see (3.21)), this means that the number of zeros of $\mathbf{u}$ has to be greater than or equal to M , the dimension of $\ker D_S(\alpha)$. ■

²For an accessible introduction, see Terence Tao's blog <https://terrytao.wordpress.com/2018/03/28/246c-notes-1-meromorphic-functions-on-riemann-surfaces-and-the-riemann-roch-theorem/> (visited on 31 August 2026) and the version of the Riemann–Roch theorem needed here is in formula (7) there.

Remarks. (1) The same argument applies in the chiral model and settles [28, Problem 14], see [5, Theorem 1].

(2) Although we invoked a basic version of the Riemann–Roch theorem, the proof that the number of poles of f_j 's has to be greater than M is explicit. If the poles are all simple, then

$$f_j(z) = \sum_{k=1}^{K(j)} \lambda_k(j) \frac{\theta'(z - a_k(j))}{\theta(z - a_k(j))} + c(j), \quad \sum_{k=1}^{K(j)} \lambda_k(j) = 0.$$

For $f_0 \equiv 1$, $f_1, \dots, f_{M-1}$ to be linearly independent, we need the number of distinct zeros $\{a_k(j) : 1 \leq k \leq K(j), 1 \leq j \leq M-1\}$ to be $\geq M$. It is not difficult to modify this construction to the case of poles of higher order.

3.4. Multiplicity two

Suppose that $\dim \ker D_S(\alpha) = 2$ and $\alpha \in \mathcal{A}$. We remark that numerical analysis of the Bistritzer–MacDonald potential (2.3) (that is, $V(z) = U(z)U(-z)$ with that U) suggest that this is the case for $\mathcal{A} \cap \mathbb{R}$.

The assumption $\dim \ker D_S(\alpha) = 2$ means that $\dim \ker H(\alpha, 0) = 4$ and consequently $\dim \ker H(\alpha, k) = 4$ for all k . Since $\dim \ker H_\ell(0, 0) \equiv \dim \ker H_\ell(\alpha, 0) \pmod{2}$, in view of (3.9), there exists $\ell \neq \pm 1$ such that $\dim \ker H_\ell(\alpha, 0) = 2$. But then (2.16) and (2.17) imply that $\ell \equiv -\ell \pmod{6}$, that is, $\ell = 0, 3$.

The kernel of $D_S(\alpha)$ is thus spanned by $\mathbf{u} \in L_0^2(\mathbb{C}/\Lambda; \mathbb{C}^2)$ and $\mathbf{v} \in L_\ell^2(\mathbb{C}/\Lambda; \mathbb{C}^2)$, ℓ is either -1 (the case of $\dim \ker H_0(\alpha, 0) = 2$) or 2 (the case of $\dim \ker H_3(\alpha, 0) = 2$). Note $\text{Wr}(\mathbf{u}, \mathbf{v}) = 0$ as otherwise we would have (3.18) and no flat bands. We note that $u_1(\pm z_S) = u_1(0) = 0$, and $v_2(\pm z_S) = v_2(0) = 0$ (for both $\ell = -1$ and $\ell = 2$). Hence

$$u_2(\pm z_S)v_1(\pm z_S) = u_2(0)v_1(0) = 0.$$

Therefore, we have two exclusive cases ($\mathbf{u}$ cannot vanish at three points as then Proposition 3.5 would contradict the multiplicity two assumptions):

$$\mathbf{u}(0) = 0 \quad \text{or} \quad \mathbf{u}(\pm z_S) = 0. \quad (3.23)$$

Suppose first that $\mathbf{u}(0) = 0$ and $\mathbf{u}(\pm z_S) \neq 0$. Then $\mathbf{v}(\pm z_S) = 0$, and we can use the following well-known assertion.

Lemma 3.6. *The unique (up to a multiplicative constant) meromorphic function with two poles at $\pm z_S$ on $\mathbb{C}/\Lambda$ and a (double) zero at 0 satisfies*

$$F(\tau z) = \tau^2 F(z), \quad \tau = e^{-\pi i/3}. \quad (3.24)$$

Proof. We first note that F is unique, as for any periodic meromorphic function, the sum of the poles is congruent to the sum of the zeros and the sum of residues adds up to 0.

We can then give an explicit formula for F :

$$\begin{aligned} F(z) &= \sum_{\gamma \in \Lambda} \frac{1}{z + \gamma - z_S} - \frac{1}{z + \gamma + z_S} - \frac{1}{\gamma - z_S} + \frac{1}{\gamma + z_S} \\ &= -2z_S \sum_{\gamma \in \Lambda} \frac{z^2 + 2z\gamma}{(z + \gamma + z_S)(z + \gamma - z_S)(\gamma - z_S)(\gamma + z_S)}. \end{aligned}$$

This shows absolute convergence and $F(z) = F(-z)$. Hence to obtain (3.24), it suffices to show that $F(\omega z) = \bar{\omega} F(z)$. In fact, $\omega z_S = z_S + \gamma_0$, $\gamma_0 \in \Lambda$, and

$$\begin{aligned} F(\omega z) &= \bar{\omega} \sum_{\gamma \in \Lambda} \frac{1}{z + \gamma - \gamma_0 - z_S} - \frac{1}{z + \gamma + \gamma_0 + z_S} - \frac{1}{\gamma - \gamma_0 - z_S} + \frac{1}{\gamma + \gamma_0 + z_S} \\ &= \bar{\omega} F(z) + \sum_{\gamma \in \Lambda} \frac{1}{z + \gamma + z_S} - \frac{1}{z + \gamma + 2\gamma_0 + z_S} - \frac{1}{\gamma + z_S} + \frac{1}{\gamma + 2\gamma_0 + z_S}. \end{aligned}$$

The sum on the right-hand side vanishes, as it converges uniformly and defines a periodic holomorphic function vanishing at 0. This completes the proof. ■

Remark. The function F can be expressed using theta functions (see [4, §3.1] for the definitions in this context)

$$F(z) = \frac{\theta(z)^2}{\theta(z - z_S)\theta(z + z_S)},$$

and the conclusion follows from [4, (3.1)] and the fact that $\bar{\omega} z_S = z_S + \gamma_0$, $\gamma_0 \in \Lambda$.

Since $F(z)\mathbf{v}$ is not a scalar multiple of $\mathbf{v}$, we have $\mathbf{u} = c_0 F(z)\mathbf{v} \in L_{\ell+2}^2$. But for $\ell = -1, 2$, we have $\ell + 2 \not\equiv 0 \pmod{6}$, which is a contradiction. Hence only the second option in (3.23) is allowed. But then (3.24) implies that $\mathbf{v} = c_0 F(z)\mathbf{u} \in L_2^2$. We summarise this with the following assertion.

Proposition 3.7. *The multiplicity of $\alpha \in \mathcal{A}$ is 2 (that is, $\dim \ker H(\alpha, k) = 4$ for all k) if and only if $\mathbf{u}(\alpha) = [2D_{\bar{z}}u(\alpha), u(\alpha)]^t$ in (3.6) vanishes at $\pm z_S$. In that case,*

$$\ker D_S(\alpha) = \mathbb{C}\mathbf{u} \oplus \mathbb{C}\mathbf{v}, \quad \mathbf{u} \in L_0^2(\mathbb{C}/\Lambda; \mathbb{C}^2), \quad \mathbf{v} \in L_2^2(\mathbb{C}/\Lambda; \mathbb{C}^2).$$

Remark. We observe that in the case of $m(\alpha) = 2$, we can write

$$\ker D_S(\alpha) = \left\{ c \frac{\theta(z - z_1)\theta(z + z_1)}{\theta(z - z_S)\theta(z + z_S)} \mathbf{u}(\alpha), z_1 \in \mathbb{C}/\Lambda, c \in \mathbb{C} \right\}.$$

The Weierstrass identity [17, (3.3)] gives

$$\theta(z - z_S)\theta(z + z_S)\theta(z_1)^2 + \theta(z_S - z_1)\theta(z_S + z_1)\theta(z)^2 = \theta(z_S)^2\theta(z - z_S)\theta(z + z_S)$$

which means that, with $\mathbf{v} = F\mathbf{u}$ ($F(z)$ defined in Lemma 3.6),

$$\frac{\theta(z - z_1)\theta(z + z_1)}{\theta(z - z_S)\theta(z + z_S)} \mathbf{u} = \frac{\theta(z_1)^2}{\theta(z_S)^2} \mathbf{u} + \frac{\theta(z_S - z_1)\theta(z_S + z_1)}{\theta(z_S)^2} \mathbf{v}.$$

3.5. Local structure of solutions

We start with some general comments. We consider the following *non-holomorphic* embedding of $\mathbb{R}^2 \simeq \mathbb{C}$ in $\mathbb{C}^2$:

$$F: \mathbb{C} \rightarrow \mathbb{C}^2, \quad F: z \mapsto (z, \bar{z}) \in \mathbb{C}^2.$$

The image of F , $X := \{(z, \bar{z}) : z \in \mathbb{C}\}$, is a totally real linear subspace (over $\mathbb{R}$) of $\mathbb{C}^2$ in the sense that $X \cap iX = \{0\}$. In particular (we use the notation $\text{neigh}_\Omega(\Omega_0)$ for an open neighbourhood of Ω_0 in Ω , $\mathcal{O}(\Omega)$ denotes holomorphic functions in Ω , $\zeta = (\zeta_1, \zeta_2 \in \mathbb{C}^2)$,

$$\begin{aligned} u \in \mathcal{O}(\text{neigh}_{\mathbb{C}^2}(X)) &\Rightarrow \begin{cases} \partial_{\zeta_1} u(\zeta)|_{\zeta=(z, \bar{z})} = \partial_z [u(z, \bar{z})], \\ \partial_{\zeta_2} u(\zeta)|_{\zeta=(z, \bar{z})} = \partial_{\bar{z}} [u(z, \bar{z})], \end{cases} \\ u \in \mathcal{O}(\text{neigh}_{\mathbb{C}^2}(X)), \quad u|_X = 0 &\Rightarrow u \equiv 0. \end{aligned} \quad (3.25)$$

With this in mind, we consider the following problem (we drop h or $\alpha = 1/h$ which can be put into V):

$$2D_w \mathbf{v}_j(z, w) = \begin{pmatrix} 0 & \alpha V(z, w) \\ \alpha & 0 \end{pmatrix} \mathbf{v}_j, \quad \mathbf{v}_j(z, \bar{z}_0) = \mathbf{e}_j, \quad (3.26)$$

where $\mathbf{e}_1 = [1, 0]^t$, $\mathbf{e}_2 = [0, 1]^t$, for z near z_0 and w near $\bar{z}_0$.

Lemma 3.8. *System (3.26) has unique solutions $\mathbf{v}_j \in \mathcal{O}(\text{neigh}_{\mathbb{C}^2}(z_0, \bar{z}_0); \mathbb{C}^2)$.*

Proof. For a fixed z , the standard theory shows existence of unique solutions which are holomorphic in w . (If $w \mapsto V(w, z)$ is entire, we have a global solution in w , see [23, §7.2].) We now see that $\partial_{\bar{z}} \mathbf{u}_j$ solves equation (3.26) but with the initial condition $\partial_{\bar{z}} \mathbf{u}_j(z, \bar{z}_0) = 0$. But that means that $w \mapsto \partial_{\bar{z}} \mathbf{u}_j(w, z) = 0$. ■

Lemma 3.9. *There exists $\mathbf{u}_j \in \mathcal{O}(\text{neigh}_{\mathbb{C}^2}(z_0, \bar{z}_0))$ solving*

$$2D_w \mathbf{u}_j(z, w) = \begin{pmatrix} 0 & \alpha V(z, w) \\ \alpha & 0 \end{pmatrix} \mathbf{u}_j,$$

such that any solution $\mathbf{u} \in C^\omega(\text{neigh}_{\mathbb{C}}(z_0))$ solving $D_S(\alpha)\mathbf{u} = 0$ (see (1.2)) is given by

$$\mathbf{u}(z, \bar{z}) = c_1(z) \mathbf{u}_1(z, \bar{z}) + c_2(z) \mathbf{u}_2(z, \bar{z}), \quad c_j \in \mathcal{O}(\text{neigh}_{\mathbb{C}}(z_0)). \quad (3.27)$$

Remark. The assumption of analyticity of V , made throughout this paper, is not necessary here. Using [18, Proposition 19.1] applied in the simplest case of $m = 1$ gives the lemma for $V \in C^\infty(\mathbb{C})$ without the necessity of complexifying $\mathbb{C}$ to $\mathbb{C}^2$. Consequently, the conclusion of this section works for a more general class of potentials. We opt for a more elementary approach under the stronger assumption of real analyticity.

Proof of Lemma 3.9. We have many different choices for $\mathbf{u}_j$, and one is given in Lemma 3.8. We note that $\mathbf{v}_j(z_0, \bar{z}_0) = \mathbf{e}_j$ and hence $\mathbf{v}_j(z, \bar{z})$ are linearly independent for $z \in \text{neigh}_{\mathbb{C}}(z_0)$. If $D_S \mathbf{u} = 0$ near z_0 , then u is real analytic in $\text{neigh}_{\mathbb{C}}(z_0) \simeq \text{neigh}_X(z_0, \bar{z}_0)$ (by the standard ellipticity result for operators with analytic coefficients). Hence it extends to $\tilde{\mathbf{u}} \in \text{neigh}_{\mathbb{C}^2}(z_0, \bar{z}_0)$ and (as in (3.25))

$$2D_w \tilde{\mathbf{u}}(z, w) = \begin{pmatrix} 0 & \alpha V(z, w) \\ \alpha & 0 \end{pmatrix} \tilde{\mathbf{u}}(z, w).$$

Hence for $(z, w) \in \text{neigh}_{\mathbb{C}^2}(z_0, \bar{z}_0)$,

$$\tilde{\mathbf{u}}(z, w) = c_1(z) \mathbf{u}_1(z, w) + c_2(z) \mathbf{u}_2(z, w),$$

and, as $\tilde{\mathbf{u}}(z, w)$ is holomorphic in both variables, the coefficients $z \mapsto c_j$ are also holomorphic. Restricting to X gives (3.27). ■

3.6. Rank-2 holomorphic vector bundle

The local structure of solutions shows the existence of the following rank-2 topologically trivial holomorphic vector bundle, $\mathcal{E}$, over $\mathbb{C}/\Lambda$. We have

$$\mathcal{E} = \mathbb{C}/\Lambda \times \mathbb{C}^2, \quad \pi: \mathcal{E} \mapsto \mathbb{C}/\Lambda.$$

To define the following covering of $\mathbb{C}/\Lambda$: for $z_0 \in \mathbb{C}^2$, we choose $U = \text{neigh}_{\mathbb{C}/\Lambda}(z_0)$ such that (3.27) holds with some linearly independent (at each $z \in U$), $\mathbf{u}_j$, $j = 1, 2$. We then define

$$\begin{aligned} g_U &= g_{U, \mathbf{u}_1, \mathbf{u}_2}: U \times \mathbb{C}^2 \rightarrow \pi^{-1}(U), \\ g_U(z, \zeta) &\mapsto (z, \zeta_1 \mathbf{u}_1(z) + \zeta_2 \mathbf{u}_2(z)) \in \mathbb{C}/\Lambda \times \mathbb{C}^2, \end{aligned}$$

noting that, as $\mathbf{u}_j(z)$ are independent, this map is an isomorphism $\mathbb{C}^2 \rightarrow \pi^{-1}(z)$. If V is another neighbourhood with a local basis of the kernel of D_S over V given by $\mathbf{v}_1, \mathbf{v}_2$, then Lemma 3.9 shows that

$$\mathbf{v}_j(z, \bar{z}) = c_{1j}(z) \mathbf{u}_1(z, \bar{z}) + c_{2j}(z) \mathbf{u}_2(z, \bar{z}), \quad c_{kj} \in \mathcal{O}(U \cap V).$$

Hence for $z \in U \cap V$,

$$g_U^{-1} \circ g_V|_{\{z\} \times \mathbb{C}^2}(\zeta) = C(z)\zeta, \quad C(z) := \begin{pmatrix} c_{11}(z) & c_{12}(z) \\ c_{21}(z) & c_{22}(z) \end{pmatrix},$$

is holomorphic.

If we now include α , we obtain a family of holomorphic vector bundles $\mathcal{E}(\alpha)$: their local sections correspond to local solutions to $D_S(\alpha) \mathbf{u} = 0$.

We can describe $\mathcal{E}(\alpha)$ at different values of α and see how it fits in the classification of holomorphic rank-2 vector bundles over tori (see [8, §4] for a gentle introduction to that theory).

We start with the following observation:

$$\text{The determinant line bundle, } \wedge^2 \mathcal{E}(\alpha), \text{ is trivial.} \quad (3.28)$$

Proof of (3.28). In our choice of $\mathbf{u}_j(z)$, defined in U , we can always arrange so that in U , $\text{Wr}(\mathbf{u}_1, \mathbf{u}_2) := \mathbf{u}_1 \wedge \mathbf{u}_2 \equiv 1$. In fact, since $D_{\bar{z}} \text{Wr}(\mathbf{u}_1, \mathbf{u}_2) = 0$, the Wronskian is holomorphic and nonvanishing, so we can replace $\mathbf{u}_1$ by $\mathbf{u}_1 / \text{Wr}(\mathbf{u}_1, \mathbf{u}_2)$. The transition matrix is given by

$$C(z) = \begin{pmatrix} \mathbf{v}_1 \wedge \mathbf{u}_2 & \mathbf{v}_2 \wedge \mathbf{u}_2 \\ -\mathbf{v}_1 \wedge \mathbf{u}_1 & -\mathbf{v}_2 \wedge \mathbf{u}_1 \end{pmatrix}, \quad z \in U \cap V.$$

But the transition factor for $\wedge^2 \mathcal{E}(\alpha)$ is then

$$\det C(z) = \det \begin{pmatrix} \mathbf{v}_1 \wedge \mathbf{u}_1 & \mathbf{v}_1 \wedge \mathbf{u}_2 \\ \mathbf{v}_2 \wedge \mathbf{u}_1 & \mathbf{v}_2 \wedge \mathbf{u}_2 \end{pmatrix} = (\mathbf{v}_1 \wedge \mathbf{v}_2)(\mathbf{u}_1 \wedge \mathbf{u}_2) \equiv 1.$$

This means that constants are global sections. ■

Case 1: $\alpha \notin \mathcal{B} \cup \mathcal{A}$. In this case, there exists a unique global section satisfying $D_S(\alpha)\mathbf{u} = 0$, $\mathbf{u} \in C^\omega(\mathbb{C}/\Lambda)$. We note that $\mathbf{u}$ is a holomorphic section of $\mathcal{E}(\alpha)$, but obviously it is *not* a holomorphic function on $\mathbb{C}/\Lambda$. Since $\alpha \notin \mathcal{A}$, $\mathbf{u}$ does not vanish and hence it defines a trivial line subbundle, $L(\alpha)$ of $\mathcal{E}(\alpha)$. The quotient line bundle $\mathcal{E}(\alpha)/L(\alpha)$ has an explicit representation based on the proof of Proposition 3.2. We showed there that for $\alpha \notin \mathcal{B} \cup \mathcal{A}$, there exists $\mathbf{v}$ such that $D_S(\alpha)\mathbf{v} = \mathbf{u}$. We also remark that $\mathbf{v}(z)$ is independent of $\mathbf{u}(z)$ at every point. In fact, in the notation of (3.15),

$$\begin{aligned} 2D_{\bar{z}}W(\mathbf{u}, \mathbf{v}) &= 2D_{\bar{z}}u_1v_2 + 2D_{\bar{z}}v_2u_1 - 2D_{\bar{z}}u_2v_1 - 2D_{\bar{z}}v_1u_2 \\ &= \alpha V u_2 v_2 + (v_1 + u_2)u_1 - u_1v_1 - (\alpha V v_2 + u_1)u_2 = 0. \end{aligned}$$

Hence dependence at some point would imply that $W(\mathbf{u}, \mathbf{v}) \equiv 0$. Arguing as in (3.19), $\mathbf{u}(z, \bar{z}) = f(z)\mathbf{v}(z, \bar{z})$ for some meromorphic function f . But then $\mathbf{u}$ would have zeros, which contradicts the assumption $\alpha \notin \mathcal{A}$ and Proposition 3.3.

We now define the following element of $C^\infty(\mathbb{C}; \mathbb{C}^2)$:

$$\tilde{\mathbf{u}}(z) := \frac{i}{2} \bar{z} \mathbf{u} - \mathbf{v}$$

which satisfies

$$D_S(\alpha)\tilde{\mathbf{u}} = 0, \quad \tilde{\mathbf{u}}(z + \gamma) = \tilde{\mathbf{u}}(z) + \frac{i}{2} \bar{\gamma} \mathbf{u}(z). \quad (3.29)$$

Since $\mathbf{u}(z)$ and $\mathbf{v}(z)$ are independent, $\mathbf{u}$ and $\tilde{\mathbf{u}}$ can be used as a basis, near z_0 and over $\mathcal{O}(\text{neigh}_{\mathbb{C}/\Lambda}(z_0))$, of local solutions to $D_S(\alpha)\mathbf{w} = 0$.

From (3.29), we see that $\tilde{\mathbf{u}}$ defines a nonvanishing section of $\mathcal{E}(\alpha)/L(\alpha)$. To see that $\mathcal{E}(\alpha)$ is indecomposable, suppose that $\mathcal{E}(\alpha) = L \oplus M$. Since $\mathcal{E}(\alpha)$ and L are trivial, so is M which consequently has a global nonvanishing section, $\mathbf{w}$. Then $[\mathbf{w}] = c[\tilde{\mathbf{u}}]$ in $\mathcal{E}(\alpha)/L$, and we can assume that $c = 1$. This means that

$$\mathbf{w}(z, \bar{z}) = \tilde{\mathbf{u}}(z, \bar{z}) + F(z, \bar{z})\mathbf{u}(z, \bar{z}).$$

Applying the equation, we see that $D_{\bar{z}}F = 0$, while (3.29) gives $F(z + \gamma) = F(z) - i\bar{\gamma}/2$. Consequently, $f(z, \bar{z}) := F(z) + i\bar{z}/2$ is periodic and solves $\partial_{\bar{z}}f = i/2$. But that is impossible as we see by integrating both sides over $\mathbb{C}/\Lambda$.

Case 2: $\alpha \in \mathcal{B}$. This is the easiest case, as the proof of Proposition 3.2 shows that $\ker D_S(\alpha)$ is spanned by two independent solutions $\mathbf{u} \in L_0^2(\mathbb{C}/\Lambda; \mathbb{C})$ and $\mathbf{v} \in L_{-1}^2(\mathbb{C}/\Lambda; \mathbb{C})$ (see (3.14)). In particular, they never vanish and define two trivial line bundles, $L_0(\alpha)$ and $L_{-1}(\alpha)$ and

$$E(\alpha) = L_0(\alpha) \oplus L_{-1}(\alpha).$$

Case 3: $\alpha \in \mathcal{A}$. In view of Proposition 3.5, the dimension of the space of holomorphic sections of $\mathcal{E}(\alpha)$ is $m(\alpha)$, which is also the number of zeros of each section. From (3.22), we see that at each point z , $\{\mathbf{w}(z) : \mathbf{w} \in \ker D_S(\alpha)\} \simeq \mathbb{C}$, and hence $\ker D_S(\alpha)$ defines a line bundle, $L(\alpha)$ over $\mathbb{C}/\Lambda$. The proof of Proposition 3.5 shows that this is the line bundle associated with the divisor given by the zeros of any fixed element of $\ker D_S(\alpha)$.

In the notation of [8, Definition 2.18], $\mathcal{E}$ (we drop α here) is decomposable if $\text{Ext}^1(\mathcal{E}/L, L) = \{0\}$. Since $\mathcal{E}$ is trivial, the degree of $\mathcal{E}/L$ is equal to $-m$, where $m > 0$ is the degree of L . We have, by Serre's duality ([8, Proposition 2.21], with trivial K_X for tori), $\text{Ext}^1(\mathcal{E}/L, L) \simeq H^1((\mathcal{E}/L)^* \otimes L) \simeq H^0((\mathcal{E}/L) \otimes L^*)^*$. But the last space is trivial, as the degree of the line bundle $(\mathcal{E}/L) \otimes L^*$ is $-2m < 0$. Hence, indeed, $\mathcal{E}$ is decomposable.

As noted in (3.28), the determinant bundle of $\mathcal{E}(\alpha)$ is trivial, and if $\mathcal{E}(\alpha) = L(\alpha) \oplus M(\alpha)$, then $M(\alpha) = \det \mathcal{E}(\alpha) \otimes L(\alpha)^* = L(\alpha)^*$. Hence we can identify $\mathcal{E}(\alpha)$ with $L(\alpha) \oplus L(\alpha)^*$.

4. WKB structure of protected states

Numerical computations indicate that the family of protected states, $u(\alpha)$, (see Proposition 3.2) has a structure of a WKB state, see Figure 6 for the case of the Bistritzer–MacDonald potential (2.3). It was shown in [13] that the (L^2 -normalised) protected state (in the scalar and chiral models) for that potential is exponentially small, that is, of size $\exp(-c_0\alpha)$ in a neighbourhood of the hexagon. Numerically, we see that outside of any neighbourhood of the 6 maxima of $|u(\alpha)|$, we have exponential decay of the solution. Mathematically, the key property behind the exponential decay near the hexagon is the vanishing of the Poisson bracket $\{q, \bar{q}\}|_{q^{-1}(0)}$, $q := (2\bar{\zeta})^2 - V(z)$ on the hexagon and its nonvanishing outside of the hexagon (except at the centre). We refer to [13] for a detailed discussion of this.

The structure changes for other potentials, and an interesting case is provided by

$$U(z) = i\overline{U_{BM}(z)}^2, \quad (4.1)$$

which satisfies conditions (2.2) but for which $\{q, \bar{q}\}_{q^{-1}(0)} \equiv 0$. That implies an existence of a global solution to the eikonal equation and suggests different WKB structure.

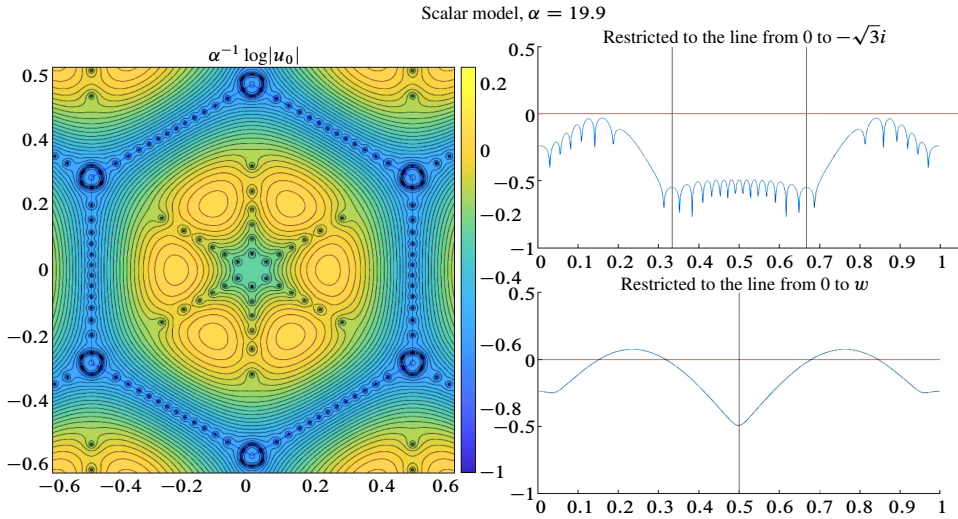


Figure 6. On the left, the contour plot of the map $z \mapsto \alpha^{-1} \log|u(\alpha, z)|$, for $\alpha = 20$, where $((2D_{\bar{z}})^2 - \alpha^2 U(z)U(-z))u(\alpha, z) = 0$ and U is the Bistritzer–MacDonald potential (2.3). We see the structure of zeros and the regions of exponential decay predicted in [13]. On the right, the plot of $z \mapsto \alpha^{-1} \log|u(\alpha, z)|$ over the interval connects the centre to the edge of the hexagon, continues through the edge and then to the translated centre (top) and over the interval connecting the centres and perpendicular to the edge of the hexagon. For an animated version with varying α , see https://math.berkeley.edu/~zworski/scalar_protected.mp4 (visited on 31 August 2026).

In this section, we will first discuss the zero creation at the corners of the hexagon for the Bistritzer–MacDonald potential. We will then present a heuristic derivation of a WKB state for the model based on (4.1) and an argument towards the proof of Theorem 1.2.

4.1. Mechanism for the creation of zeros for the Bistritzer–MacDonald potential

For the scalar model, numerical calculation show that the real elements of $\mathcal{A}$ have multiplicity 2. In view of the arguments presented in Section 3.4, this means that the protected state $\mathbf{v}(\alpha)$, $D_S(\alpha)\mathbf{v}(\alpha) = 0$ has simple zeros (in the sense of (3.20)) at the corners of the hexagon. Using (1.2), we see that

$$\mathbf{v}(\alpha, z) = \begin{pmatrix} \alpha^{-1} 2D_{\bar{z}}u(\alpha, z) \\ u(\alpha, z) \end{pmatrix}, \quad ((2D_{\bar{z}})^2 - \alpha^2 V(z))u(\alpha, z) = 0.$$

The structure of $u(\alpha, z)$ was illustrated in Figure 6.

In this section, we will describe an approximate local mechanism for the creation of zeros of u and $\partial_{\bar{z}}u$ at the corner. The symmetries of u will play a crucial role here.

We start by describing $V(z) = U_{\text{BM}}(z)U_{\text{BM}}(-z)$, where U_{BM} is given in (2.3) near the corner of the hexagon z_S :

$$V(z_S + w) = -ia\bar{w} + bw^2 + \mathcal{O}(|w|^3), \quad a = \frac{32}{3}\pi^3, \quad b = \frac{32}{9}\pi^4 + \mathcal{O}(|w|^3). \quad (4.2)$$

The model operator near the corner (writing $z = w$) is then given by

$$P = (2D_{\bar{z}})^2 + \alpha^2(ia\bar{z} - bz^2). \quad (4.3)$$

To represent the solutions using Airy functions, we write $z = -i\beta w$, $\bar{z} = i\beta\bar{w}$, $D_{\bar{z}} = -\beta^{-1}\partial_{\bar{w}}$, so that if $\beta = (2/\alpha)^{2/3}a^{-1/3}$, then

$$\begin{aligned} P &= \left(\frac{2h}{\alpha}\right)^2 \partial_{\bar{w}}^2 - a\alpha\bar{w} + b\alpha^2 w^2 = 4\left(\frac{\alpha}{2}\right)^{4/3} a^{1/3} (\partial_{\bar{w}}^2 - \bar{w} + c\alpha^{-2/3}w^2), \\ c &:= \frac{2^{2/3}b}{a^{4/3}} = \frac{2}{3}\left(\frac{16}{3}\right)^{1/3} > 0. \end{aligned}$$

We keep in mind that

$$w = i\beta(\alpha)z, \quad \beta(\alpha) := \left(\frac{2}{\alpha}\right)^{2/3} a^{-1/3}. \quad (4.4)$$

Let u denote the solution in the original variables of (1.1). We want to deduce the symmetries of (local) solutions to $Pv = 0$ pretending that the operators agree. In view of (3.8) and the fact that $\omega z_S \equiv z_S \pmod{\Lambda}$, $\bar{z}_S = -z_S \pmod{\Lambda}$, we have

$$\begin{aligned} v(w) &= u(z_S + \omega i\beta w) = u(\omega(z_S + i\beta w)) = u(z_S + \omega i\beta w) = v(\omega w), \\ v(\bar{w}) &= u(z_S + i\beta\bar{w}) = u(\overline{-z_S - i\beta w}) = \overline{u(-z_S - i\beta w)} \\ &= \overline{u(z_S + i\beta w)} = \overline{v(w)}. \end{aligned} \quad (4.5)$$

(Note that this last consequence is global in the sense that we use global properties of our solution.)

We recall some basic facts about Airy functions.³ The first Airy function is given by

$$Ai(s) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{ist + it^3/3} dt, \quad \partial_s^2 Ai(s) - sAi(s) = 0.$$

By either considering the equation satisfied by Ai or by deforming the contour in the integral, we see that $\mathbb{C} \ni \xi \mapsto Ai(\xi)$ is holomorphic and $Ai(\bar{\xi}) = \overline{Ai(\xi)}$. That allows us to define

$$Ai_j(\xi) := \sum_{k=0}^j \omega^{kj} Ai(\omega^k \xi).$$

It follows that $Ai_1 \equiv 0$ and that Ai_0 and Ai_2 give a basis of solutions to the Airy equations ($Ai_0(0) = 3Ai(0) \neq 0$, $Ai'_0(0) = 0$, $Ai_2(0) = 0$, $Ai'_2(0) = 3Ai'(0) \neq 0$). We also have the following properties:

$$\overline{Ai_j(\xi)} = Ai_j(\bar{\xi}), \quad Ai_j(\omega\xi) = \omega^{-j} Ai_j(\xi). \quad (4.6)$$

³See <https://mtaylor.web.unc.edu/wp-content/uploads/sites/16915/2018/04/airyf.pdf> (visited on 31 August 2026) for a quick but thorough introduction.

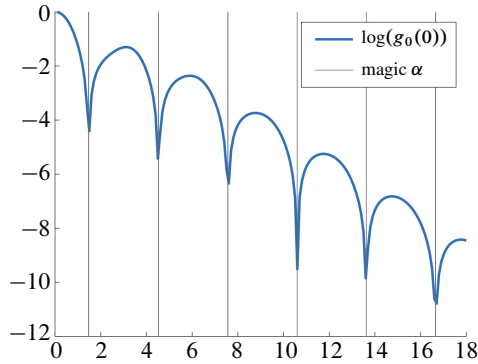


Figure 7. Illustration of $g_0(0, \alpha)$ in (4.7) and (4.8), calculated by numerically evaluating the Wronskian of the protected state with $Ai_2(\bar{w} - \alpha^{-2/3}cw^2)$. We see exponential decay explained in [13, Theorem 1] and the zeros at the magic angles of the scalar model for potential (2.3).

We now consider solutions to $Pv = 0$, where P is (4.3) but in coordinates (4.4). In view of (4.2), these solutions approximate solutions to $P(\alpha, 0)u = 0$ near the corners of the hexagon. They are given by

$$v(w, \bar{w}) = f_0(w, \alpha)Ai_0(\bar{w} - \alpha^{-2/3}cw^2) + f_2(w, \alpha)Ai_2(\bar{w} - \alpha^{-2/3}cw^2), \quad (4.7)$$

where f_j , $j = 0, 2$, are holomorphic in w . Combining (4.5) with (4.6), we obtain, with g_j , $j = 1, 2$, holomorphic,

$$f_0(w, \alpha) = g_0(w^3, \alpha), \quad f_2(w, \alpha) = wg_2(w^3, \alpha), \quad g_j(\bar{\zeta}, \alpha) = \overline{g_j(\zeta, \alpha)}. \quad (4.8)$$

Numerically, we can compute g_j 's by taking Wronskians of the actual solution to $P(\alpha, 0)u = 0$ with $Ai_j(\bar{w} - \alpha^{-2/3}cw^2)$. The resulting functions g_j are close to being holomorphic (the Airy functions provide approximate solutions for $P(\alpha, 0)$, see (4.2); they are holomorphic when solutions to $Pv = 0$ are decomposed using (4.7)) and satisfy

$$g'_0(0, \alpha) \neq 0, \quad g_2(0, \alpha) \neq 0,$$

and the zeros of $g_0(0, \alpha)$ determine the distribution of α 's. Figure 8 shows the plot of numerically computed $\log|g_0(0, \alpha)|$ as function of α (using (4.7) and (4.8)). Exponential decay and the way zeros are created suggest that

$$g_0(\zeta, \alpha) \simeq e^{-c_0\alpha}(c_1\zeta + c_2 \sin(c_3\alpha + c_4)). \quad (4.9)$$

That choice of g_0 in (4.7) and (4.8) matches the observed mechanism of zero creation, see Figure 7.

Ideally, one would like to connect solutions approximated by (4.7) with other local WKB solutions to obtain an approximation for $g_0(0, \alpha)$ coming from a global solution:

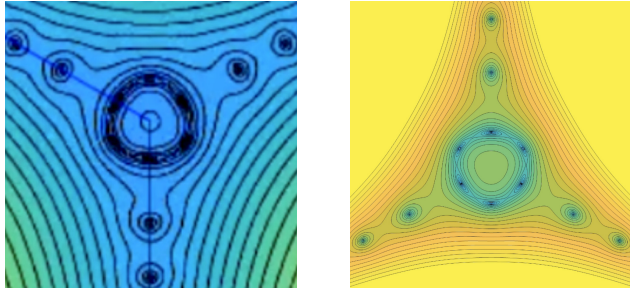


Figure 8. On the left, a snapshot from the movie referenced in Figure 6 showing the evolutions of zeros at the corner; on the right, a snapshot from https://math.berkeley.edu/~zworski/airy_movie.mp4 (visited on 31 August 2026) showing the same structure for (4.7) with g_0 given by (4.9) with $c_j = 1$, $j < 4$, $c_4 = 0$.

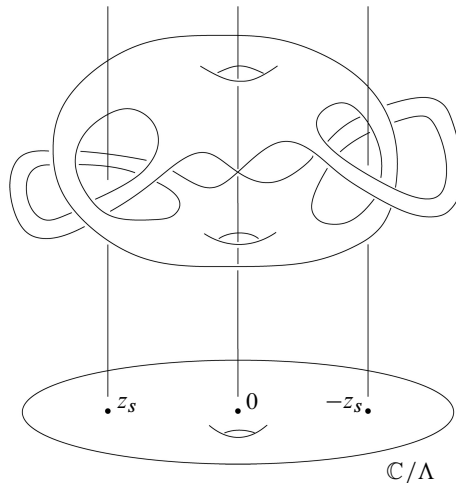


Figure 9. An attempt to illustrate the characteristic variety $q^{-1}(0)$ for the classical symbol of (1.1), $q(x, \xi) = (2\bar{\xi})^2 - V(z)$. It is a genus two surface with singular projection to the base torus above the vertices of the hexagon (the stacking points $\pm z_s$) and a single singularity above the centre of the hexagon. That singularity is almost a normal crossing $\bar{\xi}^2 - z^2 = 0$: the two surfaces, $\bar{\xi} \simeq \pm z$ cross transversally at the centre (and momentum zero) and are only $C^{2,\gamma}$, $\gamma < 1$. Elsewhere, the surfaces is real analytic.

knowing c_3 in (4.9) would then give a spacing in the values of real α 's as in (1.7). An attempt in that direction was made in [22] in a somewhat different spirit than proposed in this section. The difficulty lies in a complicated structure of the characteristic variety $q^{-1}(0)$, $q(x, \xi) = (2\bar{\xi})^2 - V(z)$, $\xi = (\xi_1 - i\xi_2)/2$, $z = x_1 + ix_2$ (this is the classical observable which is semiclassically quantised, with $\hbar = 1/\alpha$ to (1.1), see [13]). An attempt to illustrate that structure is made in Figure 9. The exponential decay of the solution near the hexagon adds to the many complications.

4.2. An example with a quantisation condition for magic alphas

We now consider U given by (4.1). Properties (2.2) are inherited from the same properties being satisfied by U_{BM} . Hence, the set of magic alphas $\mathcal{A}$ is well defined for both the corresponding chiral and scalar models. Here we will consider the latter. What changes is the structure of the characteristic variety $q^{-1}(0)$,

$$q := (2\bar{\zeta})^2 - V(z).$$

It consists of two Lagrangian tori with three high tangency common points – it is shown in Figure 10.

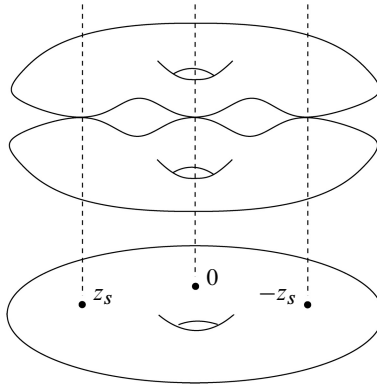


Figure 10. The characteristic variety $q^{-1}(0)$ for the classical symbol of (1.1), $q(x, \xi) = (2\bar{\zeta})^2 - V(z)$ with $V(z) = U(z)U(-z)$ and $U(z) = i\overline{U_{\text{BM}}(z)}$. It is a union of graphs of $\pm d\varphi(z)$, where φ is real and is given in (4.10). These two real Lagrangian tori touch at the edges of the hexagon and at its centre. That complicates the WKB structure of solutions which nevertheless is much simpler than that for the characteristic variety shown in Figure 9.

The simpler structure of the characteristic variety is due to the existence of a global phase function. To see it, we write the potential as

$$V(z) = W(z)^2 = U(z)U(-z), \quad U(z) := i\overline{U_{\text{BM}}(z)}^2,$$

$$W(z) := i\overline{U(z)U(-z)} = -2i\left(\frac{4\pi}{3}\right)^2 \sum_{l=0}^2 \omega^l \cos\left(\left\langle z, \omega^l \frac{4\pi i}{\sqrt{3}} \right\rangle\right).$$

The eikonal equation for $(2D_{\bar{z}}) - \alpha^2 W(z)^2$ (where we think of $\alpha = 1/h$ as a semiclassical parameter) reads

$$2\partial_{\bar{z}}\varphi(z) = W(z),$$

and it is solved by a real valued function

$$\varphi(z) := \frac{8\pi}{3\sqrt{3}} \sum_{l=0}^2 \sin\left(\left\langle z, \frac{4\pi i}{\sqrt{3}} \omega^l \right\rangle\right). \quad (4.10)$$

Hence

$$\begin{aligned} q^{-1}(0) &= \{(x, d_x \varphi(x)) : x \in \mathbb{C}/\Lambda\} \cup \{(x, -d_x \varphi(x)) : x \in \mathbb{C}/\Lambda\}, \\ q(x, \xi) &= (2\bar{\xi})^2 - W(z)^2, \quad \xi = \frac{1}{2}(\xi_1 - i\xi_2), \quad z = x_1 + ix_2. \end{aligned} \quad (4.11)$$

In particular, we have

$$\{q, \bar{q}\}|_{q^{-1}(0)} = 0, \quad (4.12)$$

and consequently there is no mechanism for exponential decay of solutions to $P(\alpha, 0)u = 0$ seen in the case of $U = U_{\text{BM}}$, see [13].

The phase function (4.10) enjoys the following natural symmetries:

$$\varphi(-z) = \varphi(\bar{z}) = -\varphi(z), \quad \varphi(\tau z) = -\varphi(z), \quad \tau = e^{-i\pi/3}.$$

For future reference, we also consider the behaviour of φ at the vertex, $z = z_S$, and at the centre, $z = 0$, of the hexagon. For that, we first note that (4.2) gives

$$W(z_S + w) = i\overline{V_{\text{BM}}(z)} = i(-iaw + b\bar{w}^2 + \mathcal{O}(|w|^3)) = aw + \mathcal{O}(|w|^2),$$

so that

$$\varphi(z_S + w) = \frac{1}{2}a|w|^2 + \mathcal{O}(|w|^3), \quad a = \frac{32}{3}\pi^3.$$

At the centre, we have

$$\begin{aligned} \varphi(z) &= \frac{8\pi}{3\sqrt{3}} \sum_{\ell=0}^2 \sin\left(\frac{2\pi i}{\sqrt{3}}(\bar{z}\omega^\ell - z\bar{\omega}^\ell)\right) = -i\frac{32\pi^4}{81} \sum_{\ell=0}^2 (\bar{z}\omega^\ell - z\bar{\omega}^\ell)^3 + \mathcal{O}(|z|^5) \\ &= -i\frac{32\pi^4}{27}(\bar{z}^3 - z^3) + \mathcal{O}(|z|^5) = -\frac{64\pi^4}{27} \text{Im } z^3 + \mathcal{O}(|z|^5). \end{aligned} \quad (4.13)$$

We will now propose an approximate solution to $P(\alpha, 0)u = 0$ satisfying the symmetries of the protected state (3.8). We will construct u over the closed parallelogram Σ spanned by ω , $-\omega^2$ (these points are Λ -congruent to 0, the centre of the hexagon), $(\omega - \omega^2)/3$ and $2(\omega - \omega^2)/3$ (vertices of the hexagon). We remark that

$$\Sigma_{\pm} := \Sigma \cap \{\pm \text{Re } z \geq 0\}, \quad \Sigma_0 := \Sigma \cap \{\text{Re } z = 0\},$$

where the $\pm$ sets are fundamental domains for the actions of the lattice Λ and $z \mapsto e^{i\pi/3}z$ while Σ_0 is the edge of the hexagon. The approximate solution is pieced from an approximate solution valid near Σ_0 (“near the edge”) and an approximate solution valid near ω , $-\omega^2$ (“near the centre”).

The ultimate goal is to count zeros on the edge as a function of α : a creation of a zero at the vertex when α is magical produces two zeros on the edge (see Figure 6). Hence counting of those zeros will determine the distribution of real magic α ’s.

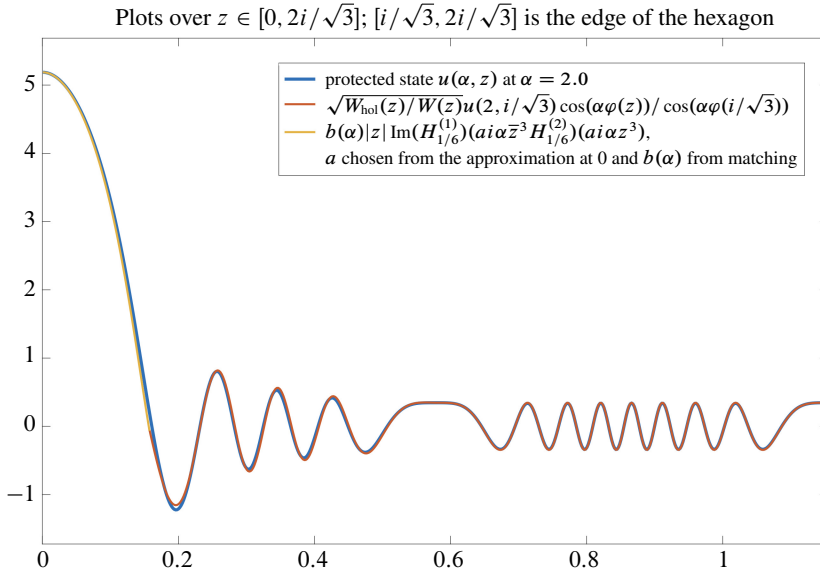


Figure 11. The protected state for potential (4.1) over a segment on the imaginary axis (it is real valued there) compared to WKB approximations near the centre of the hexagon (based on Hankel functions) and away from the centre (based on a standard WKB approximation multiplied by a suitable holomorphic function). Although we do not present a rigorous proof of this approximation, the agreement with the numerics is quite striking. For an animated version with varying α , see https://math.berkeley.edu/~zworski/on_the_edge.mp4 (visited on 31 August 2026).

4.2.1. Approximate solution near the edge. From (3.8), we see that we seek an approximate solution which is real on Σ_0 . The protected state is *not* real everywhere, but numerically we see that the imaginary part decays with α . We expect that this is related to the structure of the characteristic variety (4.11) but at the moment we do not have a rigorous argument for that.

Leading terms in a WKB solutions to the equation $P(\alpha, 0)u = 0$ are given by $W(z)^{-1/2} \exp(\pm i\varphi(z))$. The approximation of the protected state should be real and bounded on Σ_0 (the edge of the hexagon). Since we can multiply the approximate solutions by (local) holomorphic functions, this suggests the following ansatz in Σ , near $\Sigma_0 \cap \{\operatorname{Re} z = 0\}$:

$$u_E(z) = \left(\frac{W_{\text{hol}}(z)}{W(z)} \right)^{1/2} \cos(\alpha\varphi(z)), \quad W_{\text{hol}} \in \mathcal{O}(\Sigma), \quad W_{\text{hol}}|_{\Sigma_0} = W|_{\Sigma_0}. \quad (4.14)$$

In words, W_{hol} is the holomorphic extension of the real analytic function $W(z)$ from Σ_0 to Σ ,

$$W_{\text{hol}}(z) = -2i \left(\frac{4\pi}{3} \right)^2 \left(\cos\left(\frac{4\pi iz}{\sqrt{3}} \right) - \cos\left(\frac{2\pi iz}{\sqrt{3}} \right) \right),$$

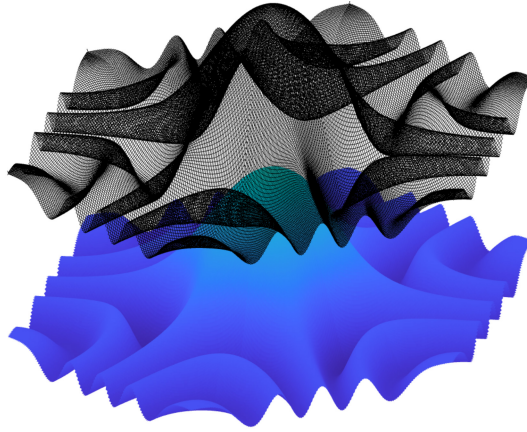


Figure 12. Bottom: the (dominant) real part of the protected state for $\alpha = 1$ over the fundamental domain given by the hexagon. Top: the matched ansatzes (4.14) and (4.18). The strong agreement in the “eyeball norm” suggests a rigorous agreement of finer approximations, see Figure 13 for some numerical comparisons. A rotating version of this figure can be found at https://math.berkeley.edu/~zworski/rotate_compare.mp4 (visited on 31 August 2026).

Ansatz (4.14) is in remarkable agreement with the protected state away from the $\pm 1/2$ that is away from the centre of the hexagon, see Figure 12. That is particularly striking on the edge continued (in a different fundamental domain) to the centre, see Figure 11. We stress that a much finer analysis would be needed to justify this agreement. That has to involve matching ansatz (4.14) with the rougher ansatz near centre (4.18).

4.2.2. Approximate solution near the centre. Near the centre, we will use a simplified version of $\varphi(z)$ obtained by using the leading term in (4.13):

$$\varphi_0(z) := ai(z^3 - \bar{z}^3), \quad W_0(z) := 2\partial_{\bar{z}}\varphi_0(z) = -6ia\bar{z}^2, \quad a := \frac{32\pi^4}{27}.$$

(Eventually, when matching, we will use the lattice to move 0 to the $\omega, -\omega^2$ vertices of Σ and vice versa.) To understand the structure of solutions to $((2D_{\bar{z}})^2 - \alpha^2 W_0(z)^2)u(z) = 0$, we note that the leading term of a WKB solution is given by

$$u_{\pm}(\bar{z}) := W_0(z)^{-1/2} e^{\pm i\varphi_0(z)} = c_0 \bar{z}^{-1} e^{\pm i\alpha a \bar{z}^3} e^{\mp i\alpha a z^3}. \quad (4.15)$$

Since we are free to multiply the approximate local solution by holomorphic functions, this suggests a substitution

$$u(\bar{z}) = \bar{z}^{-1} f(i\alpha a \bar{z}^3) \Rightarrow w^2 f''(w) + \left(w^2 + \frac{1}{9}\right) f(w) = 0, \quad w := i\alpha a \bar{z}^3. \quad (4.16)$$

The equation for f is essentially the Bessel equation: if $f(w) = w^{1/2} g(w)$, then

$$w^2 g''(w) + wg'(w) + \left(w^2 - \frac{1}{36}\right) g(w) = 0.$$

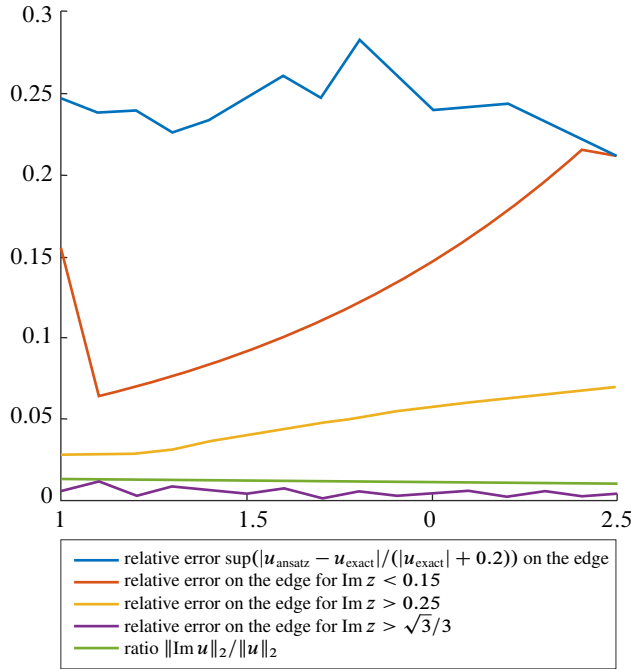


Figure 13. Numerical comparison of the rough WKB approximation and the exact solution. The global relative error is poor due to the quality of the matching; the error for the Hankel function approximation ($0 < \operatorname{Im} z < 0.15$) is also not that good, as it is based on the Taylor expansion near 0; on the other hand, the approximation on the edge of the hexagon ($\operatorname{Im} z > \sqrt{3}/3$) is excellent. Numerically, we also see that the imaginary part is small.

This is the Bessel equation, and in our case (in view of (4.15)) is it natural to consider solutions as linear combinations of Hankel functions (see [1, §9.1] for definitions and properties of these functions): $g(w) = H_{1/6}^{(j)}(w)$, $j = 1, 2$,

$$\begin{aligned} H_{1/6}^{(j)}(w) &= w^{1/6} F_1(w^2) - (-1)^j i w^{-1/6} F_2(w^2), \quad F_j(0) \neq 0, \quad F_j(w) = \overline{F_j(\bar{w})}, \\ H_{1/6}^{(j)}(w) &\simeq \left(\frac{2}{\pi}\right)^{1/2} w^{-1/2} e^{-(-1)^j i(w-\pi/3)}, \quad |w| \rightarrow \infty, \quad -\pi < \arg w < 2\pi. \end{aligned} \quad (4.17)$$

(We define the branch of $w^{1/6}$ for $-\pi < \arg w < 2\pi$, and the values on the negative axis correspond to $\arg w = \pi$.) Returning to (4.16), we see that the two standard solutions to

$$((2D_{\bar{z}})^2 - \alpha^2 W_0(z))u = 0$$

are

$$u_j(\bar{z}) = \bar{z}^{1/2} H_{1/6}^{(j)}(ia\alpha\bar{z}^3), \quad j = 1, 2, \quad a = \frac{32\pi^4}{27}.$$

We note that in view of (4.17), $\bar{z} \mapsto u_j(\bar{z})$ are entire (once we choose the correct branch of $\bar{z} \mapsto \bar{z}^{1/2}$). We also note that in view of (4.17), the complex conjugates $\overline{u_j(z)}$ also solve the equation.

We use $u_j(\bar{z})$ as approximate solutions in

$$\Omega_{\pm}^{\delta} := \pm \left\{ |z| < \delta, |\arg z| \leq \frac{\pi}{6} \right\}, \quad \Omega_{+}^{\delta} + \omega = \text{neigh}_{\Sigma}(\omega), \quad \Omega_{-}^{\delta} - \omega^2 = \text{neigh}_{\Sigma}(-\omega^2),$$

that is, near the left and right corners of Σ . That is the region where (4.14) fails to provide an approximation.

We need to take a combination of $u_j(\bar{z})$, $\overline{u_j(z)}$ with holomorphic coefficients which is smooth at 0 when extended from Ω_{+}^{δ} to a neighbourhood of 0 using the symmetry $z \mapsto e^{i\pi/3}z$ and matches ansatz (4.14). To satisfy these conditions, we arrived at the following solution to $((2D_{\bar{z}})^2 - \alpha^2 W_0(z))u = 0$:

$$u_C(z) = i|z| \left(H_{1/6}^{(1)}(i\alpha\bar{z}^3) H_{1/6}^{(2)}(i\alpha z^3) - \overline{H_{1/6}^{(2)}(i\alpha\bar{z}^3)} \overline{H_{1/6}^{(2)}(i\alpha z^3)} \right). \quad (4.18)$$

We consider u_C as an approximate solution to $((2D_{\bar{z}})^2 - \alpha^2 W(z))u = 0$ near the centre. A finer ansatz involving a more complicated equation than (4.16), depending on z as a parameter, would be needed to produce a rigorous agreement. Nevertheless, the agreement shown in Figures 11 and 12 is remarkable.

4.2.3. Quantisation rule for magic alphas. We expect real $\alpha \in \mathcal{A}$ to have multiplicity two, and the discussion in Section 3.4 then shows that at such α 's, $u(\alpha)$ and $D_{\bar{z}}u(\alpha)$ of Proposition 3.2 has simple zeros at the corners of the hexagon (in the sense of (3.20) and the discussion following it). We could use ansatz (4.14) to find α 's for which $u_C(\alpha, i/\sqrt{3}) = 0$, and that provides a good agreement (in view of the very good agreement with u on the edge of the hexagon, see Figure 11).

A more robust way, potentially applicable in other cases such as the scalar model based on the Bistritzer–MacDonald potential, is given by counting zeros of $u(\alpha)$ (the family of protected states in Theorem 1.1) on the edge of the hexagon.

Proof of Theorem 1.2. Every time a zero occurs at the corner, *two* zeros are added to the zeros on the edge when α increases, see the movies referenced in Figures 6 (for the Bistritzer–MacDonald potential) and 11 (for potential (4.1)) – this follows from the symmetries of u in (3.8). Hence if that number is given by $N(\alpha)$, then the spacing between magic alpha's, $\Delta\alpha$ is given by solving $N(\alpha + \Delta\alpha) - N(\alpha) = 2$. The approximation of $u(\alpha)$ in (4.14) is $\cos(\alpha\varphi(z))$. On the edge, we have

$$\varphi(z(t)) = \frac{8\pi}{3\sqrt{3}} \left(\sin\left(\frac{4\pi}{3}t\right) - 2\sin\left(\frac{2\pi}{3}t\right) \right), \quad z(t) = \frac{it}{\sqrt{3}}, \quad 1 \leq t \leq 2,$$

from which we see that $t \mapsto \varphi(z(t))$ is strictly increasing. Hence $N(\alpha) \simeq \alpha(\varphi(2) - \varphi(1))$ and

$$\Delta\alpha \simeq \frac{2}{\varphi(2) - \varphi(1)} = \frac{1}{4}.$$

This provides a *heuristic* argument for (1.9). ■

5. A toy model

We now turn our attention to the toy model $V(x, y) = W(x)^2$ featured in Theorem 1.3. Here W is a 1-periodic function, and we take the lattice of periodicity $\Lambda = \mathbb{Z}^2$.

5.1. Existence of protected states

Before studying the toy model, we consider the question of existence of protected states for operators of form (1.1), where V is a general potential, periodic with respect to some lattice $\Lambda \subset \mathbb{C}$, which no longer needs to satisfy symmetries (2.5).

We recall that for any $V \in L^\infty(\mathbb{C}/\Lambda; \mathbb{C})$, $k \in \mathbb{C} \setminus \Lambda^*$, $\alpha \mapsto P(\alpha, k)^{-1}$ is a meromorphic family of operators. In fact, for $k \notin \Lambda^*$,

$$(2D_{\bar{z}} + k)^{-2}: L^2(\mathbb{C}/\Lambda) \rightarrow H^2(\mathbb{C}/\Lambda)$$

exists and hence

$$P(\alpha, k) = (2D_{\bar{z}} + k)^2(I - \alpha^2(2D_{\bar{z}} + k)^{-2}V(z)).$$

Since $\alpha \mapsto \alpha^2(2D_{\bar{z}} + k)^{-2}V(z)$ is a holomorphic family of compact operators, meromorphic Fredholm theory (see [10, Theorem C.8]) gives the meromorphy of $\alpha \mapsto (I + \alpha^2(2D_{\bar{z}} + k)^{-2}V)^{-1}$ and hence of $P(\alpha, k)^{-1}$ (we have the required invertibility at one point by putting $\alpha = 0$).

Under an additional assumption, it is easy to see that there are no protected states at $k \in \Lambda^*$. That assumption, (5.1), can never hold for V satisfying symmetries (2.5).

Lemma 5.1. *Suppose that V satisfies*

$$\int_{\mathbb{C}/\Lambda} V(z) dm(z) \neq 0. \quad (5.1)$$

Then, in the notation of (1.1), and for all $k \in \mathbb{C}$, $\alpha \mapsto P(\alpha, k)^{-1}$ is a meromorphic family of operators. In particular, $\ker P(\alpha, 0) = \{0\}$ except for a discrete set of α 's.

Proof. For $k \in \Lambda^*$, we can restrict ourselves to the case of $k = 0$ (conjugation by $e^{i\langle k, z \rangle}$ reduces to this case). Let us assume for simplicity that $|\mathbb{C}/\Lambda| = 1$. We have the following Grushin problem (see [10, §C.1]) for $P(0, 0)$:

$$\begin{pmatrix} (2D_{\bar{z}})^2 & R_- \\ R_+ & 0 \end{pmatrix}: H^2(\mathbb{C}/\Lambda; \mathbb{C}) \oplus \mathbb{C} \rightarrow L^2(\mathbb{C}/\Lambda; \mathbb{C}) \oplus \mathbb{C},$$

$$R_- u_- = u_-, \quad R_+ u = \langle u, 1 \rangle.$$

The inverse is given by

$$\begin{pmatrix} E & E_+ \\ E_- & 0 \end{pmatrix}, \quad E_+ = R_-, \quad E_- = R_+, \quad Eu = (2D_{\bar{z}})^{-2}(v - \langle v, 1 \rangle).$$

We now consider $P(\alpha, 0)$ as a perturbation of $P(0, 0)$: for small values of α , the Grushin problem with the same $R_{\pm}$ remains invertible with the inverse (see [10, C.1.7])

$$\begin{pmatrix} E(\alpha) & E_{+}(\alpha) \\ E_{-}(\alpha) & E_{-+}(\alpha) \end{pmatrix}, \quad E_{-+}(\alpha) = \sum_{k=1}^{\infty} \alpha^{2k} E_{-} V (E V)^{k-1} E_{+}. \quad (5.2)$$

We recall from [10, (C.1.1)] that $P(\alpha, 0)$ is invertible if and only if $E_{-+}(\alpha) \neq 0$. The leading term in the expansion of $E_{-+}(\alpha)$ is given by $\alpha^2 \int_{\mathbb{C}/\Lambda} V(z) dm(z)$. Hence (5.1) implies that $P(\alpha, 0)^{-1}$ exists for $\alpha \in D(0, \varepsilon) \setminus \{0\}$. Since $\alpha \mapsto P(\alpha, 0)$ is a holomorphic family of Fredholm operators, we can again apply analytic Fredholm theory to obtain global meromorphy of $P(\alpha, 0)$. ■

Remark. Suppose that $\Gamma \subset \Lambda^*$ such that $0 \notin \Gamma$ and for $\gamma, \gamma' \in \Gamma$, $\gamma + \gamma' \in \Gamma$. Then consider

$$V(z) := \sum_{\gamma \in \Gamma} a_{\gamma} e^{i\langle z, \gamma \rangle} \in C^{\infty}(\Lambda/\Gamma), \quad a_{\gamma} = \mathcal{O}(|\gamma|^{-\infty}). \quad (5.3)$$

Then $\ker P(\alpha, 0) \neq \{0\}$ for all α 's. (Of course, condition (5.1) is obviously violated.) This follows from (5.2). In fact, if C_{Γ}^{∞} denotes the set of V 's of form (5.3), then

$$E: C_{\Gamma}^{\infty} \rightarrow C_{\Gamma}^{\infty}, \quad V: C_{\Gamma}^{\infty} \rightarrow C_{\Gamma}^{\infty}, \quad E_{-}: C_{\Gamma}^{\infty} \rightarrow 0.$$

(Here, as above, V as an operator denotes multiplication by V .)

5.2. Basic properties of the toy model

We first assume that the potential takes the form $V(z) = W(z)^2$, where W is periodic with respect to some lattice $\Lambda \subset \mathbb{C}$. In that case, the characteristic variety $q^{-1}(0)$, $q = (2\bar{\zeta})^2 - W(z)^2$ is given by a union of two disjoint tori $\{(x, \xi) : 2\bar{\zeta} = \pm W(z)\}$, $\zeta = (\xi_1 - i\xi_2)/2$. These tori are Lagrangian if and only if $\operatorname{Im} \partial_z W(z) \equiv 0$, but we are interested in the general case of complex trigonometric polynomials $W(z)$ satisfying the condition

$$W_0 := |\mathbb{C}/\Lambda|^{-1} \int_{\mathbb{C}/\Lambda} W(z) dm(z) \neq 0. \quad (5.4)$$

In view of Lemma 5.1, it is natural to ask about the set

$$\mathcal{A}(k) := \{\alpha : \ker P(\alpha, k) \neq \{0\}\},$$

with the multiplicity of an element α of $\mathcal{A}(k)$, $m(\alpha, k)$, defined by

$$m(\alpha, k) := \frac{1}{2\pi i} \operatorname{tr}_{L^2(\mathbb{C}/\Lambda)} \oint_{\alpha} P(\beta, k)^{-1} \partial_{\beta} P(\beta, k) d\beta, \quad (5.5)$$

where the integral is over a positively oriented circle including α as the only possible pole of $\beta \mapsto P(\beta, k)^{-1}$. If $k \in \Lambda^*$, this assumes meromorphy of $\beta \mapsto P(\beta, k)^{-1}$.

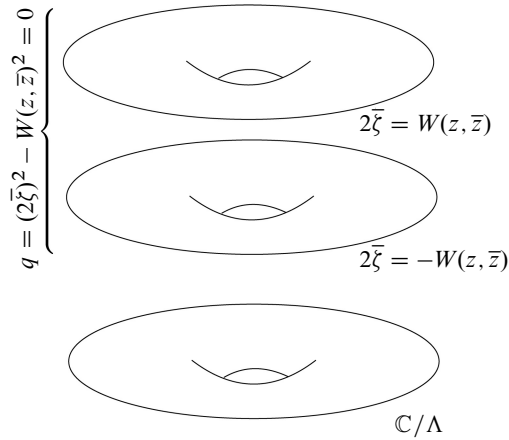


Figure 14. The characteristic variety $q^{-1}(0)$ for the classical symbol of (1.1), $q(x, \xi) = (2\bar{\xi})^2 - V(z)$ with $V(z) = W(z)^2$ and $W(z) \neq 0$. It is a union of two *disjoint* tori projecting diffeomorphically to the base torus. The tori are Lagrangian only if $\text{Im } \partial_z W(z) \equiv 0$. Flat bands do not occur for this model, but we can ask for the values of α for which $\ker_{H^2(\mathbb{C}/\Lambda)} P(\alpha, k) \neq \{0\}$. An answer in a special case is given in Theorem 1.3.

We henceforth specialize to the simplest case $\Lambda = \mathbb{Z}^2$ and $W(x, y) = W(x)$, $(x, y) \in \mathbb{R}^2$, where $W(x) = W(x + 1)$ is real analytic and in fact extends to an entire function. This amounts to the study of the operator $P_\xi(\alpha)$ defined in (1.12). We present a self-contained argument which indicates that even in the simplified setting (compared to that in Section 4), a rigorous derivation becomes quite involved.

As explained in (5.42) below, we can reduce the study of the kernel of $P_\xi(\alpha)$ on 1-periodic functions on $\mathbb{R}$ to the semiclassically rescaled operator $P(\alpha, h) := (hD_x)^2 - \alpha^2 W(x)^2$ on functions on $\mathbb{R}$ satisfying $u(x + 1) = e^{ik/h} u(x)$.

We now assume that the potential $W(x)$ extends to an entire function $W(z)$, which still has to be 1-periodic. The operator $P(\alpha, h)$ naturally extends to an operator on entire functions. This is the setting in which the analysis of this section is carried out. We emphasize that the *meaning of the letter z is different here than in the other sections*: it is the complexification of x . Accordingly, we are looking for elements of the kernel of $P(\alpha, h)$ which are *holomorphic* in z .

5.3. Semiclassical formulation

We consider the differential operator

$$P(\alpha, h) = (hD_z)^2 - \alpha^2 W(z)^2, \quad z = x + iy \in \mathbb{C}, \quad D_z = \frac{1}{2i}(\partial_x - i\partial_y),$$

where $\alpha \in \mathbb{C}$, $h > 0$, and W is an entire function. We will study the limit $h \rightarrow 0$ and assume that

$$C_0^{-1} \leq |\alpha| \leq C_0 \tag{5.6}$$

for some constant C_0 . We should mention that the problem scales and we could consider $|\alpha| \gg 1$ instead of (5.6) and $h \rightarrow 0$. It is however natural in the context of WKB to use real $h \rightarrow 0$ as the asymptotic parameter.

We study holomorphic solutions $u(z)$ to the equation

$$P(\alpha, h)u = 0. \quad (5.7)$$

We first review some basic properties of second order holomorphic ODEs. For a holomorphic function u , define the vector-valued holomorphic function

$$\mathbf{u}(z) = (u(z), hD_z u(z)) \in \mathbb{C}^2. \quad (5.8)$$

The Cauchy problem for equation (5.7) is well posed: for any $z_0 \in \mathbb{C}$, the map $u \mapsto \mathbf{u}(z_0)$ is an isomorphism from the space of solutions to (5.7) onto $\mathbb{C}^2$, see, for instance, [23, §7.2].

For $z_0, z_1 \in \mathbb{C}$, define the transition matrix $\mathcal{M}_{z_1 \leftarrow z_0}: \mathbb{C}^2 \rightarrow \mathbb{C}^2$ as follows: for each solution u to (5.7), we have

$$\mathbf{u}(z_1) = \mathcal{M}_{z_1 \leftarrow z_0} \mathbf{u}(z_0), \quad \mathcal{M}_{z_1 \leftarrow z_0} = \mathcal{M}_{z_1 \leftarrow z_0}(\alpha, h). \quad (5.9)$$

For two holomorphic functions u, v , define their Wronskian (it differs from the Wronskian used in Section 3 by a factor of h):

$$\text{Wr}(u, v) := u(hD_z v) - (hD_z u)v.$$

We will use the identity

$$hD_z \text{Wr}(u, v) = uP(\alpha, h)v - vP(\alpha, h)u. \quad (5.10)$$

If u, v are both solutions to (5.7), then (5.10) shows that $\text{Wr}(u, v)$ is constant. This implies that

$$\det \mathcal{M}_{z_1 \leftarrow z_0} = 1. \quad (5.11)$$

Finally, let $\Phi(z)$ be an antiderivative of W :

$$\partial_z \Phi = W. \quad (5.12)$$

5.4. WKB approximation

We will now study the solutions to (5.7) using the global phase function Φ .

5.4.1. A growth bound. From now on in this section, we fix a smooth non-self-intersecting curve

$$\gamma: [0, 1] \rightarrow \mathbb{C} \setminus W^{-1}(0), \quad \gamma' \neq 0. \quad (5.13)$$

We henceforth denote by C_γ a constant depending only on γ , W , and the constant C_0 in (5.6), whose precise meaning might change from place to place.

Define the real-valued function

$$\psi_{\gamma,\alpha}(t) := \operatorname{Im}(\alpha\Phi(\gamma(t))), \quad t \in [0, 1]. \quad (5.14)$$

The next lemma is a basic bound on the growth of $|\mathbf{u}|$ along γ , where u solves (5.7): it can grow exponentially at most like the integral of $|\psi'_{\gamma,\alpha}|$. This is classical, and different formulations can be found, for instance, in [23, §7.1], but we give a direct proof in our special case.

Lemma 5.2. *Assume that u solves (5.7) and $\mathbf{u}$ is given by (5.8). Then for γ satisfying (5.13), we have for $t \in [0, 1]$,*

$$|\mathbf{u}(\gamma(t))| \leq C_\gamma \exp\left(h^{-1} \int_0^t |\psi'_{\gamma,\alpha}(s)| ds\right) |\mathbf{u}(\gamma(0))|. \quad (5.15)$$

Proof. For u which solves (5.7), define the energy

$$\begin{aligned} E(z) &:= \frac{1}{2}(|h\partial_z u(z)|^2 + |\alpha W(z)u(z)|^2), \\ E_\gamma(t) &:= E(\gamma(t)). \end{aligned}$$

We may assume that u is not identically 0 so that $E_\gamma(t) > 0$ for all t . We compute

$$\begin{aligned} \partial_t \log E_\gamma(t) &= 2E_\gamma(t)^{-1} \operatorname{Re}(\partial_z E(\gamma(t))\gamma'(t)), \\ \partial_z E(z) &= ih^{-1}\alpha W(z) \operatorname{Im}(\overline{\alpha W(z)u(z)} \cdot h\partial_z u(z)) + \frac{1}{2}\bar{W}(z)\partial_z W(z)|\alpha u(z)|^2. \end{aligned}$$

For all $z = \gamma(t)$, $0 \leq t \leq 1$, we have

$$\begin{aligned} |\operatorname{Im}(\overline{\alpha W(z)u(z)} \cdot h\partial_z u(z))| &\leq E_\gamma(z), \\ \left| \frac{1}{2}\bar{W}(z)\partial_z W(z)|\alpha u(z)|^2 \right| &\leq C_\gamma E_\gamma(z). \end{aligned}$$

This implies

$$|\partial_t \log E_\gamma(t)| \leq 2h^{-1} |\operatorname{Im}(\alpha W(\gamma(t))\gamma'(t))| + C_\gamma.$$

Integrating and using the identity

$$\psi'_{\gamma,\alpha}(t) = \operatorname{Im}(\alpha W(\gamma(t))\gamma'(t)),$$

we obtain

$$|\log E_\gamma(t) - \log E_\gamma(0)| \leq C_\gamma + 2h^{-1} \int_0^t |\psi'_{\gamma,\alpha}(s)| ds \quad \text{for all } t \in [0, 1].$$

Since $C_\gamma^{-1} |\mathbf{u}(\gamma(t))|^2 \leq E_\gamma(t) \leq C_\gamma |\mathbf{u}(\gamma(t))|^2$, this shows (5.15). ■

5.4.2. WKB solutions. We next construct approximate WKB solutions, $u_N^\pm$, to equation (5.7). Let γ be a curve satisfying (5.13). Since γ does not self-intersect, we can fix a simply connected open set $\mathcal{U}_\gamma$ such that

$$\gamma \subset \mathcal{U}_\gamma \subset \mathbb{C} \setminus W^{-1}(0).$$

Our solutions have the form (where $N \geq 0$ is an integer)

$$u_N^\pm(z) = \exp\left(\frac{i\Phi_N^\pm(z; \alpha, h)}{h}\right), \quad z \in \mathcal{U}_\gamma,$$

$$\Phi_N^\pm(z; \alpha, h) = \pm\alpha\Phi(z) - ihb_N^\pm\left(z; \frac{h}{\alpha}\right),$$

where Φ is defined in (5.12). This means that

$$u_N^\pm(z) = e^{\pm i\alpha\Phi(z)/h} A_\pm\left(z, \frac{h}{\alpha}\right), \quad A_\pm(z, \tilde{h}) = e^{b_N^\pm(z; \tilde{h})}, \quad \tilde{h} := \frac{h}{\alpha}.$$

The function whose exponential gives the amplitude of $u_N^\pm$ is given by

$$b_N^\pm(z; \tilde{h}) = \sum_{j=0}^N \tilde{h}^j a_j^\pm(z), \quad (5.16)$$

where the coefficients $a_j^\pm \in \mathcal{O}(\mathcal{U}_\gamma)$ are determined by the equation for $b_N^\pm$:

$$2W(z)\partial_z b_N^\pm(z; \tilde{h}) = -\partial_z W(z) \pm i\tilde{h}(\partial_z^2 b_N^\pm(z; \tilde{h}) + (\partial_z b_N^\pm(z; \tilde{h}))^2) + \mathcal{O}(\tilde{h}^{N+1}). \quad (5.17)$$

Note that (5.17) determines the derivatives $\partial_z a_j^\pm(z)$ uniquely. Since $\partial_z a_0^\pm = -\partial_z W/(2W)$, we can take

$$a_0^\pm(z) = -\frac{1}{2} \log W(z), \quad (5.18)$$

where we fix a branch of $\log W$ on $\mathcal{U}_\gamma$.

Lemma 5.3. *In the notation of (5.14), we have for all $t \in [0, 1]$ and $0 < h < C_{\gamma, N}^{-1}$ that*

$$C_{\gamma, N}^{-1} e^{\mp \psi_{\gamma, \alpha}(t)/h} \leq |u_N^\pm(\gamma(t))| \leq C_{\gamma, N} e^{\mp \psi_{\gamma, \alpha}(t)/h}, \quad (5.19)$$

$$|P(\alpha, h)u_N^\pm(\gamma(t))| \leq C_{\gamma, N} e^{\mp \psi_{\gamma, \alpha}(t)/h} h^{N+2}. \quad (5.20)$$

Here $C_{\gamma, N}$ denotes a constant depending only on N , γ , W , and the constant C_0 in (5.6).

Proof. Bound (5.19) follows immediately from the definitions of $u_N^\pm$ and $\psi_{\gamma, \alpha}$. To see (5.20), we compute

$$\begin{aligned} P(\alpha, h)u_N^\pm(z) &= u_N^\pm(z)((\partial_z \Phi_N^\pm(z; \alpha, h))^2 - ih\partial_z^2 \Phi_N^\pm(z; \alpha, h) - \alpha^2 W(z)^2) \\ &= u_N^\pm(z)\left(\mp i\alpha h(2W(z)\partial_z b_N^\pm\left(z; \frac{h}{\alpha}\right) + \partial_z W(z))\right. \\ &\quad \left.- h^2\left(\partial_z^2 b_N^\pm\left(z; \frac{h}{\alpha}\right) + \left(\partial_z b_N^\pm\left(z; \frac{h}{\alpha}\right)\right)^2\right)\right). \end{aligned}$$

Together with (5.17), this gives (5.20). ■

We also have the bound

$$\partial_z (e^{b_N^+(z; \tilde{h}) + b_N^-(z; \tilde{h})} (2W(z) - i\tilde{h}\partial_z (b_N^+(z; \tilde{h}) - b_N^-(z; \tilde{h})))) = \mathcal{O}(\tilde{h}^{N+1}), \quad (5.21)$$

where $\tilde{h} = h/\alpha$. In fact, we have

$$\begin{aligned} \text{Wr}(u_N^+, u_N^-) &= e^{b_N^+ + b_N^-} (\partial_z (-\alpha\Phi - i\tilde{h}b_N^-) - \partial_z (\alpha\Phi - i\tilde{h}b_N^+)) \\ &= \alpha e^{b_N^+ + b_N^-} (-2W + i\tilde{h}\partial_z (b_N^+ - b_N^-)). \end{aligned}$$

On the other hand, (5.10), (5.19) and (5.20) give $\partial_z \text{Wr}(u_N^+, u_N^-) = \mathcal{O}(\tilde{h}^{N+1})$.

Next, using the WKB solutions, for a solution u to (5.7) we define the vector of modified Wronskians

$$\mathbf{Wr}_N(z) = \begin{pmatrix} u_N^+(z)^{-1} \text{Wr}(u, u_N^+)(z) \\ u_N^-(z)^{-1} \text{Wr}(u, u_N^-)(z) \end{pmatrix}, \quad z \in \mathcal{U}_\gamma.$$

Then, for all $t \in [0, 1]$,

$$\mathbf{Wr}_N(z) = B_N(z; \alpha, h) \mathbf{u}(z), \quad (5.22)$$

where

$$B_N(z; \alpha, h) = \begin{pmatrix} \partial_z \Phi_N^+(z; \alpha, h) & -1 \\ \partial_z \Phi_N^-(z; \alpha, h) & -1 \end{pmatrix} = \begin{pmatrix} \alpha W(z) + \mathcal{O}(h) & -1 \\ -\alpha W(z) + \mathcal{O}(h) & -1 \end{pmatrix}. \quad (5.23)$$

Since $W \neq 0$, both B_N and its inverse are bounded in norm uniformly in h for small h .

5.4.3. Approximating the transition matrix. We now assume that γ is a curve as in formula (5.13) and γ, α satisfy the additional condition

$$\psi'_{\gamma, \alpha}(t) \geq 0 \quad \text{for all } t \in [0, 1], \quad (5.24)$$

where the function $\psi_{\gamma, \alpha}$ was defined in (5.14). Define the diagonal matrix

$$S_{\gamma, N}(\alpha, h) := \begin{pmatrix} e^{i(\Phi_N^+(\gamma(0); \alpha, h) - \Phi_N^+(\gamma(1); \alpha, h))/h} & 0 \\ 0 & e^{i(\Phi_N^-(\gamma(0); \alpha, h) - \Phi_N^-(\gamma(1); \alpha, h))/h} \end{pmatrix}.$$

The transition matrix $\mathcal{M}_{\gamma(1) \leftarrow \gamma(0)}$ defined in (5.9) is now approximated as follows.

Lemma 5.4. *Under condition (5.24), and with B_N defined in (5.23), we have*

$$\begin{aligned} \mathcal{M}_{\gamma(1) \leftarrow \gamma(0)} &= B_N(\gamma(1); \alpha, h)^{-1} S_{\gamma, N}(\alpha, h) B_N(\gamma(0); \alpha, h) \\ &\quad + \mathcal{O}_{\gamma, N}(h^{N+1} e^{(\psi_{\gamma, \alpha}(1) - \psi_{\gamma, \alpha}(0))/h}). \end{aligned} \quad (5.25)$$

Proof. (1) Let u be a solution to (5.7). By Lemma 5.2 and (5.24), we have for all $t \in [0, 1]$,

$$|\mathbf{u}(\gamma(t))| \leq C_\gamma e^{(\psi_{\gamma, \alpha}(t) - \psi_{\gamma, \alpha}(0))/h} |\mathbf{u}(\gamma(0))|. \quad (5.26)$$

Define the Wronskians

$$\mathbf{W}_{\mathbf{r}\pm}(t) := \mathbf{W}_{\mathbf{r}}(u, u_N^{\pm})(\gamma(t)), \quad \mathbf{W}_{\mathbf{r}N}(\gamma(t)) := \begin{pmatrix} e^{-i\Phi_N^+(\gamma(t);\alpha,h)/h} \mathbf{W}_{\mathbf{r}+}(t) \\ e^{-i\Phi_N^-(\gamma(t);\alpha,h)/h} \mathbf{W}_{\mathbf{r}-}(t) \end{pmatrix}.$$

By (5.10) and since $P(\alpha, h)u = 0$, we have for all $t \in [0, 1]$,

$$\begin{aligned} |\partial_t \mathbf{W}_{\mathbf{r}\pm}(t)| &\leq C_\gamma h^{-1} |u(\gamma(t))| \cdot |P(\alpha, h)u_N^{\pm}(\gamma(t))| \\ &\leq C_{\gamma,N} h^{N+1} e^{(\mp\psi_{\gamma,\alpha}(t) + \psi_{\gamma,\alpha}(t) - \psi_{\gamma,\alpha}(0))/h} |\mathbf{u}(\gamma(0))|, \end{aligned}$$

where the last inequality follows from (5.20) and (5.26). This implies that

$$\begin{aligned} |\partial_t \mathbf{W}_{\mathbf{r}+}(t)| &\leq C_{\gamma,N} h^{N+1} e^{-\psi_{\gamma,\alpha}(0)/h} |\mathbf{u}(\gamma(0))|, \\ |\partial_t \mathbf{W}_{\mathbf{r}-}(t)| &\leq C_{\gamma,N} h^{N+1} e^{(2\psi_{\gamma,\alpha}(1) - \psi_{\gamma,\alpha}(0))/h} |\mathbf{u}(\gamma(0))|. \end{aligned}$$

Integrating in $t \in [0, 1]$, we see that

$$\begin{aligned} \mathbf{W}_{\mathbf{r}+}(1) &= \mathbf{W}_{\mathbf{r}+}(0) + \mathcal{O}_{\gamma,N}(h^{N+1} e^{-\psi_{\gamma,\alpha}(0)/h} |\mathbf{u}(\gamma(0))|), \\ \mathbf{W}_{\mathbf{r}-}(1) &= \mathbf{W}_{\mathbf{r}-}(0) + \mathcal{O}_{\gamma,N}(h^{N+1} e^{(2\psi_{\gamma,\alpha}(1) - \psi_{\gamma,\alpha}(0))/h} |\mathbf{u}(\gamma(0))|). \end{aligned} \tag{5.27}$$

(2) Multiplying both sides of (5.27) by $e^{-i\Phi_N^{\pm}(\gamma(1);\alpha,h)/h}$, we get

$$\begin{aligned} e^{-i\Phi_N^+(\gamma(1);\alpha,h)/h} \mathbf{W}_{\mathbf{r}+}(1) &= e^{(-i\Phi_N^+(\gamma(1);\alpha,h) + i\Phi_N^+(\gamma(0);\alpha,h))/h} e^{-i\Phi_N^+(\gamma(0);\alpha,h)} \mathbf{W}_{\mathbf{r}+}(t) \\ &\quad + \mathcal{O}_{\gamma,N}(h^{N+1} e^{\psi_{\gamma,\alpha}(1)/h - \psi_{\gamma,\alpha}(0)/h} |\mathbf{u}(\gamma(0))|), \\ e^{-i\Phi_N^-(\gamma(1);\alpha,h)/h} \mathbf{W}_{\mathbf{r}-}(1) &= e^{(-i\Phi_N^-(\gamma(1);\alpha,h) + i\Phi_N^-(\gamma(0);\alpha,h))/h} e^{-i\Phi_N^-(\gamma(0);\alpha,h)} \mathbf{W}_{\mathbf{r}-}(t) \\ &\quad + \mathcal{O}_{\gamma,N}(h^{N+1} e^{\psi_{\gamma,\alpha}(1)/h - \psi_{\gamma,\alpha}(0)/h} |\mathbf{u}(\gamma(0))|) \end{aligned}$$

(we used here the fact that $\text{Re}(-i\Phi_N^{\pm}(\gamma(1);\alpha,h)) = \pm\psi_{\gamma,\alpha}(1) + \mathcal{O}(h)$). That is,

$$\mathbf{W}_{\mathbf{r}N}(\gamma(1)) = S_{\gamma,N}(\alpha, h) \mathbf{W}_{\mathbf{r}N}(\gamma(0)) + \mathcal{O}_{\gamma,N}(h^{N+1} e^{(\psi_{\gamma,\alpha}(1) - \psi_{\gamma,\alpha}(0))/h} |\mathbf{u}(\gamma(0))|).$$

By (5.22), this implies

$$\begin{aligned} B_N(\gamma(1); \alpha, h) \mathbf{u}(\gamma(1)) &= S_{\gamma,N}(\alpha, h) B_N(\gamma(0); \alpha, h) \mathbf{u}(\gamma(0)) \\ &\quad + \mathcal{O}_{\gamma,N}(h^{N+1} e^{(\psi_{\gamma,\alpha}(1) - \psi_{\gamma,\alpha}(0))/h} |\mathbf{u}(\gamma(0))|). \end{aligned}$$

Therefore,

$$\begin{aligned} \mathbf{u}(\gamma(1)) &= B_N(\gamma(1); \alpha, h)^{-1} S_{\gamma,N}(\alpha, h) B_N(\gamma(0); \alpha, h) \mathbf{u}(\gamma(0)) \\ &\quad + \mathcal{O}_{\gamma,N}(h^{N+1} e^{(\psi_{\gamma,\alpha}(1) - \psi_{\gamma,\alpha}(0))/h} |\mathbf{u}(\gamma(0))|). \end{aligned}$$

Since this holds for each solution u to (5.7), we get (5.25). ■

5.5. WKB analysis in the periodic case

From now on, we assume that the potential W is 1-periodic:

$$W(z + 1) = W(z).$$

We also assume that γ is a non-self-intersecting curve as in (5.13) such that

$$\gamma(1) = \gamma(0) + 1, \quad (5.28)$$

and W has winding number 0 along γ :

$$\int_{\gamma} \frac{W'(z)}{W(z)} dz = 0. \quad (5.29)$$

5.5.1. Periodicity of WKB solutions. Let $u_N^{\pm}(z) = \exp(i\Phi_N^{\pm}(z; \alpha, h)/h)$, $z \in \mathcal{U}_{\gamma}$ (a simply connected neighbourhood of γ which satisfies (5.13) and (5.28)) be the approximate WKB solutions to (5.7) constructed in Section 5.4.2. Shrinking $\mathcal{U}_{\gamma}$ if necessary, we assume that the set $\mathcal{U}_{\gamma} \cap (\mathcal{U}_{\gamma} - 1)$ is connected.

The solutions $u_N^{\pm}$ have the following periodicity property.

Lemma 5.5. *There exist constants $c_0, \dots, c_{N+1}$ such that for all $z \in \mathcal{U}_{\gamma} \cap (\mathcal{U}_{\gamma} - 1)$,*

$$\begin{aligned} \Phi_N^{\pm}(z + 1; \alpha, h) - \Phi_N^{\pm}(z; \alpha, h) &= \pm \alpha c \left(\frac{h}{\alpha} \right), \\ c(\tilde{h}) &= \sum_{j=0}^{N+1} c_j \tilde{h}^j, \quad c_0 = \int_0^1 W(z) dz, \quad c_1 = 0. \end{aligned} \quad (5.30)$$

Proof. Since $\partial_z \Phi = W$ and W is 1-periodic, we have $\partial_z(\Phi(z + 1) - \Phi(z)) = 0$. Therefore, $\Phi(z + 1) - \Phi(z)$ is constant:

$$\Phi(z + 1) - \Phi(z) = \int_0^1 W(z) dz =: c_0.$$

Next, the functions $\partial_z a_j^{\pm}(z + 1; \tilde{h})$ and $\partial_z a_j^{\pm}(z; \tilde{h})$ (see (5.16)) satisfy the same equation derived from (5.17). Since that equation has a unique solution, we have

$$\partial_z a_j^{\pm}(z + 1) = \partial_z a_j^{\pm}(z), \quad z \in \mathcal{U}_{\gamma} \cap (\mathcal{U}_{\gamma} - 1). \quad (5.31)$$

It follows that

$$a_j^{\pm}(z + 1) - a_j^{\pm}(z) = c_j^{\pm}, \quad z \in \mathcal{U}_{\gamma} \cap (\mathcal{U}_{\gamma} - 1)$$

for some constants $c_j^{\pm}$.

Next, by (5.21) we have for all $z \in \mathcal{U}_{\gamma} \cap (\mathcal{U}_{\gamma} - 1)$ that

$$\begin{aligned} &e^{b_N^+(z+1; \tilde{h}) + b_N^-(z+1; \tilde{h})} (2W(z + 1) - i\tilde{h}\partial_z(b_N^+(z + 1; \tilde{h}) - b_N^-(z + 1; \tilde{h}))) \\ &= e^{b_N^+(z; \tilde{h}) + b_N^-(z; \tilde{h})} (2W(z) - i\tilde{h}\partial_z(b_N^+(z; \tilde{h}) - b_N^-(z; \tilde{h}))) + \mathcal{O}(\tilde{h}^{N+1}). \end{aligned}$$

By (5.31), the terms in the parentheses on both sides are the same, thus

$$e^{b_N^+(z+1;\tilde{h})+b_N^-(z+1;\tilde{h})} = e^{b_N^+(z;\tilde{h})+b_N^-(z;\tilde{h})} + \mathcal{O}(\tilde{h}^{N+1}).$$

In other words,

$$\exp\left(\sum_{j=0}^N \tilde{h}^j (c_j^+ + c_j^-)\right) = 1 + \mathcal{O}(\tilde{h}^{N+1}). \quad (5.32)$$

By (5.18), $a_0^\pm(z) = (1/2)(\log W(z))$ and hence $c_0^\pm = (1/2) \int_\gamma W'/W dz$. The winding condition (5.29) then shows that $c_0^+ + c_0^- = 0$. Expanding the exponential in (5.32) then gives

$$c_j^+ + c_j^- = 0.$$

Recalling that $\Phi_N^\pm = \pm\alpha\Phi - i \sum_{j=0}^N \tilde{h}^j a_j^\pm$ gives (5.30) with $c_j := -i c_{j-1}^+$, $j \geq 1$. In view of (5.18), $c_1 = (i/2) \int_\gamma W'(z)/W(z) dz = 0$, by assumption (5.29). ■

5.5.2. Stokes loops. We now introduce the notion of a Stokes loop. If the potential W admits a Stokes loop, then we are able to prove a quantization condition for the corresponding operator P .

Definition 5.6. Assume that W is a 1-periodic entire function on $\mathbb{C}$. Let $\gamma: [0, 1] \rightarrow \mathbb{C}$ be a smooth non-self-intersecting curve. We say that γ is a *Stokes loop* for W if γ extends to a smooth non-self-intersecting curve $\tilde{\gamma}: \mathbb{R} \rightarrow \mathbb{C}$ and the following conditions hold for all $t \in \mathbb{R}$:

$$\tilde{\gamma}(t+1) = \tilde{\gamma}(t) + 1, \quad (5.33)$$

$$W(\gamma(t))\gamma'(t) > 0. \quad (5.34)$$

Another way to formulate the condition on the curve γ is to consider it as a closed non-self-intersecting curve $\gamma: \mathbb{R}/\mathbb{Z} \mapsto \mathbb{C}/\mathbb{Z}$, where the action is $z \mapsto z + 1$. Indeed, $\tilde{\gamma}$ defines such a curve. That curve is non-self-intersecting: $\gamma([s]) = \gamma([t])$ ($t \mapsto [t]$ is the map $\mathbb{R} \rightarrow \mathbb{R}/\mathbb{Z}$) means that for some m , $\tilde{\gamma}(s) = \tilde{\gamma}(t) + m = \tilde{\gamma}(t + m)$. As $\tilde{\gamma}$ is non-self-intersecting, this implies that $s = t + m$, that is, $[s] = [t]$.

It is clear that a Stokes loop satisfies (5.13). We also have (5.29).

Lemma 5.7. Suppose that γ is a Stokes loop in the sense of Definition 5.6. Then the winding number condition (5.29) holds.

Proof. It is convenient to change the parametrisation of γ to $\rho: [0, L] \rightarrow \mathbb{C}$, $|\rho'(s)| = 1$, so that now $\gamma(L) = \gamma(0) + 1$. Similarly, it is useful to define $w(s) := W(\rho(s))/|W(\rho(s))|$ and to note that the winding numbers around 0 of $s \mapsto W(\rho(s))$ and of $s \mapsto w(s)$ are the same. Condition (5.34) becomes $w(s)\rho'(s) = 1$, that is,

$$w(s) = \overline{\rho'(s)}. \quad (5.35)$$

Condition (5.33) shows that $s \mapsto \rho'(s)$ is a smooth closed curve. Also, (5.35) shows that the winding number around zero of the curve w is the negative of the same number for the curve ρ' .

Hence it suffices to show that the winding number of $[0, L] \ni s \mapsto \rho'(s)$ is 0. As remarked after Definition 5.6, we can consider ρ as a closed curve on $\mathbb{C}/\mathbb{Z}$. Let $[0, L] \ni s \mapsto \theta(s)$ be continuously chosen angle between $\rho'(s)$ and 1 (thought of as vectors in $\mathbb{R}^2 \simeq \mathbb{C}$). The curvature of ρ is given by $\kappa_\rho(s) = \theta'(s)$. (We recall that to see this, we write $\rho'(s) = (\cos \theta(s), \sin \theta(s))$ so that $\kappa_\rho(s) = \rho''(s) \cdot \rho'(s)^\perp = \theta'(s)$). Let $\rho_1: [0, 1] \rightarrow \mathbb{C}/\mathbb{Z}$ be given by $\rho_1(s) = -s + iM \bmod 1$ with M chosen large enough so that ρ_1 and ρ are disjoint. Then the curves ρ_1 and ρ form a boundary of a bounded subset of $\mathbb{C}/\mathbb{Z}$ homeomorphic to $(\mathbb{R}/\mathbb{Z}) \times [0, 1]$ (this follows from the Jordan–Schönflies theorem (see [21, Theorem 6.7.1]) applied to the cylinder compactified at one of the infinities), and the Gauss–Bonnet theorem for a flat surface (the cylinder $\mathbb{C}/\mathbb{Z}$; see, for instance, [9, §§4 and 5] for the general case) shows that

$$\int_\rho \kappa_\rho(s) ds + \int_{\rho_1} \kappa_{\rho_1}(s) ds = 0.$$

Since $\kappa_{\rho_1} \equiv 0$, it follows that $\int_\rho \kappa_\rho(s) ds = 0$. The calculation of $\kappa_\rho(s)$ above also shows that $\kappa_\rho(s) = -i \overline{\rho'(s)} \rho''(s) = -i \rho''(s) / \rho'(s)$. The winding number of $[0, L] \ni s \mapsto \rho'(s)$ is

$$\frac{1}{2\pi i} \int_0^L \frac{\rho''(s)}{\rho'(s)} ds = \frac{1}{2\pi} \int_\rho \kappa_\rho(s) ds = 0,$$

which proves the claim. ■

Moreover, any Stokes loop γ satisfies condition (5.24) for α in the upper half-plane:

$$\psi'_{\gamma, \alpha}(t) = \operatorname{Im}(\alpha W(\gamma(t)) \gamma'(t)) \geq 0 \quad \text{for all } t \in [0, 1], \quad \operatorname{Im} \alpha \geq 0. \quad (5.36)$$

We also get that the coefficient c_0 defined in (5.30) is positive:

$$c_0 := \int_\gamma W(z) dz = \int_0^1 W(\gamma(t)) \gamma'(t) dt > 0.$$

If $W = c_0 > 0$ is a constant potential, then it has a Stokes loop $\gamma(t) = t$. The next proposition shows that perturbations of this potential have Stokes loops as well.

Proposition 5.8. *Assume that W is a 1-periodic entire function and*

$$\int_0^1 W(z) dz = c_0 > 0, \quad (5.37)$$

$$W(z) \neq 0 \quad \text{and} \quad |\operatorname{Im} W(z)| \leq \operatorname{Re} W(z) \quad \text{when } |\operatorname{Im} z| \leq 1. \quad (5.38)$$

Then W has a Stokes loop.

Proof. We are looking for $\gamma(t)$ of the form

$$\gamma(t) := t + if(t), \quad f \in C^\infty(\mathbb{R}; \mathbb{R}), \quad f(t+1) = f(t). \quad (5.39)$$

Condition (5.34) then means that

$$\begin{aligned} \operatorname{Re} W(t + if(t)) - f'(t) \operatorname{Im} W(t + if(t)) &> 0, \\ f'(t) \operatorname{Re} W(t + if(t)) + \operatorname{Im} W(t + if(t)) &= 0. \end{aligned}$$

This means that f has to solve

$$f'(t) = -\frac{\operatorname{Im} W(t + if(t))}{\operatorname{Re} W(t + if(t))},$$

and we choose $f(0) = 0$ as the initial condition. By (5.38), this Cauchy problem has a solution for $0 \leq t \leq 1$, and moreover, $|f(t)| \leq 1$ there. In the notation of (5.39), we have

$$W(\gamma(t))\gamma'(t) = \frac{|W(t + if(t))|^2}{\operatorname{Re} W(t + if(t))} > 0, \quad t \in [0, 1].$$

In particular,

$$\int_0^{1+if(1)} W(z) dz = \int_0^1 W(\gamma(t))\gamma'(t) dt > 0.$$

Since $\int_0^1 W(z) dz > 0$, we see that

$$0 = \operatorname{Im} \int_1^{1+if(1)} W(z) dz = \int_0^{f(1)} \operatorname{Re} W(1 + is) ds.$$

Since $\operatorname{Re} W(1 + is) > 0$, we see that $f(1) = 0$. But then, periodicity of W shows that $f_1(t) := f(t+1)$ solves the same equation as $f(t)$ with the same initial data. Hence $f(t) = f(t+1)$ and $\gamma(t+1) = \gamma(t) + 1$. Thus γ is a Stokes loop. ■

5.6. Multiplicity in terms of the transition matrix

For $k \in \mathbb{C}$, let $\mathcal{V}_{k,h}$ be the space of entire functions satisfying the periodicity condition

$$u(z+1) = e^{ik/h} u(z), \quad z \in \mathbb{C}. \quad (5.40)$$

We study the values of (α, k) for which the operator $P(\alpha, h)$ has a kernel on $\mathcal{V}_{k,h}$. More precisely, for α satisfying (5.6) and a fixed $k \in \mathbb{C}$, we define the multiplicity in the way similar to (2.7) (but with the roles of α and k reversed). For that, we consider

$$\begin{aligned} P_k(\alpha, h) &= P(\alpha, h): H_k^2 \rightarrow L_k^2, \\ H_k^s &:= \{w \in H_{\text{loc}}^s(\mathbb{R}) : w(x+1) = e^{ik/h} w(x)\} \end{aligned}$$

($H_k^0 = L_k^2 \simeq L^2(\mathbb{R}/\mathbb{Z})$) and then put

$$m_h(\alpha, k) = \frac{1}{2\pi i} \operatorname{tr}_{L_k^2} \oint_\alpha P_k(\beta, h)^{-1} \partial_\beta P_k(\beta, h) d\beta, \quad (5.41)$$

where the integral is over a positively oriented circle including α as the only possible pole of $\beta \mapsto P_k(\beta, h)$.

We note that, in the notation of (1.12),

$$\begin{aligned} P_k(\alpha, h) &= h^2 \tau(k) P_{k/h} \left(\frac{\alpha}{h} \right) \tau(k)^{-1}, \\ \tau(k) w(x) &:= e^{ikx/h} w(x), \quad \tau(k): H^s(\mathbb{R}/\mathbb{Z}) \rightarrow H_k^s. \end{aligned} \quad (5.42)$$

Hence, the meromorphy of $\beta \mapsto P_k(\beta, h)$ is guaranteed by Lemma 5.1, except for $k/h \in 2\pi\mathbb{Z}$ when we can have $\ker P_k(\alpha, h) \neq \{0\}$ for all α (see the remark after Lemma 5.1). In that case, we put $m_h(\alpha, k) = \infty$.

Since we consider $V(x, y) = W(x)^2$, the Fourier mode decomposition in y gives, in the notation of (1.1),

$$P(\beta, k) = \bigoplus_{m \in \mathbb{Z}} P_{2\pi mi + k}(\beta).$$

From (5.42), we then have

$$m\left(\frac{\alpha}{h}, \frac{k}{h}\right) = \sum_{m \in \mathbb{Z}} m_h(\alpha, 2\pi i m h + k),$$

where the right-hand side was defined in (5.5).

We will now relate the multiplicity defined by (5.41) to the multiplicity of zeros of an eigenvalue equation for $\mathcal{M}_{z_0+1 \leftarrow z_0}$. We note that for fixed h , the holomorphic dependence on the parameter α shows that

$$\mathbb{C} \ni \alpha \mapsto \mathcal{M}_{z_0+1 \leftarrow z_0}(\alpha, h) \quad \text{is holomorphic.}$$

Proposition 5.9. Fix $k \in \mathbb{C}$ and $h > 0$. For an arbitrary $z_0 \in \mathbb{C}$,

$$\alpha \mapsto P_k(\alpha, h)^{-1} \text{ is meromorphic} \Leftrightarrow \det(\mathcal{M}_{z_0+1 \leftarrow z_0}(\alpha, h) - e^{ik/h}) \neq 0.$$

If the determinant on the right-hand side is not identically 0, then

$$m_h(\alpha, k) = \frac{1}{2\pi i} \operatorname{tr}_{\mathbb{C}^2} \oint_{\alpha} (\mathcal{M}_{z_0+1 \leftarrow z_0}(\beta, h) - e^{ik/h})^{-1} \partial_{\beta} \mathcal{M}_{z_0+1 \leftarrow z_0}(\beta, h) d\beta, \quad (5.43)$$

where the integral is as in (5.41). In other words, the multiplicity $m_h(\alpha, k)$ is the same as the multiplicity of the zero of the eigenvalue equation

$$\beta \mapsto \det(\mathcal{M}_{z_0+1 \leftarrow z_0}(\beta, h) - e^{ik/h})$$

at $\beta = \alpha$.

Proof. (1) We start by observing that since $W(z+1) = W(z)$, $\mathcal{M}_{z_0 \leftarrow z} = \mathcal{M}_{z_0+1 \leftarrow z+1}$. Hence

$$\mathcal{M}_{z+1 \leftarrow z} = \mathcal{M}_{z+1 \leftarrow z_0+1} \mathcal{M}_{z_0+1 \leftarrow z_0} \mathcal{M}_{z_0 \leftarrow z} = \mathcal{M}_{z \leftarrow z_0} \mathcal{M}_{z_0+1 \leftarrow z_0} \mathcal{M}_{z \leftarrow z_0}^{-1} \quad (5.44)$$

and the statements for a fixed z_0 are equivalent to statements for all z .

The obstruction to the meromorphy of $P_k(\alpha, h)^{-1}$ is having a nontrivial solution $w(\alpha) \in H_k^2$ to $P_k(\alpha, h)w(\alpha) = 0$ for all α . As discussed in Section 5.2, $x \mapsto w(\alpha, x)$ extends to an entire function which then satisfies (5.40) and $P(\alpha, h)w = 0$. Definition of $\mathcal{M}_{z+1 \leftarrow z}(\alpha, h)$ in Section 5.3 then shows that $e^{ik/h}$ is an eigenvalue of $\mathcal{M}_{z+1 \leftarrow z}$ for all z . That gives $\Leftarrow$ in the first statement. To see $\Rightarrow$, we note that meromorphy of $P_k(\alpha, h)^{-1}$ implies existence of α_0 for which $\ker P_k(\alpha_0, h) = \{0\}$. If $e^{ik/h}$ were an eigenvalue of $\mathcal{M}_{z_0+1 \leftarrow z_0}(\alpha_0, h)$, then by (5.44) it would be an eigenvalue of $\mathcal{M}_{z+1 \leftarrow z}(\alpha_0, h)$ for all z and we would have an entire solution to $P(\alpha_0, h)u = 0$ satisfying (5.40). But that means that $\ker P_k(\alpha, h)$ is nontrivial.

Formula (5.43) makes the above argument quantitative, and it is easier to obtain in the setting of operators acting on periodic function:

$$\tilde{P}_k(\alpha, h) := e^{-ixk/h} P_k(\alpha, h) e^{ixk/h}: H^2(\mathbb{R}/\mathbb{Z}) \rightarrow L^2(\mathbb{R}/\mathbb{Z}).$$

The related operator acting on holomorphic functions is given by

$$\tilde{P}_k(\alpha, h) = (hD_z + k)^2 - \alpha^2 W^2.$$

The corresponding transition matrix, $\mathcal{M}_{z_1 \leftarrow z_0}^k$, is defined as in (5.9) but with (5.8) replaced by $\mathbf{u}(z) = (u(z), (hD_z + k)u(z))$. It is related to the original transition matrix as follows:

$$\mathcal{M}_{z_1 \leftarrow z_0}^k(\alpha, h) = e^{ik(z_0 - z_1)/h} \mathcal{M}_{z_1 \leftarrow z_0}(\alpha, h). \quad (5.45)$$

The final reduction is to a system

$$D_k(\alpha, h) := \begin{pmatrix} hD_z + k & -1 \\ -\alpha^2 W(z)^2 & hD_z + k \end{pmatrix}, \quad D_k(\alpha, h) \mathcal{M}_{z \leftarrow z_0}^k(\alpha, h) = 0,$$

and $\mathcal{M}_{z_0 \leftarrow z_0}^k(\alpha, h) = I_{\mathbb{C}^2}$.

Proceeding as in (2.10), we see that in the definition of $m_h(\alpha, k)$ we can replace $P_k(\alpha, h)$ by $D_k(\alpha, h)$ (now acting on periodic $\mathbb{C}^2$ -valued functions) and in view of (5.44) and (5.45), equality (5.43) is equivalent to

$$\begin{aligned} & \frac{1}{2\pi i} \operatorname{tr}_{L^2(\mathbb{R}/\mathbb{Z}; \mathbb{C}^2)} \oint_{\alpha} D_k(\beta, h)^{-1} \partial_{\beta} D_k(\beta, h) d\beta \\ &= \frac{1}{2\pi i} \operatorname{tr}_{\mathbb{C}^2} \oint_{\alpha} (\mathcal{M}_{1 \leftarrow 0}^k(\beta, h) - I)^{-1} \partial_{\beta} \mathcal{M}_{1 \leftarrow 0}^k(\beta, h) d\beta. \end{aligned} \quad (5.46)$$

(2) A systematic derivation of formula (5.46) follows for setting up a Grushin problem (see [10, §C.1]) for $D_k(\alpha, h)$ following [24, (3.12)]. To pose it, we introduce

$$\chi \in C^\infty(\mathbb{R}/\mathbb{Z}; [0, 1]), \quad \mathbb{R}/\mathbb{Z} \simeq \left(-\frac{1}{2}, \frac{1}{2}\right], \quad \operatorname{supp} \chi = \left[-\frac{1}{4}, \frac{1}{2}\right], \quad \chi|_{[-1/8, 1/4]} = 1.$$

We write $\chi' = \chi'_+ + \chi'_-$, where χ'_+ is supported near $-1/4$ and χ'_- is supported near $\pm 1/2$. The following operators are also well defined:

$$\begin{aligned} R_-: \mathbb{C}^2 &\rightarrow L^2(\mathbb{R}/\mathbb{Z}; \mathbb{C}^2), & (R_- u_-)(x) &:= \chi'_- \mathcal{M}_{x \leftarrow 0}^k u_-, \\ R_+: H^1(\mathbb{R}/\mathbb{Z}; \mathbb{C}^2) &\rightarrow \mathbb{C}^2, & R_+ \mathbf{u} &:= \mathbf{u}(0). \end{aligned} \quad (5.47)$$

We then claim that

$$\mathcal{P} := \begin{pmatrix} \frac{i}{h} D_k(\alpha, h) & R_- \\ R_+ & 0 \end{pmatrix} : H^1(\mathbb{R}/\mathbb{Z}; \mathbb{C}^2) \oplus \mathbb{C}^2 \rightarrow L^2(\mathbb{R}/\mathbb{Z}; \mathbb{C}^2) \oplus \mathbb{C}^2$$

is invertible with the inverse given by

$$\mathcal{P}^{-1} = \mathcal{E} := \begin{pmatrix} E & E_- \\ E_+ & E_{-+} \end{pmatrix}, \quad E_{-+} = \mathcal{M}_{0 \leftarrow 1}^k - I_{\mathbb{C}^2}(\alpha, h). \quad (5.48)$$

Conclusion (5.46), and hence (5.43), then follow from [24, Proposition 4.1].

(3) To show invertibility of (5.47), it is enough to construct an inverse. The operator $D_k(\alpha, h)$ is a Fredholm operator and, as

$$\mathcal{P}_k^t(\alpha, h) := \begin{pmatrix} \frac{i}{h} D_k(\alpha, h) & (1-t)R_- \\ (1-t)R_+ & tI_{\mathbb{C}^2} \end{pmatrix} : H^1(\mathbb{R}/\mathbb{Z}; \mathbb{C}^2) \oplus \mathbb{C}^2 \rightarrow L^2(\mathbb{R}/\mathbb{Z}; \mathbb{C}^2) \oplus \mathbb{C}^2$$

are finite rank perturbations of a Fredholm operator (with $D_k(\alpha, h)$ as the only non-zero entry), they are also Fredholm operators (see, for instance, [15, §2.5] for basic properties of those operators). Since $D_k(0, k)$ is invertible for $k/h \notin 2\pi\mathbb{Z}$, it follows that $\mathcal{P}_1^k$ is invertible. Hence continuity of the index shows that the index of $\mathcal{P}$ is 0 and the right inverse is automatically the left inverse.

We construct the right inverse explicitly as follows. We first consider the problem

$$\frac{i}{h} D_k \mathbf{u}(x) + \chi'_-(x) \mathcal{M}_{x \leftarrow 0}^k u_- = 0, \quad \mathbf{u}(0) = v_+ \in \mathbb{C}^2. \quad (5.49)$$

We can define multivalued matrix valued functions on $\mathbb{R}/\mathbb{Z}$ as follows:

$$I_{\pm}([x]) = \mathcal{M}_{x \leftarrow 0}^k, \quad \pm x \geq -\frac{1}{4},$$

where $x \mapsto [x]$ is the map $\mathbb{R} \rightarrow \mathbb{R}/\mathbb{Z}$. We can think of $I_+(x)$ as solving the equation forward from 0 and $I_-(x)$ backward from 0. The function $\chi(x)I_+(x)v_+, (1 - \chi(x))I_-(x)v_+$ are now well defined as functions on the circle (we drop $[x]$ notation when there is no ambiguity). We can then put

$$\mathbf{u}(x) = \chi(x)I_+(x)v_+ + (1 - \chi(x))I_-(x)v_+ \in H^1(\mathbb{R}/\mathbb{Z}; \mathbb{C}^2),$$

so that

$$\begin{aligned} \frac{i}{h} D_k \mathbf{u} &= \frac{i}{h} [D_k, \chi] I_+ v_+ - \frac{i}{h} [D_k, \chi] I_- v = \chi' I_+ v_+ - \chi' I_- v_+ \\ &= \chi'_-(x)(I_+(x)v_+ - I_-(x)v_+), \end{aligned}$$

where we used the fact that $I_+(x)v_+ = I_-(x)v_+$ on the support of χ'_+ (near 0). On the other hand, for $[x]$ on the support of χ'_- , say for x near $[1/2]$, we have

$$\begin{aligned}\chi'_-([x])I_+([x]) &= \chi'_-(x)\mathcal{M}_{x\leftarrow 0}^k = \chi'_-(x)\mathcal{M}_{x-1\leftarrow -1}^k = \chi'_-(x)\mathcal{M}_{x-1\leftarrow 0}^k\mathcal{M}_{0\leftarrow -1}^k \\ &= \chi'_-(x)I_-([x])\mathcal{M}_{1\leftarrow 0}^k.\end{aligned}$$

Hence (5.49) is solved by putting

$$\mathbf{u} = E_+v_+ := (\chi I_+ + (1 - \chi)I_-)v_+, \quad u_- = E_-v_+ := (\mathcal{M}_{0\leftarrow 1}^k - I_{\mathbb{C}^2})v_+. \quad (5.50)$$

It remains to solve

$$\frac{i}{h}D_k\mathbf{u}(x) + \chi'_-(x)\mathcal{M}_{x\leftarrow 0}^ku_- = v \in L^2(\mathbb{R}/\mathbb{Z}; \mathbb{C}^2), \quad \mathbf{u}(0) = 0,$$

and that is done similarly using the forward and backward solutions to the non-homogeneous problem. Since all that matters for us is the formula for E_{-+} in (5.50), we leave the details to the reader. This completes the proof of (5.48) and, as explained there, of (5.43). $\blacksquare$

5.7. Quantisation condition

Now, we assume that γ is a Stokes loop and $\text{Im } \alpha \geq 0$. We note that this assumption can be relaxed to $\text{Im } \alpha \geq -h^M$ for some (large) M .

We will then put $z_0 := \gamma(0)$ in Proposition 5.9, so that $\mathcal{M}_{z_0+1\leftarrow z_0} = \mathcal{M}_{\gamma(1)\leftarrow \gamma(0)}$. Putting

$$Z_h(\alpha) := \frac{1}{2} \text{tr } \mathcal{M}_{\gamma(1)\leftarrow \gamma(0)}(\alpha, h),$$

and using (5.11), the eigenvalue equation can be written as

$$0 = \frac{1}{2}e^{-ik/h} \det(\mathcal{M}_{z_0+1\leftarrow z_0}(\alpha, h) - e^{ik/h}) = \cos\left(\frac{k}{h}\right) - Z_h(\alpha), \quad (5.51)$$

and we want to find α 's for which the right-hand side vanishes.

Recalling (5.14), we note that

$$\psi_{\gamma, \alpha}(1) - \psi_{\gamma, \alpha}(0) = c_0 \text{Im } \alpha, \quad \text{where } c_0 := \int_0^1 W(z) dz > 0.$$

Now, by Lemma 5.4 (recalling (5.36))

$$\mathcal{M}_{\gamma(1)\leftarrow \gamma(0)} = B_N(\gamma(1); \alpha, h)^{-1} S_{\gamma, N}(\alpha, h) B_N(\gamma(0); \alpha, h) + \mathcal{O}_{\gamma, N}(h^{N+1} e^{c_0 \text{Im } \alpha / h}).$$

By Lemma 5.5, we have

$$B_N(\gamma(1); \alpha, h) = B_N(\gamma(0); \alpha, h), \quad S_{\gamma, N}(\alpha, h) = \begin{pmatrix} e^{-i\alpha h^{-1}c(h/\alpha)} & 0 \\ 0 & e^{i\alpha h^{-1}c(h/\alpha)} \end{pmatrix}.$$

Therefore,

$$Z_h(\alpha) = \cos\left(\alpha h^{-1} c\left(\frac{h}{\alpha}\right)\right) + R_h(\alpha), \quad R_h(\alpha) = \mathcal{O}_{\gamma, N}(h^{N+1} e^{c_0 \operatorname{Im} \alpha / h}). \quad (5.52)$$

One observation is that for the eigenvalue equation to be solvable, we need

$$|\operatorname{Im} k| \leq c_0 |\operatorname{Im} \alpha| + Ch.$$

We can also assume that $|\operatorname{Re} k| \leq \pi h$ since the spaces $\mathcal{V}_{k+2\pi h, h}$ and $\mathcal{V}_{k, h}$ defined in (5.40) are the same.

We first consider the simplified equation obtained from (5.51) and (5.52):

$$F_0(\alpha, k, h) := \cos\left(c\left(\frac{h}{\alpha}\right) \frac{\alpha}{h}\right) - \cos\left(\frac{k}{h}\right) = 0$$

which holds when

$$c\left(\frac{h}{\alpha}\right) \alpha = 2\pi n h \pm k, \quad n \in \mathbb{Z},$$

noting that the roots are double when $k/h \in \pi\mathbb{Z}$. The solutions satisfying (5.6) have asymptotic expansions (where we use the fact that $c_1 = 0$ in (5.30))

$$\begin{aligned} \alpha_{\pm}(n, k, h) &= c_0^{-1}(2\pi n h \pm k) + d_2(2\pi n h \pm k)^{-1} h^2 \\ &\quad + d_3(2\pi n h \pm k)^{-2} h^3 + \dots, \quad d_2 = -c_2, \quad \dots \end{aligned} \quad (5.53)$$

We can now compare these solutions to the solutions of (5.51), for $\operatorname{Im} \alpha \geq -h^M$, using (5.52) and Rouché's theorem applied to a contour consisting of h^N -sized discs centred at $\alpha_{\pm}(n, k, h)$. The symmetry $\alpha \rightarrow -\alpha$ then gives the result for $\operatorname{Im} \alpha \leq 0$ as well. To be consistent with this section, in comparison with Theorem 1.3 we assume that $\int_{\gamma} W(z) dz > 0$, that is, $\arg W_0 = 0$. Rescaling α , this can be removed.

Theorem 5.10. *Under the assumptions of Theorem 1.3 and when $\arg W_0 = 0$, there exist a constant C_0 and maps*

$$\tilde{\alpha}_{\pm}: \mathbb{Z} \times \mathbb{C} \rightarrow \mathbb{C},$$

such that with $\alpha_{\pm}(n, k) := \alpha_{\pm}(n, k, 1)$ (see (5.53)),

$$\tilde{\alpha}_{\pm}(n, k) - \alpha_{\pm}(n, k) = \mathcal{O}(|2\pi n \pm k|^{-\infty}),$$

and for $|\alpha| \geq C_0$, in the notation of (5.5),

$$m(\alpha, k) = |\{(n, \varepsilon) : \tilde{\alpha}_{\varepsilon}(n, k) = \alpha, \varepsilon = \pm 1\}|.$$

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