

A WKB approach to semiclassical parametrices

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Abstract. In this paper, we examine the relationship between WKB approximations and semiclassical parametrices. The fact that the two are related is a point of mathematical folk law in the field of semiclassical analysis. Here, we will set the relationship out precisely and illustrate it with a couple of examples. We will see that semiclassical parametrices for the Schrödinger propagator can be constructed directly from WKB solutions and that this approach has the advantage of allowing us to consider parametrices accurate only to highest order (in h) rather than all orders. Such parametrices are often sufficient to suit the technical needs of semiclassical analysis and the WKB method of finding approximate solutions is particularly suitable for obtaining useful explicit expressions for the top-order terms.

1. Introduction

One of the great difficulties in writing about “the WKB approximation” is that this approximation means different things to different people. So, it would be more to the point to say that this paper is about a correspondence principle between semiclassical Fourier integral operators and some WKB approximation problems. Though diverse are they, all WKB approximations share three important features.

- (1) They search for high-frequency solutions to a PDE (most commonly a Schrödinger equation).
- (2) They involve either a slowly (compared to the frequency) varying potential or slowly varying coefficients.
- (3) They involve a series solution in a semiclassical parameter, usually with very explicit solutions for the top-order terms then less explicit solutions for lower-order terms.

In this paper, we will concern ourselves with the linear time-dependent Schrödinger equation

$$\begin{cases} \left(hD_t + \frac{1}{2}h^2\Delta + V(x) \right) u(t, x) = 0, \\ u(0, x) = u_0(x), \end{cases} \quad (1.1)$$

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where

$$D_t = \frac{1}{i} \frac{\partial}{\partial t}, \quad \Delta = \sum_{j=1}^n D_{x_j}^2 = - \sum_{j=1}^n \left(\frac{\partial}{\partial x_j} \right)^2$$

and $V(x)$ is real-valued and smooth. Alongside this time-dependent version, we often want to construct solutions

$$u(t, x) = e^{\frac{it}{h} E} \psi(x),$$

where ψ satisfies the time-independent eigenfunction equation

$$P_V \psi = \left(\frac{1}{2} h^2 \Delta - E + V(x) \right) \psi = 0, \quad (1.2)$$

but typically we approach the time-independent problem by time averaging the time-dependent problem. This is because in all but a few very special settings it is impossible to find good exact expressions for solutions to eigenfunction problems. However, if ψ is a solution to (1.2), then $e^{\frac{it}{h} E} \psi$ is a solution to (1.1) but with initial condition ψ . Therefore,

$$\psi(x) = \frac{1}{T_2 - T_1} \int_{T_1}^{T_2} e^{-\frac{it}{h} E} u(t, x) dt = \frac{1}{T_2 - T_1} \int_{T_1}^{T_2} e^{-\frac{it}{h} E} e^{\frac{it}{h} P_V} \psi dt. \quad (1.3)$$

Suppose that we can obtain a parametrix solutions for $e^{\frac{it}{h} P_V}$ of the form

$$e^{\frac{it}{h} P_V} \psi = \int e^{\frac{i}{h} \phi(t, x, \xi) - \langle y, \xi \rangle} a(t, x, \xi) \psi(y) d\xi dy$$

for $\phi(t, x, \xi)$ and $a(t, x, \xi)$ smooth, real-valued functions. Then, the time averaging in (1.3) allows us to construct a reproducing operator for ψ . By studying the mapping properties of this reproducing operator, we can extract information about ψ .

One can see time averaging for eigenfunctions as a reversal one of the first PDE techniques that any mathematics students learn. In any introductory PDE class we learn to write the solution to time-dependent problems (Schrödinger, wave, heat) in terms of their stationary states and, in a few special cases where those eigenfunction equations can be explicitly solved, are able to then determine the solution to the time dependent problem. However, in the general setting, we have many more techniques to solve (at least locally) time dependent problems than time-independent ones. So, we reverse the relationship and use time averaging to extract eigenfunctions from solutions to time-dependent problems.

The main purpose of this paper is to demonstrate a correspondence between WBK solutions and the construction of parametrices for semiclassical Fourier integral operators. It is not surprising that these two constructions are linked. Both involve the construction of solutions by “solving away” powers of h . The WBK method gives an approximate solution $u(t, x)$ for specific initial conditions, while the semiclassical Fourier integral operator methods produce a parametrix for the evolution operator

$$U(t) = e^{\frac{it}{h} P_V}.$$

We link the two ideas via a decomposition of a general initial function ψ into simpler analysis functions. These analysis functions then become the initial conditions in a WKB construction. That is the following.

- (1) First, we decompose ψ via an analysis/synthesis process,

$$T[\psi](\alpha) = \int \psi(x) w_\alpha(x) dx, \quad \psi = \int T[\psi](\alpha) \tilde{w}_\alpha(x) d\mu(\alpha).$$

For instance, we may use the semiclassical Fourier transform and decompose ψ into plane waves $w_\xi = (2\pi h)^{-n/2} e^{-\frac{i}{h}\langle x, \xi \rangle}$. Here, $\tilde{w}_\xi = \bar{w}_\xi$ and $d\mu(\xi) = d\xi$. Then,

$$\begin{aligned} T[\psi](\xi) &= \mathcal{F}_h[\psi](\xi) = \frac{1}{(2\pi h)^{n/2}} \int e^{-\frac{i}{h}\langle x, \xi \rangle} \psi(x) dx, \\ \psi &= \frac{1}{(2\pi h)^{n/2}} \int e^{\frac{i}{h}\langle x, \xi \rangle} \mathcal{F}_h[\psi](\xi) d\xi. \end{aligned}$$

- (2) We then use the WKB method to find a local solution to

$$\begin{cases} \left(hD_t + \frac{h^2}{2} \Delta + V(x) \right) W_\alpha(t, x) = 0, \\ W_\alpha(t, x) = \tilde{w}_\alpha \end{cases}$$

for each value of α .

- (3) Finally, we apply the synthesis process to reproduce $u(t, x)$.

Computing purely formally if we write

$$U(t)\psi = \int T[\psi](\alpha) W_\alpha(t, x) d\mu(\alpha),$$

we see that, indeed,

$$\left(hD_t + \frac{h^2}{2} \Delta + V(x) \right) U(t)\psi = \int T[\psi](\alpha) \left(hD_t + \frac{h^2}{2} \Delta + V(x) \right) W_\alpha(t, x) d\mu(\alpha) = 0$$

and

$$U(0)\psi = \int T[\psi] W_\alpha(0, x) d\mu(\alpha) = \int T[\psi] \tilde{w}_\alpha(x) d\mu(\alpha) = \psi.$$

Therefore, provided we can make the formal computations legitimate we do, indeed, have a parametrix solution for the evolution operator $U(t)$ valid for short times. In most cases, we will be looking at initial data ψ that has some sort of localisation that does, indeed, justify the above formal computations.

Say, for example, we take the simplest approach and use the Fourier transform to develop our analysis/synthesis system. Since we are working with an inherently semiclassical problem, we use the semiclassical Fourier transform,

$$\begin{aligned} \mathcal{F}_h[f](\xi) &= \frac{1}{(2\pi h)^{n/2}} \int e^{-\frac{i}{h}\langle x, \xi \rangle} f(x) dx, \\ \mathcal{F}_h^{-1}[g](x) &= \frac{1}{(2\pi h)^{n/2}} \int e^{\frac{i}{h}\langle x, \xi \rangle} g(\xi) d\xi. \end{aligned}$$

The choice of pre-factors for this Fourier transform/inverse Fourier transform has been set to preserve the isometry properties of the Fourier transform. That is,

$$\|F_h[f]\|_{L^2} = \|f\|_{L^2}.$$

In this case, we need to solve a WKB problem

$$\begin{cases} \left(hD_t + \frac{h^2}{2}\Delta + V(x)\right)u(t, x) = 0, \\ u(0, x) = \frac{1}{(2\pi h)^{n/2}} e^{\frac{i}{h}\langle x, \xi \rangle}. \end{cases}$$

Following the classical WKB approach, we look for a solution

$$W_\xi(t, x) = \sum_{j=0}^{\infty} h^j a_{\xi,j}(t, x) e^{\frac{i}{h}\phi(t, x)},$$

where

$$\begin{cases} \phi(0, x) = \langle x, \xi \rangle, \\ a_0(0, x) = \chi(x), \\ a_j(t, x) = 0, \quad j \geq 1. \end{cases}$$

Here, $\chi(x)$ is a smooth localiser. This is a very well-known WKB problem. The only order 0 terms in the expansion come from when all derivatives hit the phase function, yielding the eikonal equation

$$\phi_t + \frac{1}{2}|\nabla\phi|^2 + V(x) = 0. \quad (1.4)$$

The $a_{\xi,j}$ can then be determined via transport equations. The solution to the eikonal equation is geometrically significant. Let $(X(t), \Xi(t))$ be the flow determined by

$$\begin{cases} X'(t) = \Xi, & X(0) = x, \\ \Xi'(t) = -\partial_x V, & \Xi(0) = \xi. \end{cases} \quad (1.5)$$

Then, via the method of characteristics, we have

$$\nabla_x \phi(t, x) = \Xi(t).$$

Therefore, solving (1.4) reduced to solving (1.5). The short time existence of a solution to (1.5) is classical (see [3, Section 3.4.2]) and in some cases can in fact be extended to global solutions. However, in most semiclassical applications, we are working with localised functions so that short time solutions are sufficient.

Then, let us consider the order h term. By collecting terms, we find that $a_{\xi,0}$ must satisfy the transport equation

$$\partial_t a_{\xi,0} + \nabla\phi \cdot \nabla a_{\xi,0} + \frac{1}{2}a_{\xi,0}\Delta\phi = 0, \quad a_{0,\xi}(0, x) = \chi(x). \quad (1.6)$$

This is another well-known PDE which has solution

$$a_{0,\xi}(t, x) = \frac{1}{\sqrt{J_t(X_t^{-1}(x))}} \chi(X_t^{-1}(x)),$$

where $X_t^{-1}(x)$ is the inverse of the flow (1.5) and

$$J_t(X_t^{-1}(x)) = \det(\nabla_x X(t, x)).$$

Lower order terms $a_{\xi,j}$ are similarly solved by solving the transport equations for each of the $a_{\xi,j}$.

Finally, we write

$$\begin{aligned} U(t) &= \int \mathcal{F}_h[u_0](\xi) W_\xi(t, x) d\xi \\ &= \frac{1}{(2\pi h)^n} \int e^{\frac{i}{h}(\phi_\xi(t, x) - \langle y, \xi \rangle)} a_\xi(t, x) u_0(y) d\xi dy, \end{aligned} \quad (1.7)$$

where

$$a_\xi(t, x) = \sum_{j=0}^{\infty} h^j a_{\xi,j}(t, x).$$

In this case, we see that our expression for $U(t)$ agrees with the well-known (see, for example, [23, Section 10.2]) semiclassical Fourier integral operator parametrix for the semiclassical Schrödinger operator. We can then use $U(t)\psi$ to express solutions to the time-dependent, or by time averaging, time independent Schrödinger problems.

While (1.7) gave us an expression for $U(t)$, it might not be the most useful one for application. The solution of the eikonal equation introduces very deep connections to the behaviour of dynamical systems and we may wish to more clearly see that in the integral representation. We may also be studying a solution where the Fourier transform is an inefficient process for analysis/synthesis. For instance, if ψ has significant special concentration, it would be better to use an analysis/synthesis system that respected that localisation (as opposed to the Fourier transform which analyses by delocalised functions). From a technical perspective, it is often not necessary to have an $O(h^\infty)$ parametrix. Particularly, it is often the case that we are working with functions that are semiclassically localised.

Definition 1. We say that a function u is semiclassically localised if there exists a compactly supported, smooth χ so that

$$\begin{aligned} u &= \chi(x, hD)u + O_{L^2}(h^\infty) \\ &= \frac{1}{(2\pi h)^n} \int e^{\frac{i}{h}\langle x-y, \xi \rangle} \chi(x, \xi) u(y) d\xi dy + O_{L^2}(h^\infty). \end{aligned}$$

The application of a localiser negates the need to have an $O(h^\infty)$ parametrix. Suppose we had a solution u such that

$$\left(hD_t + \frac{1}{2}\Delta + V(x)\right)u = 0,$$

and we apply a semiclassical localiser $\chi(x, hD)$ to u . Then,

$$\left(hD_t + \frac{1}{2}\Delta + V(x)\right)\chi(x, hD)u = \chi(x, hD)\left(hD_t + \frac{1}{2}\Delta + V(x)\right)u + hr(x, hD)u,$$

where the principal symbol of r is given by

$$\frac{1}{i}\left\{\frac{1}{2}|\xi|^2 + V(x), \chi(x, \xi)\right\}.$$

So, even if we begin with a perfect solution, the act of localizing it reduces it to only an order h solution. However, order h solutions are frequently good enough. Since our solution is now only accurate up to order h , there is no advantage to the parametrix being accurate up to $O(h^\infty)$. Therefore, it is sensible to seek $O_{L^2}(h)$ parametrices, particularly those where we can obtain very explicit representations for the top-order terms. In this case, thinking of the construction as a WKB solution is particularly valuable as the WKB method is structured around providing very explicit top-order terms (at the cost of knowledge of the lower order terms).

2. Highest-order parametrices

We want to be able to use WKB methods to obtain parametrices that are valid up to order h error (typically, we will measure that error in L^2 norms). From our $O(h^\infty)$ construction in the previous section, we see that

$$W_{\text{FT}}(t)u = \frac{1}{(2\pi h)^n} \int e^{\frac{i}{h}(\phi_\xi(t, x) - \langle y, \xi \rangle)} \frac{1}{\sqrt{J_t(X_t^{-1}(x))}} \chi(X_t^{-1}(x))u(y) d\xi dy$$

is just such an $O_{L^2}(h)$ parametrix (associated with a Fourier analysis/synthesis). However, we want to be able to think about such a construction more generally to allow for a wider range of analysis/synthesis systems. Let us now formalise the process described in the introduction.

Definition 2. We say that $(w_\alpha, \tilde{w}_\alpha)$ is an L^2 -bounded analysis/synthesis system if given

$$T_\alpha[f] = \int f(x)w_\alpha(x)dx.$$

Then, the following statements hold.

- (1) There exists a measure $\mu(\alpha)$ so that

$$f(x) = \int T_\alpha[f]\tilde{w}_\alpha(x)d\mu(\alpha).$$

- (2) There exist constants $c, C \in \mathbb{R}$ so that

$$c\|f\|_{L^2(\mathbb{R}^n)} \leq \|T_\alpha[f]\|_{L^2(\mu)} \leq C\|f\|_{L^2(\mathbb{R}^n)}.$$

Remark 1. The semiclassical Fourier transform where

$$w_\alpha = \frac{1}{(2\pi h)^{n/2}} e^{-\frac{i}{h}\langle x, \xi \rangle}, \quad \tilde{w}_\alpha = \frac{1}{(2\pi h)^{n/2}} e^{\frac{i}{h}\langle x, \xi \rangle}$$

is the most obvious example of an L^2 -bounded analysis/synthesis system, although there are many others.

Definition 3. Suppose $(w_\alpha, \tilde{w}_\alpha)$ is an L^2 -bounded analysis/synthesis system. We say that a parameter dependent family $W_\alpha(t, x)$ of WKB solutions produce a local order h Schrödinger parametrix if for $t \in [0, \varepsilon]$ for some $\varepsilon > 0$ independent of h ,

(1) we have

$$\begin{cases} \left(hD_t + \frac{h^2}{2} \Delta + V(x) \right) W_\alpha(t, x) = h r_\alpha(t, x), \\ W_\alpha(0, x) = \tilde{w}_\alpha(x). \end{cases}$$

(2) for each $t \in [0, \varepsilon]$,

$$\left\| \int T_\alpha[\psi] W_\alpha(t, x) d\mu(\alpha) \right\|_{L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n)} \lesssim 1, \quad (2.1)$$

$$\left\| \int T_\alpha[\psi] r_\alpha(t, x) d\mu(\alpha) \right\|_{L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n)} \lesssim 1. \quad (2.2)$$

In that case, we define the parametrix $W(t) : L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n)$ by

$$W(t)u = \int T_\alpha[\psi] W_\alpha(t, x) d\mu(\alpha).$$

Remark 2. The second condition usually requires some proving. It may seem like it would be efficient to place some conditions on $W_\alpha(t, x)$ and $r_\alpha(t, x)$ that would guarantee L^2 boundedness. However, in practice, there are a wide range of conditions that could ensure (2) and it is better to leave those conditions as dependent on the context. Usually, (2.1) and (2.2) are proved by the same argument. We will see a good example of this in the proof of Theorem 3.2 where the same almost orthogonality argument is used to obtain both (2.1) and (2.2) in the case of a wave-packet analysis/synthesis system.

Theorem 2.1. Suppose $W_\alpha(t, x)$ produces a local order h Schrödinger solution. Then, if

$$\left(hD_t + \frac{h^2}{2} \Delta + V(x) \right) u(t, x) = O_{L^2}(h), \quad u(0, x) = \psi$$

and $v = W^*(t)u(t, x)$, then

$$hD_t v = O_{L^2}(h), \quad v(0, x) = \psi.$$

Remark 3. Essentially, the operator $W^*(t)$ converts the Schrödinger equation to the simple equation $hD_t v = 0$. This is one of the original ideas behind the theory of Fourier

integral operators (FIOs), that they can be used to “change variables” and convert complicated variable coefficient equations to become simple constant coefficient equations. See the introduction of [4] for a discussion of this technique. The idea of using either $W(t)$ to propagate initial data or $W^*(t)$ to change variables also manifests the difference between the Schrödinger and Heisenberg pictures in quantum mechanics. In the Schrödinger picture, one propagates an initial condition ψ to obtain $U(t)\psi$, a time dependent state. In the Heisenberg picture, one propagates the observables (equivalent to the FIO idea of changing variables) and the state remains time independent.

Proof. Since

$$\left(hD_t + \frac{h^2}{2}\Delta + V(x)\right)W(t) = O_{L^2}(h),$$

we have that

$$hD_t W^*(t) + W^*(t) \left[\frac{h^2}{2}\Delta + V(x) \right] = O_{L^2}(h).$$

So, if $v = W^*(t)u(t, x)$, then

$$\begin{aligned} hD_t v &= [hD_t W^*]u(t, x) + W^*[hD_t u(t, x)] \\ &= W^*(t) \left[\frac{h^2}{2}\Delta + V(x) \right]u(t, x) + W^*[hD_t u(t, x)]. \end{aligned}$$

So, if

$$\left(hD_t + \frac{h^2}{2}\Delta + V(x)\right)u(t, x) = O_{L^2}(h),$$

then

$$hD_t v = O_{L^2}(h),$$

and hence, $W(0) = \text{Id}$ $v(0, x) = u(0, x) = \psi$. ■

3. Wave-packet transforms

In this section, we want to use WKB to develop order h parametrix that allows us to easily see the propagation in phase space properties. Roughly speaking, we see the “phase space portrait” of a function u as telling us how much of u is localised near (x, ξ) . We cannot talk about u being precisely localised at (x, ξ) . Doing so would violate the uncertainty principle. However, we can localize near a point so long as we remain within the legal bounds as instructed by the uncertainty principle. At semiclassical scale, we can localize up to a region $\Omega_x \times \Omega_\xi$ where for any $(y, \eta) \in \Omega_x \times \Omega_\xi$ and $\|u\|_{L^2} = 1$,

$$\left(\int_{\Omega_x} |x - y|^2 |u(x)|^2 dx \right)^{1/2} \left(\int_{\Omega_\xi} |\xi - \eta|^2 |\mathcal{F}_h[u](\xi)|^2 d\xi \right)^{1/2} \approx h.$$

Without any special knowledge to prefer localisation in x to ξ or vice versa, we choose a symmetric localisation where both x and ξ are localised to an $h^{\frac{1}{2}}$ region. Therefore, we

will use an analysis synthesis system based on the wave-packet transform, or windowed Fourier transform. We set our analysing functions

$$w_{(x,\xi)}(y) = \frac{1}{(2\pi h)^{\frac{3n}{4}}} e^{-\frac{i}{h}((y,\xi) - i\frac{|x-y|^2}{2})}.$$

Up to some scaling and modulation, this type of transform is also equivalent to Bargmann transform

$$Tf(x, \xi) = c_n \int e^{i\langle \xi, x-y \rangle} e^{-\frac{|x-y|^2}{2}} f(y) dy$$

used by Geba and Tataru [6] to study the wave equation (see also [11, Chapter 3] for a textbook treatment).

The exponential decay given by $e^{-\frac{|x-y|^2}{2h}}$ becomes significant when $|x - y| \approx h^{1/2}$. Beyond that point, the exponential decay is so strong that the effect is similar to cutting off in an $h^{1/2}$ region. So, these analysing functions are well localised in space near y at an $h^{1/2}$ scale. The oscillating factor $e^{-\frac{i}{h}\langle y, \xi \rangle}$ ensures they are localised in frequency near ξ . Pre-factors have been chosen to normalize L^2 (as we will see in Lemma 3.1).

Lemma 3.1. *The pair $(w, \tilde{w})$, given by*

$$\begin{aligned} w_{(x,\xi)}(y) &= \frac{1}{(2\pi h)^{\frac{3n}{4}}} e^{-\frac{i}{h}((y,\xi) - i\frac{|x-y|^2}{2})}, \\ \tilde{w}_{(x,\xi)}(y) &= \frac{1}{(2\pi h)^{\frac{3n}{4}}} e^{\frac{i}{h}((y,\xi) + i\frac{|x-y|^2}{2})}, \end{aligned}$$

give an L^2 -bounded analysis synthesis system.

Proof. Let

$$T_{(x,\xi)}[f] = \int w_{(x,\xi)}(y) f(y) dy;$$

then,

$$\begin{aligned} T_{(x,\xi)}[f] &= \frac{1}{(2\pi h)^{\frac{n}{2}}} \int \frac{e^{-\frac{i}{h}((y,\xi) - i\frac{|x-y|^2}{2})}}{(2\pi h)^{\frac{n}{4}}} f(y) dy \\ &= \mathcal{F}_h \left[\frac{e^{-\frac{|x-\cdot|^2}{2h}}}{(2\pi h)^{\frac{n}{4}}} f(\cdot) \right]. \end{aligned}$$

So, we have that

$$\frac{1}{(2\pi h)^{\frac{n}{4}}} e^{-\frac{|x-y|^2}{2h}} f(y) = \frac{1}{(2\pi h)^{\frac{n}{2}}} \int e^{\frac{i}{h}\langle y, \xi \rangle} T_{(x,\xi)}[f] d\xi \quad (3.1)$$

and

$$\|T_{(x,\cdot)}[f]\|_{L_\xi^2}^2 = \left\| \frac{e^{-\frac{|x-\cdot|^2}{2h}}}{(2\pi h)^{\frac{n}{4}}} f(\cdot) \right\|_{L_y^2}^2. \quad (3.2)$$

Multiplying (3.1) by $\tilde{w}_{(x,\xi)}$ and integrating in x yield

$$f(y) = \int T_{(x,\xi)}[f] \tilde{w}_{(x,\xi)} dx d\xi.$$

Similarly, integrating (3.2) in x yields

$$\|T_{(\cdot,\cdot)}[f]\|_{L^2_{(x,\xi)}} = \|f\|_{L^2(\mathbb{R}^n)}.$$

So, this is indeed an L^2 -bounded analysis/synthesis system. ■

We want to use this analysis/synthesis system with the WKB method to generate an order h parametrix. Therefore, we need to look at approximate WKB solutions to the following equation:

$$\begin{cases} \left(h D_t + \frac{1}{2} h^2 \Delta_y + V(y) \right) W_{(x,\xi)}(t, y) = 0, \\ W_{(x,\xi)}(0, y) = \frac{1}{(2\pi h)^{\frac{3n}{4}}} e^{\frac{i}{h}(\langle y, \xi \rangle + i \frac{|x-y|^2}{2})}. \end{cases}$$

In particular, we will focus on obtaining explicit representations for the top-order terms, allowing lower order error (of size $O_{L^2}(h)$) to be swept into one error term.

Motivated by the Fourier transform WKB solutions, we will look for a solution of the following form:

$$W_{(x,\xi)}(t, y) = \frac{1}{(2\pi h)^{\frac{3n}{4}}} e^{\frac{i}{h}(\phi_{(x,\xi)}(t, y) + i \theta_{(x,\xi)}(t, y))},$$

where ϕ and θ are smooth, real-valued functions. We further require that ϕ and θ obey the initial conditions

$$\phi_{(x,\xi)}(0, y) = \langle y, \xi \rangle, \quad \theta_{(x,\xi)}(0, y) = |x - y|^2.$$

Note that we do not look for a series solution here. We are only attempting to find an order h parametrix so it is unnecessary to deal with the lower order terms as they only introduce an $O_{L^2}(h)$ error.

As in any WKB solution, we consider first the order zero terms. At first, it may seem that we need to construct an eikonal type equation for $\phi + i\theta$, but a closer analysis shows that to do so is unnecessary. The exponential decay when $|x - y| \geq h^{1/2}$ coupled with the quadratic nature of $\theta_{(x,\xi)}$ means that terms where derivatives hit θ are really only of order $h^{1/2}$, so we will treat them separately. Therefore, again, we have that the first-order equation is the eikonal equation

$$\begin{aligned} \partial_t \phi_{(x,\xi)}(t, y) + \frac{1}{2} |\nabla_y \phi_{(x,\xi)}(t, y)|^2 + V(y) \phi_{(x,\xi)}(t, y) &= 0, \\ \phi_{(x,\xi)}(0, y) &= \langle y, \xi \rangle, \end{aligned}$$

and can solve it via the method of characteristics using the flow

$$\begin{cases} Y'(t) = \Xi(t), & Y(0) = y, \\ \Xi'(t) = -\partial_y V(t, y), & \Xi(0) = \xi. \end{cases}$$

Note that nothing in this equation depends on x so we can drop that parameter and produce a ϕ_ξ depending only on ξ . Having obtained ϕ_ξ , we now find a solution for $\theta_{(x,\xi)}$ that leaves only order h error terms. Given the quadratic nature of $\theta_{(x,\xi)}(0, y)$, we expect that the only terms to be of order $h^{1/2}$ are those where one derivative (either space or time) hits the $e^{-\frac{\theta}{h}}$ term. Grouping those terms together, we want to solve

$$\partial_t \theta_{(x,\xi)}(t, y) + \nabla_y \phi_\xi(t, y) \cdot \nabla_y \theta_{(x,\xi)}(t, y) = O(h). \quad (3.3)$$

Equation (3.3) is very similar to the transport equation (1.6) and, due to that similarity, we suggest the following ansatz:

$$\theta_{(x,\xi)}(t, y) = \frac{1}{\sqrt{J_t^{-1}(Y_t^{-1}(y))}} \frac{|x - Y_t^{-1}(y)|^2}{2}. \quad (3.4)$$

Let us now see that, indeed, this does give us an order h parametrix.

Theorem 3.2. *Suppose that ϕ_ξ satisfies the eikonal equation*

$$\partial_t \phi_\xi(t, y) + \frac{1}{2} |\nabla_y \phi_\xi(t, y)|^2 + V(y) \phi_\xi(t, y) = 0, \quad \phi_\xi(0, y) = \langle y, \xi \rangle$$

and $\theta_{(x,\xi)}$ is given by (3.4). Then, if

$$W_{(x,\xi)}(t, y) = \frac{1}{(2\pi h)^{\frac{3n}{4}}} e^{\frac{i}{h}(\phi_\xi(t, y) + i\theta_{(x,\xi)}(t, y))},$$

the family $W_{(x,\xi)}(t, y)$ gives a local order h Schrödinger parametrix.

Proof. We need first to use our ansatz to find an expression for $r_{(x,\xi)}(t, y)$. We have that

$$\begin{aligned} & \left(hD_t + \frac{\Delta}{2} + V(y) \right) W_{(x,\xi)}(t, y) \\ &= \left[\left(\partial_t \phi_\xi(t, y) + \frac{1}{2} |\nabla_y \phi_\xi(t, y)|^2 + V(y) \phi_\xi(t, y) \right) \right. \\ & \quad + \frac{1}{i} \left(\partial_t \theta_{(x,\xi)}(t, y) + \nabla_y \phi_\xi(t, y) \cdot \nabla_y \theta_{(x,\xi)}(t, y) \right) \\ & \quad \left. + \left(\frac{1}{2} |\nabla_y \theta_{(x,\xi)}(t, y)|^2 + h \frac{\Delta_y}{2} \phi_\xi(t, y) + h \frac{\Delta_y}{2} \theta_{(x,\xi)}(t, y) \right) \right] W_{(x,\xi)}(t, y). \end{aligned}$$

As we have chosen ϕ_ξ to satisfy the eikonal equation, the first line is zero. From the transport equation (1.6), we see that the second line is equal to

$$i \Delta_y \phi_\xi(t, y) \cdot \theta_{x,\xi}(t, y).$$

Therefore, we set

$$r_{(x,\xi)}(t, y) = \left[h^{-1} \left(i \Delta_y \phi_\xi(t, y) \cdot \theta_{(x,\xi)}(t, y) + \frac{1}{2} |\nabla_y \theta_{(x,\xi)}(t, y)|^2 \right) + \frac{\Delta_y}{2} \phi_\xi(t, y) + \frac{\Delta_y}{2} \theta_{(x,\xi)}(t, y) \right] W_{(x,\xi)}(t, y).$$

Let

$$R(t) = h \int r_{(x,\xi)}(t, y) T_{(x,\xi)}[u] dx d\xi.$$

From Lemma 3.1, we have

$$\|T_{(\cdot,\cdot)}[u]\|_{L_x^2 L_\xi^2} \lesssim \|u\|_{L^2}$$

so, it is enough to show that

$$\|R(t)\|_{L^2(\mathbb{R}^n)} \lesssim h \|T_{(\cdot,\cdot)}[u]\|_{L_x^2 L_\xi^2}. \quad (3.5)$$

We will use almost orthogonality to show that, viewed as an operator mapping $L_{x,\xi}^2 \rightarrow L_y^2$,

$$\|R(t)G\|_{L_y^2} \lesssim h \|G\|_{L_{x,\xi}^2},$$

which is sufficient to establish (3.5).

Almost orthogonality is a fundamental technique from harmonic analysis (see [15, Chapter 7.2] for a textbook treatment) for treating operators acting between two Hilbert spaces. If one considers a sum of operators $S_j : \mathcal{H}_1 \rightarrow \mathcal{H}_2$ (both $\mathcal{H}_1$, and $\mathcal{H}_2$ Hilbert spaces)

$$S = \sum_{j=1}^J S_j,$$

one can, of course, estimate the norm of S via the triangle inequality. However, this is often a significant overestimate. If the S_j are orthogonal ($S_j S_{j'}^* = 0$ and $S_j^* S_{j'} = 0$), then

$$\|S\|_{\mathcal{H}_1 \rightarrow \mathcal{H}_2} \leq \sup_j \|S_j\|_{\mathcal{H}_1 \rightarrow \mathcal{H}_2}.$$

Almost orthogonality provides a similar estimate in the case where the S_j are not quite orthogonal, but are close to being so. Precisely, if

$$A = \sup_j \sum_{j'} \|S_j S_{j'}^*\|_{\mathcal{H}_2 \rightarrow \mathcal{H}_2}^{1/2} < \infty, \quad B = \sup_j \sum_{j'} \|S_j^* S_{j'}\|_{\mathcal{H}_1 \rightarrow \mathcal{H}_1}^{1/2} < \infty,$$

then

$$\left\| \sum_j S_j \right\|_{\mathcal{H}_1 \rightarrow \mathcal{H}_2} \leq \sqrt{AB}.$$

Note that (while not relevant for this paper) it is not necessary for the sum to be finite. Let us now put the operator $R(t)$ into the almost orthogonality framework.

First, we rewrite $r_{(x,\xi)}(t, y)$ to separate out the oscillatory phase $e^{\frac{i}{h}\phi_\xi}$, namely,

$$r_{(x,\xi)}(t, y) = e^{\frac{i}{h}\phi_\xi(t,y)} \tilde{r}_{(x,\xi)}(t, y)$$

and in the almost orthogonality argument treat $\tilde{r}_{(x,\xi)}(t, y)$ as the symbol in any oscillatory integral calculations. We now record some regularity and boundedness properties of $\tilde{r}_{(x,\xi)}(t, y)$. We claim that

$$|D_\xi^{\gamma_1} D_y^{\gamma_2} \tilde{r}_{(x,\xi)}(t, y)| \leq C_{\gamma_1, \gamma_2} h^{-\frac{3n}{4} + \frac{1}{2}(|\gamma_1| + |\gamma_2|)}. \quad (3.6)$$

To see this, recall that

$$\theta_{(x,\xi)}(t, y) = \frac{1}{\sqrt{J_t^{-1}(Y_t^{-1}(y))}} \frac{|x - Y_t^{-1}(y)|^2}{2},$$

while $W_{(x,\xi)}(t, y)$ features an exponential decay term of the form $e^{-\frac{1}{h}\theta_{(x,\xi)}(t,y)}$. Therefore, once

$$|x - Y_t^{-1}(y)| \geq Ch^{1/2},$$

$\tilde{r}_{(x,\xi)}(t, y)$ will begin to exponentially decay. Since $\theta_{(x,\xi)}$ is quadratic, the

$$\theta_{(x,\xi)}(t, y) \quad \text{and} \quad |\nabla_y \theta_{(x,\xi)}(t, y)|^2$$

terms are of size h in the region where $\tilde{r}_{(x,\xi)}(t, y)$ does not exponentially decay. This ensures that $\tilde{r}_{(x,\xi)}(t, y)$ is bounded by $|W_{(x,\xi)}(t, y)| \leq h^{-\frac{3n}{4}}$. Now, consider what happens when we take derivatives. Any derivative that hits the

$$\theta_{(x,\xi)}(t, y) \quad \text{or} \quad |\nabla_y \theta_{(x,\xi)}(t, y)|^2$$

lowers the power of $(x - Y_t^{-1}(y))$ by one power per derivative, effectively causing an $h^{-\frac{1}{2}}$ loss per derivative. Similarly, if the derivative hits the $e^{-\frac{1}{h}\theta_{(x,\xi)}(t,y)}$ term, it generates a

$$O(|x - Y_t^{-1}(y)| e^{-\frac{1}{h}\theta_{(x,\xi)}(t,y)})$$

term which has a similar effect (a loss of $h^{-\frac{1}{2}}$ per derivative). Any derivatives hitting other terms have no loss, therefore, yielding (3.6). We are now ready to employ the almost orthogonality argument. Let $\{z\}_k \subset \mathbb{R}^n$ and $\{\eta\}_j \subset \mathbb{R}^n$ be sets of $h^{1/2}$ separated points and $R_{k,j}(t) : L_{x,\xi}^2 \rightarrow L_y^2$,

$$R_{k,j}(t)G = h \int e^{\frac{i}{h}\phi_\xi(t,y)} \tilde{r}_{(x,\xi)}(t, y) \chi(h^{-1/2}|\xi - \eta_j|) \chi(h^{-1/2}|y - z_k|) G(x, \xi) dx d\xi$$

for χ smooth compactly supported. To employ almost orthogonality, we need to establish bounds on

$$A = \sup_{(k,j)} \sum_{(k',j')} \|R_{k,j}(t) R_{k',j'}^*(t)\|_{L_y^2 \rightarrow L_y^2}^{1/2}$$

and

$$B = \sup_{(k,j)} \sum_{(k',j')} \|R_{k,j}^*(t) R_{k',j'}(t)\|_{L_{x,\xi}^2 \rightarrow L_{x,\xi}^2}^{1/2}.$$

We will do this by showing that if (k', j') and (k, j) are significantly separated, then $R_{k,j}^*(t) R_{k',j'}(t)$ and $R_{k,j}(t) R_{k',j'}^*(t)$ experience rapid decay (in norm). Where (k', j') and (k, j) are close, we can simply estimate

$$\|R_{k,j}^*(t) R_{k',j'}(t)\|_{L_{x,\xi}^2 \rightarrow L_{x,\xi}^2} \lesssim \left(\sup_{k,j} \|R_{k,j}\|_{L_{x,\xi}^2 \rightarrow L_y^2} \right)^2$$

and

$$\|R_{k,j}(t) R_{k',j'}^*(t)\|_{L_y^2 \rightarrow L_y^2} \lesssim \left(\sup_{k,j} \|R_{k,j}\|_{L_{x,\xi}^2 \rightarrow L_y^2} \right)^2.$$

We apply Schur's test. Using the support properties of χ and the exponential decay of $r_{(x,\xi)}(t, y)$, we have

$$h \sup_y \int |\tilde{r}_{(x,\xi)}(t, y)| \chi(h^{-1/2}|\xi - \eta_j|) \chi(h^{-1/2}|y - z_k|) dx d\xi \leq h^{1-\frac{3n}{4}+n} = h^{1+\frac{n}{4}}.$$

On the other hand,

$$h \sup_{(x,\xi)} \int |\tilde{r}_{(x,\xi)}(t, y)| \chi(h^{-1/2}|\xi - \eta_j|) \chi(h^{-1/2}|y - z_k|) dy \leq h^{1-\frac{3n}{4}+\frac{n}{2}} = h^{1-\frac{n}{4}},$$

which together imply that

$$\|R_{k,j}(t)\|_{L_{x,\xi}^2 \rightarrow L_y^2} \lesssim h.$$

All that remains is to prove the almost orthogonality for pairs (j, k) and (j', k') that are significantly separated. We have that

$$R_{k,j}(t) R_{k',j'}^*(t) f = \int K_{k,j,k',j'}(t, y, z) f(z) dz,$$

where

$$\begin{aligned} K_{k,j,k',j'}(t, y, z) &= h \int e^{\frac{i}{h}(\phi_\xi(t,y) - \phi_\xi(t,z))} \tilde{r}_{(x,\xi)}(t, y) \tilde{r}_{(x,\xi)}(t, z) \\ &\quad \times \chi(h^{-1/2}|\xi - \eta_j|) \chi(h^{-1/2}|y - z_k|) \\ &\quad \times \chi(h^{-1/2}|\xi - \eta_{j'}|) \chi(h^{-1/2}|z - z_{k'}|) dx d\xi. \end{aligned}$$

Clearly, if $|\eta_j - \eta_{j'}| \geq Ch^{1/2}$ (which would be the case if j and j' were separated) the support properties of χ ensure that $K_{k,j,k',j'}(t, y, z) = 0$. Now, consider the case where

$$|z_k - z_{k'}| \geq Ch^{\frac{1}{2}}.$$

For this case, we will extract decay via the method of non-stationary phase. This method recognizes that the oscillation of the phase function ϕ causes significant decay via cancellation. First, we need to establish that the phase is indeed non-stationary. We expand the phase function in y about z ,

$$\phi_\xi(t, z) - \phi_\xi(t, y) = \nabla_z \phi(t, z) \cdot (z - y) + O(|z - y|^2);$$

then,

$$\nabla_\xi(\phi_\xi(t, z) - \phi_\xi(t, y)) = \partial_{\xi, z}^2 \phi(t, z) \cdot (z - y) + O(|z - y|^2).$$

However, since

$$\phi_\xi(t, z) = \langle y, \xi \rangle + O(|t|),$$

this gives us

$$\nabla_\xi(\phi_\xi(t, z) - \phi_\xi(t, y)) = (\text{Id} + O(|t|)) \cdot (z - y) + O(|z - y|^2).$$

Since z is within $h^{1/2}$ of $z_{k'}$ and y is within $h^{1/2}$ of z_k , the condition

$$|z_k - z_{k'}| \geq Ch^{1/2}$$

ensures that z and y are well separated and in fact

$$|\nabla_\xi(\phi_\xi(t, y) - \phi_\xi(t, z))| \geq c|z_k - z_{k'}|.$$

Therefore, there is some ℓ so that

$$|\partial_{\xi_\ell}(\phi_\xi(t, y) - \phi_\xi(t, z))| \geq |z_k - z_{k'}|. \quad (3.7)$$

To extract the cancellation decay, we define the operator

$$L = \frac{h}{i(\partial_{\xi_\ell})(\phi_\xi(t, y) - \phi_\xi(t, z))} \frac{\partial}{\partial \xi_\ell}$$

and write

$$\begin{aligned} K_{k, j, k', j'}(t, y, z) &= h \int (L e^{\frac{i}{h}(\phi_\xi(t, y) - \phi_\xi(t, z))}) \tilde{r}_{(x, \xi)}(t, y) \tilde{r}_{(x, \xi)}(t, z) \\ &\quad \times \chi(h^{-1/2}|\xi - \eta_j|) \chi(h^{-1/2}|y - z_k|) \\ &\quad \times \chi(h^{-1/2}|\xi - \eta_{j'}|) \chi(h^{-1/2}|z - z_{k'}|) dx d\xi. \end{aligned}$$

Then, integrating by parts, we have

$$\begin{aligned} K_{k, j, k', j'}(t, y, z) &= h \int e^{\frac{i}{h}(\phi_\xi(t, y) - \phi_\xi(t, z))} (L^* \tilde{r}_{(x, \xi)}(t, y) \tilde{r}_{(x, \xi)}(t, z) \\ &\quad \times \chi(h^{-1/2}|\xi - \eta_j|) \chi(h^{-1/2}|y - z_k|) \\ &\quad \times \chi(h^{-1/2}|\xi - \eta_{j'}|) \chi(h^{-1/2}|z - z_{k'}|) dx d\xi. \end{aligned}$$

The lower bound (3.7) ensures that the prefactor in the definition of L is $O(h|z_k - z_{k'}|^{-1})$. We now need to account for what happens when the derivatives in ξ hit the various symbol terms. We have already seen that $\tilde{r}_{(x,\xi)}$ has only an $h^{-\frac{1}{2}}$ loss per derivative. So, any terms that arise from the derivative hitting $\tilde{r}_{(x,\xi)}$ are bounded by

$$ch^{-1/2}h|z_k - z_{k'}|^{-1} = h^{1/2}|z_k - z_{k'}|^{-1}.$$

Since $|z_k - z_{k'}| \geq Ch^{1/2}$, these terms do, indeed, experience decay. The cut-off functions in ξ have the same regularity as $\tilde{r}_{(x,\xi)}$ so similarly these experience decay. So, now we have an order $h^{1/2}|z_k - z_{k'}|^{-1}$ improvement over what we could obtain simply by applying Schur's test and using the supports of the symbols. We want better decay in $|z_k - z_{k'}|$ so we repeat the argument, integrating by parts with L again. In this step, we must also account for the possibility of the derivative hitting the inverse power of $\partial_{\xi_\ell}(\phi_\xi(t, y) - \phi_\xi(t, z))$ introduced in the first integration by parts step. However,

$$\frac{\partial}{\partial \xi_\ell} \frac{1}{(\partial_{\xi_\ell}(\phi_\xi(t, y) - \phi_\xi(t, z)))} = \frac{1}{(\partial_{\xi_\ell}(\phi_\xi(t, y) - \phi_\xi(t, z))^2)} \partial_{\xi_\ell \xi_\ell}^2 (\phi_\xi(t, y) - \phi_\xi(t, z)).$$

Since $|z_k - z_{k'}| \geq Ch^{1/2}$, the lower bound (3.7) ensures that this also creates no more than an $h^{-1/2}$ loss per derivative. We keep repeating this argument N times, for some $N \in \mathbb{N}$ assumed to be large, and obtain

$$|K_{k,j,k',j'}(t, y, z)| \leq h \left(\frac{h^{1/2}}{|z_k - z_{k'}|} \right)^N \int \tilde{\chi}(t, y, z, \xi, x) dx d\xi,$$

where $\tilde{\chi}$ is a product of functions with the same support properties as those in the definition of $K_{k,j,k',j'}(t, y, z)$. Another application of Schur's test gives

$$\|R_{k,j}(t)R_{k',j'}^\star(t)\|_{L_y^2 \rightarrow L_y^2} \lesssim h \left(\frac{h^{1/2}}{|z_k - z_{k'}|} \right)^N.$$

Therefore, if we select N sufficiently large, we can ensure that the significant contributions to A come from the case where (k, j) and (k', j') are close, specifically,

$$|\eta_j - \eta_{j'}| \leq Ch^{1/2} \quad \text{and} \quad |z_k - z_{k'}| \leq Ch^{1/2}.$$

However, there are only a fixed (independent of h) number of such terms so we can conclude that

$$A \lesssim h.$$

We argue in a similar way to obtain estimates on $R_{k,j}^\star(t)R_{k',j'}(t)$, namely,

$$R_{k,j}^\star(t)R_{k',j'}(t)g = \int \tilde{K}_{k,j,k',j'}(x, \xi, z, \eta)g(z, \eta)dz d\eta$$

with

$$\begin{aligned}\tilde{K}_{k,j,k',j'}(x, \xi, z, \eta) &= h \int e^{-\frac{i}{h}(\phi_\xi(t,y) - \phi_\eta(t,y))} \tilde{r}_{(x,\xi)}(t, y) \tilde{r}_{(z,\eta)}(t, y) \\ &\quad \times \chi(h^{-1/2}|\xi - \eta_j|) \chi(h^{-1/2}|y - z_k|) \\ &\quad \times \chi(h^{-1/2}|\eta - \eta_{j'}|) \chi(h^{-1/2}|y - z_{k'}|) dy.\end{aligned}$$

In this case, $\tilde{K}_{k,j,k',j'}(x, \xi, z, \eta) = 0$ if $|z_k - z_{k'}| > Ch^{1/2}$ and a similar expansion of the phase function yields

$$\begin{aligned}\nabla_y(\phi_\xi(t, y) - \phi_\eta(t, y)) &= \partial_{\xi,y}^2 \phi(t, y) \cdot (\xi - \eta) + O(|\xi - \eta|^2) \\ &= (\text{Id} + O(|t|)) \cdot (\xi - \eta) + O(|\xi - \eta|^2).\end{aligned}$$

Therefore, if $|\eta_j - \eta_{j'}| \geq Ch^{1/2}$ the phase is non-stationary in y and via integration by parts, we can obtain

$$\|R_{k,j}^*(t)R_{k',j'}(t)\|_{L^2 \rightarrow L^2} \lesssim h \left(\frac{h^{1/2}}{|z_k - z_{k'}|} \right)^N$$

for any $N \in \mathbb{N}$. Therefore,

$$B \leq h.$$

So, by Cotlar–Stein almost orthogonality,

$$\|R(t)\|_{L^2 \rightarrow L^2} \lesssim h.$$

Since $W(t)$ can be written as an oscillatory integral operator with the same phase and with a symbol with the same regularity properties as $\tilde{r}$ a similar almost orthogonality argument gives L^2 boundedness for $W(t)$,

$$W(t) = \int W_{(x,\xi)}(t, y) T_{(x,\xi)}[u] dx d\xi. \quad \blacksquare$$

So, we have now obtained another expression for an order h Schrödinger parametrix. That is,

$$\begin{aligned}W_{\text{WP}}(t) &= \int T_{(x,\xi)} W_{(x,\xi)}(t, y) [u] dx d\xi \\ &= \frac{1}{(2\pi h)^{\frac{3n}{2}}} \int e^{\frac{i}{h}(\phi_\xi(t,y) - \langle z, \xi \rangle)} e^{-\frac{i}{h}(\theta_{(x,\xi)}(t,y) + \frac{|x-z|^2}{2})} u(z) dx d\xi dz.\end{aligned}$$

Note that since

$$W_{\text{FT}}(t)u = \frac{1}{(2\pi h)^n} \int e^{\frac{i}{h}(\phi_\xi(t,x) - \langle y, \xi \rangle)} \frac{1}{\sqrt{J_t(X_t^{-1}(x))}} \chi(X_t^{-1}(x)) u(y) d\xi dy$$

is also an order h parametrix, these two expressions must agree up to lower order error. They do, however, look quite different and each has advantages/disadvantages when it comes to applications.

Propagation along bicharacteristics

Using the $W_{\text{WP}}(t)$ parametrix, the importance of the characteristic flow is immediately apparent. If we start with initial data that has some localisation, for instance, if $|T_{(x,\xi)}[u]|$ is negligible outside a set Ω we can see that the data will flow out to any points that can be reached by the bicharacteristic flow with initial conditions in Ω . The same information is encoded in W_{FT} but it is not as easy to see. Geba and Tataru [6], for instance, make use of this kind of wave-packet decomposition and look at an adapted class of pseudodifferential operators (compared to the usual class which is adapted to the Fourier transform). Their work builds on previous developments in the same vein by Córdoba–Fefferman [1], Folland [5], and Tataru [20, 21]. They use, as we do, a continuous version of a wave-packet transform but discrete versions are also available. See, for example, Smith [14]. Notably, these works focus on L^2 type results where the full advantages of orthogonality/almost orthogonality can be felt.

Resolution of small-scale features

When studying potential concentrations to solutions of Schrödinger equations (or of eigenfunctions) we often want to study the growth of L^p norms (see, for example, [2, 9, 10, 16, 18, 19, 22]) rather than simply study the L^2 norm. By looking at L^p norms with $p > 2$, we are able to measure small scale features that may be “invisible” to L^2 . By “invisible”, we mean that the feature is localised on a region Ω_h so that $\|u\|_{L^2(\Omega_h)} \rightarrow 0$. A classic example of this is the L^p growth rate of eigenfunctions (or quasimodes) for high p , saturated near a point. On spheres where such examples are explicit, the zonal spherical harmonics are saturated near the poles. For quasimode constructions in the general case, see [17]. For all such examples, the L^p norms are saturated in a radius h ball at which the eigenfunction is $\approx h^{-\frac{n-1}{2}}$ (and so invisible to L^2). Therefore, if you are studying L^p norms of Schrödinger solutions or eigenfunctions via a parametrix, the W_{FT} form has a significant advantage as it is expressed as an oscillatory integral with smooth symbol. The W_{WP} parametrix can only be understood as an oscillatory integral with a symbol that has $h^{-1/2}$ regularity loss per derivative. In Theorem 3.2, the regularity loss was not problematic as we were working entirely in L^2 , and therefore, could use Hilbert space properties and reduce the problem to an almost orthogonality proof. However, outside of L^2 , in general Banach spaces, we no longer have Hilbert space techniques and the analysis becomes more difficult. Unfortunately, it is because the parametrix W_{WP} gives us a good picture of what happens in phase space that it is less useful for L^p estimates. It gives accuracy up to an order $h^{1/2}$ error in both spatial and frequency variables so the uncertainty principle prevents it from resolving features at order h special scale.

Wave-packet-based norms

As discussed above, when studying L^p norms, it is necessary to work with W_{FT} or other parametrices that allow for special localisation all the way down to the minimum scale h . However, following from Smith’s [14] work on L^1 spaces, L^p wave-packet-based norms

	Propagation	Small-scale resolution	Wave-packet norms
W_{WP}	The role of the underlying flow is easily seen. The analysis is often purely L^2 -based, where almost-orthogonality arguments are available.	Resolves only down to the $h^{1/2}$ scale, and is therefore not suitable for full small-scale resolution.	Well adapted to norms such as H_{FIO}^p , which themselves only resolve down to the $h^{1/2}$ scale.
W_{FT}	Information about the flow is present, but not immediately visible in the representation.	Resolves down to the h scale (the minimum scale as determined by the uncertainty principle), making it suitable for all small-scale theories.	Can be used with wave-packet-based norms, but is not naturally adapted to them.

Table 1. Comparison of wave-packet W_{WP} and Fourier-transform W_{FT} based parametrices.

have recently been developed. In particular, the $\mathcal{H}_{FIO}^p$ spaces were introduced by Hassell–Portal–Rozendaal [7] and used to study wave equations and pseudodifferential operators with rough coefficients [8, 12, 13]. While the $\mathcal{H}_{FIO}^p$ spaces were designed for the wave, rather than Schrödinger, equations, half-wave parametrices can also be constructed in the same fashion as for the Schrödinger ones. Roughly speaking, the $\mathcal{H}_{FIO}^p$ spaces first apply a wave-packet transform and then consider an $L_x^p L_\xi^p$ norm of $T_{(x,\xi)}[f]$. Since wave packet transform creates a natural $h^{1/2}$ uncertainty blurring, the $\mathcal{H}_{FIO}^p$ spaces do not see the fine scale behaviour below an $h^{1/2}$ scale. Therefore, for these types of norms W_{WP} is a more useful parametrix as it closely matches the decompositions used to define the norm and, unlike the L^p case, no longer loses necessary fine scale detail.

Table 1 summarised the differences between W_{WP} and W_{FT} in these applications.

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