

On the WKB method in linear scattering theory: A survey on the Isozaki–Kitada parametrix

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Abstract. This survey is devoted to the main WKB aspects of the Isozaki–Kitada parametrix and to the applications thereof in scattering theory.

1. Introduction

The WKB approximation, named after Wentzel, Kramers and Brillouin, is a famous ansatz that allows us to capture the oscillations of solutions to Schrödinger equations. Originally introduced for one-dimensional problems, the acronym WKB now refers to various settings that use the essence of the historical construction. The WKB ansatz describes highly oscillatory functions which, in the simplest case, can be modelled by plane waves $e^{i(x-x_0)\cdot\xi}$ in a regime where $|\xi| \rightarrow \infty$ (or proportional to $\hbar^{-1}$ for some effective Planck constant $\hbar \rightarrow 0$); in scattering theory, which is the purpose of this note, the oscillations come primarily from letting $|x - x_0| \rightarrow \infty$.

Our purpose here is thus to review results illustrating why and how WKB type techniques are used in scattering theory. Specifically, we focus on the Isozaki–Kitada construction and its applications. Our wish with this survey is both to show to what extent this is genuinely a WKB approach and to help non-experts in the field to grasp the main aspects of the construction of Isozaki–Kitada modifiers. Our goal will be fulfilled if we can give an effective access to the related literature, in particular to the applications surveyed in Section 4.

The Isozaki–Kitada modifiers, also called Isozaki–Kitada construction or parametrix throughout this paper, were introduced in the 80s to deal with long-range scattering theory for Schrödinger operators, more specifically as time-independent modifiers (see [19]). We recall informally that for short range potentials V , say smooth real-valued functions such that $V(x) = O(\langle x \rangle^{-\rho})$ with $\rho > 1$, one can define (outgoing and incoming) wave operators associated with $P_0 = -\Delta$ and $P = -\Delta + V$ on $L^2(\mathbb{R}^n)$ as the strong limits

$$W_{\pm} = s - \lim_{t \rightarrow \pm\infty} e^{itP} e^{-itP_0}.$$

Under fairly general conditions, one can prove asymptotic completeness, namely the invertibility of $W_{\pm}$ as operators between L^2 and the absolutely continuous subspace of P , leading to the scattering asymptotics

$$e^{-itP}u_0 \sim e^{-itP_0}W_{\pm}^{-1}u_0, \quad t \rightarrow \pm\infty, \quad (1.1)$$

whenever u_0 belongs to the absolutely continuous subspace of P . When V is of long-range, i.e., if $\rho \in (0, 1]$, the long-time asymptotics of e^{-itP} are no longer given in term e^{-itP_0} as in (1.1), but by a suitably modified dynamics (see Theorem 4.1 below for an example of precise statement). The Isozaki–Kitada modifiers are an answer to this problem as a certain type of Fourier integral operators (FIO). Beyond these motivations coming from scattering theory, the Isozaki–Kitada construction can be reinterpreted as a general microlocal normal form theorem. Basically, they allow us to give an approximation of the form

$$e^{-itP}\chi_{\pm}(x, D) \approx J_{\pm}(a)e^{-itP_0}J_{\pm}(b)^*,$$

which is valid for *long* times; here $\chi_{\pm}$ are pseudodifferential operators supported in outgoing/incoming areas (see (3.4)), $J_{\pm}(a)$ are elliptic Fourier integral operators microlocalized in such areas, and $J_{\pm}(b)$ are such that $J_{\pm}(a)J_{\pm}(b)^* \approx \chi_{\pm}(x, D)$. In a sense, this normal form is a long-time analogue to the local one used by Hörmander for pseudodifferential operators of real principal type (see [16, Section XXVI.1]).

The plan of the paper is the following. In Section 2, we consider basic examples of scattering theory on the half-line to allow the non-expert reader to get a feeling of the issues related to long-range perturbations and motivate the use of WKB techniques in this context. Section 3 is devoted to the main WKB aspects of the Isozaki–Kitada parametrix, namely the resolution of the eikonal and transport equations. In Section 4, we survey various applications of the Isozaki–Kitada parametrix, illustrating how strong and natural this tool is.

2. A one-dimensional motivation

Given a real-valued potential V , say, on the half-line (we are only interested in the behaviour at infinity), we wish to find generalized eigenfunctions of $-\frac{d^2}{dr^2} + V(r)$ with a prescribed behaviour at infinity, namely solutions $u = u(k, r)$ of

$$-\frac{d^2u}{dr^2} + V(r)u = k^2u, \quad u \sim e^{ikr} \quad \text{as } r \rightarrow +\infty \quad (2.1)$$

when k is real. The procedure and result are well known; they basically rest on the observation that the equation can be written

$$\left(\frac{d}{dr} + ik\right)\left(\frac{d}{dr} - ik\right)u = Vu. \quad (2.2)$$

Proposition 2.1. *If $V \in L^1$ and $k \in \mathbb{R} \setminus 0$, there exists a unique C^1 bounded solution u to the fixed point problem*

$$u(r) = e^{ikr} - \int_r^\infty \frac{\sin k(r-s)}{k} V(s)u(s)ds, \quad (2.3)$$

and it satisfies (2.1) in the distributions sense. Moreover, $u' - iku \rightarrow 0$ as $r \rightarrow +\infty$. It is called the Jost solution.

We refer, e.g., to [24, Theorem XI.56] for more details; basically, the integrability of V allows us to show that the integral operator in the right-hand side of (2.3) is a contraction on $C^0([R, \infty)) \cap L^\infty([R, \infty))$ if R is large enough. Note in passing that we do not use the fact that V is real valued here.

We can be more specific about the behaviour of Jost solutions when V belongs to the symbol class $S^{-\rho}$, namely if for all $j \in \mathbb{N}$,

$$|V^{(j)}(r)| \lesssim r^{-\rho-j}, \quad r \gg 1.$$

Proposition 2.2. *If V belongs to $S^{-\rho}$ with $\rho > 1$, then the solution to (2.3) is of the form*

$$u(r) = e^{ikr} a(r) \quad \text{with } a - 1 \in S^{1-\rho}.$$

Proof. Let $a(r) := e^{-ikr} u(r)$. We infer from (2.3) that $a(r) = 1 + O(r^{1-\rho})$ as $r \rightarrow \infty$. We next study the decay of $a'(r)$. It is not hard to get, from the equation on u , that

$$\begin{aligned} a'(r) &= - \int_r^\infty e^{2ik(s-r)} V(s)a(s)ds \\ &= \underbrace{\frac{V(r)a(r)}{2ik} + \int_r^\infty \frac{e^{2ik(s-r)}}{2ik} V'(s)a(s)ds}_{=O(r^{-\rho})} + \int_r^\infty \frac{e^{2ik(s-r)}}{2ik} V(s)a'(s)ds. \end{aligned} \quad (2.4)$$

This implies that

$$\left(1 - \frac{1}{2|k|} \int_r^\infty |V(s)|ds\right) \sup_{s \geq r} |a'(s)| \leq C r^{-\rho}$$

and thus, if r is large enough, that $a'(r) = O(r^{-\rho})$ since the integral in the left-hand side is less than $\frac{1}{2}$ for r large. Iteration of this argument by differentiating (2.4) yields the result. $\blacksquare$

Proposition 2.2 means that the oscillations (as $r \rightarrow \infty$) of the Jost solution are well factorized by e^{ikr} .

What happens when V is no longer integrable? To keep the discussion simple and parallel to further sections in higher dimensions, we assume from now on that V is smooth and that

$$V \in S^{-\rho} \quad \text{for some } \rho > 0.$$

Note that in Proposition 2.1, no smoothness on V was required. We want to construct generalized eigenfunctions of $-\frac{d^2}{dr^2} + V(r)$ in which we can factorize the oscillations relatively explicitly. The WKB method turns out to be very useful for constructing the analogues to Jost solutions in this case. Defining $\varphi = \varphi(r, k)$ by

$$\varphi(r, k) = \int_A^r \sqrt{k^2 - V(s)} ds, \quad r \gg 1$$

(where A is an arbitrary constant large enough so that $k^2 - V$ is positive on $[A, \infty)$), which solves the eikonal equation

$$(\varphi')^2 + V(r) = k^2,$$

we look for functions u satisfying

$$-\frac{d^2 u}{dr^2} + V(r)u = k^2 u, \quad u(r) \sim e^{i\varphi(r, k)} \quad \text{as } r \rightarrow +\infty. \quad (2.5)$$

Of course, $-\varphi$ also solves the eikonal equation, but we only illustrate our purpose with φ here. Mimicking what we obtain when V is integrable, we also require that $u - i\varphi' u \rightarrow 0$ as $r \rightarrow +\infty$. Our choice of φ ensures in particular that

$$\varphi''(r) = -\frac{V'(r)}{2\sqrt{k^2 - V(r)}} \in S^{-1-\rho} \quad \text{hence is integrable at infinity}$$

and that the (generalized) eigenfunction equation can be written

$$\left(\frac{d}{dr} + i\varphi'\right)\left(\frac{d}{dr} - i\varphi'\right)u = -i\varphi''u. \quad (2.6)$$

Since φ'' is integrable, we are in a situation analogous to (2.2) and have a similar result.

Proposition 2.3. *The fixed point equation in u*

$$\begin{aligned} e^{-i\varphi(r)}u(r) &= 1 - \frac{e^{-2i\varphi(r)}}{2\varphi'(r)} \int_r^{+\infty} e^{i\varphi(t)}\varphi''(t)u(t)dt \\ &\quad + \int_r^{+\infty} \frac{e^{-i\varphi(t)}}{2\varphi'(t)}\varphi''(t)u(t)dt \\ &\quad + \int_r^{+\infty} e^{-2i\varphi(s)} \frac{\varphi''(s)}{2\varphi'(s)^2} \left(\int_s^{+\infty} e^{i\varphi(t)}\varphi''(t)u(t)dt \right) ds \end{aligned} \quad (2.7)$$

has a unique solution in $L^\infty(R, \infty) \cap C^0([R, \infty))$ if R is large enough. This solution satisfies (2.5). Moreover,

$$a - 1 := e^{-i\varphi}u - 1 \quad \text{belongs to } S^{-\rho}.$$

Sketch of proof. Solving the fixed point equation follows from the integrability of φ'' , and we omit the details. To prove that $a - 1$ belongs to $S^{-\rho}$, we observe first that (2.7) implies after elementary calculations that

$$\begin{aligned} a' &= i e^{-2i\varphi(r)} \int_r^\infty e^{2i\varphi(t)} \varphi''(t) a(t) dt \\ &= -\frac{\varphi''(r)}{2\varphi'(r)} a(r) - e^{-2i\varphi(r)} \int_r^{+\infty} e^{2i\varphi(t)} \left(\frac{\varphi''}{2i\varphi'} \right)'(t) a(t) dt \\ &\quad - e^{-2i\varphi(r)} \int_r^{+\infty} e^{2i\varphi(t)} \frac{\varphi''(t)}{2i\varphi'(t)} a'(t) dt \end{aligned}$$

by integration by parts. Since $a \sim 1$ at infinity and φ'' belongs to $S^{-\rho-1}$, the first line of the second inequality is $O(r^{-1-\rho})$ while the second line can be absorbed in the left-hand side as in the proof of Proposition 2.2. This leads to $a'(r) = O(r^{-1-\rho})$. Estimates of higher-order derivatives follow by iteration of this argument. ■

3. Isozaki–Kitada modifiers

We work in a general framework where we replace the Laplacian of Section 2 by a constant coefficients pseudodifferential operator $p_0(D)$ on $\mathbb{R}^d$ and the potential by a pseudodifferential perturbation $V(x, D)$.

We assume that p_0 is real valued, smooth and goes to $+\infty$ as $|\xi| \rightarrow \infty$. We also assume that p_0 is an order function, i.e., for each β , $|\partial_\xi^\beta p_0(\xi)| \lesssim_\beta \langle p_0(\xi) \rangle$, and that it is of polynomial growth. All these conditions are, for instance, satisfied by $|\xi|^2$ or real positive powers of $1 + |\xi|^2$.

Regarding the perturbation, we assume that for some $\rho > 0$,

$$|\partial_x^\alpha \partial_\xi^\beta V(x, \xi)| \lesssim_{\alpha\beta} \langle p_0(\xi) \rangle \langle x \rangle^{-\rho-|\alpha|}. \quad (3.1)$$

We assume that $p_0(D) + V(x, D)$ is selfadjoint on the domain

$$\{u \in L^2(\mathbb{R}^d) \mid p_0(D)u \in L^2(\mathbb{R}^d)\}.$$

Note that in most applications in this context, this selfadjointness can be easily verified, either in the semiclassical case $p_0(hD) + V(x, hD)$ when h is small enough, or in the non-semiclassical one if, for instance, $|\partial_\xi^\beta p_0(\xi)| \lesssim_\beta \langle p_0(\xi) \rangle \langle \xi \rangle^{-|\beta|}$, provided in both cases one has a global ellipticity condition $|p_0(x, \xi) + V(x, \xi)| \gtrsim |p_0(\xi)|$. This condition, however, will not play any role below (up to the existence of $\exp\{it(p_0(D) + V(x, D))\}$). In the sequel, we shall assume that there exists a real-valued symbol v such that

$$|\partial_x^\alpha \partial_\xi^\beta (V(x, \xi) - v(x, \xi))| \lesssim \langle p_0(\xi) \rangle \langle x \rangle^{-\rho-1-|\alpha|} \quad (3.2)$$

and define $p(x, \xi)$ by

$$p(x, \xi) = p_0(\xi) + v(x, \xi)$$

which we shall call later the principal symbol of $p_0(D) + V(x, D)$. As pointed out in the very first paragraph of the introduction, we focus on the decay in x , not on that in ξ which, in this section, will belong to a relatively compact subset of $\mathbb{R}^d$. Thus p is a principal (or leading) symbol in the sense that the full symbol of $p_0(D) + V(x, D)$ minus p decays faster with respect to x than p (or than v which is $O(\langle x \rangle^{-\rho})$ according to (3.1)) and (3.2).

In this framework, microlocal WKB techniques will be used to put $p_0(D) + V(x, D)$ into normal form, namely to get an approximation

$$Ap_0(D)B^* \approx (p_0(D) + V(x, D))AB^*, \quad (3.3)$$

where A and B are Fourier integral operators, which will lead to a (microlocal) long-time approximation of the propagator

$$e^{it(p_0(D)+V(x,D))}AB^* \approx Ae^{itp_0(D)}B^*.$$

We refer to Proposition 3.10 for an exact statement; informally for the moment, approximation (3.3) means that the difference between the two sides decays fast in x . This is analogous to Section 2, where we obtain faster decay in r in the right-hand side of (2.6) upon introducing the ‘modifier’ $e^{i\varphi}$.

The construction will be localized in energy, so we consider an open interval I_{nc} which is non-critical for p_0 , namely

$$\nabla_{\xi} p_0(\xi) \neq 0 \quad \text{at any } \xi \in p_0^{-1}(I_{\text{nc}}).$$

Approximation (3.3) is not global but, for most scattering purposes (see Section 4 for the applications), it is sufficient to get it on outgoing (+) or incoming (−) regions, namely open sets of the form

$$\begin{aligned} \Gamma^{\pm}(I, R, \sigma) \\ := \{(x, \xi) \in \mathbb{R}^{2d} \mid p_0(\xi) \in I, |x| > R, \pm x \cdot \nabla_{\xi} p_0(\xi) > \sigma|x||\nabla_{\xi} p_0(\xi)|\}, \end{aligned} \quad (3.4)$$

where I is a subinterval of I_{nc} , $R > 0$ is large and $\sigma \in (-1, 1)$. Occasionally, we shall also denote for $\omega \in \mathbb{S}^{n-1}$,

$$\Gamma_{\omega}^{\pm}(I, \sigma) := \{\xi \in \mathbb{R}^d \mid p_0(\xi) \in I, \omega \cdot \nabla_{\xi} p_0(\xi) > \sigma|\nabla_{\xi} p_0(\xi)|\}, \quad (3.5)$$

which allows us to get the following characterization:

$$(x, \xi) \in \Gamma^{\pm}(I, R, \sigma) \Leftrightarrow |x| > R \quad \text{and} \quad \xi \in \Gamma_{x/|x|}^{\pm}(I, \sigma). \quad (3.6)$$

3.1. Construction of the phase

To construct FIO satisfying (3.3), we mainly need phases. This is the purpose of the following theorem.

Theorem 3.1. Assume that $0 < \rho < 1$. Let I_3 be an open relatively compact subinterval of I_{nc} . Let $\sigma_3 \in (-1, 1)$. Then there exist $R_3 > 0$ and smooth real-valued functions $\varphi_{\pm}$ on $\mathbb{R}^{2n}$ satisfying

$$p(x, \nabla_x \varphi_{\pm}(x, \xi)) = p_0(\xi), \quad (x, \xi) \in \Gamma^{\pm}(I_3, R_3, \sigma_3) \quad (3.7)$$

and

$$|\partial_x^{\alpha} \partial_{\xi}^{\beta} (\varphi_{\pm}(x, \xi) - x \cdot \xi)| \leq C_{\alpha\beta} \langle x \rangle^{1-\rho-|\alpha|}, \quad (x, \xi) \in \mathbb{R}^{2n}. \quad (3.8)$$

Since these phases capture the WKB essence of the Isozaki–Kitada approach, we devote this section to their construction. We refer to the original papers by Isozaki and Isozaki–Kitada [18, 19] and the book [8] both for Schrödinger operators, and to the paper by Robert [29] in a similar context to ours. Here we propose a slightly different approach which we feel is more intuitive.

Below, we denote by $\phi^t = (x_t, \xi_t) = (x_t, \xi_t)(x, \xi)$ the Hamiltonian flow of p , namely the solution to

$$\frac{dx_t}{dt} = \frac{\partial p}{\partial \xi}(x_t, \xi_t), \quad \frac{d\xi_t}{dt} = -\frac{\partial p}{\partial x}(x_t, \xi_t) \quad (3.9)$$

with initial condition $(x_0, \xi_0) = (x, \xi)$.

Lemma 3.2. Let $I_0 \Subset I_{\text{nc}}$ and $\sigma \in (-1, 1)$. There exist $R, C > 0$ large enough such that for all $\pm t \geq 0$ and $(x, \xi) \in \Gamma^{\pm}(I_0, R, \sigma)$,

$$|x_t| \geq C^{-1}(|x| + |t|) \quad \text{and} \quad |\xi_t - \xi| < C \langle x \rangle^{-\rho}. \quad (3.10)$$

The first inequality of (3.10) says that characteristics escape to infinity forward (resp. backward) in time in outgoing (resp. incoming) regions; the second one suggests that $\xi \mapsto \xi_t$ remains close to the identity and thus can be inverted, which is a crucial input for solving eikonal equations. The next proposition gives estimates on derivatives of the flow.

Proposition 3.3. Let $I_0 \Subset I_{\text{nc}}$ and $\sigma \in (-1, 1)$. If R is large enough, then for each α, β ,

$$|\partial_x^{\alpha} \partial_{\xi}^{\beta} (x_t - x)| \leq C_{\alpha\beta} |t| \langle x \rangle^{-|\alpha|}, \quad |\partial_x^{\alpha} \partial_{\xi}^{\beta} (\xi_t - \xi)| \leq C_{\alpha\beta} \langle x \rangle^{-\rho-|\alpha|}, \quad (3.11)$$

for all $(x, \xi) \in \Gamma^{\pm}(I_0, R, \sigma)$ and $\pm t \geq 0$.

The second inequality of (3.11) improves (3.10) in that $\xi \mapsto \xi_t$ is C^1 -close to the identity when x is large (in outgoing/incoming regions). To see that it is indeed a diffeomorphism between appropriate open sets, we use the following lemma.

Lemma 3.4. Let $I_2 \Subset I_1 \Subset I_0 \Subset I_{\text{nc}}$ be open intervals. Let $1 > \sigma_2 > \sigma_1 > \sigma_0 > -1$. There exists $\epsilon > 0$ such that, for all $\omega \in \mathbb{S}^{d-1}$ and all maps $F_{\omega}: \Gamma_{\omega}(I_0, \sigma_0) \rightarrow \mathbb{R}^d$ satisfying

$$\sup_{\xi \in \Gamma_{\omega}(I_0, \sigma_0)} |F_{\omega}(\xi) - \xi| < \epsilon \quad \text{and} \quad \sup_{\xi \in \Gamma_{\omega}(I_0, \sigma_0)} \|d_{\xi} F_{\omega} - I\| < \epsilon, \quad (3.12)$$

one has the following properties:

- F_{ω} is a diffeomorphism between $\Gamma_{\omega}(I_1, \sigma_1)$ and its open range,
- $F_{\omega}(\Gamma_{\omega}(I_1, \sigma_1))$ contains $\Gamma_{\omega}(I_2, \sigma_2)$.

The proofs of Lemmas 3.2 and 3.4 and of Proposition 3.3 can be found in Appendix A. We now proceed to the proof of Theorem 3.1. Using the second estimates of (3.10) and (3.11), we see that $F_\omega := \xi_t(|x|\omega, \cdot)$ satisfies (3.12) for $|x|$ large enough. Note that here F_ω is seen as a function of ξ depending not only on $\omega = \frac{x}{|x|}$ but also on the additional parameters t and $|x|$; we do not mention this dependence for the sake of coherence with the notation of Lemma 3.4. Denoting by $\eta_t(|x|\omega, \cdot) := F_\omega^{-1}$, it follows from Lemma 3.4 and (3.6) that we have well-defined maps

$$\Gamma^\pm(I_2, R_2, \sigma_2) \ni (x, \xi) \mapsto (x, \eta_t(x, \xi)) \in \Gamma^\pm(I_1, R_2, \sigma_1), \quad \pm t \geq 0,$$

(with $R_2 > R_1 \gg 1$) which are smooth and satisfy, using Proposition 3.3,

$$|\partial_x^\alpha \partial_\xi^\beta (\eta_t(x, \xi) - \xi)| \lesssim \langle x \rangle^{-\rho - |\alpha|}. \quad (3.13)$$

By the classical Hamilton–Jacobi theory (see, e.g., [1, Section 46], [10, Section 3.3] and [27, Section IV.5]), the function

$$\psi(t, x, \xi) := x \cdot \eta_t(x, \xi) + \int_0^t (p - x \cdot \partial_x p) \circ \phi^s(x, \eta_t(x, \xi)) ds$$

is well defined on $\Gamma^\pm(I_2, R_2, \sigma_2)$ and solves the problem

$$\partial_t \psi = p(\partial_\xi \psi, \xi) = p(x, \partial_x \psi), \quad \psi(0, x, \xi) = x \cdot \xi. \quad (3.14)$$

In addition,

$$\partial_\xi \psi(t, x, \xi) = x_t(x, \eta_t(x, \xi)), \quad \partial_x \psi(t, x, \xi) = \eta_t(x, \xi). \quad (3.15)$$

Using (3.10) and (3.13), we see that, if x is large enough,

$$\begin{aligned} x_s(x, \eta_t) &= x_s(x, \xi) + \int_0^1 (\partial_\xi x_s)(x, \xi + u(\eta_t - \xi)) du (\eta_t - \xi) \\ &\gtrsim |s| - O(s \langle x \rangle^{-\rho}) \gtrsim |s| \end{aligned}$$

and then, by differentiating in time the identity $\xi_t(x, \eta_t) = \xi$ and using the second equation of (3.9) in which $\partial_x p = \partial_x v$, we get

$$\partial_t \eta_t = (\partial_\xi \xi_t)(x, \eta_t)^{-1} (\partial_x v)(x_t(x, \eta_t), \xi) = O((t + |x|)^{-1-\rho}) \quad (3.16)$$

so that $\partial_x \psi(t, x, \xi)$ converges as $t \rightarrow \pm\infty$. Therefore, letting $t \rightarrow \pm\infty$ in (3.14) yields

$$p_0(\xi) = \lim_{t \rightarrow \pm\infty} p(\partial_\xi \psi, \xi) = \lim_{t \rightarrow \pm\infty} p(x, \partial_x \psi) = p(x, \lim_{t \rightarrow \pm\infty} \partial_x \psi). \quad (3.17)$$

This suggests to seek $\varphi_\pm$ such that $\partial_x \varphi_\pm = \lim_{t \rightarrow \pm\infty} \partial_x \psi$. It is hopeless to define $\varphi_\pm$ as $\lim_{t \rightarrow \pm\infty} \psi(t, x, \xi)$ since these limits do not exist ($\partial_\xi \psi$ diverges as $t \rightarrow \pm\infty$). However, by choosing appropriately $\tilde{\psi}(t, \xi)$ independent of x , we may hope that $\psi(t, x, \xi) - \tilde{\psi}(t, \xi)$ converges to good candidates for $\varphi_\pm(x, \xi)$ as $t \rightarrow \pm\infty$. To construct $\tilde{\psi}(t, \xi)$, we shall use the following lemma whose proof is also given in Appendix A.

Lemma 3.5. Assume without loss of generality that ρ is irrational.¹ For $\xi \in p_0^{-1}(I_2)$, let us define

$$x_{1,s,t} = s \partial_\xi p_0(\xi), \quad \xi_{1,s,t} = \xi,$$

and for $k \geq 1$,

$$\xi_{k+1,s,t} = \xi + \int_s^t (\partial_x v)(x_{k,r,t}, \xi_{k,r,t}) dr, \quad x_{k+1,s,t} = \int_0^s (\partial_\xi p)(x_{k,r,t}, \xi_{k+1,r,t}) dr.$$

Then, for $0 \leq \pm s \leq \pm t$ and $(x, \xi) \in \Gamma^\pm(I_2, R_2, \sigma_2)$,

$$|\xi_s(x, \eta_t(x, \xi)) - \xi_{k,s,t}| \leq C \langle s \rangle^{-k\rho}, \quad (3.18)$$

$$|x_s(x, \eta_t(x, \xi)) - x_{k,s,t}| \leq C(\langle x \rangle + \langle s \rangle)^{1-k\rho} \quad (3.19)$$

and

$$|\partial_t x_{k,s,t}| \leq C \langle t \rangle^{-1-\rho}, \quad |\partial_t \xi_{k,s,t}| \leq C \langle t \rangle^{-1-\rho}. \quad (3.20)$$

Using this lemma, we can define for k such that $1 - k\rho < 0$,

$$\tilde{\psi}(t, \xi) := \int_0^t p(x_{k,s,t}, \xi) ds.$$

By the first equalities of (3.14) and (3.15) together with Lemmas 3.2 and 3.5, we see that locally uniformly in x ,

$$\begin{aligned} \partial_t(\psi - \tilde{\psi}) &= p(x_t(x, \eta_t), \xi) - p(x_{k,t,t}, \xi) + \int_0^t (\partial_x v)(x_{k,s,t}, \xi) \partial_t x_{k,s,t} ds \\ &= O(\langle t \rangle^{-1-\rho}(|x_t(x, \eta_t) - x_{k,t,t}|)) + \int_0^t O(\langle s \rangle^{-1-\rho} \langle t \rangle^{-1-\rho}) ds \\ &= O(\langle t \rangle^{-1-\rho}) \end{aligned}$$

using in the second line that $|x_t(x, \eta_t)| \gtrsim_x \langle t \rangle$. This implies that $\psi - \tilde{\psi}$ converges as $t \rightarrow \pm\infty$ and allows us to define

$$\tilde{\varphi}_\pm(x, \xi) = \lim_{t \rightarrow \pm\infty} \psi(t, x, \xi) - \tilde{\psi}(t, \xi)$$

on $\Gamma^\pm(I_2, R_2, \sigma_2)$, where it is continuous by local uniform convergence. It is also C^1 with respect to x since $\partial_x(\psi - \tilde{\psi}) = \partial_x \psi$ converges uniformly as $t \rightarrow +\infty$ since $\partial_t \partial_x \psi = O((|x| + t)^{-1-\rho})$ by (3.15) and (3.16). By construction and trick (3.17), it satisfies the eikonal equation (3.7) on $\Gamma^\pm(I_2, R_2, \sigma_2)$. Also,

$$\begin{aligned} (\partial_x \tilde{\varphi}_\pm)(x, \xi) &= \partial_x \psi(0, x, \xi) + \int_0^{+\infty} \partial_t \partial_x \psi(t, x, \xi) dt \\ &= \xi + \int_0^\infty O((t + |x|)^{-1-\rho}) dt = \xi + O(\langle x \rangle^{-\rho}). \end{aligned} \quad (3.21)$$

¹So that $1 - k\rho$ never vanishes for $k \in \mathbb{N}$.

By picking $R > R_2$ and using the above expansion, we can write

$$\begin{aligned}\tilde{\varphi}_{\pm}(x, \xi) &= \tilde{\varphi}_{\pm}\left(R \frac{x}{|x|}, \xi\right) + \int_R^{|x|} (\partial_x \tilde{\varphi}_{\pm})\left(t \frac{x}{|x|}, \xi\right) dt \frac{x}{|x|} \\ &= \tilde{\varphi}_{\pm}\left(R \frac{x}{|x|}, \xi\right) + x \cdot \xi - R\xi \cdot \frac{x}{|x|} + \int_R^{|x|} O(\langle t \rangle^{-\rho}) dt,\end{aligned}$$

where, since $\rho < 1$, the integral is $O(\langle x \rangle^{1-\rho})$. This shows that

$$\tilde{\varphi}_{\pm}(x, \xi) = x \cdot \xi + O(\langle x \rangle^{1-\rho}) \quad (3.22)$$

within $\Gamma^{\pm}(I_2, R_2, \sigma_2)$. To define $\varphi_{\pm}$ globally on $\mathbb{R}^{2n}$, we introduce $\chi_{\pm}$ equal to 1 near $\Gamma^{\pm}(I_3, R_3, \sigma_3)$ and supported in $\Gamma^{\pm}(I_2, R_2, \sigma_2)$ so that we can set

$$\varphi_{\pm}(x, \xi) = x \cdot \xi + \chi_{\pm}(x, \xi)(\tilde{\varphi}_{\pm}(x, \xi) - x \cdot \xi). \quad (3.23)$$

It follows from (3.22) that (3.8) holds when $\alpha = \beta = 0$; it also holds when $|\alpha| = 1$ and $\beta = 0$ by (3.21). We omit the details on the estimates of other derivatives, which follow basically from Proposition 3.3 and (3.13), together with checking analogues to (3.18) and (3.19) for ξ derivatives of $x_{k,s,t}$, $\xi_{k,s,t}$. This completes the proof of Theorem 3.1.

3.2. The parametrix

In this section, we give a precise meaning to the approximation (3.3) to emphasize in particular the second WKB aspect of this parametrix, namely the relevant transport equations.

We use the space $S^{\mu,m}$ of symbols q such that $|\partial_x^{\alpha} \partial_{\xi}^{\beta} q(x, \xi)| \lesssim \langle x \rangle^{\mu-|\alpha|} \langle \xi \rangle^m$. We also denote $S^{\mu,-\infty} = \bigcap_{m \in \mathbb{R}} S^{\mu,m}$. Since p_0 is of polynomial growth, (3.1) and (3.2) imply that, for some m , p belongs to $S^{0,m}$, v, V belong to $S^{-\rho,m}$ and $v - V$ belongs to $S^{-\rho-1,m}$. We also consider an interval $I_3 \Subset I_{\text{nc}}$ and $\sigma_3 \in (-1, 1)$. Using the phases $\varphi_{\pm}$ given by Theorem 3.1, we consider the Fourier integral operators defined by

$$(J_{\varphi_{\pm}}(a)u)(x) = (2\pi)^{-n} \iint e^{i\varphi_{\pm}(x, \xi) - iy \cdot \xi} a(x, \xi) u(y) d\xi dy,$$

where, for our purpose, considering symbols in $S^{v,-\infty}$ with $v \leq 0$ will be sufficient.

Proposition 3.6. *If $R \gg 1$, then for all $q \in S^{\mu,m}$ and $a \in S^{v,-\infty}$ ($\mu, v, m \in \mathbb{R}$), with a supported in $\Gamma^{\pm}(I_3, R, \sigma_3)$, we have*

$$q(x, D)J_{\varphi_{\pm}}(a) = J_{\varphi_{\pm}}\left(\sum_{k=0}^N (q \# a)_k\right) + J_{\varphi_{\pm}}(r_N)$$

with $r_N \in S^{\mu+v-N,-\infty}$ and $(q \# a)_k \in S^{\mu+v-k,-\infty}$ given by

$$(q \# a)_k(x, \xi) = i^{-k} \sum_{|\alpha|=k} \sum_{\alpha_1 + \alpha_2 = \alpha} \frac{1}{\alpha_1! \alpha_2!} \partial_z^{\alpha_1} ((\partial_{\xi}^{\alpha} q)(x, \Xi(z, x, \xi)))|_{z=0} \partial_x^{\alpha_2} a(x, \xi)$$

with

$$\Xi(z, x, \xi) = \int_0^1 (\nabla_x \varphi_{\pm})(x + tz, \xi) dt.$$

More explicitly,

$$(q\#a)_0(x, \xi) = q(x, \partial_x \varphi_{\pm})a(x, \xi), \quad (q\#a)_1(x, \xi) = i^{-1}(\partial_{\xi} q)(x, \varphi_{\pm}) \cdot (\partial_x a)(x, \xi)$$

and, if $k \geq 2$, $(q\#a)_k$ is a linear combinations of terms of the form

$$(\partial_{\xi}^{\beta} d_{\xi}^j q)(x, \partial_x \varphi_{\pm})(\partial_x^{\gamma_1} \nabla_x \varphi_{\pm}) \cdots (\partial_x^{\gamma_j} \nabla_x \varphi_{\pm}) \partial_x^{\alpha_2} a(x, \xi)$$

with

$$\alpha_1 + \alpha_2 = \alpha, \quad \gamma_1 + \cdots + \gamma_j = \alpha_1.$$

We omit the proof of this classical result (see, e.g., [27, Theorem IV-19]).

We can now start to give a precise meaning to (3.3) and prove it. For notational simplicity, we consider the outgoing case. We look for $A_+ = J_{\varphi_+}(a_+)$ for some symbol

$$a_+ = a_0 + \cdots + a_N, \quad a_j \in S^{-j, -\infty},$$

such that, by setting $P_0 = p_0(D)$ and $P = p_0(D) + V(x, D)$,

$$PJ_{\varphi_+}(a_+) - J_{\varphi_+}(a_+)P_0 = J_{\varphi_+}(R_a),$$

where R_a has a ‘small’ contribution (see Proposition 3.8 for the final statement). On the one hand, we have the simple exact formula

$$J_{\varphi_+}(a_0 + \cdots + a_N)p_0(D) = J_{\varphi_+}((a_0 + \cdots + a_N)p_0).$$

On the other hand, by Proposition 3.6 with $q(x, D) = P$ which we write $p(x, D) + w(x, D)$ with $w \in S^{-1-\rho, m}$ according to (3.2), we have an expansion

$$(p(x, D) + w(x, D))J_{\varphi_+}(a_+) \sim \sum_{j,k} J_{\varphi_+}((p\#a_j)_k + (w\#a_j)_k)$$

in which, if the symbols $a_0, \dots, a_N$ are supported in $\Gamma^+(I_3, R, \sigma_3)$, the eikonal equation (3.7) shows that

$$(p\#a_j)_0 = p(x, \partial_x \varphi_+)a_j = p_0 a_j.$$

Thus, trying to cancel the terms of the expansion of

$$(p(x, D) + w(x, D))J_{\varphi_+}(a_0 + \cdots + a_N) - J_{\varphi_+}(a_0 + \cdots + a_N)p_0(D)$$

leads to the transport equations (see Proposition 3.6 for $\#_1$)

$$(p\#a_0)_1 + (w\#a_0)_0 = 0, \tag{3.24}$$

$$(p\#a_J)_1 + (w\#a_J)_0 = b_J := \sum_{j \leq J-1} (p\#a_j)_{J+1-j} + (w\#a_j)_{J-j}, \quad J \geq 1. \tag{3.25}$$

To solve these equations, we shall use the following proposition.

Proposition 3.7. *The characteristics of the vector field $(\partial_\xi p)(x, \varphi_+(x, \xi)) \cdot \partial_x$ are*

$$\tilde{x}_t := x_t(x, \partial_x \varphi_+(x, \xi)), \quad (x, \xi) \in \Gamma^+(I_3, R, \sigma_3), \quad t \geq 0.$$

Proof. Denote $\tilde{\xi}_t := \xi_t(x, \partial_x \varphi_+(x, \xi))$, so that $\phi^t(x, \partial_x \varphi_+(x, \xi)) = (\tilde{x}_t, \tilde{\xi}_t)$, and denote as well $\tilde{\xi}_t := (\partial_x \varphi_+)(\tilde{x}_t, \xi)$. We show that $\tilde{\xi}_t = \xi_t$. They coincide at $t = 0$ and, componentwise, their derivatives satisfy

$$\frac{d\tilde{\xi}_{t,j}}{dt} = -(\partial_{x_j} p)(\tilde{x}_t, \tilde{\xi}_t), \quad \frac{d\tilde{\xi}_{t,j}}{dt} = (\partial_{x_j} \partial_x \varphi_+)(\tilde{x}_t, \xi) \cdot (\partial_\xi p)(\tilde{x}_t, \tilde{\xi}_t). \quad (3.26)$$

On the other hand, differentiating the eikonal equation (3.7) with respect to x_j yields

$$-(\partial_{x_j} p)(x, \partial_x \varphi_+(x, \xi)) = (\partial_\xi p)(x, \partial_x \varphi_+(x, \xi)) \cdot \partial_{x_j} \partial_x \varphi_+(x, \xi). \quad (3.27)$$

Evaluating (3.27) at $(\tilde{x}_t, \xi)$ yields $-(\partial_{x_j} p)(\tilde{x}_t, \tilde{\xi}_t) = (\partial_{x_j} \partial_x \varphi_+)(\tilde{x}_t, \xi) \cdot (\partial_\xi p)(\tilde{x}_t, \tilde{\xi}_t)$ so that, using (3.26),

$$\left| \frac{d\tilde{\xi}_t}{dt} - \frac{d\tilde{\xi}_t}{dt} \right| \lesssim |\tilde{\xi}_t - \tilde{\xi}_t|.$$

By the Gronwall lemma, $\tilde{\xi}_t$ and ξ_t coincide for $t \geq 0$. Then

$$\frac{d\tilde{x}_t}{dt} = (\partial_\xi p)(\tilde{x}_t, \tilde{\xi}_t) = (\partial_\xi p)(\tilde{x}_t, \xi_t) = (\partial_\xi p)(\tilde{x}_t, \partial_x \varphi_+(\tilde{x}_t, \xi))$$

completes the proof of the proposition. $\blacksquare$

Seeking $a_0 \rightarrow 1$ at infinity (to get the ellipticity), the standard method of characteristics leads to select the following solution a_0 to (3.24):

$$a_0(x, \xi) = \exp\left(i \int_0^{+\infty} w(\tilde{x}_s, \partial_x \varphi_+(\tilde{x}_s, \xi)) ds\right). \quad (3.28)$$

Note that the integral is convergent since, if $|x|$ is large enough, $(x, \partial_x \varphi_+(x, \xi))$ is outgoing if (x, ξ) is, so $|\tilde{x}_s| \gtrsim |x| + |s|$ by (3.10). For $J \geq 1$, seeking a_J in $S^{-J, -\infty}$, hence going to 0 at infinity, the solution to (3.25) is given by

$$a_J(x, \xi) = -i \int_0^{+\infty} b_J(\tilde{x}_t, \xi) \exp\left(i \int_0^t w(\tilde{x}_s, \partial_x \varphi_+(\tilde{x}_s, \xi)) ds\right) dt, \quad (3.29)$$

which is well defined too since $b_J(\tilde{x}_t, \xi)$ is integrable in time, using that

$$(p\#a_j)_{J+1-j} \in S^{-J-1, -\infty} \quad \text{and} \quad (w\#a_j)_{J-j} \in S^{-J-\rho, -\infty},$$

where both $J+1$ and $J+\rho$ are greater than 1.

Now we have to observe an important fact: as they are defined, functions (3.28) and (3.29) make sense for $(x, \xi) \in \Gamma^+(I_3, R, \sigma_3)$ (with R large enough) but they are not

supported in this area so, to define them globally, they have to be multiplied by cutoffs (similarly to (3.23)). We thus define, for $J \geq 0$,

$$a_J^+(x, \xi) = a_J(x, \xi)\chi_+(x, \xi), \quad (3.30)$$

where $\chi_+ \in S^{0, -\infty}$ is supported in $\Gamma^+(I_3, R, \sigma_3)$ and is equal to 1 on a smaller area $\Gamma^+(I_4, R_4, \sigma_4)$. Let us summarize what we have obtained so far.

Proposition 3.8. *Let $I_4 \Subset I_3$ and $\sigma_4 > \sigma_3$. If R_3 and R_4 are large enough, with $R_4 > R_3$,*

$$PJ_{\varphi_+}(a_0^+ + \dots + a_N^+) - J_{\varphi_+}(a_0^+ + \dots + a_N^+)P_0 = J_{\varphi_+}(r_N^+) + J_{\varphi_+}(\check{a}_N^+), \quad (3.31)$$

with $r_N^+ \in S^{-N, \infty}$ and $\check{a}_N^+ \in S^{0, -\infty}$, $\check{a}_N^+$ supported in $\Gamma^+(I_3, R_3, \sigma_3) \setminus \Gamma^+(I_4, R_4, \sigma_4)$.

Note that $\check{a}^+$ does not belong to $S^{-N, -\infty}$ but only to $S^{0, -\infty}$, so at this stage it is not clear why its contribution will give some decay in x . It is actually not the case in (3.31), but it will provide the good behaviour at the final stage (3.33) using non-stationary phase arguments.

Having constructed a non-trivial A for (3.3), we now want to construct B so that AB^* can be an arbitrary microlocalization (within an outgoing area). To do so, we need the following.

Proposition 3.9. *Let $I_5 \Subset I_4$ and $\sigma_5 > \sigma_4$. If R_5 is large enough, then for every $c^+ \in S^{0, -\infty}$ supported in $\Gamma^+(I_5, R_5, \sigma_5)$, there exist*

$$b_j^+ \in S^{-j, -\infty} \text{ supported in } \Gamma^+(I_4, R_4, \sigma_4), \quad j = 0, \dots, N,$$

and $\tilde{r}_N^+ \in S^{-N, -\infty}$ such that

$$J_{\varphi_+}(a_0^+ + \dots + a_N^+)J_{\varphi_+}(b_0^+ + \dots + b_N^+)^* = c^+(x, D) + \tilde{r}_N^+(x, D). \quad (3.32)$$

Sketch of proof. This follows from the fact that, for general symbols a, b , $J_{\varphi_+}(a)J_{\varphi_+}(b)^*$ is a pseudodifferential operator with top order symbol²

$$a(x, \xi(x, \eta))\bar{b}(x, \xi(x, \eta)) \det|\partial_\eta \xi(x, \eta)|$$

(see [27, Theorem IV.21]), where $\eta \mapsto \xi(x, \eta)$ is the inverse of $\xi \mapsto \partial_x \varphi_+(x, \xi)$. In other words, we can determine b_0^+ in (3.32) by

$$\overline{b_0^+(x, \xi)} = \frac{1}{a_0^+(x, \xi)} c^+(x, \partial_x \varphi_+(x, \xi)) |\det \partial_{x, \xi}^2 \varphi_+(x, \xi)|,$$

which is well defined since a_0^+ is elliptic, more precisely close to 1, near the support of c^+ . Higher-order terms b_j^+ are defined similarly in an iterative fashion by examination of the full symbol of $J_{\varphi_+}(a)J_{\varphi_+}(b)^*$. ■

Summarizing, we have obtained the following assertion.

²In the sense of asymptotic expansion in increasing powers of $\langle x \rangle^{-1}$.

Proposition 3.10 (Isozaki–Kitada parametrix). *Let*

$$I_5 \Subset I_4 \Subset I_3 \Subset I_{\text{nc}} \quad \text{and} \quad 1 > \sigma_5 > \sigma_4 > \sigma_3 > -1.$$

Then, if $R_5 \gg R_4 \gg R_3 \gg 1$, for all $N \in \mathbb{N}$ and all symbols $c_{\pm} \in S^{0,-\infty}$ that are supported in $\Gamma^{\pm}(I_5, R_5, \sigma_5)$, one can find

- $a^{\pm} \in S^{0,-\infty}$, supported in $\Gamma^{\pm}(I_3, R_3, \sigma_3)$,
- $b^{\pm} \in S^{0,-\infty}$, supported in $\Gamma^{\pm}(I_4, R_4, \sigma_4)$,
- $r_N^{\pm}, \tilde{r}_N^{\pm} \in S^{-N,-\infty}$,
- $\tilde{a}_N^{\pm} \in S^{0,-\infty}$, supported in $\Gamma^{\pm}(I_3, R_3, \sigma_3) \setminus \Gamma^{\pm}(I_4, R_4, \sigma_4)$,

such that, for all $t \in \mathbb{R}$,

$$e^{-itP} c^{\pm}(x, D) = J_{\varphi_{\pm}}(a_{\pm}) e^{-itP_0} J_{\varphi_{\pm}}(b_{\pm})^* + \mathcal{R}_N^{\pm}(t), \quad (3.33)$$

where

$$\mathcal{R}_N^{\pm}(t) = -e^{-itP} \tilde{r}_N^{\pm}(x, D) - i \int_0^t e^{-i(t-s)P} J_{\varphi_{\pm}}(r_N^{\pm} + \tilde{a}_N^{\pm}) e^{-isP_0} J_{\varphi_{\pm}}(b_{\pm}) ds. \quad (3.34)$$

Proof. It follows by integration between 0 and t of

$$\frac{d}{dt} (e^{itP} J_{\varphi_{\pm}}(a_{\pm}) e^{-itP_0} J_{\varphi_{\pm}}(b_{\pm})^*) = i e^{itP} (P J_{\varphi_{\pm}}(a_{\pm}) - J_{\varphi_{\pm}}(a_N^{\pm}) P_0) e^{-itP_0} J_{\varphi_{\pm}}(b_{\pm})^*$$

together with (3.31) and (3.32) (or their incoming analogues). ■

We will not explain here how to exploit the decay in x of the symbols $r_N^{\pm}$ and $\tilde{r}_N^{\pm}$ in the various applications described in the next section; this would be too long and go beyond the topic of this survey since this is related to propagation estimates (or positive commutator estimates), while all the WKB flavour of the parametrix is contained in Proposition 3.10. Nevertheless, we still have to explain why the non-decaying symbols $\tilde{a}_N^{\pm}$ (recall they belong to $S^{0,-\infty}$, not to $S^{-N,-\infty}$) do provide decay.

Proposition 3.11. *The construction of symbols from Proposition 3.10 can be done in such a way that, in addition to the conclusion of this proposition, one has: for all $M \geq 0$,*

$$\|\langle x \rangle^M J_{\varphi_{\pm}}(\tilde{a}_N^{\pm}) e^{-isP_0} J_{\varphi_{\pm}}(b_{\pm}) \langle x \rangle^M\|_{L^2 \rightarrow L^2} \leq C_M (1 + |s|)^{-M}, \quad \pm s \geq 0.$$

Proof. For simplicity and coherence with the previous proofs, we consider the outgoing case only and thus assume that $s \geq 0$. Basically, the symbol $\tilde{a}_N^+$ is supported in $\text{supp}(\nabla \chi_+)$, where χ_+ is the cutoff used in (3.30) (it is the contribution of derivatives on the terms a_j^+ which do not all fall on a_J). The cutoff χ_+ can be chosen of the form

$$\chi_+(x, \xi) = \chi_1(p_0(\xi)) \chi_2(|x|) \chi_3\left(\frac{x \cdot \nabla p_0(\xi)}{|x| |\nabla p_0(\xi)|}\right),$$

where $\chi_1 \equiv 1$ near I_4 and is supported in I_3 , $\chi_2 \equiv 1$ on $[R_4, +\infty)$ and is supported in $[R_3, +\infty)$ and, more importantly, $\chi_3 \equiv 1$, say, near $[\sigma, +\infty)$ with $\sigma \in (\sigma_3, \sigma_4)$ and is supported in $(\sigma_3, +\infty)$. The terms corresponding to derivatives falling on $\chi_2(|x|)$ can be discarded, for they are compactly supported in x and can thus be added to the contribution of r_N^+ in (3.34). Now observe that the Schwartz kernel of $J_{\varphi+}(\tilde{a}_N^+)e^{-isP_0}J_{\varphi+}(b_+)$ is the oscillatory integral

$$(2\pi)^{-n} \int e^{i(\varphi_+(x,\xi) - s p_0(\xi) - \varphi_+(y,\xi))} \tilde{a}_N^+(x, \xi) \overline{b_+(y, \xi)} d\xi. \quad (3.35)$$

The terms of $\tilde{a}_N^+$ corresponding to derivatives falling on $\chi_1(p_0(\xi))$ have zero contribution in (3.35) since b_+ is supported in $\{p_0(\xi) \in I_4\}$ on which derivatives of $\chi_1(p_0(\xi))$ vanish. We may thus assume that on the support of $\tilde{a}_N^+(x, \xi) \overline{b_+(y, \xi)}$, we have

$$x \cdot \nabla p_0(\xi) < \sigma|x||\nabla p_0(\xi)|, \quad y \cdot \nabla p_0(\xi) > \sigma_4|y||\nabla p_0(\xi)|,$$

from which we can infer that the norm of the gradient of the phase

$$\partial_\xi \varphi_+(x, \xi) - s \nabla p_0(\xi) - \partial_\xi \varphi_+(y, \xi) \quad (3.36)$$

is bounded from below by (a positive constant times) $|x| + |y| + s$. A proof of this fact is given in [6, Lemma 4.1 and (4.18)] but we recall that, intuitively, it is based on the following facts: since (y, ξ) is outgoing, then so is $(y + s \nabla p_0(\xi), \xi)$ for s positive and $|y + s \nabla p_0(\xi)| \gtrsim |y| + s|\nabla p_0(\xi)|$; then, since (x, ξ) is incoming, $(-x, \xi)$ is outgoing and so is $(-x + s \nabla p_0(\xi) + y, \xi)$ in which $|-x + s \nabla p_0(\xi) + y| \gtrsim |x| + |y| + s|\nabla p_0(\xi)|$. To get this lower bound rigorously, one uses that $\sigma_4 - \sigma > 0$ (see [6, Lemma 4.1]). To conclude that the norm of (3.36) is bounded below by $|x| + |y| + s$, we use the fact that $\partial_\xi \varphi_+(x, \xi) = x + O(|x|^{1-\rho})$ (and likewise for y) by (3.8). Using this lower bound, we can use non-stationary phase integrations by parts in the integral to see that, for any M ,

$$|(3.35)| \leq C_M(|x| + |y| + s + 1)^{-M}, \quad s \geq 0.$$

The result follows. ■

4. Applications

4.1. Wave operators and scattering matrices

In this paragraph, we consider applications of the Isozaki–Kitada construction to fundamental objects of scattering theory, namely wave operators, scattering matrices and (generalized) scattering phases.

We start with the case of Schrödinger operators, i.e., assume that $p_0(\xi) = |\xi|^2$ and $V = v = V(x) \in S^{-\rho}$ is a real-valued potential (with the notation of Section 3). In particular, any interval contained in $(0, +\infty)$ is non-critical for p_0 .

The first result below states that the operators $J_{\varphi_{\pm}}(a_{\pm})$ provide good modifiers to study the scattering theory for the pair of operators $(P_0, P) = (-\Delta, -\Delta + V)$. Recall indeed that when V is of short range, i.e., when $\rho > 1$, the standard wave operators

$$s - \lim_{t \rightarrow +\infty} e^{itP} e^{-itP_0}$$

exist, e.g., according to Cooks method ([24, Theorem XI.4], [15, Theorem 14.4.6]). When $\rho \in (0, 1]$, the problem is more delicate, and one has to modify the free dynamics e^{-itP_0} to get the asymptotics of e^{-itP} as $t \rightarrow \pm\infty$; Isozaki–Kitada modifiers were introduced to this end. Below, we denote by $E_I(P)$ (resp. $E_I(P_0)$) the spectral projection on the interval I associated with P (resp. P_0), namely $f(P)$ (resp. $f(P_0)$) with f the characteristic function of I .

Theorem 4.1 ([19]). *Let $I \Subset \tilde{I} \Subset (0, +\infty)$ and $-1 < \sigma < \sigma < 1$. Let $a_{\pm} \in S^{0,\infty}$ be supported in $\Gamma^{\pm}(\tilde{R}, \tilde{I}, \tilde{\sigma})$ and equal to 1 on $\Gamma^{\pm}(R, I, \sigma)$ with $\tilde{R} > R \gg 1$. Then*

$$W_{\pm}(I) := s - \lim_{t \rightarrow \pm\infty} e^{itP} J_{\varphi_{\pm}}(a_{\pm}) e^{-itP_0} E_I(P_0)$$

are well-defined bounded operators on $L^2(\mathbb{R}^n)$. In addition, they are complete, i.e., are isometries from $\text{Ran}(E_I(P_0))$ to $\text{Ran}(E_I(P))$, and this range is contained in the absolutely continuous subspace of P .

The scattering operator S is then defined as

$$S = W_+(I)^* W_-(I).$$

It follows easily from their definition that the wave operators intertwine the dynamics, namely $W_{\pm}(I) e^{itP_0} = e^{itP} W_{\pm}(I)$, which implies that S commutes with e^{itP_0} for all t , hence with P_0 . To define the scattering matrices, one introduces the unitary transform

$$\mathcal{U}: L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}_{\lambda}^+, L^2(\mathbb{S}_{\omega}^{n-1}))$$

defined formally by

$$(\mathcal{U}f)(\lambda) = [\omega \mapsto (2\pi)^{-n/2} \sqrt{2}^{-1} \lambda^{(n-2)/4} \hat{f}(\sqrt{\lambda}\omega)].$$

The above expression makes sense for f in the Schwartz space, $\mathcal{U}$ is defined on L^2 by density. It diagonalizes the Laplacian in the sense that $\mathcal{U}P_0\mathcal{U}^*$ is the multiplication operator by λ . As a commuting operator with P_0 , the scattering operator behaves well with respect to $\mathcal{U}$, namely there are bounded operators $S(\lambda)$ on $L^2(\mathbb{S}^{n-1})$ defined for a.e. $\lambda \in I^3$ such that

$$(\mathcal{U}Sf)(\lambda) = S(\lambda)(\mathcal{U}f)(\lambda) \tag{4.1}$$

³Actually, for a.e. $\lambda \in \mathbb{R}^+$ but, since the wave operators are spectrally localized in I , only those $\lambda \in I$ are relevant in this definition.

(see Appendix B for a proof). The operators $S(\lambda)$ are called the scattering matrices. In other words, if f is, say, a Schwartz function with Fourier transform supported in $\{|\xi|^2 \in I\}$, then the Fourier inversion formula yields a decomposition of f into generalized eigenfunctions of P_0 through

$$f(x) = (2\pi)^{-n} \int e^{ix \cdot \xi} \hat{f}(\xi) d\xi = (2\pi)^{-n} \int_I \frac{\lambda^{(n-2)/2}}{2} \left(\int_{\mathbb{S}^{n-1}} e^{i\sqrt{\lambda}x \cdot \omega} \hat{f}(\sqrt{\lambda}\omega) d\omega \right) d\lambda,$$

so that

$$(Sf)(x) = (2\pi)^{-n} \int_0^\infty \frac{\lambda^{(n-2)/2}}{2} \left(\int_{\mathbb{S}^{n-1}} e^{i\sqrt{\lambda}x \cdot \omega} (S(\lambda) \hat{f}(\sqrt{\lambda}\cdot))(\omega) d\omega \right) d\lambda.$$

Formally, but more explicitly, if $\psi \in C^\infty(\mathbb{S}^{n-1})$ and

$$\Psi_\lambda(x) := \int_{\mathbb{S}^{n-1}} e^{i\sqrt{\lambda}x \cdot \omega} \psi(\omega) d\omega$$

is a generalized eigenfunction of P_0 (at energy λ), then

$$(S\Psi_\lambda)(x) = \int_{\mathbb{S}^{n-1}} e^{i\sqrt{\lambda}x \cdot \omega} (S(\lambda)\psi)(\omega) d\omega,$$

which is formal since Ψ_λ is not L^2 (unless $\psi \equiv 0$).

In the next theorem, we gather two different applications of the Isozaki–Kitada construction to the analysis of scattering matrices. They are given again in the case of perturbations of the Laplacian by long-range potentials, but in case (2) one considers the semiclassical case, namely $P_0 = -h^2 \Delta$ and $P = -h^2 \Delta + V(x)$ with $h \in (0, 1]$. One also assumes in this case that the potential is analytic in a conic neighbourhood of $\mathbb{R}^n$ in the sense that $V(x) = O((1 + |x|)^{-\rho})$ within a region of the form

$$\{x \in \mathbb{C}^n \mid |\operatorname{Re}(x)| > R_0 \text{ and } |\operatorname{Im}(x)| \leq \epsilon_0(1 + |\operatorname{Re}(x)|)\}, \quad (4.2)$$

with $R_0 > 0$ large enough and $\epsilon_0 > 0$ small enough. This analyticity assumption allows in particular to consider the resonances associated with $-h^2 \Delta + V$ defined as the poles of the resolvent $(h^2 \Delta + V - z)^{-1}$ acting on appropriate distorted spaces (see [14, 17] for details); the point to keep in mind is that the resonances are intrinsic, i.e., independent of the distortion, and are the analogues of the eigenvalues (for operators with discrete spectrum) in this context where operators have continuous spectrum on $[0, \infty)$.

Theorem 4.2 (Scattering matrices). (1) *If $\rho > \frac{1}{2}$, then the family of scattering matrices $(S(\lambda))_{\lambda>0}$ determine the potential [20].*

(2) *If $\rho > 0$ and V is analytic in (4.2), in the semiclassical case, the map $\lambda \mapsto S(\lambda)$ can be extended meromorphically to a complex neighbourhood of $(0, +\infty)$ which is a conic at infinity and its poles are the resonances of $-h^2 \Delta + V$ [12].*

To prove item (2), Gérard–Martinez extend the Isozaki–Kitada construction to the complex domain.

We next consider another application to generalized scattering phases. We present the results in the case when

$$P = -\Delta_g + A(x) \cdot D_x + D_x \cdot A(x) + V(x),$$

for simplicity. Here $g = (g_{jk})$ is a Riemannian metric on $\mathbb{R}^n$, $A = (A_1, \dots, A_n)$ belongs to $C^\infty(\mathbb{R}^n, \mathbb{R}^n)$ and corresponds to a magnetic field; we assume

$$|\partial_x^\alpha (g_{jk}(x) - \delta_{jk})| + |\partial_x^\alpha A_j(x)| + |\partial_x^\alpha V(x)| \leq C_\alpha \langle x \rangle^{-\rho-|\alpha|}.$$

When $\rho > n$, one can define (uniquely modulo a constant) the spectral shift function s_1 by

$$-\int f'(\lambda) s_1(\lambda) d\lambda = \text{tr}(f(P) - f(P_0)),$$

for all f in the Schwartz space on $\mathbb{R}$. The right-hand side defines s'_1 as a distribution, but it is a general fact that s_1 is actually a locally integrable function [33]. In our context where operators have continuous spectrum, this is the analogue of the eigenvalues counting function for operators on compact manifolds. For instance, we have a high-energy Weyl law

$$s_1(\lambda) = \frac{\omega_n}{(2\pi)^n} \lambda^{n/2} \int (\det(g_{jk}(x))^{1/2} - 1) dx + O(\lambda^{(n-1)/2})$$

as proved in [28] (ω_n is the volume of the unit Euclidean ball). The terminology *scattering phase* is in particular due to the Birman–Krein formula [33] saying that the determinant of the scattering matrix has modulus 1 with phase given (mod 2π) by s_1 . For $\rho > 0$, $f(P) - f(P_0)$ is not of trace class in general, so s_1 is not defined. However, following a regularization introduced by Koplienko [22], by selecting $p \in \mathbb{N}$ such that $\rho p > n$, one can define a distribution s_p by considering

$$P_s := P_0 + s(P - P_0)$$

and

$$\langle s'_p, f \rangle := \text{tr} \left(f(P) - \sum_{j=0}^{p-1} \frac{1}{j!} \frac{d^j}{ds^j} f(P_s) \Big|_{s=0} \right), \quad f \in \mathcal{S}(\mathbb{R}). \quad (4.3)$$

The next theorem records results on these scattering phases which are analogous to what was proved before for s_1 (i.e., when $\rho > n$) or even more roughly to what happens for the counting function of eigenvalues for second order operators on compact manifolds. It turns out that the Isozaki–Kitada parametrix is a powerful technical tool to prove these results.

Theorem 4.3 (Generalized scattering phases, [2]). *Let $\rho > 0$ and $p \in \mathbb{N}$ be such that $p\rho > n$. Then*

- (1) *Formula (4.3) makes sense, i.e., the operator inside the trace is indeed trace class and depends continuously on f in the Schwartz class.*
- (2) *The distribution s_p so defined, up to an additive constant, is smooth on $(0, +\infty)$.*
- (3) *As $\lambda \rightarrow +\infty$, we have a Weyl law*

$$s_p(\lambda) = \frac{\omega_n}{(2\pi)^n} \lambda^{n/2} \text{vol}_p(g) + O(\lambda^{(n-1)/2}),$$

where, if we define (for s close to 0 or equal to 1) the metric $g_s = (g_{jk,s}(x))$ by $g_s = (I + s(g^{-1} - I))^{-1}$,

$$\text{vol}_p(g) = \int_{\mathbb{R}^n} \left(\det(g_1)^{1/2} - \sum_{j=0}^{p-1} \frac{1}{j!} \frac{d^j}{ds^j} \det(g_s)^{1/2} \Big|_{s=0} \right) (x) dx.$$

- (4) *If in addition, the geodesic flow associated with the metric g is non-trapping,⁴ then one has a complete asymptotic expansion of s'_p as $\lambda \rightarrow +\infty$, i.e., there are coefficients $(a_j)_{j \geq 0}$ such that for all N ,*

$$s'_p(\lambda) = \lambda^{n/2-1} \sum_{j=0}^{N-1} a_j \lambda^{-j/2} + O(\lambda^{(n-N)/2-1}).$$

Item (1) is very general and holds for a much wider class of operators (no asymptotically Euclidean condition is required). Items (2), (3) and (4) are much more delicate and are proved using the Isozaki–Kitada parametrix. To analyze s_p through its Fourier transform, we need a good *long-time* knowledge about

$$e^{-itP} - \sum_{j=0}^{p-1} \frac{1}{j!} \frac{d^j}{ds^j} e^{-it(P_0 + s(P - P_0))} \Big|_{s=0}.$$

The relevant estimates are obtained in [2] by using the Isozaki–Kitada construction for the s dependent operators $P_s = P_0 + s(P - P_0)$; in particular, we have to study the dependence on s of the phases given by Theorem 3.1 for such operators.

We also point out that the version of (2) in [2] states that s_p is smooth away from the embedded eigenvalues of P , but it is now known that they are absent for such long-range perturbations of the Euclidean Laplacian according to [21]. Likewise, item (3) was proved conditionally to rough exponential upper bounds on the weighted resolvent of P on the real axis, but such estimates are known to hold from [7], at least in the context of Theorem 4.3.

⁴That is, all geodesics with non-zero speed escape to infinity.

4.2. Propagation and Strichartz estimates

In this paragraph, we consider more PDE type applications of the Isozaki–Kitada parametrix.

The propagator of the free semiclassical Laplacian on $\mathbb{R}^n$, i.e., $e^{ith\Delta}$, satisfies

$$\|\langle x \rangle^{-s} e^{ith\Delta} f(-h^2\Delta) \langle x \rangle^{-s}\|_{L^2 \rightarrow L^2} \leq C_s \langle t \rangle^{-s}, \quad h \in (0, 1], \quad t \in \mathbb{R}, \quad (4.4)$$

whenever f belongs to $C_0^\infty(0, +\infty)$ and $s \geq 0$ is a real number. Similar estimates for long-range perturbations V (as in the previous paragraph for simplicity) can be obtained from resolvent estimates: if $\nu > \frac{1}{2}$ and if the classical flow is non-trapping on the energy region $p^{-1}(I)$, namely if characteristics of the Hamiltonian flow of $p(x, \xi) = |\xi|^2 + V(x)$ with energy in the interval I escape to infinity as time goes to infinity, one can show that for any integer $k \geq 0$,

$$\|\langle x \rangle^{-\nu-k} (-h^2\Delta + V - \lambda \pm i\epsilon)^{-k} \langle x \rangle^{-\nu-k}\|_{L^2 \rightarrow L^2} \leq C_k h^{-1-k}, \quad \lambda \in \text{supp}(f) \quad (4.5)$$

(see [11, 30] for a proof). Then, using integrations by parts in the Stone formula [25, p. 237]

$$f(-h^2\Delta + V) e^{-it(-h^2\Delta + V)/h} = \frac{1}{\pi} \int_{\mathbb{R}} f(\lambda) e^{-it\lambda/h} \text{Im}(-h^2\Delta + V - \lambda - i0)^{-1} d\lambda,$$

together with (4.5), we obtain

$$\|\langle x \rangle^{-\nu-k} f(-h^2\Delta + V) e^{-it(-h^2\Delta + V)/h} \langle x \rangle^{-\nu-k}\|_{L^2 \rightarrow L^2} \leq C_k h^{-1} \langle t \rangle^{-k}.$$

Compared to (4.4), the above estimate exhibits a loss in h and non-optimal weights. Recovering the optimal estimates can be achieved using the Isozaki–Kitada parametrix as shown by the following theorem of Wang.

Theorem 4.4 ([32]). *If I is a non-trapping interval for $|\xi|^2 + V(x)$ and $f \in C_0^\infty(I)$,*

$$\|\langle x \rangle^{-s} f(-h^2\Delta + V) e^{-it(-h^2\Delta + V)/h} \langle x \rangle^{-s}\|_{L^2 \rightarrow L^2} \leq C_s \langle t \rangle^{-s}, \quad h \in (0, 1], \quad t \in \mathbb{R}.$$

Another instance of application of the Isozaki–Kitada techniques to PDE is given by Strichartz estimates for the Schrödinger equation. Recall that in the free case, the solution to the Schrödinger equation

$$i \partial_t u + \Delta u = 0, \quad u(0) \in L^2(\mathbb{R}^n)$$

satisfies

$$\|u\|_{L^p(\mathbb{R}, L^q(\mathbb{R}^n))} \leq C_{n,p,q} \|u(0)\|_{L^2}, \quad u(0) \in L^2(\mathbb{R}^n)$$

provided the exponents p, q satisfy

$$\frac{2}{p} + \frac{n}{q} = \frac{n}{2}, \quad p, q \geq 2, \quad (p, q) \neq (2, \infty). \quad (4.6)$$

Here is what can be said when the Euclidean metric is replaced by an asymptotically Euclidean one.

Theorem 4.5 ([5]). *Let g be a Riemannian metric on $\mathbb{R}^n$ which is a long-range perturbation of the Euclidean metric,*

$$|\partial_x^\alpha (g_{jk}(x) - \delta_{jk})| \leq C_\alpha \langle x \rangle^{-\rho-|\alpha|}, \quad (4.7)$$

and denote by u the solution to $i\partial_t u + \Delta_g u = 0$ with initial condition $u(0) \in L^2$.

- (1) *Let χ_R be the indicator function of $B(0, R)$. If R is large enough, for each pair of exponents (p, q) satisfying (4.6), there exists C such that for all $u(0) \in L^2$,*

$$\|(1 - \chi_R)u\|_{L^p([0,1], L^q(\mathbb{R}^n))} \leq C \|u(0)\|_{L^2}. \quad (4.8)$$

- (2) *If in addition the metric has no trapped geodesic, (4.8) holds with $\chi_R \equiv 0$.*

Both items of this result were proved using the Isozaki–Kitada parametrix (in the semiclassical form), to derive dispersion estimates on the propagator microlocalized in outgoing/incoming regions. We can mention that (2) was obtained first in [26, 31], but the localization at infinity of the estimates given in (1) was original and a fairly direct consequence of the Isozaki–Kitada parametrix. This is yet another illustration of the strength of this parametrix.

More recently, analogues to the Isozaki–Kitada parametrix have been constructed on Riemannian manifolds with ends (M, g) which are either asymptotically hyperbolic or asymptotically conic in the following sense. For some compact subset $K \Subset M$, $M \setminus K$ is diffeomorphic to $(R_0, +\infty) \times S$ for some closed manifold S and the metric on $M \setminus K$ is of the form $g = dr^2 + w(r)^{-2}h(r)$ with some positive function w and $(h(r))_{r>R_0}$ a family of Riemannian metrics on S approaching a fixed one h as $r \rightarrow +\infty$, more precisely $\partial_r^k(h(r) - h) = O(r^{-\rho-k})$ (similarly to (4.7)); the case of asymptotically hyperbolic manifolds corresponds to $w(r) = e^{-r}$ while asymptotically conical manifolds correspond to $w(r) = r^{-1}$. Compared to approximation (3.33), the free Hamiltonian in this geometric context is now the Laplacian $-\frac{\partial^2}{\partial r^2}$ on the line, and the parametrix takes the form

$$e^{it\Delta_g} Op(\chi_\pm) \approx J_{\varphi_\pm}(a^\pm) e^{it\frac{\partial^2}{\partial r^2}} J_{\varphi_\pm}(b^\pm),$$

where $Op(\chi_\pm)$ is a pseudodifferential operator with symbol supported in a *strongly* outgoing (+)/incoming (-) area defined as follows: if $p = p(r, \theta, \rho, \eta) := \rho^2 + w(r)^2 h^{jk}(r) \eta_j \eta_k$ is the principal symbol of $-\Delta_g$, $\kappa: S \supset U \rightarrow \Theta \subset \mathbb{R}^{n-1}$ is a coordinates chart on S , $I \Subset (0, +\infty)$ is an open interval, $R \gg 1$ and $0 < \varepsilon \ll 1$, the associated strongly outgoing/incoming area is

$$\Gamma_{\text{st}}^\pm(R, \Theta, I, \varepsilon) = \{(r, \theta, \rho, \eta) \mid r > R, \theta \in \Theta, p \in I, |\rho \mp p^{1/2}| < \varepsilon p^{1/2}\}.$$

Basically, they are the analogues of (3.5) when $\sigma = 1 - \varepsilon$ is close to 1 (up to the additional (angular) localization in Θ). The operators $J_{\varphi_\pm}(c)$ are Fourier integral operators associated with real phases $\varphi_\pm$ solving the eikonal equation

$$p(r, \theta, \partial_r \varphi_\pm, \partial_\theta \varphi_\pm) = \rho^2$$

in strongly outgoing/incoming areas. By revisiting the Isozaki–Kitada constructions in such geometric contexts, we were able to obtain the following results.

Theorem 4.6. (a) *If (M, g) is asymptotically hyperbolic in the above sense, results (1) and (2) of Theorem 4.5 still hold in this context [3].*

(b) *If (M, g) is asymptotically conic in the above sense and of dimension $n \geq 3$, we have the following results [4]:*

- *If the weighted resolvent of $-\Delta_g$ is of polynomial growth at high energy, i.e., for some $N \gg 1$,*

$$\|\langle r \rangle^{-N} (-\Delta_g - \lambda \pm i0)^{-1} \langle r \rangle^{-N}\|_{L^2(M) \rightarrow L^2(M)} \leq C \lambda^N, \quad \lambda \gg 1,$$

then item (1) of Theorem 4.5 still holds in this context but globally in time, i.e., with $\mathbb{R}$ instead of $[0, 1]$ in (4.8).

- *If the geodesic flow is non-trapping, then item (2) of Theorem 4.5 also holds in this context but globally in time.*

The local in time version of the first item of (b), in similar geometric frameworks, was first proved in [13, 23]. Regarding the global-in-time results, we point out they require to understand the *low frequency* regime, hence to get appropriate long-time parametrices of $f(-\frac{\Delta_g}{\epsilon})e^{it\Delta_g}$ as $\epsilon \rightarrow 0$, within the limits allowed by the uncertainty principle, i.e., far away at spatial infinity ($r \gtrsim \epsilon^{-1/2}$). This led to the development of a low frequency version of the Isozaki–Kitada parametrix in [4] in a geometric framework (see [9] for another low frequency parametrix for perturbation by potentials on $\mathbb{R}^n$). Finally, we mention that we were also able to prove global-in-time Strichartz estimates (second item of (b)) even in the presence of few hyperbolic trapped trajectories, but we refrain from quoting this more technical result here.

A. Proofs

In all proofs below, we consider the outgoing case, the incoming one being similar.

A.1. Proof of Lemma 3.2

We start with general computations without any assumptions on the initial data (x, ξ) :

$$\begin{aligned} \frac{d^2}{dt^2} \frac{|x_t|^2}{2} &= \frac{d}{dt} (x_t \cdot (\nabla_\xi p)(x_t, \xi_t)) \\ &= x_t \cdot ((\nabla_{x\xi}^2 v)(x_t, \xi_t)(\nabla_\xi p)(x_t, \xi_t) - (\nabla_{\xi\xi}^2 p)(x_t, \xi_t)(\nabla_x v)(x_t, v_t)) \\ &\quad + |(\nabla_\xi p)(x_t, \xi_t)|^2, \end{aligned}$$

where

$$|(\nabla_\xi p)(x_t, \xi_t)|^2 = |(\nabla_\xi p_0)(\xi_t)|^2 + 2(\nabla_\xi p_0)(\xi_t) \cdot (\nabla_\xi v)(x_t, \xi_t) + |(\nabla_\xi v)(x_t, \xi_t)|^2.$$

Thus, for some constant C_1 depending only on seminorms of p_0 and v , we have

$$\left| \frac{d^2}{dt^2} \left(\frac{|x_t|^2}{2} \right) - |(\nabla_\xi p_0)(\xi_t)|^2 \right| \leq C_1 \langle p_0(\xi_t) \rangle^2 \langle x_t \rangle^{-\rho},$$

and therefore

$$\begin{aligned} \frac{|x_t|^2}{2} &\geq \frac{|x|^2}{2} + tx \cdot (\nabla_\xi p)(x, \xi) + \frac{t^2}{2} |(\nabla_\xi p_0)(\xi)|^2 - C_1 \frac{t^2}{2} \sup_{s \in [0, t]} \langle p_0(\xi_s) \rangle^2 \langle x_s \rangle^{-\rho} \\ &\quad - \frac{t^2}{2} \sup_{s \in [0, t]} ||(\nabla_\xi p_0)(\xi)|^2 - |(\nabla_\xi p_0)(\xi_s)|^2|. \end{aligned}$$

If we assume that

$$tx \cdot (\nabla_\xi p_0)(\xi) \geq -|\sigma t||x| |(\nabla_\xi p_0)(\xi)|, \quad (\text{A.1})$$

then

$$\begin{aligned} \frac{|x|^2}{2} + tx \cdot (\nabla_\xi p_0)(\xi) + \frac{t^2}{2} |(\nabla_\xi p_0)(\xi)|^2 &\geq \frac{|x|^2}{2} - |\sigma t||x| |(\nabla_\xi p_0)(\xi)| + \frac{t^2}{2} |(\nabla_\xi p_0)(\xi)|^2 \\ &\geq \frac{1-|\sigma|}{2} (|x|^2 + t^2 |(\nabla_\xi p_0)(\xi)|^2). \end{aligned}$$

Summarizing, we have proved the following.

Lemma A.1. *If (A.1) holds, then we have*

$$|x_t|^2 \geq (1-|\sigma|)(|x|^2 + t^2 |(\nabla_\xi p_0)(\xi)|^2) - 2R(t, x, \xi), \quad (\text{A.2})$$

where

$$\begin{aligned} R(t, x, \xi) &= C_1 \frac{t^2}{2} \sup_{s \in [0, t]} \langle p_0(\xi_s) \rangle^2 \langle x_s \rangle^{-\rho} + \frac{t^2}{2} \sup_{s \in [0, t]} ||(\nabla_\xi p_0)(\xi)|^2 - |(\nabla_\xi p_0)(\xi_s)|^2| \\ &\quad + |t||x| |\nabla_\xi v(x, \xi)|. \end{aligned}$$

Intuitively, if $|x_s|$ and $|x|$ are large and $\xi_s - \xi$ is small, then $R(t, x, \xi) = o(|x|^2 + t^2)$ and we get Lemma 3.2. We shall see that a classical bootstrap argument justifies this intuition if (x, ξ) belongs to an outgoing region.

Let $\tilde{I}_0$ be such that $I_0 \Subset \tilde{I}_0$ and $\tilde{I}_0$ is still non-critical for p_0 . Since p_0 is proper by assumption, we see by uniform continuity that for some $\delta > 0$,

$$p_0(\xi) \in I_0 \quad \text{and} \quad |\xi - \xi'| < \delta \Rightarrow p_0(\xi') \in \tilde{I}_0.$$

By non-criticality and by choosing $\tilde{I}_0$ close enough to I_0 , we can select next $\kappa > 0$ such that

$$p_0(\xi) \in \tilde{I}_0 \Rightarrow |\nabla_\xi p_0(\xi)|^2 \geq \kappa^2.$$

To get Lemma 3.2, we shall prove by a bootstrap argument that if R is large enough, we have

$$|x_t|^2 \geq \frac{1-|\sigma|}{2} (|x|^2 + \kappa^2 t^2), \quad |\xi_t - \xi| < \delta, \quad (\text{A.3})$$

for all $t \geq 0$ and $(x, \xi) \in \Gamma^+(I_0, R, \sigma)$. To guess the suitable R to bootstrap the argument, let us see what we get if estimate (A.3) holds over a bounded time interval $[0, t_0]$. First, for some constant A_{I_0} depending on I_0 and a seminorm of v , we have

$$|t||x||\partial_\xi v(x, \xi)| \leq A_{I_0}(t^2 + |x|^2)\langle x \rangle^{-\rho}.$$

Also, using that $|\xi - \xi_s| < \delta$ and $|x_s| \geq (\frac{1-|\sigma|}{2})^{1/2}|x|$, we have for some B_{δ, σ, I_0} ,

$$C_1 \frac{t^2}{2} \sup \langle p_0(\xi_s) \rangle^2 \langle x_s \rangle^{-\rho} \leq B_{\delta, \sigma, I_0} t^2 \langle x \rangle^{-\rho}.$$

It remains to estimate the contribution of the second term of $R(t, x, \xi)$. Using the Taylor formula, we start from

$$|(\nabla_\xi p_0)(\xi)|^2 - |(\nabla_\xi p_0)(\xi_s)|^2 \leq C_{\delta, I_0} |\xi_s - \xi|.$$

We then estimate $|\xi_s - \xi|$ by using the motion equation, namely

$$\xi_s - \xi = - \int_0^s (\nabla_x v)(x_r, \xi_r) dr$$

from which we infer, using (A.3) on $[0, t_0]$,

$$|\xi_s - \xi| \leq D_{\delta, I_0, \sigma, \kappa} \int_0^s (|x| + r)^{-1-\rho} dr \leq \tilde{D}_{\delta, I_0, \sigma, \kappa} \langle x \rangle^{-\rho}.$$

All this shows that, for some constant C depending only on $I_0, \delta, \kappa, \sigma$ and some seminorms of p_0 and v ,

$$(A.3) \quad \text{and} \quad (A.2) \Rightarrow |x_t|^2 \geq (1 - |\sigma|)(|x|^2 + \kappa^2 t^2) - C \langle x \rangle^{-\rho} (|x|^2 + t^2),$$

as well as

$$(A.3) \quad \text{and} \quad (A.2) \Rightarrow |\xi_s - \xi| \leq C \langle x \rangle^{-\rho}, \quad s \in [0, t].$$

Thus, by picking $|x| \geq R$ large enough, we see that

$$(A.3) \quad \text{and} \quad (A.2) \Rightarrow |x_t|^2 \geq \frac{2}{3}(1 - |\sigma|)(|x|^2 + \kappa^2 t^2) \quad \text{and} \quad |\xi_t - \xi| < \frac{\delta}{2},$$

for $t \in [0, t_0]$. This then implies that (A.3) holds beyond t_0 and the bootstrap follows.

A.2. Proof of Proposition 3.3

Let z denote either x or ξ . Using the motion equation (3.9), we have

$$\begin{aligned} \partial_z(\xi_t - \xi) &= - \int_0^t \frac{\partial^2 v}{\partial \xi \partial x} |_{(x_s, \xi_s)} \partial_z(\xi_s - \xi) ds - \int_0^t \frac{\partial^2 v}{\partial x^2} |_{(x_s, \xi_s)} \partial_z(x_s - x) ds \\ &\quad - \int_0^t \frac{\partial^2 v}{\partial \xi \partial x} |_{(x_s, \xi_s)} \partial_z \xi + \frac{\partial^2 v}{\partial x^2} |_{(x_s, \xi_s)} \partial_z x ds, \end{aligned}$$

so, using Lemma 3.2,

$$|\partial_z(\xi_t - \xi)| \lesssim \int_0^t (|x| + s)^{-1-\rho} |\partial_z(\xi_s - \xi)| ds + \int_0^t \frac{s}{(|x| + s)^{2+\rho}} \frac{|\partial_z(x_s - x)|}{s} ds \\ + |x|^{-\rho} |\partial_z \xi| + |x|^{-1-\rho} |\partial_z x|.$$

Then, by Gronwall's inequality along with $\int_0^t \frac{s}{(|x| + s)^{2+\rho}} ds \leq |x|^{-\rho} \int_0^{+\infty} \frac{r}{(1+r)^{2+\rho}} dr$, we get

$$|\partial_z(\xi_t - \xi)| \lesssim \left(|x|^{-\rho} |\partial_z \xi| + |x|^{-1-\rho} |\partial_z x| + |x|^{-\rho} \sup_{s \in [0, t]} \left| \frac{\partial_z(x_s - x)}{s} \right| \right). \quad (\text{A.4})$$

Now, using the motion equation on x_t , we have similarly

$$\partial_z(x_t - x) = \int_0^t \frac{\partial^2 v}{\partial x \partial \xi} |_{(x_s, \xi_s)} \partial_z(x_s - x) ds + \int_0^t \frac{\partial^2 p}{\partial \xi^2} |_{(x_s, \xi_s)} \partial_z(\xi_s - \xi) ds \\ + \int_0^t \frac{\partial^2 v}{\partial x \partial \xi} |_{(x_s, \xi_s)} \partial_z x ds + \frac{\partial^2 p}{\partial \xi^2} |_{(x_s, \xi_s)} \partial_z \xi ds$$

and by Lemma 3.2 again,

$$|\partial_z(x_t - x)| \lesssim \int_0^t (|x| + s)^{-1-\rho} |\partial_z(x_s - x)| + |\partial_z(\xi_s - \xi)| ds \\ + t |x|^{-1-\rho} |\partial_z x| + t |\partial_z \xi|.$$

By Gronwall's inequality,

$$|\partial_z(x_t - x)| \lesssim t |x|^{-1-\rho} |\partial_z x| + t |\partial_z \xi| + t \sup_{s \in [0, t]} |\partial_z(\xi_s - \xi)|.$$

The last term of the right-hand side can now be estimated by means of (A.4), and we obtain

$$\frac{|\partial_z(x_t - x)|}{t} \lesssim |x|^{-1-\rho} |\partial_z x| + |\partial_z \xi| + |x|^{-\rho} \sup_{s \in [0, t]} \frac{|\partial_z(x_s - x)|}{s}.$$

The left-hand side can be replaced by its supremum over $[0, t]$, so if $|x|$ is large enough, we can absorb the last term of the right-hand side to get

$$\sup_{t \geq 0} \frac{|\partial_z(x_t - x)|}{t} \lesssim |x|^{-1-\rho} |\partial_z x| + |\partial_z \xi|. \quad (\text{A.5})$$

Plugging this inequality inside (A.4), we obtain

$$\sup_{t \geq 0} |\partial_z(\xi_t - \xi)| \lesssim |x|^{-1-\rho} |\partial_z x| + |x|^{-\rho} |\partial_z \xi|. \quad (\text{A.6})$$

Estimates (A.5) and (A.6) give the result when $|\alpha + \beta| = 1$. Higher-order derivatives are studied similarly by induction. To help the reader wishing to work out the details,

we record that if $|\alpha + \beta| \geq 2$ and the result holds true for lower-order derivatives, a careful inspection of the contribution of lower-order derivatives shows that

$$\begin{aligned} \partial_x^\alpha \partial_\xi^\beta \dot{\xi}_t &= O((|x| + t)^{-\rho-2}) \partial_x^\alpha \partial_\xi^\beta (x_t - x) + O((|x| + t)^{-\rho-1}) \partial_x^\alpha \partial_\xi^\beta (\xi_t - \xi) \\ &\quad + O((|x| + t)^{-\rho-1} |x|^{-|\alpha|}), \\ \partial_x^\alpha \partial_\xi^\beta \dot{x}_t &= O((|x| + t)^{-\rho-1}) \partial_x^\alpha \partial_\xi^\beta (x_t - x) + O(1) \partial_x^\alpha \partial_\xi^\beta (\xi_t - \xi) + O(|x|^{-|\alpha|}). \end{aligned}$$

One can then repeat the arguments used when $|\alpha + \beta| = 1$. This completes the proof.

A.3. Proof of Lemma 3.4

Let $\chi_I \in C_0^\infty(\mathbb{R})$ be supported in I_0 and equal to 1 near I_1 . Let $\chi_\sigma \in C^\infty(\mathbb{R})$ be supported in (σ_0, ∞) and equal to 1 near $[\sigma_1, \infty)$. We also request that both χ_I and χ_σ take values between 0 and 1. Set then

$$\kappa_\omega(\xi) = \chi_I(p_0(\xi)) \chi_\sigma \left(\omega \cdot \frac{\partial_\xi p_0(\xi)}{|\partial_\xi p_0(\xi)|} \right)$$

and

$$\tilde{F}_\omega(\xi) = \xi + \kappa_\omega(\xi)(F_\omega(\xi) - \xi), \quad \xi \in \mathbb{R}^n.$$

Note that $\tilde{F}_\omega = F_\omega$ near $\Gamma_\omega(I_1, \sigma_1)$. Also,

$$\sup_{\xi \in \mathbb{R}^n} |\tilde{F}_\omega(\xi) - \xi| \leq \sup_{\Gamma_\omega(I_0, \sigma_0)} |F_\omega(\xi) - \xi| \quad (\text{A.7})$$

and

$$\sup_{\xi \in \mathbb{R}^n} \|d_\xi \tilde{F}_\omega(\xi) - I\| \leq \|\nabla \kappa_\omega\|_\infty \left(\sup_{\Gamma_\omega(I_0, \sigma_0)} |F_\omega(\xi) - \xi| \right) + \sup_{\Gamma_\omega(I_0, \sigma_0)} \|d_\xi F_\omega - I\|. \quad (\text{A.8})$$

If ϵ is small enough, (A.8) implies that $\tilde{F}_\omega$ is a global diffeomorphism from $\mathbb{R}^n$ to itself. In particular, it is injective on $\Gamma_\omega(I_1, \sigma_1)$ and thus so is F_ω . On the other hand, if $\epsilon < 1$, the second condition of (3.12) implies that F_ω is a local diffeomorphism near any point of $\Gamma_\omega(I_0, \sigma_0)$. Therefore, F_ω is a diffeomorphism from $\Gamma_\omega(I_1, \sigma_1)$ onto its range. Now, by possibly taking ϵ smaller, we can ensure that for all $\omega \in \mathbb{S}^{d-1}$, one has the property

$$|\xi - \eta| < \epsilon \quad \text{and} \quad \eta \in \Gamma_\omega(I_2, \sigma_2) \Rightarrow \xi \in \Gamma_\omega(I_1, \sigma_1).$$

Using such an ϵ in the first condition of (3.12) and then using (A.7), it follows that if $\tilde{F}_\omega(\xi)$ belongs to $\Gamma_\omega(I_2, \sigma_2)$, then ξ belongs to $\Gamma_\omega(I_1, \sigma_1)$. Now since $\tilde{F}_\omega$ coincides with F_ω on $\Gamma_\omega(I_1, \sigma_1)$, we conclude that $F_\omega(\Gamma_\omega(I_1, \sigma_1))$ contains $\Gamma_\omega(I_2, \sigma_2)$.

A.4. Proof of Lemma 3.5

Let us prove (3.18) and (3.19) for $k = 1$. We start with

$$\xi_s(x, \eta_t) - \xi = \int_s^t (\partial_x v)(x_r(x, \eta_t), \xi_r(x, \eta_t)) dr = \int_s^t O(\langle x_r(x, \eta_t) \rangle^{-\rho-1}) = O(\langle s \rangle^{-\rho})$$

since $|x_r(x, \eta_t)| \gtrsim \langle r \rangle$ by (3.10). Then

$$\begin{aligned} x_s(x, \eta_t) - s \partial_\xi p_0(\xi) \\ &= x + \int_0^s \partial_\xi p_0(\xi_r(x, \eta_t)) - \partial_\xi p_0(\xi) + \partial_\xi v(x_r(x, \eta_t), \xi_r(x, \eta_t)) dr \\ &= x + \int_0^s O(\xi_r(x, \eta_t) - \xi) + O(\langle x_r(x, \eta_t) \rangle^{-\rho}) dr = x + O(s^{1-\rho}) \end{aligned}$$

if we assume that $1 - \rho > 0$; otherwise, $1 - \rho < 0$ and the integral is bounded with respect to s . The estimates for $k \geq 2$ are obtained similarly by induction; assuming ρ irrational allows us to have either $1 - k\rho > 0$ or $1 - k\rho < 0$. We now prove (3.20). The result is trivial for $k = 1$. Next, by assumption we assume it holds at rank $k \geq 1$ and then $\partial_t \xi_{k+1, s, t}$ reads

$$\begin{aligned} (\partial_x v)(x_{k, t, t}, \xi_{k, t, t}) + \int_s^t (\partial_x^2 v)(x_{k, r, t}, \xi_{k, r, t}) \partial_t x_{k, r, t} + (\partial_{x\xi}^2 v)(x_{k, r, t}, \xi_{k, r, t}) \partial_t \xi_{k, r, t} dr \\ &= O(\langle t \rangle^{-1-\rho}) + \int_s^t O(\langle r \rangle^{-2-\rho}) O(\langle t \rangle^{-1-\rho}) + O(\langle r \rangle^{-1-\rho}) O(\langle t \rangle^{-1-\rho}) dr \\ &= O(\langle t \rangle^{-1-\rho}) \end{aligned}$$

using the induction assumption and the fact that $|x_{k, t, t}| \gtrsim \langle t \rangle$. The estimate on $\partial_t x_{k, s, t}$ is obtained similarly.

B. Proof of (4.1)

Since S commutes with the propagator of P_0 , it commutes with any spectral projection, and thus for all subintervals $J \in I$,

$$\chi_J(P_0)S = S\chi_J(P_0), \quad (\text{B.1})$$

where χ_J is the characteristic function of J . On the other hand, it follows from the definition of $\mathcal{U}$ that

$$\mathcal{U}\chi_J(P_0) = \chi_J(\lambda)\mathcal{U}. \quad (\text{B.2})$$

Now, for fixed $f, g \in L^2(\mathbb{S}^{n-1})$, consider the map

$$T_{f, g}: L^2(I) \times L^2(I) \ni (\theta, \gamma) \mapsto (S\mathcal{U}^*(\theta \otimes f), \mathcal{U}^*(\gamma \otimes g))_{L^2(\mathbb{R}^n)}.$$

It is linear in γ , conjugate linear in θ , and satisfies the estimates

$$|T_{f, g}(\theta, \gamma)| \leq \|S\|_{L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n)} \|\theta\|_{L^2(I)} \|\gamma\|_{L^2(I)} \|f\|_{L^2(\mathbb{S}^{n-1})} \|g\|_{L^2(\mathbb{S}^{n-1})}.$$

On the other hand, if A is the operator associated with $T_{f, g}(\theta, \gamma)$ by $T_{f, g}(\theta, \gamma) = (\theta, A\gamma)$, it follows from (B.1) and (B.2) that A commutes with multiplication operators, hence is

itself a multiplication operator: there exists a unique function $m_{f,g} \in L^\infty(I)$ (given by A1) such that, for all θ, γ ,

$$T_{f,g}(\theta, \gamma) = \int \overline{\theta(\lambda)} m_{f,g}(\lambda) \gamma(\lambda) d\lambda$$

and

$$\|m_{f,g}\|_{L^\infty(I)} \leq \|S\|_{L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n)} \|f\|_{L^2(\mathbb{S}^{n-1})} \|g\|_{L^2(\mathbb{S}^{n-1})}.$$

By definition of $T_{f,g}$, it follows that $(f, g) \mapsto m_{f,g}$ is linear in g and conjugate linear in f , so we expect that we can write for a.e. $\lambda \in I$,

$$m_{f,g}(\lambda) = (S(\lambda)f, g)_{L^2(\mathbb{S}^{n-1})}, \quad f, g \in L^2(\mathbb{S}^{n-1}), \quad (\text{B.3})$$

for some operator $S(\lambda)$ on $L^2(\mathbb{S}^{n-1})$ such that

$$\|S(\lambda)\|_{L^2(\mathbb{S}^{n-1}) \rightarrow L^2(\mathbb{S}^{n-1})} \leq \|S\|_{L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n)}.$$

Let us explain how to handle the almost everywhere definition issue. If each $m_{f,g}$ was a continuous function, we could define $S(\lambda)$ for all $\lambda \in I$ by (B.3). Let $\chi_\epsilon(\lambda) = \epsilon^{-1} \chi(\frac{\lambda}{\epsilon})$ be an approximation of identity in C_0^∞ , non-negative of integral 1. Then

$$\chi_\epsilon * m_{f,g}(\lambda) = \int \epsilon^{-1} \chi\left(\frac{\lambda - \mu}{\epsilon}\right) m_{f,g}(\mu) d\mu$$

is sesquilinear continuous in (f, g) , hence can be represented by some $S_\epsilon(\lambda)$ of operator norm bounded by the operator norm of S . Since $\chi_\epsilon * m_{f,g} \rightarrow m_{f,g}$ in $L^1_{\text{loc}}(I)$ as $\epsilon \rightarrow 0$ for all f, g , we can find a sequence $\epsilon_\ell \rightarrow 0$ and a set of zero measure $\mathcal{N}$ such that for all functions φ_j, φ_k of an orthonormal basis of $L^2(\mathbb{S}^{n-1})$,

$$(S_{\epsilon_\ell}(\lambda) \varphi_j, \varphi_k)_{L^2(\mathbb{S}^{n-1})} \rightarrow m_{\varphi_k, \varphi_j}(\lambda) \quad \text{as } \ell \rightarrow \infty, \quad \lambda \in I \setminus \mathcal{N}, \quad j, k \in \mathbb{N}. \quad (\text{B.4})$$

Now we claim that for all $f, g \in L^2(\mathbb{S}^{n-1})$ and all $\lambda \in I \setminus \mathcal{N}$,

$$(S_{\epsilon_\ell}(\lambda) f, g)_{L^2(\mathbb{S}^{n-1})} \rightarrow m_{f,g}(\lambda), \quad \ell \rightarrow \infty.$$

Indeed, since there is convergence in L^1_{loc} , it suffices to check that the left-hand side above is a Cauchy sequence. This follows from the fact that, since $S_\epsilon(\lambda)$ has norm bounded uniformly in λ and ϵ_ℓ , we see that for all $\varepsilon > 0$ we can find finite linear combinations $f_\varepsilon, g_\varepsilon$ of elements of the base $(\varphi_k)_k$ such that

$$|(S_{\epsilon_\ell}(\lambda) f, g)_{L^2(\mathbb{S}^{n-1})} - (S_{\epsilon_\ell}(\lambda) f_\varepsilon, g_\varepsilon)_{L^2(\mathbb{S}^{n-1})}| < \varepsilon.$$

According to formula (B.4), $((S_{\epsilon_\ell}(\lambda) f_\varepsilon, g_\varepsilon)_{L^2(\mathbb{S}^{n-1})})_\ell$ is a Cauchy sequence, thus so is $((S_{\epsilon_\ell}(\lambda) f, g)_{L^2(\mathbb{S}^{n-1})})_\ell$.

Overall, we can define $S(\lambda)$ as the weak limit of $S_{\epsilon_\ell}(\lambda)$ as $\ell \rightarrow \infty$ for all $\lambda \in I \setminus \mathcal{N}$, and we obtain

$$(S\mathcal{U}^*(\theta \otimes f), \mathcal{U}^*(\gamma \otimes g))_{L^2(\mathbb{R}^n)} = \int_I \overline{\theta(\lambda)} \gamma(\lambda) (S(\lambda)f, g)_{L^2(\mathbb{S}^{n-1})} d\lambda,$$

which is what we expect from the operators $S(\lambda)$.

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References

- [1] V. I. Arnold, *Mathematical methods of classical mechanics*. 2nd edn., Grad. Texts in Math. 60, Springer, New York, 1989 Zbl 1571.70001 MR 997295
- [2] J.-M. Bouclet, *Spectral distributions for long range perturbations*. *J. Funct. Anal.* **212** (2004), no. 2, 431–471 Zbl 1088.35041 MR 2064934
- [3] J.-M. Bouclet, *Strichartz estimates on asymptotically hyperbolic manifolds*. *Anal. PDE* **4** (2011), no. 1, 1–84 Zbl 1230.35027 MR 2783305
- [4] J.-M. Bouclet and H. Mizutani, *Global in time Strichartz inequalities on asymptotically flat manifolds with temperate trapping*. *Mém. Soc. Math. Fr. (N.S.)* **182** (2024), 112 pp. Zbl 1573.35003 MR 4809177
- [5] J.-M. Bouclet and N. Tzvetkov, *Strichartz estimates for long range perturbations*. *Amer. J. Math.* **129** (2007), no. 6, 1565–1609 Zbl 1154.35077 MR 2369889
- [6] J.-M. Bouclet and N. Tzvetkov, *On global Strichartz estimates for non-trapping metrics*. *J. Funct. Anal.* **254** (2008), no. 6, 1661–1682 Zbl 1168.35005 MR 2396017
- [7] F. Cardoso and G. Vodev, *Uniform estimates of the resolvent of the Laplace–Beltrami operator on infinite volume Riemannian manifolds. II*. *Ann. Henri Poincaré* **3** (2002), no. 4, 673–691 Zbl 1021.58016 MR 1933365
- [8] J. Dereziński and C. Gérard, *Scattering theory of classical and quantum N -particle systems*. Texts Monogr. Phys., Springer, Berlin, 1997 Zbl 0899.47007 MR 1459161
- [9] J. Dereziński and E. Skibsted, *Scattering at zero energy for attractive homogeneous potentials*. *Ann. Henri Poincaré* **10** (2009), no. 3, 549–571 Zbl 1213.81210 MR 2519820
- [10] L. C. Evans, *Partial differential equations*. Grad. Stud. Math. 19, American Mathematical Society, Providence, RI, 1998 Zbl 0902.35002 MR 1625845
- [11] C. Gérard and A. Martinez, *Principe d’absorption limite pour des opérateurs de Schrödinger à longue portée*. *C. R. Acad. Sci. Paris Sér. I Math.* **306** (1988), no. 3, 121–123 Zbl 0672.35013 MR 929103
- [12] C. Gérard and A. Martinez, *Prolongement méromorphe de la matrice de scattering pour des problèmes à deux corps à longue portée*. *Ann. Inst. H. Poincaré Phys. Théor.* **51** (1989), no. 1, 81–110 Zbl 0711.35097 MR 1029851
- [13] A. Hassell, T. Tao, and J. Wunsch, *Sharp Strichartz estimates on nontrapping asymptotically conic manifolds*. *Amer. J. Math.* **128** (2006), no. 4, 963–1024 Zbl 1177.58019 MR 2251591
- [14] B. Helffer and J. Sjöstrand, *Résonances en limite semi-classique*. *Mém. Soc. Math. France (N.S.)* **24–25** (1986), iv+228 Zbl 0631.35075 MR 871788
- [15] L. Hörmander, *The analysis of linear partial differential operators. II. Differential operators with constant coefficients*. Grundlehren Math. Wiss. 257, Springer, Berlin, 2005 Zbl 0521.35002 MR 705278
- [16] L. Hörmander, *The analysis of linear partial differential operators. IV. Fourier integral operators*. Grundlehren Math. Wiss. 275, Springer, Berlin, 2009 Zbl 0612.35001 MR 781537
- [17] W. Hunziker, *Distortion analyticity and molecular resonance curves*. *Ann. Inst. H. Poincaré Phys. Théor.* **45** (1986), no. 4, 339–358 Zbl 0619.46068 MR 880742

- [18] H. Isozaki, [On the generalized Fourier transforms associated with Schrödinger operators with long-range perturbations](#). *J. Reine Angew. Math.* **337** (1982), 18–67 Zbl [0486.35026](#) MR [676041](#)
- [19] H. Isozaki and H. Kitada, [Modified wave operators with time-independent modifiers](#). *J. Fac. Sci. Univ. Tokyo Sect. IA Math.* **32** (1985), no. 1, 77–104 Zbl [0582.35036](#) MR [783182](#)
- [20] H. Isozaki and H. Kitada, [Scattering matrices for two-body Schrödinger operators](#). *Sci. Papers College Arts Sci. Univ. Tokyo* **35** (1986), no. 2, 81–107 Zbl [0615.35065](#) MR [847881](#)
- [21] H. Koch and D. Tataru, [Carleman estimates and absence of embedded eigenvalues](#). *Comm. Math. Phys.* **267** (2006), no. 2, 419–449 Zbl [1151.35025](#) MR [2252331](#)
- [22] L. S. Koplienko, [Trace formula for nontrace-class perturbations](#). *Sib. Math. J.* **25** (1984), 735–743 Zbl [0574.47021](#)
- [23] H. Mizutani, [Strichartz estimates for Schrödinger equations on scattering manifolds](#). *Comm. Partial Differential Equations* **37** (2012), no. 2, 169–224 Zbl [1244.35020](#) MR [2876829](#)
- [24] M. Reed and B. Simon, *Methods of modern mathematical physics. III. Scattering theory*. Academic Press, New York, 1979 Zbl [0405.47007](#) MR [529429](#)
- [25] M. Reed and B. Simon, *Methods of modern mathematical physics. I: Functional analysis*. 2nd edn., Academic Press, New York, 1980 Zbl [0459.46001](#) MR [751959](#)
- [26] L. Robbiano and C. Zuily, [Strichartz estimates for Schrödinger equations with variable coefficients](#). *Mém. Soc. Math. Fr. (N.S.)* **101–102** (2005), vi+208 Zbl [1097.35002](#) MR [2193021](#)
- [27] D. Robert, *Autour de l'approximation semi-classique*. Progr. Math. 68, Birkhäuser, Boston, MA, 1987 Zbl [0621.35001](#) MR [897108](#)
- [28] D. Robert, [Asymptotique de la phase de diffusion à haute énergie pour des perturbations du second ordre du laplacien](#). *Ann. Sci. Éc. Norm. Supér. (4)* **25** (1992), no. 2, 107–134 Zbl [0801.35100](#) MR [1169349](#)
- [29] D. Robert, [Relative time-delay for perturbations of elliptic operators and semiclassical asymptotics](#). *J. Funct. Anal.* **126** (1994), no. 1, 36–82 Zbl [0813.35073](#) MR [1305063](#)
- [30] D. Robert and H. Tamura, [Semiclassical estimates for resolvents and asymptotics for total scattering cross-sections](#). *Ann. Inst. H. Poincaré Phys. Théor.* **46** (1987), no. 4, 415–442 Zbl [0648.35066](#) MR [912158](#)
- [31] G. Staffilani and D. Tataru, [Strichartz estimates for a Schrödinger operator with nonsmooth coefficients](#). *Comm. Partial Differential Equations* **27** (2002), no. 7–8, 1337–1372 Zbl [1010.35015](#) MR [1924470](#)
- [32] X. P. Wang, [Time-decay of scattering solutions and classical trajectories](#). *Ann. Inst. H. Poincaré Phys. Théor.* **47** (1987), no. 1, 25–37 Zbl [0641.35018](#) MR [912755](#)
- [33] D. R. Yafaev, *Mathematical scattering theory: General theory*. Transl. Math. Monogr. 105, American Mathematical Society, Providence, RI, 1992 Zbl [0761.47001](#) MR [1180965](#)

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