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Groups and Dynamics: Topology, Measure, and Borel Structure

Organized by
David Kerr, Münster
Anush Tserunyan, Montréal
Robin Tucker-Drob, Gainesville
Tianyi Zheng, San Diego

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Abstract. Over the last couple of decades a major branch of study in groups and dynamics has blossomed around the basic structural themes of measure, topology, and Borelness. While several different currents of research have come to shape this theory, the picture that has emerged is one of a constellation of ideas and methods that cut across all three categories in an often surprisingly coordinated way, with constantly expanding connections to probability and geometric group theory. The workshop aimed to further advance this picture with a focus on a number of topics including hyperfiniteness, ergodic theorems, subgroup dynamics, random walks and the Poisson boundary, percolation, entropy, rigidity, Borel complexity, and genericity and classification in spaces of actions.

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Introduction by the Organizers

Following the precedent of the 2022 MFO workshop with the same title, this meeting aimed to promote interaction between investigators working on related or complementary problems across the fields of topological, measured, and Borel dynamics and their connections with group-theoretic phenomena of a geometric and/or combinatorial nature. This time the program was augmented by a substantially greater input from probability theory, whose ideas and techniques continue to exert an ever-growing impact on this sphere of research. The workshop gave

participants the opportunity to engage with what turned out to be a very wide spectrum of topics, all united by a number of interrelated themes of broad interest.

Topological dynamics appeared in various forms and contexts across a number of different talks. Questions of generic behaviour in spaces of actions of free groups were addressed by Tsankov and Geffen, while Sabok discussed classifiability for symbolic systems with specification and Chen investigated the subtle interplay between topological and Borel structure in groupoids. Bontemps used subgroup dynamics to investigate the existence of amenable actions for generalized Baumslag–Solitar groups, Bernsteyn employed descriptive-set-theoretic combinatorics to establish the existence of minimal subdynamics in very general situations, Cortez spoke about the stabilized automorphism groups of odometer actions, and Li presented results in local entropy theory with applications to Banach space geometry.

The Poisson boundary of random walks, and more generally of stationary systems, formed a prominent theme at the workshop. Chawla presented a breakthrough result on the non-realizability of the Poisson boundary that in particular implies that there exist infinite entropy random walks on the free group whose Poisson boundary is not isomorphic to the natural hyperbolic boundary equipped with the hitting measure, answering a longstanding problem dating back to the 1980s. Furman gave a lecture about the range of Furstenberg entropy for quotients of the Poisson boundary of a free group, in connection with embeddings of the free group into higher rank simple Lie groups, and explained with great clarity the use of spectral gap in the study of stationary measures. Quotients of the Poisson boundary, also known as μ-boundaries, were the central object in the talk of Cascioli on finding such boundaries in the space of amenable subgroups. Silva presented interesting results on using the Poisson boundary to prove continuity of asymptotic entropy when varying the driving random walk step distributions. Gekhtman's talk concerned the geometry of quotients of hyperbolic groups, in particular the injectivity radius on such quotient spaces along a random walk; in his approach, stationary random subgroups associated with such random walks play a key role. Abért gave a talk on how to define and study uniform random walks. In his lecture, entropy considerations and Doob transform techniques were related to the boundary theory of random walks.

Advances in ergodic theory, particularly concerning boundary actions and generalized notions of mixing, formed another central theme of the workshop. Nevo presented an approach to pointwise ergodic theorems for actions of semisimple algebraic groups and non-amenable discrete groups, utilizing horospherical decompositions and their associated equivalence relations. Zomback introduced *k-chaining*, a strengthening of weak mixing for measure-class-preserving actions of countable groups, and situated it within the mixing hierarchy before demonstrating its equivalence to 1-chaining for probability-measure-preserving actions and to strict irreducibility for Markov boundary actions of free groups. Expanding on this shared theme, Zimmermann discussed the generalization of strict irreducibility to continuous-state Markov chains, an approach that facilitates the treatment

of a broader class of skew extensions and paves the way for future applications to ergodic theorems for group actions and random dynamical systems.

Another prominent theme centered on point processes, random partitions, and percolation, exploring their deep connections to the geometry of the underlying space and amenability of naturally arising equivalence relations. Wilkens discussed the ideal Poisson–Voronoi tessellation, leveraging its wild dynamical behavior on certain geometric spaces to establish a deterministic relationship between a manifold's volume and the generators of its fundamental group, as well as to prove uniqueness thresholds in related percolation models. Terlov presented progress on the heavy repulsion of clusters in Bernoulli percolation on quasi-transitive graphs, focusing primarily on the nonunimodular case where recent advances from the measure-class-preserving (mcp) setting made it possible to prove that the set of vertices in one infinite cluster within distance 1 of another almost surely has finite total weight. Mellick examined the weak limits of finitary invariant random equivalence relations, demonstrating that whether or not these are amenable holds exactly for exact groups, correcting prior intuition and highlighting the depth of recent fixed price results.

The final theme of the workshop delved into hyperfiniteness, treeability, and the descriptive complexity of group actions and equivalence relations. Bowen established a geometric characterization of measure-hyperfiniteness, proving that a one-ended, locally finite measurable graph on a standard probability space (X, μ) admits a measurable one-ended spanning subtree if and only if it is measure-hyperfinite, yielding applications to constructing Følner sets with nearly connected boundaries. Karpinski further explored Borel hyperfiniteness in geometric group theory, demonstrating that orbit equivalence relations induced by the boundary actions of graphical small cancellation groups are Borel hyperfinite. Shifting to the higher complexity class of treeable equivalence relations, Poulin exploited double coset orbit equivalence relations for locally compact second countable groups, utilizing stabilizer groups to study lattice action complexity and transfer non-treeability results from the pmp to the mcp setting.

Although the subject of operator algebras was not officially part of the program, it is related in many ways to the style of research in groups and dynamics covered by the meeting and inevitably registered its presence in one form or another. This was explicit in the first and last talks, given, respectively, by Houdayer and Horbez, who addressed rigidity questions in group von Neumann algebras, and in Geffen's talk, which was motivated by a problem in C*-algebras.

The workshop program consisted of 27 talks, 11 of which were delivered by Ph.D. students and postdocs. The stimulating mixture of research directions and the atmosphere of the superb MFO facilities made for a great deal of lively and fruitful interaction between participants. Wednesday afternoon featured the usual hike to St. Roman and copious amounts of black forest cake.

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Workshop: Groups and Dynamics: Topology, Measure, and Borel Structure

Table of Contents

Cyril Houdayer (joint with Adrian Ioana)
Weyl groups and rigidity of von Neumann algebras 1035

Todor Tsankov (joint with Michal Doucha and Julien Melleray)
Comeager isomorphism classes in zero-dimensional dynamics 1037

María Isabel Cortez (joint with Vicente Urria)
Stabilized automorphism groups of odometers given by group actions. . . 1038

Shirly Geffen (joint with Jamie Bell, David Kerr)
Stable rank one in nonnuclear crossed products 1039

Amos Nevo
How to prove a pointwise ergodic theorem for non-amenable groups 1040

Kunal Chawla (joint with Joshua Frisch)
Non-realizability of the Poisson boundary 1043

Alex Furman (joint with Sam Dodds)
Quotients of Poisson boundaries, their entropies, and the use of spectral gap 1043

Marcin Sabok (joint with Konrad Deka, Dominik Kwietniak and Bo Peng)
Bowen's Problem 32 and the conjugacy problem for systems with specification 1044

Sam Mellick (joint with Héctor Jardón-Sánchez, Antoine Poulin, Konrad Wróbel)
Characterising exactness via invariant random equivalence relations 1047

Rachel Skipper (joint with James Hyde, Matthew C. B. Zaremsky)
Rigidity of embeddings between groups acting on spaces with dimension ≤ 1 1048

Grigory Terlov (joint with Sasha Bell, Tasmin Chu, Owen Rodgers, and Anush Tserunyan)
Heavy repulsion of clusters in Bernoulli percolation 1050

Chris Karpinski (joint with Damian Osajda, Koichi Oyakawa)
Graphical small cancellation and hyperfiniteness of boundary actions . . . 1051

Damien Gaboriau (joint with Antoine Poulin, Anush Tserunyan, Robin Tucker-Drob, Konrad Wróbel)
On the Measure Equivalence classification of $\mathrm{BS}(p,q)$*,* $\mathrm{Aut}(T_{p,q})$*, and lcsc treeable groups* 1053

Anton Bernshteyn (joint with Joshua Frisch)
Minimal subdynamics 1055

Anna Cascioli (joint with Martín Gilabert Vio, Eduardo Silva)
Stationary boundaries on the space of amenable subgroups 1058

Sasha Bontemps
GBS groups, class $\mathcal{A}$, and relative property (T) 1060

Miklós Abért (joint with Adam Arras, Jaelin Kim)
The uniform random walk: minimal entropy random walk and its limits 1062

Jenna Zomback (joint with Anush Tserunyan)
Chaining in measurable dynamics 1062

Eduardo Silva
Continuity of asymptotic entropy on groups 1064

Elias Zimmermann (joint with Pablo Lummerzheim, Felix Pogorzelski)
Strict irreducibility of Markov chains and random ergodic theorems for group actions 1067

Ruiyuan Chen
Componentwise topologies on group actions, equivalence relations, and groupoids 1068

Amanda Wilkens (joint with Mikolaj Fraczyk, Sam Mellick, and Matteo D'Achille, Jan Grebik, Ali Khezeli, Konstantin Recke)
The ideal Poisson–Voronoi tessellation and some applications 1070

Antoine Poulin
On double cosets in Borel complexity 1071

Ilya Gekhtman
Critical exponents and growth of injectivity radius in quotients of hyperbolic groups 1073

Hanfeng Li (joint with Kairan Liu)
Local entropy theory and combinatorics 1074

Camille Horbez
Rigidity of graph products of groups 1077

Abstracts

Weyl groups and rigidity of von Neumann algebras

Cyril Houdayer
(joint work with Adrian Ioana)

Let k be a local field of characteristic zero, that is, k is equal to $\mathbb{R}$, $\mathbb{C}$ or a finite extension of $\mathbb{Q}_p$ for some prime number p. Let $\mathbf{G}$ be a Zariski connected simply connected k-isotropic almost k-simple algebraic k-group and $\mathbf{S} < \mathbf{G}$ a maximal k-split torus. The dimension of $\mathbf{S}$ is called the k-rank of $\mathbf{G}$ and is denoted by $\mathrm{rk}_k(\mathbf{G})$. Since $\mathbf{G}$ is k-isotropic, we have $\mathrm{rk}_k(\mathbf{G}) \geq 1$. The centralizing k-subgroup $\mathbf{H} = \mathscr{Z}_{\mathbf{G}}(\mathbf{S})$ is the Zariski connected component of the normalizing k-subgroup $\mathscr{N}_{\mathbf{G}}(\mathbf{S})$. The finite group $\mathscr{W}_{\mathbf{G}} = \mathscr{N}_{\mathbf{G}}(\mathbf{S})/\mathscr{Z}_{\mathbf{G}}(\mathbf{S})$ is called the *Weyl group* of $\mathbf{G}$ relative to k. Every coset of $\mathscr{N}_{\mathbf{G}}(\mathbf{S})/\mathscr{Z}_{\mathbf{G}}(\mathbf{S})$ is represented by an element rational over k, that is, $\mathscr{N}_{\mathbf{G}}(\mathbf{S}) = \mathscr{N}_{\mathbf{G}}(\mathbf{S})(k) \cdot \mathscr{Z}_{\mathbf{G}}(\mathbf{S})$. In the case when $\mathbf{G} = \mathrm{SL}_n$ for $n \geq 2$, we may assume that $\mathbf{S} < \mathbf{G}$ is the group of diagonal matrices, we have $\mathbf{H} = \mathbf{S}$ and $\mathscr{W}_{\mathbf{G}} \cong \mathfrak{S}_n$.

Set $G = \mathbf{G}(k)$ and $H = \mathbf{H}(k)$. Endow the homogeneous space G/H with its unique G-invariant measure class. Since G and H are both unimodular, the homogeneous space G/H carries a G-invariant σ-finite infinite measure m whose measure class coincides with the unique G-invariant measure class on G/H. Let $\Gamma < G$ be a lattice and set $\Lambda = \Gamma/\mathscr{Z}(\Gamma)$. Since $\mathscr{Z}(\Gamma) < H \cap \mathscr{Z}(G)$, we have $\mathscr{Z}(\Gamma) < \ker(\Gamma \curvearrowright G/H)$ and so we may consider the well-defined nonsingular action $\Lambda \curvearrowright G/H$. Using the terminology on algebraic groups as above and since $H < G$ is noncompact and $\Gamma < G$ is a lattice, Moore's ergodicity theorem implies that $\Lambda \curvearrowright G/H$ is ergodic. Moreover, $\Lambda \curvearrowright G/H$ is essentially free.

As a consequence of the main results of [1] any Λ-equivariant nonsingular automorphism of G/H is automatically G-equivariant and so we have the following isomorphisms

$$\mathrm{Aut}_{\Lambda}(G/H) = \mathrm{Aut}_{\Gamma}(G/H) = \mathrm{Aut}_{G}(G/H) \cong \mathscr{W}_{\mathbf{G}}.$$

The aim of our talk is to present a noncommutative analogue of the above isomorphism result. Before stating our main theorem, we need to introduce some further notation.

We denote by $\sigma : \Lambda \curvearrowright \mathrm{L}^{\infty}(G/H)$ the von Neumann algebraic action corresponding to the nonsingular action $\Lambda \curvearrowright G/H$. Denote by $M = \mathrm{L}(\Lambda) = \{u_\gamma \mid \gamma \in \Lambda\}''$ the group von Neumann algebra. Denote by $\mathscr{M} = \mathrm{L}(\Lambda \curvearrowright G/H)$ the group measure space von Neumann algebra associated with the nonsingular action $\Lambda \curvearrowright G/H$ together with its Cartan subalgebra $\mathscr{A} = \mathrm{L}^{\infty}(G/H)$ and regard $M \subset \mathscr{M}$ as a von Neumann subalgebra. For every $w \in \mathscr{W}_{\mathbf{G}}$, choose $n \in \mathscr{N}_{\mathbf{G}}(\mathbf{S})(k)$ so that $w = n\mathbf{H} \in \mathscr{W}_{\mathbf{G}}$ and define $\theta_w \in \mathrm{Aut}(\mathscr{A})$ as the unique unital normal $*$-automorphism that satisfies $\theta_w(F)(gH) = F(gHn^{-1})$ for every $F \in \mathscr{A}$ and almost every $gH \in G/H$. Then we have $\theta_w \circ \sigma_\gamma = \sigma_\gamma \circ \theta_w$ for every $\gamma \in \Lambda$ and every $w \in \mathscr{W}_{\mathbf{G}}$.

Denote by $\mathrm{Aut}_M(\mathscr{M})$ the group of all unital normal $*$-automorphisms $\Theta \in \mathrm{Aut}(\mathscr{M})$ such that $\Theta|_M = \mathrm{id}_M$. For every $w \in \mathscr{W}_{\mathbf{G}}$, define $\Theta_w \in \mathrm{Aut}_M(\mathscr{M})$ as the unique unital normal $*$-automorphism that satisfies $\Theta_w|_M = \mathrm{id}_M$ and $\Theta_w|_{\mathscr{A}} = \theta_w$. Then $\rho : \mathscr{W}_{\mathbf{G}} \to \mathrm{Aut}_M(\mathscr{M}) : w \mapsto \Theta_w$ is an injective group homomorphism.

Our main result is the following novel rigidity phenomenon for the inclusion of von Neumann algebras $M = \mathrm{L}(\Lambda) \subset \mathrm{L}(\Lambda \curvearrowright G/H) = \mathscr{M}$ associated with the nonsingular action $\Lambda \curvearrowright G/H$.

Theorem 1 ([2])**.** *The group homomorphism $\rho : \mathscr{W}_{\mathbf{G}} \to \mathrm{Aut}_M(\mathscr{M})$ is an isomorphism. In particular, we have*

$$\mathscr{W}_{\mathbf{G}} \cong \mathrm{Aut}_M(\mathscr{M}).$$

The striking feature of Theorem 1 is that even though the inclusion $M = \mathrm{L}(\Lambda) \subset \mathrm{L}(\Lambda \curvearrowright G/H) = \mathscr{M}$ tends to forget the nonsingular action $\Lambda \curvearrowright G/H$, it still retains the Weyl group $\mathscr{W}_{\mathbf{G}}$ via the group isomorphism $\mathscr{W}_{\mathbf{G}} \cong \mathrm{Aut}_M(\mathscr{M})$.

For every $j \in \{1, 2\}$, let k_j be a local field of characteristic zero, $\mathbf{G}_j$ a Zariski connected simply connected k_j-isotropic almost k_j-simple algebraic k_j-group and $\mathbf{S}_j < \mathbf{G}_j$ a maximal k_j-split torus. Set $G_j = \mathbf{G}_j(k_j)$ and $H_j = \mathscr{Z}_{\mathbf{G}_j}(\mathbf{S}_j)(k_j)$. Let $\Gamma_j < G_j$ be a lattice and set $\Lambda_j = \Gamma_j/\mathscr{Z}(\Gamma_j)$. Following Theorem 1, denote by $\rho_j : \mathscr{W}_{\mathbf{G}_j} \to \mathrm{Aut}_{\mathrm{L}(\Lambda_j)}(\mathrm{L}(\Lambda_j \curvearrowright G_j/H_j))$ the corresponding group isomorphism. We derive the following consequence of Theorem 1 which gives new insight towards Connes' rigidity conjecture for higher rank lattices.

Corollary 2 ([2])**.** *Let $\Psi : \mathrm{L}(\Lambda_1 \curvearrowright G_1/H_1) \to \mathrm{L}(\Lambda_2 \curvearrowright G_2/H_2)$ be a surjective unital normal $*$-isomorphism such that $\Psi(\mathrm{L}(\Lambda_1)) = \mathrm{L}(\Lambda_2)$. Then the mapping $\mathscr{W}_{\mathbf{G}_1} \to \mathscr{W}_{\mathbf{G}_2} : w \mapsto \rho_2^{-1}(\Psi \circ \rho_1(w) \circ \Psi^{-1})$ is a group isomorphism.*

Let us say a few words about the proof of Theorem 1. Let $\Theta \in \mathrm{Aut}_M(\mathscr{M})$. In order to prove that there exists $w \in \mathscr{W}_{\mathbf{G}}$ such that $\Theta = \Theta_w$, it is enough to show that $\Theta(\mathscr{A}) = \mathscr{A}$, where $\mathscr{A} = \mathrm{L}^\infty(G/H)$. To do this, we proceed in three steps. Firstly, we consider the equivariant normal ucp map $\Phi = \mathrm{E} \circ \Theta|_{\mathscr{A}} : \mathscr{A} \to \mathscr{A}$, where $\mathrm{E} : \mathscr{M} \to \mathscr{A}$ is the canonical faithful normal conditional expectation. By interpreting the main result of [1] as a rigidity statement for equivariant normal ucp maps, we conclude that $\Phi = \sum_{w \in \mathscr{W}_{\mathbf{G}}} \alpha_w \theta_w$ is a convex combination of the finite group $\{\theta_w \mid w \in \mathscr{W}_{\mathbf{G}}\}$. Secondly, we develop a convexity/maximality argument to show that there exist $w \in \mathscr{W}_{\mathbf{G}}$ and $u \in \mathscr{U}(\mathscr{M})$ such that $\Theta = \mathrm{Ad}(u) \circ \Theta_w$ on $\mathscr{A}$. This second step is purely noncommutative and is reminiscent of the intertwining method in Popa's deformation/rigidity theory [3]. Thirdly, by exploiting the essential freeness of the nonsingular action $\Lambda \times \mathscr{W}_{\mathbf{G}} \curvearrowright G/H$, we conclude that $u \in \mathscr{U}(\mathscr{A})$, which finally implies that $\Theta = \Theta_w$. We refer to [2] for further details.

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Comeager isomorphism classes in zero-dimensional dynamics

Todor Tsankov

(joint work with Michal Doucha and Julien Melleray)

It is often possible to parametrize a given class of dynamical systems by elements of a Polish space and then it becomes natural to ask what properties hold "generically", i.e., on a comeager set of systems. The most extreme situation is when there is a single comeager isomorphism class: that is, the generic properties are captured by a single system. This does not usually happen in ergodic theory but can occur in zero-dimensional topological dynamics. For example, it is a result of Kechris and Rosendal that there is a generic action of $\mathbf{Z}$ on the Cantor space and of Kwiatkowska that there is such a generic action of the free group $\mathbf{F}_n$. Doucha extended this work to free products of finite groups and connected it to symbolic dynamics. However, from a dynamical perspective, these generic systems are somewhat pathological, as they are, for example, not topologically transitive.

In [1], we are interested in minimal dynamical systems and develop a model-theoretic framework, which combined with some new results in symbolic dynamics, allows us to show the following.

Theorem 1. *Let G be a finitely generated free group and let A be a finite alphabet. Then the set of sofic minimal subshifts is dense in the set of all minimal subshifts of A^G.*

As a consequence, there is a comeager conjugacy class in the space of all minimal actions of G on the Cantor space, and its elements are isomorphic to the Fraïssé limit of the sofic minimal subshifts of G.

Theorem 2. *Let G be a finitely generated free group and let A be a finite alphabet. Then the set of finite subshifts is dense in the set of all minimal, probability measure-preserving subshifts of G.*

As a consequence, there is a comeager conjugacy class in the space of all minimal, probability measure-preserving actions of G on the Cantor space, and its elements are isomorphic to the universal profinite action of G (which is the Fraïssé limit of all finite actions).

Our techniques do not apply to actions of other groups. In particular, the following question remains open.

Question 3. *Is there a comeager conjugacy class in the space of all minimal actions of $\mathbf{Z}^d$ on the Cantor space for $d \geq 2$?*

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Stabilized automorphism groups of odometers given by group actions.

María Isabel Cortez
(joint work with Vicente Urria)

The notion of the stabilized automorphism group was introduced by Hartman, Kra, and Schmieding in [2] for subshifts of finite type arising from $\mathbb{Z}$-actions. Subsequently, Schmieding proved in [6] that two full $\mathbb{Z}$-shifts have isomorphic stabilized automorphism groups if and only if the cardinalities of their alphabets share the same prime decomposition. Later, Jones-Baro [4] initiated the study of these algebraic invariants in the setting of minimal Cantor systems, providing explicit descriptions of the stabilized automorphism groups for odometers and Toeplitz subshifts, and establishing that these groups form complete invariants for conjugacy within certain classes of odometers. This line of work was further developed by Espinoza and Jones-Baro in [3], who extended the analysis to minimal and certain transitive systems, showing in particular that equality of rational eigenvalues is a necessary condition for the corresponding stabilized automorphism groups to be isomorphic. More recently, Salo [5] introduced the concept of stabilized automorphism groups for actions of residually finite groups and described these groups in the context of shifts of finite type.

In [1], we investigate the stabilized automorphism groups of odometers equipped with a group structure, where the acting group is assumed to be residually finite. In this framework, we prove that the stabilized automorphism group coincides with the topological full group associated to the action of the odometer on itself by right multiplication. Moreover, we show that two odometers have isomorphic stabilized automorphism groups if and only if they admit isomorphic clopen subgroups of the same finite index. As a consequence, continuously orbit equivalent odometers necessarily have isomorphic stabilized automorphism groups, although the converse implication fails in general.

For $\mathbb{Z}^d$-odometers, the existence of an isomorphism between stabilized automorphism groups implies orbit equivalence. However, beyond the one-dimensional case, neither continuous orbit equivalence nor orbit equivalence is equivalent to the isomorphism of stabilized automorphism groups. Indeed, we construct examples of $\mathbb{Z}^2$-odometers that are not continuously orbit equivalent yet have isomorphic stabilized automorphism groups, as well as examples of orbit equivalent $\mathbb{Z}^2$-odometers whose stabilized automorphism groups are not isomorphic.

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Stable rank one in nonnuclear crossed products

Shirly Geffen
(joint work with Jamie Bell, David Kerr)

Considering a simple reduced crossed product C^*-algebra $C(X) \rtimes_r G$ arising from an action of a discrete countable group on a compact metrizable space X, we investigate when the invertible elements are dense in the algebra, a property known as *stable rank one*.

A necessary condition, due to Cuntz, is that the space of invariant Borel probability measures $\mathrm{Prob}^G(X)$ is nonempty; this ensures that the C^*-algebra is stably finite. Whether this condition is also sufficient remains an open problem.

When G is amenable, the condition $\mathrm{Prob}^G(X) \neq \emptyset$ holds automatically, and most existing positive results are obtained in this setting. The most general results in this direction are due to Li–Niu and were later recovered and extended in joint work with Bell and Kerr.

Our goal is to extend these results to actions of nonamenable groups. To this end, we introduce a dynamical property called *square divisibility*, which guarantees stable rank one. We verify this condition for a generic class of actions of free groups (and more generally, free product groups) on the Cantor set X.

Specifically, we consider two classes of actions. The first is $\mathrm{WA}(F_d, X)$, consisting of topologically free, minimal, weakly mixing actions $F_d \curvearrowright X$ on the Cantor set such that $\mathrm{Prob}^{F_d}(X) \neq \emptyset$. The second is $A^*(F_d, X)$, the class of topologically free actions $F_d \curvearrowright X$ on the Cantor set with $\mathrm{Prob}^{F_d}(X) \neq \emptyset$, which are minimal and spectrally aperiodic on each standard generator of F_d.

In both settings, we show that a generic action in the class yields a crossed product C^*-algebra with stable rank one. Moreover, for $A^*(F_d, X)$, we prove that the conjugacy class of each action is meager, thereby identifying a large collection (in the sense of Baire category) of pairwise nonisomorphic actions for which the result holds.

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How to prove a pointwise ergodic theorem for non-amenable groups

Amos Nevo

The purpose of this note is twofold. First, we briefly explain the ingredients of an alternative proof of the mean, maximal and pointwise ergodic theorems for probability-measure-preserving actions of $SL_2(\mathbb{R})$. The proof is based solely on the classical arguments underlying the proofs of the ergodic theorems for actions of $\mathbb{R}$, together with a variant of the classical method of rotations. Unlike the first proofs of these ergodic theorems in [5], it applies also in the function space $L(\log L)$ (which is optimal), and furthermore the role of representations theory in the proof is limited solely to the use of the Howe-Moore mixing theorem. The proof extends to simple groups of real-rank one, and was established in joint work with Lewis Bowen in [2].

Second, we recall Kostant's and Silberger's horospherical decomposition theorem for general semisimple algebraic groups, and Kostant's problem regarding the density of Haar measure associated with the decomposition. While this problem is far from solved, we note that a suitable property of the density will allow the use of the method of rotations to prove ergodic theorems, following the steps indicated in the proof for $SL_2(\mathbb{R})$. We formulate a sufficient condition, namely polynomial-like behavior of the density, and show that it holds in the case of $SL_{\ell+1}(\mathbb{K})$ for locally-compact totally-disconnected fields $\mathbb{K}$. We conclude that the maximal and pointwise ergodic theorems hold for the actions in $L(\log L)^{\ell}$, a result which can be expected to be optimal. This is part of an on-going joint work with Bertrand Remy and Bartosz Trojan.

1. The pointwise ergodic theorem for $SL_2(\mathbb{R})$

Notation: Let $K = SO_2(\mathbb{R})$, $N = \{n_t\,;\, t \in \mathbb{R}\}$=the upper unipotent group, and $A = \{a_r\,;\, r \in \mathbb{R}\}$=the diagonal group, $G/K = \mathbb{H}^2$=upper-half plane, with $[K] = i$ the origin, D the Riemannian metric.

Geodesic polar coordinates. for $z = u + iv \in \mathbb{H}^2$, write $z = k_\theta a_r \cdot i$, with $r = D(z, i)$. In particular $n_t \cdot i = k_{\theta(t)} a_{r(t)} \cdot i$, and so $K n_t K = K a_{r(t)} K$, with $D(n_t \cdot i, i) = D(t+i, i) = r(t)$. The distance (from i) formula is given by $\cosh D(n_t \cdot i, i) = 1 + \frac{t^2}{2}$.

Horospherical decomposition and volume formula. It follows that $K n_t K = K a_r K \iff \cosh r = 1 + \frac{t^2}{2}$, for $r, t \geq 0$, and in particular : $G = KNK$ (since $G = KAK$). We have the classical volume formula for Haar measure in Polar coordinates:

$$\int_G f(g) dm_G(g) = c \int_K \int_{A_+} \int_K f(k a_r k') \sinh r \, dr \, dk \, dk',$$

from which it is straight forward to conclude the formula :

Haar measure in horospherical coordinates:

$$\int_G f(g) dm_G(g) = \int_K \int_{N_+} \int_K f(k n_t k') \, t \, dt \, dk \, dk'$$

We conclude that Haar measure in horospherical coordinates is given by a *polynomial density* on $N_+ \cong \mathbb{R}_+$!!!!

The averages $\eta_T = \frac{2}{T^2}\int_0^T \delta_{n_t}\, t\, dt$ are clearly a *weighted variant* of the standard interval averages on $N_+ \cong \mathbb{R}_+$, the weight being t. It is straightforward to see that they satisfy the standard mean, maximal and pointwise ergodic theorems in any ergodic p.m.p action of N.

The ball averages β_R on G, with Haar-uniform density supported on $B_R = \{g \in G\,;\, D(g \cdot i, i) \leq r\}$ are given by

$$\beta_R = m_K * \eta_T * m_K \text{ (with } T = T(R) = 2\sinh(R/2)).$$

Method of Rotations: It follows that analysis of β_R amounts to version of classical method of rotations applied to averaging the translates $\delta_k * \eta_T * \delta_{k'}$ over $K \times K$. We can conclude the following:

Proof of the ergodic theorems for ball averages on $SL_2(\mathbb{R})$ (Bowen and Nevo, 2017)[2] :

- *Strong maximal inequalities* for β_R in L^p, $1 < p \leq \infty$ follow from
$$\sup_{R \geq R_0} |\beta_R f(x)| \leq m_K \left(\sup_{T \geq T_0} \eta_T(m_K(|f|)) \right)(x)$$
- *Pointwise a.e. convergence* of η_T implies the same for β_R for bounded function, using the bounded convergence theorem.
- The *mean ergodic theorem* is a consequence of vanishing of matrix coefficients namely the Howe-Moore mixing theorem, which implies that N is mixing in ergodic G-actions.
- The weak-type $(1,1)$-maximal inequality for η_T is compromised by the integration on $K \times K$. For β_R the maximal inequality, and hence also pointwise a.e convergence, hold only in $L \log L$.

Optimality: The weak-type maximal inequality and pointwise a.e. convergence *fails in* L^1 *for general actions* of $SL_2(\mathbb{R})$ (Bowen and Burton, [1]). Hence the $L \log L$ result is *optimal*.

2. Simple higher rank groups and Kostant's problem

Kostant's decomposition for semisimple groups. Every connected semisimple Lie group G with finite center has the following decompositions:

(1) *Cartan polar coordinates decomposition:* $G = KAK = K\overline{A_+}K$.
(2) *Iwasawa coordinates decomposition :* $G = KAN$.
(3) *Kostant horospherical coordinates decomposition:* $G = KNK$.

Note that (3) is a consequence of *Kostant's convexity theorem* (1973) [4].

Recall that $\dim(K \setminus G/K) = \dim A =$ real-rank (G). But in general $\dim N > \dim A$, so Kostant's decomposition is highly redundant.

Silberger (1975) [6] proved that $G = KNK$ for connected simply-connected semisimple algebraic groups over locally compact totally disconnected non-discrete fields.

Kostant's problem : Compute the density Ω of Haar measure on G associated with the decomposition :

$$\int_G F(g) dm_G(g) = \int_K \int_N \int_K F(knk')\, \Omega(n)\, dk\, dn\, dk' \ (*)\,.$$

How to prove ergodic theorems for higher rank groups:

- Suppose we can choose ℓ one-parameter unipotent subgroups so that $\exp(t_1 X_1)\exp(t_2 X_2)\cdots\exp(t_\ell X_\ell)$ is section of $K \setminus G/K$.
- Suppose also that the associated decomposition has density which is comparable to a product of polynomial-like expressions in each of the variables $t_1, t_2, \ldots, t_\ell$.
- Then define the averages $\eta_{\overrightarrow{T}}$ with $\overrightarrow{T} = (T_1, \ldots, T_\ell)$, $T_i \geq 1$, supported on boxes $0 \leq t_i \leq T_i$ in N and weighted by the density, and also the averaged operators $\beta_{\overrightarrow{T}} = m_K * \eta_{\overrightarrow{T}} * m_K$.
- The mean, maximal and pointwise ergodic theorems for $\eta_{\overrightarrow{T}}$ in ergodic actions of N follow using the polynomial-like behavior of the density, and the multi-parameter Dunford-Zygmund theorem [3][7].
- Applying integration on $K \times K$ implies the ergodic theorems for $\beta_{\overrightarrow{T}}$, which constitute a rich family of K-bi-invariant averages on G, in ergodic actions of G.

Special linear group over t.d fields. This program raises significant challenges and is far from complete. The case of *algebraic groups over t.d. fields* is much more accessible than the case of connected semisimple groups. In on-going joint work with Bertrand Remy and Bartosz Trojan we have recently proved for the groups $SL_{\ell+1}(\mathbb{K})$, $\mathbb{K}$ a local field :

Theorem 1 (N+Remy+Trojan, 2026). The operators $\beta_{\overrightarrow{T}}$ satisfy the *unrestricted* maximal inequality in L^p, $1 < p \leq \infty$, as well as the weak-type maximal inequality in $L(\log L)^\ell$, the mean ergodic theorem in L^p, $1 \leq p < \infty$, and the pointwise ergodic theorem in $L(\log L)^\ell$ (which contains every L^p-space, $1 < p \leq \infty$).

This result is expected to be *optimal*, generalizing the case $\ell = 1$.

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Non-realizability of the Poisson boundary

Kunal Chawla
(joint work with Joshua Frisch)

Given a countable group G equipped with a probability measure μ, one can define a random walk on G as the Markov process whose increments are iid sampled by μ. The large-scale properties of this random walk can exhibit a plethora of exotic behaviours, relating to the algebraic and geometric structure of G.

One way of capturing the large-scale behaviour is via the Poisson boundary of (G, μ), a canonical (abstract) measure space associated with the Markov chain. This object has been studied intensely over the decades. In particular, mathematicians have found concrete 'realizations' of this measure space as topological boundaries for the group (i.e. the Gromov boundary of a hyperbolic group). In 1983, Kaimanovich and Vershik asked whether the Poisson boundary can always be realized in this way [2].

I will describe the complete resolution of this problem in the negative. This is joint work with Joshua Frisch.

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Quotients of Poisson boundaries, their entropies, and the use of spectral gap

Alex Furman
(joint work with Sam Dodds)

The basic setup consists of a countable discrete finitely generated group Γ equipped with a probability μ on Γ that is aperiodic (not supported on a coset) and has exponential moment. The Poisson boundary (B, ν) of (Γ, μ) is a measurable Γ-space with $\mu * \nu = \nu$ such that the Furstenberg-Poisson transform $L^\infty(B, \nu) \to H^\infty(\Gamma, \mu)$ is an isomorphism, where $H^\infty(\Gamma, \mu)$ is the space (algebra) of bounded μ-harmonic functions. The subject of study is the order-theoretic lattice of Γ-equivariant measurable quotients of (B, ν) with the strictly monotonic ranking by Furstenberg entropy defined by

$$h(C, \eta) = \int_\Gamma \int_C -\log \frac{d\gamma_*^{-1}\eta}{d\eta} d\eta d\mu.$$

The results concerned the free groups, surface groups, and established the richness of this lattice at low entropy (i.e., with $h(C,\eta) \in (0, h_0)$). The construction involved looking at representations $\rho\colon \Gamma \to G = \mathrm{SL}_2(\mathbb{C})$ (with dense image) for which the harmonic measure ν_ρ on $S^2 = \widehat{\mathbb{C}}$ is in the Lebesgue class.

Bowen's Problem 32 and the conjugacy problem for systems with specification

Marcin Sabok

(joint work with Konrad Deka, Dominik Kwietniak and Bo Peng)

The methods of mathematical logic can be useful in establishing impossibility results. For example, a descriptive set-theoretic complexity argument was used by Wojtaszczyk and Bourgain who solved Problem 49 from the Scottish book by establishing the non-existence of certain types of Banach space. In a similar vein, the theory of complexity of Borel equivalence relations can be used to demonstrate the impossibility of classifying certain mathematical objects. An equivalence relation E on a standard Borel space X is **smooth** if there exists a Borel assignment $f\colon X \to Y$ of elements of another standard Borel space Y to elements of X which provides a complete classification of the equivalence relation E, i.e., two elements $x, y \in X$ are E-related if and only if $f(x) = f(y)$. The definition is broad enough to justify a Borel version of the Church–Turing thesis, namely that an isomorphism relation admits a concrete classification if and only if it is smooth.

Smooth equivalence relations are actually only at the beginning of a larger hierachy of descriptive set-theoretic complexity. Given two equivalence relations E and F on standard Borel spaces X and Y, respectively, we say that E is **Borel-reducible** to F, written $E \leq_B F$ if there exists a Borel map $f\colon X \to Y$ such that for $(x_1, x_2) \in X \times X$ we have $(x_1, x_2) \in E$ if and only if $(f(x_1), f(x_2))) \in F$. The Borel complexity of an isomorphism problem measures how complicated the problem is in comparison with other equivalence relations, when we compare equivalence relations using Borel reductions. Equivalence relations which are Borel-reducible to the equality $=$ on the real numbers, or anything that can be coded by real numbers, are exactly the smooth ones. However, the hierarchy goes much higher. The next step is formed by the **hyperfinite** equivalence relations, that is those which are induced by Borel actions of the group $\mathbb{Z}$. Another successor of $=$ is defined in terms of the Friedman–Stanley version of the Turing **jump**, and is denoted by $=^+$. The jump can be iterated and all of the countable iteration of $^+$ on $=$ are induced by Borel actions of the group S_∞ of permutations of $\mathbb{N}$. The class of equivalence relations reducible to actions of S_∞ is quite large (cf. the recent result of Paolini and Shelah [11]) but not every Borel equivalence relation is in that class. Hjorth developed the theory of turbulence and showed that turbulent group actions are not Borel reducible to any Borel S_∞-action. The theory of complexity of equivalence relations also continues in the class of equivalence relations induced by actions of more general groups than S_∞ (cf. [5, 10, 12, 14]). A lot of effort has been put in measuring the complexity of problems arising in dynamical systems,

where the classification problems and their complexity are understood in the same way as in descriptive set theory. The breakthrough for measure-preserving systems came with the results of Foreman and Weiss [8] and of Foreman, Rudolph and Weiss [9]. In [8] Foreman and Weiss showed that the conjugacy relation of ergodic transformations is turbulent and in [9] Foreman, Rudolph and Weiss showed that the conjugacy relation of ergodic transformations is not Borel, and thus the classification problem in ergodic theory is intractable. In topological dynamics, Camerlo and Gao proved that the conjugacy of Cantor systems is the most complicated among equivalence relation induced by actions of S_∞ and for minimal Cantor systems it is shown in [6] that the relation is not Borel.

In contrast, the conjugacy relation of symbolic systems is quite simple from the point of view of descriptive set theory. Since every isomorphism between symbolic systems is given by a block code, the conjugacy of symbolic systems is a **countable Borel equivalence relation**, i.e. a Borel equivalence relation whose equivalence classes are countable. Clemens proved that the topological conjugacy of symbolic subshifts is a universal countable Borel equivalence relation. Gao, Jackson and Seward generalized it to G-subshifts any countable group G which is not locally finite, while for a locally finite group G they showed that the conjugacy of symbolic G-subshifts is hyperfinite. In case of minimal systems, Gao, Jackson and Seward proved that the conjugacy relation of minimal symbolic G-subshifts is not smooth for any countable infinite group G, and Thomas showed that the conjugacy relation of Toeplitz symbolic subshifts is not smooth. It remains unknown whether the conjugacy of minimal symbolic subshifts is hyperfinite, which is connected to a conjecture of Thomas on the isomorphism of complete groups. In fact, it is not known [13, Question 1.3] whether the conjugacy relation restricted to Toeplitz systems is hyperfinite or not.

A special class of symbolic systems is formed by systems with **specification**, considered by Bowen [1]. A dynamical system satisfies the specification property if for every $\varepsilon > 0$ we can find $k \in \mathbb{N}$ such that given any collection of finite fragments of orbits, there exists a point which is ε-closely following these orbit segments and takes k steps to switch between consecutive orbit segments. Nowadays, the specification property in symbolic systems for general discrete groups goes also under the name of **strong irreducibility**. Around 1970's Bowen wrote an influential list of open problems, which is now maintained on the webpage [3]. Problem 32 [2] asks to

classify symbolic systems with specification.

Several results in the direction of a classification have been obtained, as reported on the website [2]. For example Bertrand proved that every symbolic system with specification is synchronized and Thomsen found a connection with the theory of countable state Markov chains. However, a complete classification has not been obtained. Buhanan and Kwapisz proved a result suggesting that systems with specification are quite complicated. They considered the cocyclic shift spaces, which is a countable family of symbolic systems with specification and proved that the problem of equality of cocyclic shift spaces is undecidable. However, the

equality relation is a much simpler relation than the conjugacy on such systems. In in talk we discuss the following result, which shows that a classification with any concrete invariants is impossible.

Theorem 1. *The conjugacy relation of symbolic systems with the specification property is not smooth.*

In fact,we prove that the conjugacy relation of symbolic systems with specification is not hyperfinite and essentially the same proof shows that it is not treeable.

Next, we look the conjugacy relation of pointed systems with the specification property and consider its complexity. It turns out that in order to compute the complexity for pointed Cantor systems with the specification property, we need to solve a problem posed in a paper of Ding and Gu [7].

In [7] Ding and Gu consider the equivalence relation E_{cs} defined on the space of metrics on $\mathbb{N}$, where two metrics are equivalent if the identity map on $\mathbb{N}$ extends to a homeomorphism of the completions of $\mathbb{N}$ with respect to those two metrics. The restriction of E_{cs} to the set of metrics whose completion is compact is denoted by E_{csc}. This is a natural equivalence relation from the point of view of descriptive set theory, and it is interesting to ask what is its complexity. Indeed, Ding and Gu ask [7, Question 4.11] whether for a given countable ordinal α and a natural number n, the restriction of E_{csc} to the metrics whose completion is homeomorphic to $\omega^{1+\alpha} \cdot n + 1$ is Borel-reducible to $=^+$. Even though this question does not seem directly connected to the topological conjugacy of systems with specification, we find a connection between the relation E_{csc} and Cantor systems, using a construction coming from the work of Williams and the work of Kaya and we answer it in the positive, by showing a slightly stronger statement. By $\mathbb{X}_{\text{0-dim}}$ we denote the set of metrics on $\mathbb{N}$ whose completion is zero-dimensional. The result below implies in particular a positive answer to [7, Question 4.11].

Theorem 2. *The relation E_{csc} restricted to $\mathbb{X}_{\text{0-}dim}$ is Borel bi-reducible with $=^+$.*

Finally, in [4] Bruin and Vejnar also studied the conjugacy relation of pointed transitive systems and asked [4, Table 1, Question 5.6] about the complexity of

	homeomorphisms	pointed transitive homeomorphisms
interval	Borel-complete	$\emptyset$
circle	Borel-complete	$=$
Cantor set	Borel-complete	$=^+$
Hilbert cube	complete orbit e.r.	?

TABLE 1. Source: [4, Table 1].

the conjugacy of pointed transitive homeomorphisms of the Hilbert cube. It turns out that the conjugacy of pointed transitive homeomorphisms of the Hilbert cube has the same complexity as the conjugacy relation of Hilbert cube systems with the specification property and in this paper we answer this question as follows.

Theorem 3. *The conjugacy relation of pointed transitive Hilbert cube systems is Borel bireducible with a turbulent group action.*

In particular, the conjugacy of pointed transitive Hilbert cube systems is not classifiable by countable structures, as opposed to most other relations considered in [4].

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Characterising exactness via invariant random equivalence relations

Sam Mellick

(joint work with Héctor Jardón-Sánchez, Antoine Poulin, Konrad Wróbel)

Invariant random equivalence relations (IREs) are exactly what they sound like: random equivalence relations on a group with a shift invariant distribution. They provide a common generalisation for coset decompositions, invariant random subgroups, and invariant percolation processes. In this talk, I will discuss finitary IREs (FIREs) and their weak limits. It's not so hard to characterise amenability in terms of FIREs: a group is amenable if and only if the full IRE (the relation where all elements are equivalent) is a weak limit of FIREs. In light of this, it's natural to imagine that in general, the classes of a weak limit of FIREs satisfy some sort of amenability property.

It turns out (contrary to the intuition of several experts in the field) that the statement is false in general. In fact, this falsity leads to a characterisation of the property known as exactness (or boundary amenability, or amenability at infinity, or many other names). In practice, most groups one encounters are exact, so the failure of the intuition doesn't usually make a difference. This is also helpful: the direction using exactness has applications in proving fixed price.

In this talk, I will define all the relevant notions and outline the structure of the proofs. No prior experience with IREs or exactness will be assumed.

Rigidity of embeddings between groups acting on spaces with dimension ≤ 1

Rachel Skipper
(joint work with James Hyde, Matthew C. B. Zaremsky)

For a group described via an action on a space, a (vague) general question is:

Question 1. *How fully do the dynamics of the action determine the algebra of the group?*

For some groups and dynamically rich actions, the action completely determines the group. A famous example is Rubin's theorem.

Recall that an action of group G on a space X is called a *Rubin action* if

(1) the space X is locally compact, Hausdorff, and has no isolated points and
(2) for each open set $U \subseteq X$ and each $p \in U$, the closure of the orbit of p under $G_U := \{g \in G \mid supp t(g) \subseteq U\}$ contains a neighborhood of U.

Here $supp t(g) := \{x \in X \mid g(x) \neq x\}$.

Then Rubin's Reconstruction Theorem says that any automorphism of a group G with such an action actually arises via conjugation in the group of homeomorphisms of X [4]. An equivalent formulation is the following.

Theorem 2 (Rubin)**.** *If a group G has a Rubin action on two topological spaces X and Y, then there exists a G-equivariant homeomorphism $\phi : X \to Y$, i.e., a homeomorphism satisfying $\phi(gx) = g\phi(x)$.*

Rubin's Theorem provides an example of a *semi-conjugacy* between two actions. Suppose G acts on spaces X and Y, then we say the actions are semi-conjugate if there exists a surjective G-equivariant continuous function $X \to Y$.

For these types of groups and actions, one may ask how these sorts of rigidity phenomena pass to subgroups. We consider specifically the case of *Thompson's group V*. Thompson's group V is the group of homeomorphisms h of Cantor space $\mathfrak{C} = \{0,1\}^{\mathbb{N}}$ that are locally given by prefix replacements, i.e., for all $\kappa \in \mathfrak{C}$ there exist a finite prefix w of κ and another finite string u such that $h(w\kappa') = u\kappa'$ for all $\kappa' \in \mathfrak{C}$. First introduced by Richard Thompson in handwritten, never published notes, this group, together with its subgroup T, gave the first known example of a finitely presented infinite simple group, and a significant amount of research has been dedicated to understanding it. At the same time, Thompson

introduced a third group F as the group of piecewise-linear orientation-preserving homeomorphisms of the interval $[0,1]$ that have breakpoints in $\mathbb{Z}[\frac{1}{2}]$ and slopes powers of 2. Using the usual lexicographic order on $\mathfrak{C}$ coming from the order $0<1$ on $\{0,1\}$, one can view F as isomorphic to the subgroup of V of all order preserving elements. A wealth of background information on these groups can be found for example in [1]. Note that for $\kappa=\epsilon_1\epsilon_2\epsilon_3\cdots\in\mathfrak{C}$ $(\epsilon_i\in\{0,1\})$, the map sending κ to the real number $\frac{\epsilon_1}{2}+\frac{\epsilon_2}{4}+\frac{\epsilon_3}{8}+\cdots$ gives a semi-conjugacy for F from $\mathfrak{C}\to[0,1]$.

Our main theorem shows that for many subgroups E of $\mathrm{Homeo}_+[0,1]$, for every embedding into V, the induced action of E on $\mathfrak{C}$ is semi-conjugate to the natural action. For our purposes, it is helpful to take the action of E on $S^1=[0,1]/0\sim 1$ to have a unique global fixed point.

Theorem 3 (Hyde, S., Zaremsky)**.** *Let $E\leq Homeo_+([0,1])$ be finitely generated, locally moving with abelian germs for every $a\in[0,1]$, and FG-filtered. Then for any abstract embedding of E into V, the resulting action of E on $\mathfrak{C}$ is* always *semi-conjugate to the standard action of E on S^1.*

The group $E\leq\mathrm{Homeo}_+([0,1])$ is *locally moving* if for all $a<b$ in $[0,1]$ the group $E[a,b]=\{g\in E\mid suppt(g)\subseteq[a,b]\}$ has no global fixed points in (a,b). The *group of germs of E* at $a\in[0,1]$ is the quotient of E by the subgroup consisting of elements which restrict to the identity on some neighborhood of a. And finally, E is *FG-filtered* if for all $a<b$ in $[0,1]$ the group $(E[a,b])'$ is contained in a finitely generated subgroup of $(E[\ell,r])'$ for some $\ell<r$ in $[0,1]$ with $\ell<a<b<r$.

This theorem applies for instance to Thompson's group F and the *Stein group* $F_{2,3}$, first studied in depth by Stein in [5] along with a broad family of related groups. The Stein group is the group of all piecewise-linear orientation-preserving homeomorphisms of $[0,1]$ with breakpoints in $\mathbb{Z}[1/6]$ and slopes of the form 2^m3^n $(m,n\in\mathbb{Z})$.

Corollary 4. *For any abstract embedding of Thompson's group F into V, the resulting action of F on $\mathfrak{C}$ is semi-conjugate to the standard action of F on S^1.*

Corollary 5. *The Stein group $F_{2,3}$ does not embed in V.*

This second corollary comes from some additional analysis on the germs at 1 and answers a question of the third author. In [3], Lodha proved that $F_{2,3}$ cannot abstractly embed as a subgroup of F (see also [2] for another proof by the first author and Moore), so Corollary 5 is a strengthening of this.

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Heavy repulsion of clusters in Bernoulli percolation

Grigory Terlov

(joint work with Sasha Bell, Tasmin Chu, Owen Rodgers, and Anush Tserunyan)

Percolation theory studies random subgraphs of a given graph. One of the classical models in this field is Bernoulli(p) percolation, in which each edge of the underlying graph is retained independently with probability $p \in (0,1)$. Originating in statistical mechanics, this model has been studied in a wide variety of settings for more than 60 years, yet several fundamental questions remain open even on $\mathbb{Z}^d$.

In 1996, Benjamini and Schramm laid the groundwork for modern percolation theory by extending its scope to all connected, locally finite, quasi-transitive graphs. Much of the subsequent theory has been developed under the additional assumption of unimodularity, a condition satisfied in particular by Cayley graphs. Their counterpart, nonunimodular quasi-transitive graphs, exhibit an inherent asymmetry, quantified by a relative weight function on the vertices. One central phenomenon in percolation theory is the phase transition in the sizes of connected components (called clusters): namely, the transition from all clusters being finite to the existence of an infinite cluster. In the unimodular case, the relative weight function provides no additional information: finite clusters have finite total weight, and vice versa. In the nonunimodular case, however, while finite clusters still have finite total weight, an infinite cluster may have either finite or infinite total weight. This yields an additional phase transition, now in the total weight of infinite clusters, whose properties are still being actively investigated.

In 1999, Häggström, Peres, and Schonmann [4] introduced the notion of **cluster repulsion**: *a graph G is said to exhibit cluster repulsion at $p \in [0,1]$ if any two infinite clusters come within unit distance from each other in at most finitely many places $\mathbb{P}_p$-almost surely.* They then asked the following question:

Question 1 ([4, p. 87])**.** *Does a quasi-transitive graph necessarily exhibit cluster repulsion for any value of $p \in [0,1]$?*

In 2006, Timár gave a positive answer to this question under the assumption of unimodularity [5]. The result presented in this talk gives the first progress on the nonunimodular case and establishes. More precisely, we show the following extension of Timár's result.

Theorem 2. *Suppose G is a connected locally finite quasi-transitive graph. Then any two clusters are within unit distance of each other only on a set of vertices of finite total weight, $\mathbb{P}_p$-almost surely.*

Our proof builds on Timár's approach, but requires the development of weighted analogues of several classical unimodular results. This presents nontrivial challenges, since in the nonunimodular setting, an invariant subforest with infinitely

many ends may be hyperfinite. To overcome this, we use tools from measured group theory, a closely related field in which a setting with a relative weight function also arises naturally: there, it is given by the Radon–Nikodym cocycle associated with a nonsingular, or measure-class-preserving (mcp), group action. In particular, recent progress in understanding the interplay between this cocycle and the geometry of mcp graphs has proved useful in studying percolation on nonunimodular graphs [1, 2, 3]. As an intermediate step, we establish a weighted generalization of a theorem of Benjamini, Lyons, and Schramm on the existence of an invariant random subgraph with positive weighted Cheeger constant, a result of independent interest.

We end with a discussion of the remaining open part of the cluster repulsion question, as well as related open problems on the measured group theoretic side of the interplay. A preprint of the article is available on arXiv [6].

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Graphical small cancellation and hyperfiniteness of boundary actions

Chris Karpinski

(joint work with Damian Osajda, Koichi Oyakawa)

1. Hyperfiniteness of boundary actions: General background

A (Gromov) hyperbolic metric space is a metric space in which geodesic triangles are uniformly slim. Hyperbolic metric spaces have a boundary at infinity, known as the *Gromov boundary*, which has a natural metrizable topology. See [2] for the definition and properties of hyperbolic metric spaces and their Gromov boundaries. A group acting by isometries on a hyperbolic metric space induces an action by homeomorphisms on the Gromov boundary. Given a Gromov hyperbolic space equipped with an action of a group by isometries, one can study the orbit equivalence relation of the induced action of the group on the Gromov boundary of the space. For many classes of groups with hyperbolic Cayley graphs, the induced action of the group on the boundary of its Cayley graph turns out to have the property that the orbits can be arranged into lines in a consistent manner, a

property known as *hyperfiniteness*. For instance, as an immediate consequence of [1], the action of any countable free group on the boundary of its Cayley graph is hyperfinite. A wide class of groups with "negative curvature" features generalizing free groups are *hyperbolic groups*, which are finitely generated groups having hyperbolic Cayley graphs. In [3], Marquis and Sabok prove that the action of hyperbolic groups on the Gromov boundaries of their Cayley graphs is hyperfinite.

Since [3], many other classes of "negatively curved" groups have been shown to induce hyperfinite orbit equivalence relations on the boundaries of natural hyperbolic graphs that they act on. See, for instance [5] for mapping class groups acting on the boundaries of their arc and curve graphs, and [6] for groups acting on trees.

An even wider class of "negatively curved groups" generalizing hyperbolic groups, mapping class groups and many groups acting on trees are the *acylindrically hyperbolic groups* introduced by Osin in [4]. These are groups having hyperbolic Cayley graphs (with respect to infinite generating sets in general) on which the action satisfies a certain finiteness property called *acylindricity* (see [4] for details). We call a hyperbolic Cayley graph *acylindrical* if the action of the group on the Cayley graph is acylindrical. In [7], Oyakawa proved that for any countable acylindrically hyperbolic group, there exists an acylindrical Cayley graph such that the action of the group on the boundary of the Cayley graph is hyperfinite. The question remains open as to whether for any countable acylindrically hyperbolic group, the action of the group on the boundary of *every one* of its acylindrical Cayley graphs is hyperfinite.

Question 1. *Let G be a countable acylindrically hyperbolic group. For each acylindrical Cayley graph Γ of G, is the induced boundary action $G \curvearrowright \partial\Gamma$ on the Gromov boundary $\partial\Gamma$ of Γ hyperfinite?*

2. Graphical small cancellation groups

Graphical small cancellation groups are a class of groups constructed from labeled graphs that often have many exotic properties. See [9] for background on graphical small cancellation groups.

Graphical small cancellation groups have natural hyperbolic Cayley graphs called "coned off" Cayley graphs, which are obtained by collapsing certain subsets of the Cayley graph of the group with respect to a finite generating set to complete subgraphs (i.e. "coning off" these subgraphs).

In joint work with Damian Osajda and Koichi Oyakawa, we show that under certain conditions on the underlying labeled graphs, the action of a graphical small cancellation group on the Gromov boundary of its coned-off Cayley graph induces a hyperfinite equivalence relation.

A graph Γ is called *extremely fine* if there exists $K \in \mathbb{N}$ such that for every edge e in Γ, there are at most K simple closed paths γ containing e. We refer to [8] for background and definitions in Theorem 2 below.

Theorem 2 (K, Damian Osajda, Koichi Oyakawa 2026, [8])**.** *Let G be a group defined by a graphical $C'(1/10)$ presentation $\langle S|\Gamma\rangle$ with S countable and with the*

graph Γ having countably many connected components $(\Gamma_n)_{n\in\mathbb{N}}$, all of which are finite. Suppose that Γ is extremely fine. Let Y denote the associated coned-off Cayley graph. Then the action $G \curvearrowright \partial Y$ is hyperfinite.

The extreme fineness property is an essential ingredient in the proof of Theorem 2. However, we do not know if it is a necessary condition.

Question 3. *Is Theorem 2 still true without the assumption of the extreme fineness of the underlying graph?*

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On the Measure Equivalence classification of BS(p, q), Aut$(T_{p,q})$, and lcsc treeable groups

Damien Gaboriau

(joint work with Antoine Poulin, Anush Tserunyan, Robin Tucker-Drob, Konrad Wróbel)

The Baumslag-Solitar groups $\mathrm{BS}(p,q) = \langle b, t \mid tb^p t^{-1} = b^q \rangle$ $(1 \le p \le q)$ naturally fall into three families:

(F1) (Amenable) $1 = p \le q$ solvable;
(F2) (Non-amenable "unimodular") $1 < p = q$ virtually $\mathbb{Z} \times \mathbf{F}$ (w. $\mathbf{F}$ f.g. free);
(F3) (Non-amenable "non-unimodular") $1 < p < q$.

Their classification up to quasi-isometry (QI), after the classification of (F1) by Farb and Mosher [2], was completed in 2001 when Whyte proved the surprising result that (F3) forms a single QI family [5].

The question of their classification up to measure equivalence (ME) arose immediately. In fact, (F1) forms a single class since all infinite amenable countable groups are ME, by Ornstein–Weiss [4]. Similarly for (F2), since they are all virtually isomorphic to the direct product of $\mathbb{Z}$ with a finitely generated non-abelian free group. Concerning (F3) it had remained unresolved until very recently.

With Antoine Poulin, Anush Tserunyan, Robin Tucker-Drob, & Konrad Wróbel, we complete the classification by proving:

Theorem 1 (G. – Poulin – Tserunyan –Tucker-Drob – Wróbel 2025)**.** *All the* $\mathrm{BS}(p,q)$ *with* $1 < p < q$ *are ME to each other, i.e. (F3) forms a single ME class.*

I will explain the unexpected paths we had to take: switching from

- probability measure preserving (pmp) actions $\rightsquigarrow$ measure class preserving (mcp) equivalence relations;
- discrete groups $\rightsquigarrow$ lcsc groups;
- unimodular groups $\rightsquigarrow$ non-unimodular ($\rightsquigarrow \mathrm{Aut}(T_{p,q})$).

By a result of Cornulier–Valette [1], $\mathrm{BS}(p,q)$ is a lattice in $\mathrm{Aut}(T_{p,q}) \ltimes_{mod} \mathbf{R}$ (where $T_{p,q}$ is the directed Bass-Serre of $\mathrm{BS}(p,q)$ and $\mathrm{Aut}(T_{p,q})$ acts on $\mathbf{R}$ via its modular homomorphism).

We found that it is enough to show that all the $\mathrm{Aut}(T_{p,q})$ $(1 < p < q)$ are ME to each other, since then all their modular product $\mathrm{Aut}(T_{p,q}) \ltimes_{mod} \mathbf{R}$ would be ME.

Theorem 2 (G. – Poulin – Tserunyan –Tucker-Drob – Wróbel 2025)**.** *There are exactly four ME classes among the* $\mathrm{Aut}(T_{p,q})$ *groups:*

(1) $\mathrm{Aut}(T_{1,1}) = \mathbb{Z}$ *(amenable unimodular);*
(2) $\mathrm{Aut}(T_{1,q})$*, for* $q \geq 2$ *(amenable nonunimodular);*
(3) $\mathrm{Aut}(T_{p,p})$ *for* $p \geq 2$ *(nonamenable unimodular);*
(4) $\mathrm{Aut}(T_{p,q})$*, for* $1 < p < q$ *(nonamenable nonunimodular).*

In fact, we prove more generally:

Theorem 3 (G. – Poulin – Tserunyan –Tucker-Drob – Wróbel)**.** *All nonunimodular nonamenable treeable lcsc groups are pairwise ME.*

Together with former results, this leads to a complete classification of treeable locally compact groups.

Corollary 4 (G. – Poulin – Tserunyan –Tucker-Drob – Wróbel)**.** *There are exactly six ME classes of treeable locally compact groups:*

(1) *compact, e.g. the trivial group;*
(2) *unimodular amenable non-compact, e.g.* $\mathbf{Z}$ *(Ornstein–Weiss* [4]*);*
(3) *unimodular nonamenable finite cost, e.g.* $\mathbf{F}_2$ *(Hjorth* [3]*);*
(4) *unimodular infinite cost, e.g.* $\mathbf{F}_\infty$*;*
(5) *nonunimodular amenable, e.g.* $\mathrm{Aut}(T_{1,2})$*;*
(6) *nonunimodular nonamenable treeable, e.g. all* $\mathrm{Aut}(T_{p,q})$*, for* $1 < p < q$*.*

I will in particular present a crucial tool we introduce: **scalar potentials** for measure class preserving equivalence relations.

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Minimal subdynamics

Anton Bernshteyn
(joint work with Joshua Frisch)

Topological dynamics studies continuous group actions on compact spaces. In this talk, we focus on the case of countable discrete groups. Historically the earliest and by far the best understood case is that of the additive group $\mathbb{Z}$. For Abelian groups, especially $\mathbb{Z}^n$, the theory is also fairly well developed. By contrast, topological dynamics of general non-Abelian groups remains a much more enigmatic field, with even the most basic questions often posing a considerable challenge. That being said, recent years saw the arrival of new powerful techniques, based on tools from descriptive set theory and combinatorics, that were able to significantly expand our knowledge of the dynamical behavior exhibited by countable groups regardless of any assumptions on their structure. This talk is about one such recent development, namely the construction of flows with minimal subdynamics [1].

Let Γ be an infinite countable group. A Γ-*flow* is a nonempty compact Hausdorff space X equipped with a continuous action of Γ. A Γ-flow X is *minimal* if it has no proper subflows (i.e., closed Γ-invariant subsets); equivalently, X is minimal if every Γ-orbit in X is dense. A straightforward argument using Zorn's lemma shows that every Γ-flow has a minimal subflow. Unfortunately, this non-constructive proof gives no further insight into the properties of minimal Γ-flows, which leads to many intriguing open problems. The question we address in this talk is:

Question 1 (Minimal subdynamics problem)**.** *Suppose $\Delta \leq \Gamma$ is an infinite subgroup. Does there exist a free Γ-flow that is minimal as a Δ-flow?*

In certain special cases, such flows are easy to come by. For instance, an irrational rotation action of $\mathbb{Z}$ on a circle is minimal with respect to every nontrivial subgroup; a similar construction works for $\mathbb{Z}^n$ as well. Zucker [9] answered Question 1 positively when Γ decomposes as $\Gamma = \Delta \times \Delta'$; one can say that in this situation there is still enough "Abelian-ness" that can be exploited. The most general previous result, due to Frisch, Seward, and Zucker [4], pushes this idea further and applies whenever the subgroup Δ is *normal*. Moreover, it can handle countably many normal subgroups at once.

Theorem 2 (Frisch–Seward–Zucker [4])**.** *If $(\Delta_n)_{n\in\mathbb{N}}$ is a countable family of infinite normal subgroups of Γ, then there exists a free Γ-flow that is minimal as a Δ_n-flow for all $n \in \mathbb{N}$.*

On the other hand, the following very modest special case of Question 1 remained open prior to our work:

Question 3. *Let $\mathbb{F}_2$ be the free group of rank 2, and let $\gamma \in \mathbb{F}_2$ be a non-identity element. Does there exist a free $\mathbb{F}_2$-flow X such that γ acts minimally on X?*

We solve Question 1 in complete generality. In particular, the answer to Question 3 is "yes"; actually, there exists a free $\mathbb{F}_2$-flow on which *all* non-identity group elements act minimally. Not only do we remove the normality assumption from Theorem 2, we also remove the assumption that each Δ_n is a *subgroup*. Namely, we say that a Γ-flow X is *S-minimal* for a subset $S \subseteq \Gamma$ if for each point $x \in X$, the partial orbit $Sx = \{\gamma x : \gamma \in S\}$ is dense.

Theorem 4 (Main result [1])**.** *If $(S_n)_{n\in\mathbb{N}}$ is a countable family of infinite subsets of Γ, then there exists a free Γ-flow X that is S_n-minimal for all $n \in \mathbb{N}$. Furthermore, X can be taken to be a subflow of the Bernoulli shift 2^Γ.*

Since a countable group has countably many finitely generated subgroups, we immediately obtain the following corollary:

Corollary 5. *If Γ has countably many infinite locally finite subgroups, then there exists a free Γ-flow that is Δ-minimal for every infinite subgroup $\Delta \leq \Gamma$.*

On the other hand, the conclusion of Corollary 5 fails for some locally finite groups with uncountably many infinite subgroups, e.g., for $\Gamma = \bigoplus_{n\in\mathbb{N}}(\mathbb{Z}/2\mathbb{Z})$ [4]. The following natural problem remains open:

Problem 6. Characterize the countable groups Γ that admit a free Γ-flow which is Δ-minimal with respect to every infinite subgroup $\Delta \leq \Gamma$.

To illustrate the power of Theorem 4, let us describe its application to the study of disjoint flows, which was initiated by Furstenberg [7].

A *joining* of two Γ-flows X and Y is a subflow $Z \subseteq X \times Y$ that projects onto X and Y. We say X and Y are *disjoint*, in symbols $X \perp Y$, if their only joining is $X \times Y$ itself. We are interested in the following natural problem:

Question 7 (Existence of disjoint flows)**.** *For a given Γ-flow X, does there exist a free Γ-flow Y such that $X \perp Y$?*

Some important results related to Question 7 were obtained by Glasner, Tsankov, Weiss, and Zucker [8]. Generalizing a theorem of Furstenberg [7], they showed that the answer is "yes" when $X = 2^\Gamma$ and, more generally, when X is a strongly irreducible subshift; in these cases, Y can be taken to be any free minimal Γ-flow. They also gave the positive answer when X is minimal and metrizable. (The assumption that X is metrizable cannot be dropped here.)

On the other hand, there exist Γ-flows X for which the answer is negative. The simplest example is the one-point compactification Γ^* of Γ, where Γ acts on itself

by left multiplication and fixes the point at infinity. The issue is the presence of *wandering points*, i.e., points x with a neighborhood U such that $U \cap \gamma U = \emptyset$ for all but finitely many $\gamma \in \Gamma$. In Γ^*, all points except the point at infinity are wandering, and they can be used to find nontrivial joinings of Γ^* with all nontrivial Γ-flows.

Using Theorem 4, we provide a complete answer to Question 7 in the metrizable case by showing that the existence of wandering points is the only obstruction:

Theorem 8. *The following statements are equivalent for a metrizable Γ-flow X:*

(1) *X is disjoint from some infinite Γ-flow,*
(2) *X is disjoint from some free minimal subflow of 2^Γ,*
(3) *X has no wandering points.*

Let us explain how Theorem 4 enters the proof of the most difficult part of Theorem 8, namely the implication (3) $\implies$ (2). Compact metrizable spaces are second-countable, so we can fix a countable basis $(U_n)_{n\in\mathbb{N}}$ for X consisting of nonempty open sets. Since X has no wandering points, a standard Baire category argument shows that for each $n \in \mathbb{N}$, there is a point $x_n \in U_n$ such that the set

$$S_n = \{\gamma \in \Gamma : \gamma x_n \in U_n\}$$

is infinite. Theorem 4 yields a free subflow $Y \subseteq 2^\Gamma$ that is S_n-minimal for all $n \in \mathbb{N}$. It is then not hard to show that $X \perp Y$, as desired.

As mentioned earlier, the proof of Theorem 4 relies on a combination of tools from topological dynamics, descriptive set theory, and combinatorics. Without going into any detail, let us briefly mention some of the relevant keywords and important references.

Existence via genericity: We argue that the Γ-flows with desired properties form a comeager set in a certain class. Our approach is inspired by [4, 5, 6].

Asymptotic separation index: This is a parameter introduced in [3] to study the relationship between descriptive set-theoretic constructions and large-scale geometry. Its adaptation for the continuous setting plays a crucial role in our proof.

Lovász Local Lemma is a powerful tool in probabilistic combinatorics, which has recently found a number of applications in set theory and dynamics. We rely on its relationship with asymptotic separation index that was studied in [2].

As a final remark, we note that, while our ultimate goal is to find a continuous action of Γ on a *compact* space with certain properties, our proof mostly involves studying dynamically rich *non-compact* Polish Γ-spaces. This suggests that topological dynamics of non-compact spaces is a worthy subject of investigation, both in its own right and due to its potential to shed new light on the compact case.

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Stationary boundaries on the space of amenable subgroups

Anna Cascioli
(joint work with Martín Gilabert Vio, Eduardo Silva)

The space $\mathrm{Sub}(G)$ of subgroups of a discrete countable group G is naturally identified with a closed subset of $\{0,1\}^G$ endowed with the product topology. The induced topology, known as the *Chabauty topology*, makes $\mathrm{Sub}(G)$ a compact metrizable space. The conjugation action of G on its subgroups induces an action by homeomorphisms $G \curvearrowright \mathrm{Sub}(G)$. The set $\mathrm{Sub}_{\mathrm{am}}(G)$ of amenable subgroups of G is closed in $\mathrm{Sub}(G)$, and a reason to study the dynamics of $G \curvearrowright \mathrm{Sub}_{\mathrm{am}}(G)$ is its connection with the C*-simplicity of G.

Recall that a measurable action of G on a probability space (X,η) is *μ-stationary* if $\eta = \sum_{g\in G}\mu(g)g_*\eta$. A μ-stationary probability measure on $\mathrm{Sub}_{\mathrm{am}}(G)$ is called an *amenable μ-stationary random subgroup* (or *amenable μ-SRS)*. This notion turns out to be useful to characterize C*-simplicity of countable groups by work of Hartman-Kalantar [3], who proved that a countable group G is C*-simple if and only if there exists a non-degenerate probability measure $\mu \in \mathrm{Prob}(G)$ such that $\mathrm{Sub}_{\mathrm{am}}(G)$ admits a unique μ-stationary probability measure (which must be $\delta_{\{e_G\}}$). We address the existence of probability measures with the opposite property.

Question 1. *Let G be a countable group. Does there exist a non-degenerate probability measure μ on G that admits an amenable μ-SRS distinct from $\delta_{\{e_G\}}$?*

It follows from [3] that Question 1 is of interest only when G is C*-simple. Our results deal with the following class of amenable SRSs.

Definition 2. Let μ be a probability measure on a countable group G and denote by $\mathbb{P}_\mu$ the law of the right μ-random walk on G. A μ-stationary probability measure η on the space $\mathrm{Sub}_{\mathrm{am}}(G)$ is called an *amenable boundary μ-stationary random subgroup* if, for $\mathbb{P}_\mu$-almost every trajectory $(w_n)_{n\geq 0} \in G^{\mathbb{N}}$, the sequence $(w_{n*}\eta)_{n\geq 0}$ converges in the weak*-topology to a Dirac mass in $\mathrm{Sub}_{\mathrm{am}}(G)$.

Our first main theorem provides a condition that guarantees the existence of such a measure.

Theorem 3. *Let G be a countable group. Suppose that there exists a non-trivial subgroup H such that for all finite subsets $Q, Z \subseteq G$ there is $b \in G$ such that*

$$bZ, bZ^{-1} \subseteq \left\{ g \in G : Q \cap H = Q \cap gHg^{-1} \right\}.$$

Then there exists a non-degenerate, symmetric and finite-entropy probability measure μ on G such that $\overline{\mathrm{Orb}_G(H)}$ supports a μ-boundary SRS distinct from $\delta_{\{e_G\}}$.

In particular, if H is amenable then G admits an amenable boundary μ-SRS distinct from $\delta_{\{e_G\}}$.

The condition appearing in the statement can be viewed as a strong form of recurrence for the action of G on the orbit of H. The strategy we use to construct μ is the method of record times used by Frisch-Hartman-Tamuz-Vahidi Ferdowsi [2] to prove that any non-hyper-FC-central group admits non-degenerate probability measures with a non-trivial Poisson boundary.

The criterion in Theorem 3 applies to (permutational) wreath products, which include groups that are C*-simple, and to Thompson's group F, whose C*-simplicity is equivalent to its non-amenability [4], and therefore remains an open problem.

Corollary 4. *There exists a non-degenerate, symmetric and finite-entropy probability measure μ on Thompson's group F such that $\mathrm{Sub}_{\mathrm{ab}}(F) = \{H \in \mathrm{Sub}(F) : H \text{ is abelian}\}$ supports a non-atomic μ-boundary.*

Following [1], we say that a subgroup H of G is called *normalish* if $\bigcap_{z \in Z} zHz^{-1}$ is infinite for every finite subset $Z \subseteq G$. Our second main theorem connects this concept to Question 1.

Theorem 5. *Let μ be a non-degenerate probability measure on a countable group G and let η be a μ-stationary probability measure on $\mathrm{Sub}(G)$. Suppose that $(\mathrm{Sub}(G), \eta)$ is a μ-boundary of G, and that η is not a Dirac mass on a finite normal subgroup of G. Then η is supported on normalish subgroups of G.*

As a consequence, we obtain the next result.

Corollary 6. *Let G be a countable group with no finite normal subgroups and no amenable normalish subgroups. Then, for every non-degenerate $\mu \in \mathrm{Prob}(G)$, the action of G on the Poisson boundary of (G, μ) is essentially free.*

Finally, we observe that one cannot expect a converse to Theorem 5. More precisely, we show that many Baumslag-Solitar groups have amenable normalish subgroups but their space of amenable subgroups is countable. Thus, they admit no non-trivial amenable μ-SRSs for any non-degenerate μ. This is the case in particular for $\mathrm{BS}(2,3) = \langle a, t \mid ta^2t^{-1} = a^3 \rangle$.

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GBS groups, class $\mathcal{A}$, and relative property (T)

Sasha Bontemps

A **generalized Baumslag-Solitar group** (GBS group) of rank $d \in \mathbb{N}^*$ is a group that acts cocompactly on an oriented tree with vertex and edge stabilizers isomorphic to $\mathbb{Z}^d$. GBS groups form a large class of groups that include for instance Baumslag-Solitar groups, fundamental groups of torus knots, semi-direct products $\mathbb{Z}^d \rtimes_\rho \mathbb{F}_r$ (where $r \in \mathbb{N}^*$ and $\rho : \mathbb{F}_r \to \mathrm{GL}_d(\mathbb{Z})$ is a homomorphism), Leary-Minasyan groups...

In [GM06], Glasner and Monod introduced the **class** $\mathcal{A}$ of countable groups that admit an amenable, faithful, and transitive action. Here, "amenable" has to be understood in the sense of Greenleaf (*i.e.* an action is **amenable** if it admits a Følner sequence). Faithfulness and transitivity are required to avoid trivialities.

Of course amenable groups are in $\mathcal{A}$, but there are plenty of other examples, such as free groups (*cf.* [vDou90]), many free products ([GM06]), and some HNN extensions and amalgamated free products (see [Fim12] and [Moo11]).

The space of subgroups $\mathrm{Sub}(G)$ of any infinite countable group G can be endowed with a topology called the **Chabauty topology**; it is the one inherited from the canonical inclusion of $\mathrm{Sub}(G)$ into $\{0,1\}^G$. Equipped with this topology, $\mathrm{Sub}(G)$ is a closed subset of the Cantor space, on which the G-conjugation induces an action by homeomorphisms. In [LM24, Example 10.31], Le Maître observed that the belonging of free groups to $\mathcal{A}$ can be understood as a consequence of the following result (see [LM24, Example 9.38]):

Theorem 1. *For every $r \geq 2$, the conjugation action of $\mathbb{F}_r$ on its space of infinite index subgroups has a dense orbit.*

In particular, there exists a conjugacy class whose closure contains both a coamenable subgroup of $\mathbb{F}_r$ that has infinite index, and the trivial subgroup. Any subgroup H in this conjugacy class yields an $\mathbb{F}_r$-action $H\backslash\mathbb{F}_r \curvearrowleft \mathbb{F}_r$ that is transitive, amenable, and faithful. In [AG25], Azuelos and Gaboriau obtained many new examples using this kind of dynamical methods, that include for instance virtually compact special groups.

On the other hand, one obstruction for being in $\mathcal{A}$ is relative property (T) to a normal subgroup of infinite exponent (*cf.* [GM06, Proposition 1.7 (viii)]). Recall that a pair (G, H) (where $H \leq G$) has **relative property** (T) if, for any unitary representation of G that has almost invariant vectors, there exists a non-zero H-invariant vector.

At the end of their article, Glasner and Monod ask which groups acting on trees are in $\mathcal{A}$.

We provide some new elements to answer this question in the context of GBS groups. The fact that the second derived subgroup of any GBS group of rank 1 is free (as observed in [Kro90]), combined with hereditary properties obtained in [GM06] proves that all GBS groups of rank 1 are in $\mathcal{A}$. In higher dimension, we provide some partial results based on the dynamical study of the conjugation action on the space of subgroups of GBS groups performed in [Bon25]. Our investigation relies on the following result, which is a consequence of [Bon25, Theorem 5.2, Theorem 6.5]):

Theorem 2 (B. 2025)**.** *Let G be a non-amenable GBS group that is not a semi-direct product $\mathbb{Z}^d \rtimes \mathbb{F}_r$. For any subgroup $H \leq G$ that acts non-cocompactly on the Bass-Serre tree $\mathcal{T}$ and any vertex v of $\mathcal{T}$, there exists a conjugacy class of some subgroup that accumulates both on H and on $H \cap \mathrm{Stab}(v)$*

Applying this theorem to any co-amenable subgroup that contains a vertex stabilizer and that acts non-cocompactly on the Bass-Serre tree leads to the following corollary:

Corollary 3. *Let G be a GBS groups that acts **faithfully** on its Bass-Serre tree. Then $G \in \mathcal{A}$.*

"Generic" GBS groups satisfy the assumption of Corollary 3. More specifically, any GBS group G of rank d comes equipped with a **modular homomorphism** $\Delta : G \to \mathrm{GL}_d(\mathbb{Q})$. The faithfulness of the action on the Bass-Serre tree is equivalent to the fact that $\Delta(G)$ does not stabilize any non-trivial rational lattice.

On the other hand, when $\Delta(G)$ is virtually solvable, then hereditary properties of the class $\mathcal{A}$ (*cf.* [GM06, Proposition 1.7 (vi)]) imply that $G \in \mathcal{A}$.

These two extreme cases rule out the case when $d = 2$:

Theorem 4. *Let G be a GBS groups of rank 2. Then:*

- *either $G \in \mathcal{A}$; or*
- *G has a normal subgroup isomorphic to $\mathbb{Z}^2$ relative to which it has property (T).*

In higher dimension, the situation becomes more subtle and is the subject of an ongoing joint work with Yves Cornulier.

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The uniform random walk: minimal entropy random walk and its limits

Miklós Abért
(joint work with Adam Arras, Jaelin Kim)

We define the uniform random walk on a locally finite connected infinite graph as the weak limit of uniform walks of length n. We analyze the existence and phase transitions for various classes of graphs, including loop perturbed Cayley graphs. It turns out that one can use the global spectral theory of graphings to study this notion.

Chaining in measurable dynamics

Jenna Zomback
(joint work with Anush Tserunyan)

A Borel action of a countable group on a standard probability space, $\Gamma \curvearrowright^{\alpha} (X, \mu)$ is **ergodic** if the only measurable, α-invariant subsets of X are null or conull. Equivalently, for all measurable sets A and B of positive measure, for some $\gamma \in \Gamma$, $\mu(\gamma A \cap B) > 0$.

More restrictively, α is called **weakly mixing** if the diagonal action with any μ-preserving action β is ergodic. A classic example of an ergodic but not weak mixing action is the irrational rotation of the circle. Indeed, the set of pairs (x, y) of any bounded distance forms a set which is invariant under the diagonal action $\alpha \times \alpha$.

There are a plethora of conditions on group actions that are known to imply weak mixing, including double ergodicity (DE) and metric ergodicity (ME).

Definition 1. α is **doubly ergodic** if the diagonal action of α with itself is ergodic.

Definition 2. α is **metrically ergodic** if every measurable equivariant function $\varphi : X \to Z$ where (Z, d) is a separable metric space and $\Gamma \curvearrowright Z$ by isometries is constant μ-a.e..

In the measure-class preserving (mcp) setting, we have the following implications due to Glasner and Weiss in [1]:

$$\text{DE} \implies \text{ME} \implies \text{WM}$$

Since weak mixing trivially implies double ergodicity in the probability-measure-preserving (pmp) setting, in this setting all three of the above conditions are equivalent. Glasner and Weiss show that in the mcp setting, metric ergodicity does not imply double ergodicity. On the other hand, it is open whether weak mixing implies metric ergodicity.

We introduce the following notion, and clarify its place in the weak mixing heirarchy.

Definition 3. We say that α is **k-chaining** (kC) if for any sets $A, B \subseteq X$ of positive measure, there are group elements $\gamma_1, \ldots, \gamma_k \in \Gamma$ such that the sets $A \cap \gamma_1^{-1}(A)$, $\gamma_i^{-1}(A) \cap \gamma_{i+1}^{-1}(A)$, and $\gamma_k^{-1}(A) \cap B$ all have positive measure, for each $i < k$. We say that α is **chaining** (C) if it is k-chaining for some $k \in \mathbb{N}$.

We also define a weaker notion of chaining which we call **essentially chaining** (Ess-C), which says that α is chaining on a complete section. Our main result is the following.

Theorem 4. *Let (X, μ) be a standard probability space, and let $\Gamma \curvearrowright^{\alpha} (X, \mu)$ be an mcp measurable action of the countable group Γ.*

(1) *If α is doubly ergodic, then α is 1-chaining.*
(2) *If α is essentially chaining, then α is metrically ergodic.*

Therefore, the heirarchy in the mcp setting is as follows.

$$\text{DE} \implies \text{1C} \implies \cdots \implies \text{C} \implies \text{eC} \implies \text{ME} \implies \text{WM}$$

This immediately yields the following corollary.

Corollary 5. *The following are equivalent for any Borel **pmp** action of a countable group $\Gamma \curvearrowright (X, \mu)$:*

(1) *α is weakly mixing.*
(2) *α is chaining (i.e., k-chaining for some k).*
(3) *α is 1-chaining.*

We show that in the mcp setting, 1-chaining does not imply double ergodicity, and that essential chaining does not imply chaining. We also show that the essential chaining heierarchy is strict, i.e., for all $k \in \mathbb{N}$, essential $k+1$ chaining does not imply essential k chaining. However, in the null-preserving setting, we do not know whether weak mixing is equivalent to chaining, nor whether 1-chaining is equivalent to chaining.

Question 6. *Is chaining equivalent to weak mixing for measure-class-preserving actions?*

Question 7. *Are there measure-class-preserving actions that are chaining but not 1-chaining?*

Finally, we have the following result for boundary actions of free groups, where the boundary action is equipped with a Markov measure induced by a strictly positive stationary distribution π and a transition matrix P. Below, we equate weak

mixing of the boundary action to strict irreducibility of P, a condition introduced by Bufetov in [2].

Definition 8. A matrix indexed by I is **irreducible** for all $i, j \in I$, for some $n \in \mathbb{N}^+$ $P(i,j) > 0$.

Definition 9. P is **strictly irreducible** if P and PP^t are irreducible. Equivalently, P and P^tP are irreducible.

We remark that in general, the boundary actions are not mcp, since if $P(a,b) = 0$, then $\mu([b]) = \pi(b) > 0$ but $\mu([ab]) = \pi(a)P(a,b) = 0$.

Theorem 10. *The following are equivalent for a stationary Markov measure on $\mathbb{F}_r$, $r < \infty$:*

(1) *The boundary action is weakly mixing.*
(2) *The boundary action is $2r-1$-chaining.*
(3) *P is strictly irreducible.*

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Continuity of asymptotic entropy on groups

Eduardo Silva

1. Introduction and motivation

Let μ be a probability measure on a countable group G. A function $f : G \to \mathbb{R}$ is called μ-harmonic if $f(g) = \sum_{h \in G} f(gh)\mu(h)$ for all $g \in G$. The space $H^\infty(G,\mu)$ of bounded μ-harmonic functions is a nice space of functions on which the group acts and which we can use to study the geometry of G. We care about the following vague question:

Question 1 (vague)**.** *Let G be a countable group. How "sensitive" is the space $H^\infty(G,\mu)$ of bounded μ-harmonic functions to variations in μ?*

Remark 2. *In the continuous setting, one has the following result due to Furstenberg '60s: take a connected semisimple Lie group G with finite center and no compact factor. Let K be a maximal compact subgroup and P a minimal parabolic subgroup of G. Think $G = \mathrm{SL}_n(\mathbb{R})$, take $K = SO_n(\mathbb{R})$ and $P \leqslant G$ the subgroup of upper triangular matrices. Let μ be a probability measure that is spherical: absolutely continuous with respect to the Haar measure, and bi-K-invariant. Then*

$$H^\infty(G,\mu) \cong L^\infty(G/P, m),$$

where m is the unique K-invariant measure on the G/P. Note that the right-hand side does not depend on μ!

In the discrete setting of course we cannot expect to have *exactly* the same space of bounded harmonic functions for two different measures. The way that we will quantify the sensitivity of $H^\infty(G,\mu)$ is through asymptotic entropy.

Definition 3. Let μ a non-degenerate probability measure on a countable group G, and such that $H(\mu) = -\sum_{g\in G}\mu(g)\log\mu(g) < \infty$, so that the *asymptotic entropy*

$$h(\mu) = \lim_{n\to\infty}\frac{H(\mu^{*n})}{n}$$

is well-defined and finite.

This quantity was first introduced by Avez in the early 70's. It turns out that asymptotic entropy fully captures, in one numerical quantity, whether the pair (G,μ) admits non-constant bounded harmonic functions.

Theorem 4 (Derriennic '80, Kaimanovich-Vershik '83)**.**

$$H^\infty(G,\mu) \text{ contains non-constant functions} \iff h(\mu) > 0.$$

From now on in the talk I will concentrate in the next question.

Question 5 (concrete)**.** *Let G be a countable group, $(\mu_i)_i, \mu$ probability measures on G with finite entropy. Suppose that*

- $\lim_{i\to\infty}\mu_i(g) = \mu(g)$ *for all* $g \in G$*, and*
- $\lim_{i\to\infty} H(\mu_i) = H(\mu)$.

Is it true that $\lim_{i\to\infty} h(\mu_i) = h(\mu)$*?*

I will denote the two convergences above as $\mu_i \xrightarrow[i\to\infty]{} \mu$ to save space.

Some history and motivation:

- Historically, the first example of discontinuity is due to Kaimanovich-Vershik '83. They exhibited measures with finite entropy and non-trivial Poisson boundary on the group $\mathrm{FSym}(\mathbb{Z})$ of finitary permutations of $\mathbb{Z}$. Now of course we know that one can always do this on a non-trivial group with an ICC quotient, thanks to Frisch-Hartman-Tamuz-Vahidi Ferdowsi '19.
- For random walks with a finite first moment on a finitely generated group, another related quantity is the asymptotic drift $\ell(\mu) := \lim_{n\to\infty} E[|w_n|]/n$ of the random walk $(w_n)_n$ with respect to a word metric. In many classes of groups, the ratio $h(\mu)/\ell(\mu)$ coincides with the Hausdorff dimension of the hitting measure of the μ-random walk on a geometric boundary of the group G. This is the case for free groups (Kaimanovich '98) and hyperbolic groups (Le Prince '08).

2. The results

The following result reduces the problem of continuity of asymptotic entropy to that of identifying the Poisson boundary in a Polish space, together with showing the weak convergence of harmonic measures.

Theorem 6. [Sil25] *Let G, $(\mu_i)_i, \mu$ non-degenerate as above with $\mu_i \to \mu$. Suppose there is a Polish G-space X and $(\nu_i)_i, \nu \in \mathrm{Prob}(X)$ such that (X, ν_i) is the Poisson boundary of (G, μ_i) for each i, and such that (X, ν) is the Poisson boundary of (G, μ).*

- *If ν_i converges weakly to ν, then $\lim_{i\to\infty} h(\mu_i) = h(\mu)$.*
- *Suppose that X is further supposed to be compact and uniquely μ-stationary. Then it always holds that ν_i converges weakly to ν.*

Hence, the difficulties associated with showing the continuity of entropy are replaced by the difficulties of identifying the Poisson boundary of the random walks on the group. The above result proves continuity in the next contexts:

- Acylindrically hyperbolic groups. In this case, the continuity of asymptotic entropy had been established previously, with different sets of hypotheses on the measures, by Erschler-Kaimanovich, Gouëzel-Matheus-Macourant, and Choi. All of their proofs are rather technical and rely on the moment hypotheses that they consider (except Choi's proof, which has no moment hypotheses).
- $\mathrm{SL}_d(\mathbb{Z})$ with $d \geq 3$. In this case, the continuity of asymptotic entropy is completely new, even if you only consider probability measures supported on a fix generating set of the group.

I refer to the citations on the paper for the references of identifications of Poisson boundaries that are used for the above applications, in addition to the recent preprint of Chawla-Frisch-Forghani-Tiozzo '25 for lattices in semisimple Lie groups.

The second theorem I want to speak about is the following.

Theorem 7. [Sil25] *Let A be a countable group and consider the wreath product $G = A \wr \mathbb{Z}^d = \bigoplus_{\mathbb{Z}^d} A \rtimes \mathbb{Z}^d$ with $d \geq 3$. Then asymptotic entropy is continuous among all non-deg. probability measures with finite entropy. More generally, instead of non-deg., you can just ask that all probability measures have a projection to $\mathbb{Z}^d$ with support of at least cubic growth.*

This theorem is particularly interesting since at the generality of finite entropy measures, there is no identification of the Poisson boundary for $A \wr \mathbb{Z}^d$. Further, work in progress of Josh Frisch and I shows that there cannot exist a "universal realization". We can prove that there does not exist a Polish space X on which the Poisson boundary of all finite entropy non-deg. random walks on $A \wr \mathbb{Z}^d$ is modeled.

One should also remark that the theorem is false is $d = 1, 2$, and this goes back to examples of discontinuity of asymptotic entropy on wreath products due to Erschler.

You can think of entropy on a wreath product as bits of information being encoded by the lamps visited by the walker. Hence, the entropy is directly related to the asymptotic amount of distinct positions visited by the random walk on the base group. A key step in the proof of the above theorem is showing that in general, on groups with at least cubic growth, this quantity is continuous.

3. Some discussion and next questions

I want to spend the last minutes of the talk motivating a situation in which none of the theorems apply.

Let $G \leqslant \mathrm{Homeo}_+(S^1)$ so that $G \curvearrowright S^1$ is minimal and proximal. Deroin, Kleptsyn and Navas '07 showed that for any $\mu \in \mathrm{Prob}(G)$ there is a unique μ-stationary probability measure ν on S^1, and (S^1, ν) is a μ-boundary, i.e., a G-equivariant quotient of the Poisson boundary of (G, μ). If (S^1, ν) were to be the Poisson boundary, then one could apply our first theorem to conclude continuity of entropy. This is the case, for example, for cocompact lattices in $\mathrm{PSL}_2(\mathbb{R})$. However, there is a large class of groups for which this approach will fail. The following theorem is the main result of [GKS25].

Theorem 8 (Gilabert Vio, Kravaris, S. '25)**.** *In the above context, suppose further that $G \curvearrowright S^1$ is topologically non-free, i.e., there is a non-trivial element that acts trivially on an open subset of S^1. Then for any $\mu \in \mathrm{Prob}(G)$ with $H(\mu) < \infty$, the space (S^1, ν) is **not** the Poisson boundary of (G, μ).*

This result covers Thompson's group T. This motivates the following question.

Question 9. *Is asymptotic entropy continuous in Thompson's group T?*

This would be interesting even when restricting asymptotic entropy to the space of probability measures with support exactly equal to a given fixed finite symmetric generating set of T.

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Strict irreducibility of Markov chains and random ergodic theorems for group actions

Elias Zimmermann

(joint work with Pablo Lummerzheim, Felix Pogorzelski)

We consider a stationary and ergodic stochastic process $(\xi_n)_{n=1}^{\infty}$ taking values in a Borel group G and ask under which conditions every pmp action $\{T_g \colon g \in G\}$ of G on a standard probability space (X, μ) gives rise to an ergodic theorem in the sense that the random averages

$$\frac{1}{n} \sum_{i=0}^{n-1} f \circ T_{\xi_i} \cdots \circ T_{\xi_0}$$

converge μ-almost surely to the projection onto the fixspace of $\{T_g\}$. In the case that $(\xi_n)_{n=1}^{\infty}$ is an iid process (the increment process of a random walk on G) the above statement holds true by a classical result of Kakutani, see [5]. In [6]

we generalize this to the setting of Markov chains. We introduce the notion of strict irreducibility and show that Markov chains enjoying this property satisfy the random ergodic theorem for every pmp action of G. This generalizes a result of Bufetov for finite state Markov chains, cf. [2], and contains Kakutani's random ergodic theorem as a special case.

A stationary Markov chain $(\xi_n)_{n=1}^{\infty}$ with values in G is given by a Markov kernel $\pi\colon G \times \mathrm{Bor}(G) \to [0,1]$ together with a π-invariant probability measure p on G. The Markov chain is ergodic if π is irreducible with respect to p. We call a set $B \in \mathrm{Bor}(G)$ *deterministic* if for p-almost all $g \in G$ we have $\pi(g, B) \in \{0, 1\}$. The Markov chain is called *strictly irreducible* if there are only trivial deterministic sets. The notion of strict irreducibility was originally introduced by Bufetov in [2, 3] for finite state Markov chains and arises implicitly in the work of Grigorenko and Bezhaeva-Oseledets, see [4] and[1].

It is well known that in the iid setting the random ergodic theorem holds true not only for measures, which are invariant under the action of G, but also for the more general class of stationary measures, see [5]. In joint work in progress we try to extend this result to the setting of Markov chains.

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Componentwise topologies on group actions, equivalence relations, and groupoids

Ruiyuan Chen

Let $G \curvearrowright X$ be a Borel action of a Polish group G on a standard Borel space X. On each orbit $G \cdot x \subseteq X$, we have a canonical Polish topology, given by the identification $G/G_x \cong G \cdot x$ (recall that stabilizers G_x of Borel actions of Polish groups are closed). We call a subset $B \subseteq X$ **orbitwise open** if its restriction to each orbit $G \cdot x$ is open in this quotient topology, and write $\mathcal{O}_G(X)$ for the collection of all such sets, i.e., the disjoint union of the quotient topologies on each $G \cdot x$.

It is well-known folklore in the theory of Polish group actions that the orbitwise topology $\mathcal{O}_G(X)$, and especially its Borel part $\mathcal{BO}_G(X) := \mathcal{B}(X) \cap \mathcal{O}_G(X)$, plays a pivotal role in the subtle interactions between the Borel and topological

structure. For instance, the idealisticity of Polish group actions [7] boils down to the uniformity of the orbitwise meager ideal. Similarly, the Vaught transform [9] of a subset $B \subseteq X$ is its saturation mod orbitwise meager.

The following two fundamental results make precise the canonicity and meaning of $\mathcal{BO}_G(X)$:

Theorem 1 (Pettis [8])**.** *A Borel set $B \subseteq X$ is in $\mathcal{BO}_G(X)$ iff its preimage under the action $G \times X \to X$ is a countable union of Borel rectangles.*

Theorem 2 (Becker–Kechris [1])**.** *A Borel set $B \subseteq X$ is in $\mathcal{BO}_G(X)$ iff it is open in some compatible Polish topology on X making the action $G \curvearrowright X$ continuous.*

These results were not originally stated in these forms, but are easily seen to be equivalent. In [3], we gave a new proof of the Becker–Kechris theorem that emphasized this role of $\mathcal{BO}_G(X)$.

In [4], further developing the techniques introduced in [3], we proved the following counterpart of the Becker–Kechris theorem, showing that the "orbitwise topology" makes sense even without an *a priori* given Polish group action, and still captures key features of such actions:

Theorem 3. *Let $E \subseteq X^2$ be a Borel equivalence relation on a standard Borel space. Suppose there exists a "uniformly Borel" assignment of Polish topologies $\mathcal{O}([x]_E)$ on each equivalence class $[x]_E$. Then E is Borel bireducible with the orbit equivalence relation of a free Polish group action $G \curvearrowright Y$.*

Here the precise meaning of "uniformly Borel" is that the second-countability of the topologies $\mathcal{O}([x]_E)$ may be witnessed uniformly in x (by countably many Borel sets $U_i \subseteq E$ whose vertical fibers over each $x \in X$ yield a countable basis for $\mathcal{O}([x]_E)$), and moreover nonemptiness of open sets in $\mathcal{O}([x]_E)$ may be detected uniformly in x (for example, one could simply require each basic fiberwise open U_i above to have nonempty fibers).

The proof of theorem 3 works by treating E as a groupoid; the arguments are not simplified in any substantial way by E being an equivalence relation. The more general result is the following:

Theorem 4. *Let $G \rightrightarrows X$ be a standard Borel groupoid. Suppose there exists a right-translation-invariant, "uniformly Borel" assignment of Polish topologies on each fiber of the source map $G \to X$. Then G is Borel equivalent to the action groupoid of a Polish group action $H \curvearrowright Y$.*

The proof of this requires several steps, but the core can be seen as a common generalization of the Pettis theorem 1 and Becker–Kechris theorem 2. Under slightly stronger hypotheses than above, we show that the **componentwise open** Borel sets $\mathcal{BO}_G(G)$, defined as the Borel sets $B \subseteq G$ whose preimage under the multiplication $G \times_X G \to G$ is a countable union of rectangles of fiberwise open Borel sets, turn G into a "sigma-topological groupoid". Standard techniques from point-free topology [6] then allow one to obtain cofinally many second-countable (*quasi*-Polish [5]) groupoid topologies within $\mathcal{BO}_G(G)$. The reduction from Polish

groupoids to Polish group actions follows from previous work [2]; while Baire category arguments are used to improve the quasi-Polish topology to Polish, as well as to obtain the initial strengthened hypotheses required for $\mathcal{BO}_G(G)$ to be well-behaved.

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The ideal Poisson–Voronoi tessellation and some applications

Amanda Wilkens

(joint work with Mikolaj Fraczyk, Sam Mellick, and Matteo D'Achille, Jan Grebik, Ali Khezeli, Konstantin Recke)

We explore some recent results involving a new random object, the ideal Poisson–Voronoi tessellation (IPVT), which inherently detects some of the geometry of its underlying space. In joint work with Fraczyk and Mellick, we use the IPVT to prove a deterministic result on the relationship between the volume of a manifold and the number of generators of its fundamental group. In joint work with D'Achille, Grebik, Khezeli, and Recke, we use the IPVT to prove a result on the uniqueness threshold in a related percolation model. In both cases, we use the IPVT's wild behavior on spaces with a certain geometry (specifically, products of trees and symmetric spaces of higher rank semisimple Lie groups), originally discovered via a dynamical argument. We discuss this behavior and give some intuition for the main results.

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On double cosets in Borel complexity

Antoine Poulin

In this talk, we give a short review of Borel complexity, especially applied to recent interests in the complexity of Archimedean orders. This is the subject of yet unreleased work.

Borel complexity is the abstract study of classification problems, using formalism from descriptive set theory. A specific case of this is when G is a Polish group and $G \times X \to X$ is a Borel action on a standard Borel space. In this case, we denote E_G^X for the induced orbit equivalence relation on X, i.e.

$$x \, E_G^X \, y \iff \exists g \in G, y = g.x.$$

The notion of Borel reduction allows for multiple benchmarks for the complexity of a relation, such as smoothness, essential hyperfiniteness and essential treeability, i.e., admitting a Borel reduction to equality, to a free Borel action of $\mathbb{Z}$ or to a free Borel action of $\mathbb{F}_2$ (respectively). For actions of countable groups, we omit the word "essential". See [7, 10] for standard references.

Recently, a lot of attention has been given on the complexity of orders on groups, see [2, 3, 4, 5, 6, 9, 12, 13].

Definition 1. An **Archimedean order** on a countable group Γ is a strict linear order $<$ such that:

- $<$ is left-invariant: for all $g, h, k \in \Gamma$,
$$h < k \iff gh < gk.$$
- $<$ satisfies Archimedes' property: for all positive elements $g, h > \mathrm{id}$, there is $n \in \mathbb{N}$ with $g < h^n$.

There is a Polish space of Archimedean orders $\mathrm{Ar}(\Gamma)$ and an action of $\mathrm{Aut}(\Gamma)$ on $\mathrm{Ar}(\Gamma)$.

A classical theorem of Hölder states that if Γ is Archimedean-orderable, then there is an order-preserving injective homomorphism $\phi : \Gamma \to \mathbb{R}$. Thus, we focus on groups of the form R^n, where $\mathbb{Z} \subset R \subset \mathbb{Q}$, typically $\mathbb{Z}, \mathbb{Q}$ or $\mathbb{Z}[p^{-1}]$.

In [2], Calderoni, Marker, Motto-Ros and Shani initiated the study of complexity of Archimedean orders. They prove that $E^{\mathrm{Ar}\,\mathbb{Q}^2}_{\mathrm{GL}_2(\mathbb{Q})}$ is not smooth. A deeper look at their argument shows that $E^{\mathrm{Ar}\,\mathbb{Z}^2}_{\mathrm{GL}_2(\mathbb{Z})}$ is not smooth but hyperfinite. They ask the following questions:

Question 2. [2] *Is $E^{\mathrm{Ar}\,\mathbb{Q}^2}_{\mathrm{GL}_2(\mathbb{Q})}$ hyperfinite? What is the complexity of $E^{\mathrm{Ar}\,\mathbb{Q}^n}_{\mathrm{GL}_n(\mathbb{Q})}$ for $n \geq 3$?*

It was shown by the author in [12] that $E^{\mathrm{Ar}\,\mathbb{Z}^n}_{\mathrm{GL}_n(\mathbb{Z})}$ is not hyperfinite for $n \geq 3$ and not treeable for $n \geq 4$, using the theory of Zimmer-amenability and work of [11]. This results also holds for $\mathbb{Q}$, giving a strong anticlassification answer to the latter half of Question 2. However, the matter of treeability for $n = 3$ was still out of reach, as well as the hyperfiniteness for $n = 2$. Towards this, the author asked the following:

Question 3. [12] *Let p be a prime number. Is $E^{\mathrm{Ar}\,\mathbb{Z}^2[p^{-1}]}_{\mathrm{GL}_2(\mathbb{Z}[p^{-1}])}$ hyperfinite? Is $E^{\mathrm{Ar}\,\mathbb{Z}^3}_{\mathrm{GL}_3(\mathbb{Z})}$ treeable?*

In this talk, we answer both of these questions in the negative. In fact, we show neither is treeable. This also implies $E^{\mathrm{Ar}\,\mathbb{Q}^2}_{\mathrm{GL}_2(\mathbb{Q})}$ is not treeable, answering Question 2 and 3. To do so, we exploit a trick on double cosets, which we phrase in term of Borel complexity:

Lemma 4. *If G is a Polish group and $H, H' < G$ are closed subgroups, then $E^{G/H'}_{H}$ and $E^{G/H}_{H'}$ are classwise Borel isomorphic.*

This is known to experts and appear in the literature a few times, such as in [14, Proof of Prop. 2.2.2 and 3.2.13], [1, Proof of Prop. 2.4] and [11, Proof of Prop. 3.3 and Prop 5.1]. Coupled with an antitreeability criterion for groups inspired by [8], this allows us to deduce the following:

Theorem 5 (P. '26+)**.** *Let p be a prime number. The orbit equivalence relations of the following actions are not treeable:*

- $\mathrm{PSL}_2(\mathbb{Z}[p^{-1}])$ *acting on* $\mathbb{P}^1(k)$*, for* $k = \mathbb{R}, \mathbb{Q}_p$*,*
- $\mathrm{SL}_2(\mathbb{Z}[p^{-1}])$ *acting on* $\mathbb{R}^2$*,*
- $\mathrm{SL}_3(\mathbb{Z})$ *acting on* $\mathbb{P}^2(\mathbb{R})$ *or* $\mathbb{R}^3$*,*
- $\mathrm{PSL}_2(\mathbb{Q})$ *acting on* $\mathbb{P}^1(k)$*, for* $k = \mathbb{R}, \mathbb{Q}_p$*,*
- $\mathrm{SL}_2(\mathbb{Q})$ *acting on* $\mathbb{R}^2$*,*
- $\mathrm{GL}_2(\mathbb{Z}[p^{-1}])$ *acting on* $\mathrm{Ar}(\mathbb{Z}[p^{-1}]^2)$*,*
- $\mathrm{GL}_2(\mathbb{Q})$ *acting on* $\mathrm{Ar}(\mathbb{Q}^2)$*,*
- $\mathrm{GL}_3(\mathbb{Z})$ *acting on* $\mathrm{Ar}(\mathbb{Z}^3)$*.*

These actions are chosen for their relevance in the study of the Archimedean orders. The formal link between them follows from Hölder's theorem, as explained in [12].

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Critical exponents and growth of injectivity radius in quotients of hyperbolic groups

Ilya Gekhtman

Let X be a geodesic Gromov hyperbolic space whose isometry group contains a uniform lattice Γ. Let H be a discrete subgroup of isometries of X with critical exponent $\delta(H)$ less than $\delta(\Gamma)/2$. We show that the injectivity radius of X/H grows linearly along almost every geodesic in X.

Local entropy theory and combinatorics

Hanfeng Li
(joint work with Kairan Liu)

This talk is based on results in [9]. Let $\Gamma \curvearrowright X$ be a continuous action of a countably infinite group on a compact metrizable space X.

The group Γ is called *amenable* if, for any nonempty finite set $K \subseteq \Gamma$ and any $\varepsilon > 0$, there is a nonempty finite set $F \subseteq \Gamma$ such that $|KF\Delta F| < \varepsilon|F|$. When Γ is amenable, one has the *topological entropy* $\mathrm{h}_{\mathrm{top}}(X) \in [0, \infty]$ defined as follows. For any finite open cover $\mathcal{U}$ of X, denote by $N(\mathcal{U})$ the minimal number of elements of $\mathcal{U}$ needed to cover X. For finite open covers $\mathcal{U}$ and $\mathcal{V}$ of X, the join $\mathcal{U} \vee \mathcal{V}$ is the cover $\{U \cap V : U \in \mathcal{U}, V \in \mathcal{V}\}$ of X. For any finite open cover $\mathcal{U}$ of X, there is a value $\mathrm{h}_{\mathrm{top}}(\mathcal{U}) \in [0, \infty)$ satisfying that for any $\delta > 0$, there are some nonempty finite set $K \subseteq \Gamma$ and some $\varepsilon > 0$ so that one has $\left| \frac{1}{|F|} \log N(\bigvee_{s \in F} s^{-1}\mathcal{U}) - \mathrm{h}_{\mathrm{top}}(\mathcal{U}) \right| < \delta$ for every nonempty finite set $F \subseteq \Gamma$ satisfying $|KF\Delta F| < \varepsilon|F|$. Then $\mathrm{h}_{\mathrm{top}}(X)$ is defined as the supremum of $\mathrm{h}_{\mathrm{top}}(\mathcal{U})$ for $\mathcal{U}$ ranging over finite open covers of X.

The local entropy theory was initiated by Blanchard. See [6, 5] for surveys on the theory. One of the key notions in this theory is that of IE-tuples. Let $k \in \mathbb{N}$ and let $\boldsymbol{A} = (A_1, \dots, A_k)$ be a k-tuple of subsets of X. We say a nonempty finite set $F \subseteq \Gamma$ is an *independence set* for $\boldsymbol{A}$, if for every map $\sigma : F \to [k] = \{1, \dots, k\}$ there is an $x_\sigma \in X$ such that $sx_\sigma \in A_{\sigma(s)}$ for all $s \in F$. The *independence density* of $\boldsymbol{A}$ is defined as the largest $c \geq 0$ satisfying that for every nonempty finite set $F \subseteq \Gamma$ there is a $J \subseteq F$ with $|J| \geq c|F|$ so that J is an independence set for $\boldsymbol{A}$. A k-tuple $\boldsymbol{x} = (x_1, \dots, x_k) \in X^k$ is called an *IE-tuple* if for every product neighborhood $U_1 \times \cdots \times U_k$ of $\boldsymbol{x}$ in X^k, the independence density of the tuple $(U_1, \dots, U_k)$ is strictly positive. We write $\mathrm{IE}_k(X)$ for the set of all IE-k-tuples of X. This is a Γ-invariant closed subset of X^k. One of the classical results in the local entropy theory is that when Γ is amenable, $\mathrm{h}_{\mathrm{top}}(X) > 0$ if and only if there is an $(x_1, x_2) \in \mathrm{IE}_2(X)$ with $x_1 \neq x_2$.

Denote by $\mathcal{M}(X)$ the space of all Borel probability measures on X. This is a compact metrizable convex space under the weak*-topology, and the Γ-action on X induces a continuous affine Γ-action on $\mathcal{M}(X)$. For $\Gamma = \mathbb{Z}$, when $\mathrm{h}_{\mathrm{top}}(X) > 0$, one has $\mathrm{h}_{\mathrm{top}}(\mathcal{M}(X)) = \infty$ [3]. This indicates that the dynamics of $\Gamma \curvearrowright \mathcal{M}(X)$ is much more complicated than that of $\Gamma \curvearrowright X$. In 1995, Glasner and Weiss showed the striking result that for amenable Γ, when $\mathrm{h}_{\mathrm{top}}(\mathcal{M}(X)) = 0$, one has $\mathrm{h}_{\mathrm{top}}(X) = 0$ [7].

In terms of IE-pairs, the Glasner–Weiss result says that when Γ is amenable, $\mathrm{IE}_2(X)$ contains non-diagonal pairs if and only if $\mathrm{IE}_2(\mathcal{M}(X))$ contains non-diagonal pairs. Our main result describes the precise relation between $\mathrm{IE}_k(X)$ and $\mathrm{IE}_k(\mathcal{M}(X))$ for every $k \in \mathbb{N}$. First of all, one may treat X as a closed Γ-invariant subset of $\mathcal{M}(X)$ via identifying each $x \in X$ with the Dirac measure δ_x at x. Then $\mathrm{IE}_k(X) \subseteq X^k \subseteq \mathcal{M}(X)^k$ and $\mathrm{IE}_k(\mathcal{M}(X)) \subseteq \mathcal{M}(X)^k$. Note that $\mathcal{M}(X)^k$ is a compact convex set in the dual of the direct sum of k copies of $C(X)$, where $C(X)$ is the Banach space of all continuous $\mathbb{R}$-valued functions on X. For each

$\mu \in \mathcal{M}(X^k)$ and $j \in [k]$, we denote by $\mu^{(j)}$ the push-forward of μ in $\mathcal{M}(X)$ under the j-th coordinate map $X^k \to X$.

Theorem 1. *Assume that Γ is amenable. For each $k \in \mathbb{N}$, the following hold:*

(1) $\mathrm{IE}_k(\mathcal{M}(X))$ *is the closed convex hull of* $\mathrm{IE}_k(X)$ *in* $\mathcal{M}(X)^k$.
(2) $\mathrm{IE}_k(X) = \mathrm{IE}_k(\mathcal{M}(X)) \cap X^k$ *is the set of extreme points of* $\mathrm{IE}_k(\mathcal{M}(X))$.
(3) $\mathrm{IE}_k(\mathcal{M}(X)) = \{(\mu^{(1)}, \dots, \mu^{(k)}) : \mu \in \mathcal{M}(\mathrm{IE}_k(X))\}$.

The Glasner–Weiss result follows from the case $k = 2$ of Theorem 1. For each $N \in \mathbb{N}$, denote by $\mathcal{M}_N(X)$ the set of $\mu \in \mathcal{M}(X)$ whose support has at most N points. This is a closed Γ-invariant subset of $\mathcal{M}(X)$, and one has $\mathrm{IE}_k(\mathcal{M}_N(X)) \subseteq \mathrm{IE}_k(\mathcal{M}(X))$ trivially for every $k \in \mathbb{N}$. The naive guess $\mathcal{M}_N(X)^k \cap \mathrm{IE}_k(\mathcal{M}(X)) = \mathrm{IE}_k(\mathcal{M}_N(X))$ fails for every amenable Γ. From Theorem 1 one has the following consequence.

Corollary 2. *Assume that Γ is amenable. Let $k \in \mathbb{N}$. For any $N_1, \dots, N_k \in \mathbb{N}$, setting $N = \prod_{j=1}^k N_j$, one has*

$$(\mathcal{M}_{N_1}(X) \times \cdots \times \mathcal{M}_{N_k}(X)) \cap \mathrm{IE}_k(\mathcal{M}(X)) \subseteq \mathcal{M}_k(\mathcal{M}_N(X)).$$

The main tool for the proof of Theorem 1 is a new combinatorial lemma. For a finite set Z and $k \in \mathbb{N}$, we denote by $\mathscr{R}(Z, k)$ the set of maps $\psi : Z \to [k]$ which are as equidistributed as possible in the sense that $||\psi^{-1}(j)| - |Z|/k| < 1$ for all $j \in [k]$. For a function $f : Z \to \mathbb{R}$ on a finite set Z and $1 \le p \le \infty$, we write $\|f\|_p$ for the standard ℓ_p-norm of f, i.e. $\|f\|_p = \left(\sum_{z \in Z} |f(z)|^p\right)^{1/p}$ when $1 \le p < \infty$ and $\|f\|_p = \max_{z \in Z} |f(z)|$ when $p = \infty$. The following is the key combinatorial lemma.

Lemma 3. *Let $k \in \mathbb{N}$ and $0 < r < R \le C$. Also let $1 < p \le \infty$ and $1 \le q < \infty$ such that $1/p + 1/q = 1$. Then there exist $c, N > 0$ and a finite set $T \subseteq \mathbb{R}^k$ depending only on k, r, R, C and p such that the following hold:*

(1) $\frac{1}{k} \sum_{j \in [k]} t_j \ge r$ *for every $(t_1, \dots, t_k) \in T$, and*
(2) *for any finite set Z with $|Z| \ge N$, if for each $\psi \in \mathscr{R}(Z, k)$ we take an $f_\psi : Z \to \mathbb{R}$ with $\|f_\psi\|_p \le C$ and $\frac{1}{|Z|^{1/q}} \sum_{z \in Z} f_\psi(z) \ge R$, then there are some $J \subseteq Z$ with $|J| \ge c|Z|$ and $(t_1, \dots, t_k) \in T$ such that every map $\sigma : J \to [k]$ extends to a $\psi \in \mathscr{R}(Z, k)$ so that $f_\psi(z) \ge t_j |Z|^{-1/p}$ for all $j \in [k]$ and $z \in \sigma^{-1}(j)$.*

Answering a question of Erdős, the Sauer-Perles-Shelah lemma [11, 12] is a classical result in the theory of extremal combinatorics. The asymptotic form of the Karpovsky-Milman generalization [8] of this lemma has wide applications in dynamical systems, and says that for any integer $k \ge 2$ and any $\lambda \in (k-1, k]$, there is a $c > 0$ depending only on k and λ such that for all $n \in \mathbb{N}$ and $S \subseteq [k]^{[n]}$ with $|S| \ge \lambda^n$, there exists a $J \subseteq [n]$ satisfying that $|J| \ge cn$ and $S|_J = [k]^J$. The statement of Lemma 3 has some similarity with but is much more complicated than this asymptotic form. The proof of Lemma 3 uses this asymptotic form.

In fact the proof of Theorem 1 uses only the case $p = \infty$ of Lemma 3. The full range $1 < p \le \infty$ of Lemma 3 enables us to obtain an application in the local

theory of Banach spaces. This theory, also called asymptotic geometric analysis, studies Banach spaces via their finite-dimensional subspaces or quotients [1, 2, 10]. For two finite-dimensional real Banach spaces V and W of the same dimension, we denote by $d_{\mathrm{BM}}(V,W)$ the (multiplicative) *Banach-Mazur distance* between them, defined as the infimum of $\|\Phi\| \cdot \|\Phi^{-1}\|$ for Φ ranging over linear isomorphisms between them. For a constant $C \geq 1$, we say V and W are *C-isomorphic* if $d_{\mathrm{BM}}(V,W) \leq C$. Given $C > 1$ and $1 \leq q' \leq \infty$, using the concentration of measure phenomenon on the Euclidean spheres, Figiel, Lindenstrauss and Milman found the asymptotic bounds for m and n such that ℓ_2^n is C-isomorphic to a linear subspace of $\ell_{q'}^m$ [4]. Then showed that the optimal value for n is $c \log m$ when $q' = \infty$, $cm^{2/q'}$ when $2 \leq q' < \infty$, and cm when $1 \leq q' \leq 2$. Using Lemma 3 we prove the following result.

Theorem 4. *Let $C > 1$ and $1 \leq q < \infty$. Then there is some $c > 0$ depending only on C and q such that if ℓ_q^n is C-isomorphic to a linear subspace of ℓ_∞^m for some $m \geq 2$, then $n \leq c \log m$.*

The case $1 \leq q \leq 2$ of Theorem 4 follows from the results of Figiel-Lindenstrauss-Milman. When $2 < q < \infty$, the results of Figiel-Lindenstrauss-Milman yield a bound $n \leq c(\log m)^{q/2}$, which is weaker than the bound in Theorem 4.

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Rigidity of graph products of groups

Camille Horbez

This talk is based on joint works with Amandine Escalier [7, 8] and Adrian Ioana [11].

The notion of graph products of groups was introduced by Green in [10]. Let Γ be a finite simple graph (i.e. with no loop-edge and no multiple edges between two vertices) with at least two vertices, and let $(G_v)_{v\in V\Gamma}$ be a family of non-trivial groups, indexed by the vertices of Γ. The *graph product* over Γ with vertex groups $(G_v)_{v\in V\Gamma}$ is the group

$$G := (*_{v\in V\Gamma} G_v)/\langle\langle [G_v, G_w] = 1 \text{ when } vw \in E\Gamma\rangle\rangle.$$

Graph products encompass free products (when Γ is edgeless), direct products (when Γ is complete), right-angled Artin groups A_Γ (when all vertex groups are isomorphic to $\mathbb{Z}$), and right-angled Coxeter groups (when all vertex groups are isomorphic to $\mathbb{Z}/2\mathbb{Z}$).

The talk was centered on the classification of graph products of groups, and their rigidity properties, from three perspectives: algebraic (group isomorphisms), measurable (orbit and measure equivalence) and operator algebraic (W^*-equivalence).

Group isomorphisms. A theorem of Droms [6] asserts that two right-angled Artin groups A_Γ and A_Λ are isomorphic if and only if the graphs Γ and Λ are isomorphic. A theorem of Laurence [12], confirming a conjecture proved in various cases by Servatius [13], gives a generating set for $\mathrm{Aut}(A_\Gamma)$, consisting of four types of automorphisms: graph automorphisms, inversions (sending one of the generators, associated to a vertex v of Γ, to its inverse v^{-1}), transvections (sending v to vw, provided the link of v is contained in the star of w), and partial conjugations (conjugating a connected component of $\Gamma \setminus \mathrm{st}(v)$ by v). Here the *link* $\mathrm{lk}(v)$ is the subgraph of Γ spanned by all vertices that are adjacent to v, and its *star* $\mathrm{st}(v)$ is the subgraph spanned by v and $\mathrm{lk}(v)$. Note in particular that $|\mathrm{Out}(A_\Gamma)| < +\infty$ if and only if Γ has no transvection (i.e. no two distinct vertices v, w with $\mathrm{lk}(v) \subseteq \mathrm{st}(w)$) and no separating star – this happens for instance whenever Γ is an n-gon with $n \geq 5$.

A group can sometimes split as a graph product in two different ways. Indeed, assume that G splits as a graph product over Γ, and that some vertex group G_v splits as a graph product over a graph Γ_v. Then G also splits as a graph product over the graph $\tilde{\Gamma}$ obtained from Γ by replacing v by Γ_v, and joining every vertex of Γ_v to every vertex of Γ contained in $\mathrm{lk}(v)$.

For this reason, when considering the isomorphism problem for graph products, one should only consider decompositions over the minimal possible graph. We say that a subgraph $\Lambda \subseteq \Gamma$ is *collapsible* if for any two vertices $v, w \in V\Lambda$, we have $\mathrm{lk}(v) \cap (\Gamma \setminus \Lambda) = \mathrm{lk}(w) \cap (\Gamma \setminus \Lambda)$. In the above example, the subgraph $\Gamma_v \subseteq \tilde{\Gamma}$ is collapsible. A graph is *strongly reduced* if it does not contain any proper collapsible subgraph on at least two vertices.

Theorem 1 (Escalier–Horbez [8]). *Let Γ, Λ be strongly reduced finite simple graphs, with $|V\Gamma|, |V\Lambda| \geq 3$. Let G be a graph product over Γ with non-trivial vertex groups $(G_v)_{v\in V\Gamma}$, and H be a graph product over Λ with non-trivial vertex groups $(H_w)_{w\in V\Lambda}$.*

Then $G \simeq H$ if and only if there exists a graph isomorphism $\theta : \Gamma \to \Lambda$ such that for every $v \in V\Gamma$, one has $G_v \simeq H_{\theta(v)}$.

We are then interested in the classification of graph products for weaker notions than isomorphism.

Orbit equivalence. Two countable groups G, H are *orbit equivalent* (denoted by $G \sim_{\mathrm{OE}} H$) if they admit free measure-preserving actions on a standard probability space X with the same orbits.

Theorem 2 (Escalier–Horbez [7]). *Let Γ, Λ be finite simple graphs, not reduced to one vertex, with no transvection and no separating star. Let G be a graph product over Γ with countably infinite vertex groups $(G_v)_{v\in V\Gamma}$, and H be a graph product over Λ with countably infinite vertex groups $(H_w)_{w\in V\Lambda}$.*

Then $G \sim_{\mathrm{OE}} H$ if and only if there exists a graph isomorphism $\theta : \Gamma \to \Lambda$ such that for every $v \in V\Gamma$, one has $G_v \sim_{\mathrm{OE}} H_{\theta(v)}$.

This is also equivalent to G and H being *measure equivalent*, in the sense of having free, measure-preserving commuting actions on a standard measure space, with finite measure fundamental domains.

W^*-equivalence. Given a countable group G, consider the left regular representation $\lambda : G \to \mathcal{U}(\ell^2 G)$. The *group von Neumann algebra* $L(G)$ is the von Neumann subalgebra of $\mathcal{B}(\ell^2 G)$ generated by $\lambda(G)$. It is a *factor* (i.e. its center is reduced to the scalars) if and only if G is *ICC* (i.e. the conjugacy class of every non-trivial element of G is infinite). In joint work with Adrian Ioana, we obtained the following two classification theorems for von Neumann algebras of graph products of groups.

Theorem 3 (Horbez–Ioana [11]). *Let Γ, Λ be two transvection-free right-angled Artin groups. Then $L(A_\Gamma) \simeq L(A_\Lambda)$ if and only if $A_\Gamma \simeq A_\Lambda$.*

Theorem 4 (Horbez–Ioana [11]). *Let Γ, Λ be two finite simple graphs of girth at least 5, with no vertices of valence 0 or 1. Let G be a graph product over Γ with non-trivial ICC vertex groups $(G_v)_{v\in V\Gamma}$, and let H be a graph product over Λ with non-trivial ICC vertex groups $(H_w)_{w\in V\Lambda}$.*

Then $L(G) \simeq L(H)$ if and only if there is a graph isomorphism $\theta : \Gamma \to \Lambda$ such that for every vertex $v \in V\Gamma$, one has $L(H_{\theta(v)}) \simeq L(G_v)$.

If one further assumes that Γ has no separating star, then we get a description of the automorphism group of $L(G)$, namely:

$$\mathrm{Aut}(L(G)) \simeq (\oplus_{v\in V\Gamma} \mathrm{Out}(L(G_v))) \rtimes \mathrm{Aut}_{L(G)}(\Gamma),$$

where $\mathrm{Aut}_{L(G)}(\Gamma) \subseteq \mathrm{Aut}(\Gamma)$ is the subgroup preserving the isomorphism types of the von Neumann algebras $L(G_v)$. This gives a von Neumann analogue to a theorem of Genevois–Martin [9] regarding the automorphism group of G.

Theorems 3 and 4 cover two different settings: in the first the vertex algebras $L(G_v)$ are abelian, in the second they are II_1 factors. In fact, Caspers and Fima introduced a notion of graph product of tracial von Neumann algebras [2], and our results are phrased in this larger framework. In particular, Theorem 4 is about graph products of II_1 factors. The proofs of these theorems are in the framework of Popa's deformation/rigidity theory. Other results regarding the classification and rigidity of graph product von Neumann algebras include [3, 4, 1, 5], among others.

A word on the proofs. The proofs of all theorems presented above exploit the tension between the presence of direct products inside G, and the fact that G acts on trees, namely the Bass–Serre trees of the amalgamated free product decompositions

$$G = G_{\mathrm{st}(v)} *_{G_{\mathrm{lk}(v)}} G_{\Gamma\setminus\{v\}},$$

with v varying over $V\Gamma$. This enables to "locate" maximal direct products inside graph products. Under suitable assumptions on the graphs Γ, Λ (including transvection-freeness), this is key for proving that any isomorphism (or orbit equivalence, or W^*-isomorphism) between G and H has to "send" (in an appropriate sense) any vertex group to a conjugate of a vertex group.

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Reporter: Miguel Donoso-Echenique

Participants

Prof. Dr. Miklos Abert
HUN-REN Alfred Renyi Institute of Mathematics
P.O. Box 127
1364 Budapest
HUNGARY

Dr. Nathalie Aubrun
Universite Paris-Saclay
Laboratoire de Recherche en Informatique
Bat 650 Ada Lovelace
91405 Orsay
FRANCE

Pénélope Azuelos
Department of Mathematics
University of Bristol
Woodland Road
Bristol BS8 1UG
UNITED KINGDOM

Dr. Sebastián Barbieri
Departamento de matemática y ciencia de la computación
Universidad de Santiago de Chile
Avda. Bernardo O'Higgings 3363
9160000 Estación Central Santiago
CHILE

Dr. Anton Bernshteyn
Department of Mathematics
UCLA
Math Sciences Building 6363
520 Portola Plaza
Los Angeles, CA 90095-1555
UNITED STATES

Sasha Bontemps
Dept. de Mathématiques, U.M.P.A.
École Normale Superieure de Lyon
46, Allee d'Italie
69364 Lyon Cedex 07
FRANCE

Dr. Matthew Bowen
Mathematical Institute
Oxford University
Andrew Wiles Building
Woodstock Road
Oxford OX2 6GG
UNITED KINGDOM

Anna Cascioli
Mathematisches Institut
Universität Münster
Einsteinstr. 62
48149 Münster
GERMANY

Kunal Chawla
Department of Mathematics
Princeton University
Fine Hall
Washington Road
Princeton, NJ 08544-1000
UNITED STATES

Dr. Ruiyuan Chen
Institute of Mathematics
University of Warsaw
ul. Banacha 2
02-097 Warszawa
POLAND

Clinton T. Conley
Department of Mathematical Sciences
Carnegie Mellon University
7121 Wean Hall
5000 Forbes Avenue
Pittsburgh, PA 15213
UNITED STATES

Prof. Dr. Maria Isabel Cortez
Facultad de Matemáticas
Pontificia Universidad Católica de Chile
Avda. Vicuña Mackenna 4860
Santiago 8940000
CHILE

Miguel Donoso-Echenique
Kolde-Ring 60
Mathematisches Institut
Universität Münster
Einsteinstrasse 62
48149 Münster
GERMANY

Amandine Escalier
Université Claude Bernard Lyon 1
43 Blvd. du 11. Novembre 1918
69622 Villeurbanne Cedex
FRANCE

Joshua Frisch
Department of Mathematics
University of California, San Diego
9500 Gilman Drive
La Jolla, CA 92093-0112
UNITED STATES

Prof. Dr. Alex Furman
Department of Mathematics, Statistics
and Computer Science
University of Illinois at Chicago
851 S. Morgan Street
Chicago IL 60607-7045
UNITED STATES

Prof. Dr. Damien Gaboriau
UMPA, UMR 5669, CNRS
École Normale Supérieure de Lyon
46, allée d'Italie
69364 Lyon Cedex 07
FRANCE

Prof. Dr. Shirly Geffen
Mathematisches Institut
Universität Münster
Einsteinstr. 62
48149 Münster
GERMANY

Boglárka Gehér
Alfred Renyi Institute of Mathematics
Hungarian Academy of Sciences
P.O. Box 127
1364 Budapest
HUNGARY

Prof. Dr. Ilya Gekhtman
Department of Mathematics
Technion – Israel Institute of Technology
Amado Building
Haifa 32000
ISRAEL

Martín Gilabert Vio
Institut Camille Jordan
Batiment Braconnier
43 boulevard du 11 novembre 1918
69622 Villeurbanne Cedex
FRANCE

Prof. Łukasz Grabowski
Augustusplatz
Mathematisches Institut
Universität Leipzig
04103 Leipzig
GERMANY

Dr. Camille Horbez
Laboratoire de Mathématiques
Université Paris-Saclay
Batiment 307
91405 Orsay Cedex
FRANCE

Prof. Dr. Cyril Houdayer
Département de mathématiques et applications
École normale supérieure
45 rue d'Ulm
75230 Paris Cedex
FRANCE

Christopher Karpinski
Department of Mathematics & Statistics
McGill University
805, Sherbrooke Street West
Montréal QC H3A 2K6
CANADA

Prof. Dr. David Kerr
Mathematisches Institut
Universität Münster
Einsteinstraße 62
48149 Münster
GERMANY

Anett Kocsis
Alfred Renyi Institute of Mathematics
Hungarian Academy of Sciences
13-15 Realtanoda u.
1053 Budapest
HUNGARY

Prof. Dr. François Le Maître
Institut de Mathématiques
Université de Bourgogne
BP 47970
21078 Dijon Cedex
FRANCE

Prof. Dr. Hanfeng Li
College of Mathematics and Statistics
Chongqing University
Chongqing 401331
CHINA

Samuel Mellick
Institute of Mathematics
Jagiellonian University
Lojasiewicza 6
30-348 Kraków
POLAND

Prof. Dr. Amos Nevo
Department of Mathematics
Technion – Israel Institute of Technology
Haifa 32000
ISRAEL

Hanna Oppelmayer
Institut für Mathematik
Universität Innsbruck
Technikerstr. 13
6020 Innsbruck
AUSTRIA

Spyridon Petrakos
Department of Mathematical Sciences
University of Gothenburg
Chalmers Tvärgata 3
41296 Göteborg
SWEDEN

Antoine Poulin
Dept. of Mathematics and Statistics
McGill University
805, Sherbrooke Street West
Montréal QC H3A 2K6
CANADA

Konstantin Recke
Mathematical Institute
University of Oxford
Andrew Wiles Building
Radcliffe Observatory Quarter
Woodstock Road
Oxford OX2 6GG
UNITED KINGDOM

Dr. Marcin Sabok
Department of Mathematics and Statistics
McGill University
Burnside Hall, Rm.#916
805 Sherbrooke Street West
Montréal, Québec H3A 0B9
CANADA

Forte Shinko
Department of Mathematics
University of California, Berkeley
970 Evans Hall
Berkeley CA 94720-3840
UNITED STATES

Dr. Eduardo Silva
Institut für Mathematische Stochastik
Universität Münster
Orléans-Ring 10
48149 Münster
GERMANY

Dr. Rachel Skipper
Department of Mathematics
University of Utah
155 South 1400 East
Salt Lake City, UT 84112-0090
UNITED STATES

Dr. Grigory Terlov
Department of Statistics and Operations Research
University of North Carolina at Chapel Hill
Hanes Hall
Chapel Hill NC 27599-3250
UNITED STATES

Prof. Dr. Andreas B. Thom
Institut für Geometrie
Fakultät für Mathematik
Technische Universität Dresden
Postfach 01062
01062 Dresden
GERMANY

Prof. Dr. Todor Tsankov
Institut Camille Jordan
Université Claude Bernard - Lyon 1
43, boulevard du 11 novembre 1918
69622 Villeurbanne Cedex
FRANCE

Prof. Dr. Anush Tserunyan
McGill University
Dept. of Mathematics and Statistics
805 Sherbrooke St W
Montréal QC H3A 2K6
CANADA

Amanda Wilkens
Department of Mathematical Sciences
Carnegie Mellon University
6113 Wean Hall
Pittsburgh PA 15213
UNITED STATES

Konrad Wróbel
University of Texas in Austin
Department of Mathematics
2515 Speedway
Austin 78712
UNITED STATES

Prof. Dr. Tianyi Zheng
Department of Mathematics
University of California, San Diego
9500 Gilman Drive, #0112
La Jolla, San Diego CA 92093
UNITED STATES

Elias Zimmermann
Mathematisches Institut
Universität Leipzig
Postfach 10 09 20
04009 Leipzig
GERMANY

Dr. Jenna Zomback
Department of Mathematics
Amherst College
Amherst, MD 20742-4015
UNITED STATES