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Complex Geometry and Dynamical Systems

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Abstract. The workshop focused on recent developments in complex geometry, several complex variables, and holomorphic dynamics. The topics of the talks included complex Monge–Ampère and Hessian equations, pluripotential theory and the theory of currents, Bergman and Szegő kernels together with their asymptotic and spectral aspects, as well as complex dynamics in higher dimensions. Further themes were random and asymptotic questions in complex geometry, hyperbolicity, non-Kähler geometry, period maps, moduli problems, and interactions with birational geometry, arithmetic geometry, and mathematical physics.

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Introduction by the Organizers

The workshop Complex Geometry and Dynamical Systems was organized by Simion Filip (Chicago), Jaehyun Hong (Daejeon), George Marinescu (Cologne), and Valentino Tosatti (New York). It continued and expanded the scientific program of the 2022 Oberwolfach meeting with the same title, whose aim had been to bring together researchers in complex geometry, several complex variables, holomorphic dynamics, and neighboring areas. As in the first edition, the 2026 workshop focused on recent developments at the interface of complex analysis, complex differential and algebraic geometry, pluripotential theory, and dynamical systems, with special emphasis on the interplay between these subjects and on analytic techniques with applications to geometry and dynamics.

Complex geometry continues to be one of the most active areas of contemporary mathematics, both because of its internal developments and because of its deep connections with algebraic geometry, arithmetic geometry, foliation theory, and mathematical physics. The proposal for this workshop singled out four principal themes: complex Monge–Ampère equations and related complex PDEs; Bergman and Szegő kernels and their applications; pluripotential theory and the theory of currents; and complex dynamics in higher dimensions. It also highlighted several emerging directions, among them zeros of random holomorphic sections, hyperbolicity questions, Levi-flat geometry, functional transcendence, and interactions with non-Archimedean and arithmetic geometry.

Reflecting this broad perspective, the scientific program devoted considerable attention to asymptotic analytic methods in complex geometry. Berndtsson discussed valuations on convex bodies and their relation to Kähler geometry. Ma presented a work on Bergman kernels on non-compact manifolds and Poincaré series, while Savale treated a quantitative Weyl law for Toeplitz operators. Zhu explained semiclassical heat kernel asymptotics on complex manifolds with boundary, giving a heat-kernel approach to holomorphic Morse inequalities in the boundary setting. Finski gave an analytic perspective on Kodaira–Iitaka dimension and multiplicity, illustrating once more how asymptotic analytic tools can be used to study birational invariants. These contributions aligned particularly well with the Bergman kernel, spectral, and L^2-theoretic themes emphasized in the proposal.

A second major theme of the workshop concerned pluripotential theory, positive currents, and complex Monge–Ampère equations. Kołodziej reported on complex Hessian equations on Hermitian manifolds and on new developments concerning Hessian measures and polar sets. Vu discussed the relation between higher Lelong numbers and full Monge–Ampère mass, motivated in part by questions around singularities of plurisubharmonic functions and Demailly's program. Viet-Anh Nguyen presented a work on Siu's analyticity theorem for positive pluriharmonic currents, obtaining an important extension of classical analyticity phenomena to a broader setting. Păun discussed plurisubharmonic upper envelopes and nef divisors with isolated log-canonical centres, in connection with extension problems and birational geometry. Koike presented work on the pluripotential geometry of semi-positive effective divisors of numerical dimension one. Altogether, these contributions highlighted the continuing vitality of pluripotential methods and the theory of currents, one of the principal themes of the workshop from the outset.

The workshop also featured several talks on projective and Kähler geometry in a broader sense. Hwang discussed results on fundamental forms and infinitesimal symmetries of projective varieties, with applications to rigidity questions for Euler-symmetric varieties. Bakker presented recent work on Baily–Borel compactifications of period images and the b-semiampleness conjecture, connecting Hodge theory, moduli, and birational geometry. Yang Li reported on the metric SYZ conjecture, explaining how non-Archimedean geometry and pluripotential methods enter the study of collapsing Calabi–Yau metrics. Cupit-Foutou described a construction of non-Kähler compact complex manifolds in the spherical and

horospherical setting. Kim discussed rigidity and recognition problems for submanifolds in homogeneous spaces via free rational curves. Together, these talks highlighted the broad geometric scope of the meeting, extending from projective geometry and moduli spaces to non-Kähler geometry and mirror symmetry.

Another prominent theme was the distribution of zeros and degeneracy sets of holomorphic sections, together with related asymptotic questions in complex geometry. Coman presented a work on equidistribution results for Grassmannian embeddings, extending Tian-type asymptotics and proving almost sure convergence results for random degeneracy sets. This topic was explicitly anticipated in the proposal, where zeros of random holomorphic sections and their interactions with dynamics and complex geometry were identified as an important direction.

The short talks by Junyu Cao and Duc-Bao Nguyen highlighted two striking recent developments. Cao studied canonical currents and Archimedean height pairings in parabolic dynamics on K3 surfaces, while Nguyen introduced a quantitative pluripotential approach to complex hyperbolicity based on liftable currents and hyperbolicity indices.

Another major theme of the scientific program was holomorphic dynamics in higher dimensions. Dujardin discussed multiplier rigidity for complex Hénon maps, giving strong rigidity results for polynomial automorphisms of $\mathbf{C}^2$. Kaufmann presented effective equidistribution results for periodic points in holomorphic dynamics on projective space. Raissy described perturbations of parabolic endomorphisms in dimension two and a higher-dimensional analogue of Lavaurs' theory. Bianchi reported on the dynamics of Hénon-like maps. These talks highlighted the continuing development of several strands of higher-dimensional dynamics, including rigidity, equidistribution, and local bifurcation phenomena. Collectively, they underscored the interplay between geometric and pluripotential techniques.

Finally, the workshop also included topics at the interface with moduli theory, arithmetic geometry, and Teichmüller-type questions. Möller spoke on completed volumes and the double ramification cycle, while Jonsson presented recent work on the Yau–Tian–Donaldson conjecture for constant scalar curvature Kähler metrics. These lectures, together with the other contributions mentioned above, showed in a striking way how ideas from complex geometry now interact with a broad range of questions across geometry and dynamics.

The meeting brought together senior and junior researchers from a broad range of backgrounds, and the diversity of perspectives represented in the program fostered lively discussions both during the talks and in informal interactions. As in the 2022 edition, one of the workshop's defining features was the close exchange between experts in several complex variables, complex geometry, pluripotential theory, and holomorphic dynamics. The scientific program once again demonstrated that these areas continue to evolve in close contact, with many of the most significant recent advances emerging precisely at their intersections.

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Abstracts

Valuations on convex bodies and Kähler geometry

Bo Berndtsson

A convex body in $\mathbb{R}^n$ is a non-empty compact and convex set. We denote by $\mathcal{K}$ the set of all convex bodies in $\mathbb{R}^n$. If Γ is an additive semigroup, a *valuation* with values in Γ is a map V from $\mathcal{K}$ to Γ satisfying

$$V(K \cup L) + V(K \cap L) = V(K) + V(L),$$

provided that $K \cup L$ is convex. Here we will discuss mainly real or complex valued valuations. Such a valuation is continuous if it is continuous with respect to the Hausdorff metric on $\mathcal{K}$ and it is translation invariant if

$$V(K + a) = V(K)$$

for all $K \in \mathcal{K}$ and $a \in \mathbb{R}^n$. We will only consider valuations that are continuous and translation invariant, so 'valuation' in this abstract always means continuous and translation invariant valuation. The theory of valuations has been intensively studied in the last decades. Here we only mention the recent paper [2], which contains, apart from an important recent result, some background and an extensive list of references. This lecture is based on a chapter in [1]

A convex body, K, is uniquely determined by its support function

$$h_K(x) := \sup_{y \in K} x \cdot y,$$

which is a 1-homogeneous (finite valued) convex function. Conversely, any 1-homogeneous convex function is the support function of some convex body. Hence, a valuation can be seen as a function on the space of 1-homogeneous convex functions. If ϕ is any convex function on $\mathbb{R}^n$ we can extend it to a function on $\mathbb{C}^n$ by

$$\phi(x + i\xi) = \phi(x).$$

We then define

$$\omega_\phi := dd^c\phi = \sum \phi_{jk} dx_j \wedge d\xi_k,$$

where $d^c = (i/2)(\bar{\partial} - \partial)$ is the standard d^c-operator in Kähler geometry. If $\phi = h_K$, the support function of a convex body, we also write $\omega_{h_K} = \omega_K$. Using that $h_{K+a}(x) = h_K(x) + x \cdot a$, we see that

$$\omega_{K+a} = \omega_K.$$

It is not hard to see that the converse of this also holds; if $\omega_K = \omega_L$ (where K and L are convex bodies), then $L = K + a$ must be a translate of K. This is the basic observation of this lecture.

Any translation invariant valuation V (or, actually, any translation invariant function on $\mathcal{K}$) must therefore be a function of ω_K,

$$V(K) = F(\omega_K).$$

The main result of this lecture is an explicit representation of this type.

We first need to determine which forms appear as ω_K for some $K \in \mathcal{K}$. Let $E = \sum x_i \partial/\partial x_i$ be the Euler vector field on $\mathbb{R}^n$, which we consider as a vector field on $\mathbb{C}^n$. Let also $E^\# = \sum x_i \partial/\partial \xi_i$. Let δ be contraction by the vector field E and $\delta^\#$ be contraction by $E^\#$. These operators act on differential forms and lower their degrees by 1.

Lemma 1. *A differential form (or current)*

$$\omega = \sum \omega_{jk} dx_j \wedge d\xi_k$$

can be written $\omega = \omega_K$ for some convex body K if and only if

$$d\omega = 0,\ \delta^\# \omega = 0, \text{ and the matrix } [\omega_{jk}] \text{ is positive semidefinite.}$$

We can now state one version of our main result. We consider first differential forms (or currents)

$$\Theta_p = \sum \Theta_{JK}(x) dx_J \wedge d\xi_K,$$

where we sum over (increasing) multiindices of length p. We say that such a form (or current) is of real bidegree (p, p). Then look at

$$\Theta = \sum_p \Theta_p. \tag{1}$$

Theorem 2. *Any smooth translation invariant valuation can be written*

$$V(K) = \int_{\mathbb{C}^n} e^{\omega_K} \wedge \Theta \tag{2}$$

for a unique smooth (outside the origin) Θ of the form (0.1) satisfying $d\Theta = 0$, $\delta^\# \Theta = 0$.

The notion of smooth valuation was introduced by Alesker; a valuation is smooth if

$$V(gK)$$

is a smooth function of $g \in GL(\mathbb{R}, n)$.

Some comments are in order: The integrand in (0.2) depends only on x and not on ξ. Therefore the integral with respect to ξ is interpreted as *Berezin integration*; by definition

$$\int d\xi_1 \wedge ... d\xi_n = (-1)^{n(n-1)/2}.$$

Moreover, the forms that we integrate have a singularity at the origin, so the wedge product has to be interpreted with care; for this we use a variant of the technique of Bedford and Taylor. (Actually, the integrand turns out to be a Dirac mass at the origin.) Alternatively, we can use Stokes formula to rewrite the integral as an integral over a contour surrounding the origin. Finally, there is also a variant of the representation formula for valuations that are not smooth, but increasing in the sense that $K \subset L$ implies $V(K) \leq V(L)$.

One should notice the similarity of (0.2) with the Fourier transform. The map

$$K \to e(K) := e^{\omega_K}$$

satisfies

$$e(K+L) = e(K) \wedge e(L),$$

(where $K+L$ is Minkowski addition), so it can be viewed as a current-valued character.

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Bergman kernels on non-compact manifolds and Poincaré series

Xiaonan Ma

(joint work with Louis Ioos, Wen Lu, George Marinescu)

We show that the Bergman kernel of a finite-volume quotient of a Hermitian manifold $\widetilde{X}$ with bounded geometry by a discrete group Γ of its isometries is the same as the averaging over Γ of the Bergman kernel on $\widetilde{X}$. We then use these results when $\widetilde{X}$ is a Hermitian symmetric space to show that a large class of relative Poincaré series does not vanish.

Let $(\widetilde{X}, \widetilde{J}, g^{T\widetilde{X}})$ be a complete Kähler manifold of complex dimension n, equipped with a holomorphic Hermitian line bundle $(\widetilde{L}, h^{\widetilde{L}})$ whose Chern curvature satisfies

$$\frac{\sqrt{-1}}{2\pi} R^{\widetilde{L}}(\cdot, \widetilde{J}\cdot) = g^{T\widetilde{X}}\,, \tag{1}$$

and let also Γ be a discrete group acting properly discontinuously and effectively on $(\widetilde{X}, \widetilde{J}, g^{T\widetilde{X}})$ by biholomorphic isometries lifting to a holomorphic Hermitian action on $(\widetilde{L}, h^{\widetilde{L}})$, so that the quotient space $X := \widetilde{X}/\Gamma$ has a natural structure of a Kähler orbifold (X, J, g^{TX}), equipped with a positive orbifold line bundle (L, h^L) whose Chern curvature satisfies (1) for g^{TX} in the same way. For any $p \in \mathbb{N}^*$, we write $\widetilde{L}^p := \widetilde{L}^{\otimes p}$ for the p-th tensor power of $\widetilde{L}$ over $\widetilde{X}$, and for each $x,\, y \in \widetilde{X}$, we consider the associated *Bergman kernel* $\widetilde{P}_p(x,y) \in \widetilde{L}^p_x \otimes (\widetilde{L}^p_y)^*$ defined as the Schwartz kernel with respect to the Riemannian volume form of $(\widetilde{X}, g^{T\widetilde{X}})$ of the orthogonal projection operator $\widetilde{P}_p : L^2(\widetilde{X}, \widetilde{L}^p) \to H^0_{(2)}(\widetilde{X}, \widetilde{L}^p)$ from the space of square integrable sections to the subspace $H^0_{(2)}(\widetilde{X}, \widetilde{L}^p)$ of square-integrable holomorphic sections of $\widetilde{L}^p$. Similarly, we consider for each $x,\, y \in X$ the associated Bergman kernel $P_p(x,y) \in L^p_x \otimes (L^p_y)^*$ defined analogously as the Schwartz kernel of the orthogonal projection operator $P_p : L^2(X, L^p) \to H^0_{(2)}(X, L^p)$ on square-integrable holomorphic sections of L^p.

We assume $(\widetilde{X}, \widetilde{J}, g^{T\widetilde{X}})$ has positive injectivity radius, the derivatives of any order of $R^{\widetilde{E}}$, $R^{\widetilde{L}}$, $\widetilde{J}$ and $g^{T\widetilde{X}}$ are uniformly bounded on X in the norm induced by $g^{T\widetilde{X}}$, $h^{\widetilde{E}}$ and $h^{\widetilde{L}}$.

The first main result of this talk is the following.

Theorem 1. *If the volume of $X := \widetilde{X}/\Gamma$ is finite, there exists $p_0 \in \mathbb{N}^*$ such that for any $p \geq p_0$, we have*

$$\sum_{g\in\Gamma}(g^{-1},1).\widetilde{P}_p(g.\widetilde{x},\widetilde{y}) = P_p(\pi(\widetilde{x}),\pi(\widetilde{y}))\ , \tag{2}$$

where for any compact set of $K \subset \widetilde{X}$, the convergence is uniform and absolute in $\widetilde{x},\widetilde{y} \in K\times\widetilde{X}\cup\widetilde{X}\times K$, and where $(g^{-1},1)\in\Gamma\times\Gamma$ acts on $\widetilde{L}^p\boxtimes(\widetilde{L}^p)^*$ over $\widetilde{X}\times\widetilde{X}$ for all $g\in\Gamma$.

When $X = \widetilde{X}/\Gamma$ is smooth and compact, Theorem 1 has been established in [4, Theorem 2] under the more general hypotheses, and in the general symplectic case, and also by Lu and Zelditch in [3, Theorem 2]. The proof of Theorem 1 is based on the exponential decay of the Bergman kernel of $\widetilde{X}$ away from the diagonal established by Ma-Marinescu in 2015:

Theorem 2. *There exist $\boldsymbol{c} > 0$, $\boldsymbol{p}_0 > 0$, which can be determined explicitly from the geometric data such that for any $k\in\mathbb{N}$, there exists $C_k > 0$ such that for any $p \geqslant \boldsymbol{p}_0$, $x, x' \in X$, we have*

$$\left|\widetilde{P}_p(g.\widetilde{x},\widetilde{y}\right|_{\mathscr{C}^k} \leqslant C_k\, p^{n+\frac{k}{2}}\ \exp(-\boldsymbol{c}\sqrt{p}\, d(\widetilde{x},\widetilde{y}))\,. \tag{3}$$

In the case the relative Poincaré series associated with $SU(n,1)$, where the discrete subgroup $\Gamma\subset SU(n,1)$ acts on the unit ball $\widetilde{X} = B_n$ in $\mathbb{C}^n$ and $\Gamma_0\subset\Gamma$ is the subgroup generated by an *loxodromic element* $g_0\in\Gamma$, the underlying Γ_0-invariant submanifold again coincides with the closed geodesic of $X := B_n/\Gamma$ generated by g_0, and Barron showed in [1, § 2.2] that this geodesic satisfies the Bohr-Sommerfeld condition if all eigenvalues of $g_0\in\Gamma$ are real and if the endpoints x, $y\in\partial B_n$ of the geodesic $\widetilde{\gamma}\subset B_n$ generated by g_0 belong to $\mathbb{R}^n\subset\mathbb{C}^n$. We then obtain the following result as a consequence of Theorem 1, which shows once again that these relative Poincaré series do not vanish as soon as their weight $p\in\mathbb{N}^*$ is large enough.

Theorem 3. *Let $\Gamma\subset SU(n,1)$ be a discrete subgroup acting effectively and satisfying $Vol(B_n/\Gamma) < +\infty$, let*

$$g_0 = \begin{pmatrix} A & b \\ c^T & d\end{pmatrix}\in\Gamma,$$

with $A\in M_n(\mathbb{C})$, b, $c\in\mathbb{C}^n$ et $d\in\mathbb{C}$, be a loxodromic element with real eigenvalues, and let $\Gamma_0\subset\Gamma$ be the subgroup generated by g_0. Let $\widetilde{\gamma}:\mathbb{R}\mathbb{R}ightarrowB_n$ be the geodesic generated by g_0, and let x, $y\in\partial B_n$ be the two points of the boundary of B_n joined by $\widetilde{\gamma}$, and assume that x, $y\in\mathbb{R}^n$. Then there exists $p_0\in\mathbb{N}^*$ such that for all $p\geq p_0$, the convergent series

$$\sum_{[g]\in\Gamma_0\backslash\Gamma}\left(\frac{(c.z+d)^{-2}}{(\langle g.z,x\rangle-1)(\langle g.z,y\rangle-1)}\right)^{(n+1)p} \tag{4}$$

does not vanish identically in $z\in B_n$, where $\langle\cdot,\cdot\rangle$ is the canonical Hermitian product of $\mathbb{C}^n$ and $c.z := \sum_{j=1}^n c_j z_j$.

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Fundamental forms and infinitesimal symmetries of projective varieties

Jun-Muk Hwang

(joint work with Qifeng Li)

For a finite-dimensional vector space W, a finite-dimensional graded subspace

$$S = \oplus_{i=0}^{r} S^i \ \subset \ \mathrm{Sym}W^* := \oplus_{i\geq 0}\mathrm{Sym}^i W^*$$

is called a *symbol system on* W if $S^0 = \mathbb{C}, S^1 = W^*$ and S is stable under the contraction by any $w \in W$. A classical result due to E. Cartan says that for any nondegenerate projective variety $Z \subset \mathbb{P}^n$, there is a dense open subset $Z^{\mathrm{Cartan}} \subset Z$ such that the system of projective fundamental forms at any $x \in Z^{\mathrm{Cartan}}$ is a symbol system S_x on T_xZ with $\dim S_x = n+1$. Conversely, the following result was proved in [1].

Theorem [Fu-Hwang]. For any symbol system $S \subset \mathrm{Sym}W^*$ of dimension $n+1$, there is a nondegenerate projective variety $Z_S \subset \mathbb{P}^n$ with a dense open subset $O \subset Z_S^{\mathrm{Cartan}}$ such that for each $x \in O$

(1) the system of fundamental forms $S_x \subset \mathrm{Sym}T_x^*Z_S$ is isomorphic to $S \subset \mathrm{Sym}W^*$ and

(2) there is a multiplicative subgroup $\mathbb{C}^* \subset \mathrm{GL}(n+1)$ that acts on Z_S with an isolated fixed point at x with the induced action on T_xZ_S equal to scalar multiplications.

Furthermore, such $Z_S \subset \mathbb{P}^n$ is unique up to projective isomorphism.

We call $Z_S \subset \mathbb{P}^n$ in the above theorem the *Euler-symmetric variety* associated to S. Examples of Euler symmetric varieties include all equivariant embeddings of Hermitian symmetric spaces.

For a projective variety $Z \subset \mathbb{P}^n$, let $\mathfrak{aut}(Z) \subset \mathfrak{sl}(n+1)$ be the Lie algebra of linear automorphisms of Z. Our main result is the following.

Theorem [Hwang-Li]. Let $Z \subset \mathbb{P}^n$ be a nondegenerate projective variety.

(i) Then $\dim \mathfrak{aut}(Z) \leq \dim \mathfrak{aut}(Z_{S_x})$ for any $x \in Z^{\mathrm{Cartan}}$.

(ii) If the equality in (i) holds for some $x \in Z^{\mathrm{Cartan}}$, then $Z \subset \mathbb{P}^n$ is projectively isomorphic to the Euler-symmetric variety $Z_{S_x} \subset \mathbb{P}^n$.

As an application, we have the following rigidity result for Euler-symmetric varieties.

Corollary. Let $Z_S \subset \mathbb{P}^n$ be the Euler-symmetric variety associated to a symbol system S. Let

$$\{Z_t \subset \mathbb{P}^n \mid t \in \Delta\}$$

be a flat family of subschemes such that the subscheme $Z_t \subset \mathbb{P}^n$ is projectively isomorphic to $Z_S \subset \mathbb{P}^n$ for all $t \neq 0$. If the reduction of Z_0 contains a component Z such that at some point $x \in Z^{\mathrm{Cartan}}$, the system $S_x \in \mathrm{Sym}T_x^*Z$ is isomorphic to S, then $Z = Z_0$ and $Z_0 \subset \mathbb{P}^n$ is projectively isomorphic to $Z_S \subset \mathbb{P}^n$.

This corollary was proved in some special cases previously (for example, [2, Proposition 2.3]) using methods of differential geometry (special geometric structures on Z_S) and complex analysis (Hartogs-type extension theorem). Using Theorem [Hwang-Li], we have an algebro-geometric proof for all S.

The main idea in the proof of Theorem [Hwang-Li] is to find a suitable filtrations in $\mathfrak{aut}(Z)$ and $\mathfrak{aut}(Z_{S_x})$ such that there is a natural inclusion between their graded objects. This yields the inequality in (i) immediately. If the equality holds, then we can produce a suitable $\mathbb{C}^*$-action on Z, which shows (ii) by Theorem [Fu-Hwang]. The ideas of the filtrations on $\mathfrak{aut}(Z)$ and $\mathfrak{aut}(Z_{S_x})$ come from the work [3].

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Solutions of complex Hessian equations on Hermitian manifolds

Sławomir Kołodziej

(joint work with Cuong Ngoc Nguyen)

We study the complex m-hessian equation with a Hermitian background form in two settings - domains in $\mathbb{C}^n$ and on compact Hermitian manifolds.

Let us first define the m-hessian equation in its classical form.

Let $\lambda = (\lambda_1, \lambda_2, ..., \lambda_n)$ be the eigenvalues of $n \times n$ Hermitian matrix A. $S_m(A)$ denotes the elementary symmetric polynomial of degree m in λ:

$$S_m(A) = \sum_{0<j_1<\ldots<j_m\leq n} \lambda_{j_1}\lambda_{j_2}, ...\lambda_{j_m}.$$

If f is a given function on $\Omega \subset \mathbb{C}^n$, and A is the complex hessian of the unknown u of class C^2 then

$$S_m\big((u_{z_j\bar{z}_k}(z))\big) = f(z)$$

is the m-hessian equation for $1 \le m \le n$.

In terms of differential forms ($d = \partial + \bar{\partial}, d^c := i(\bar{\partial} - \partial)$):

$$(dd^c u)^m \wedge \beta^{n-m} = f\beta^n, \quad \beta = dd^c|z|^2.$$

The equation is elliptic on the set of m-subharmonic functions (m-sh). A $\mathcal{C}^2(\Omega)$ function u is called m-sh if

$$(dd^c u)^k \wedge \beta^{n-k} \ge 0, \ k = 1, \cdots, m. \qquad (*)$$

Now, let ω be a Hermitian form on a complex manifold X. We wish to find a constant c and (m, ω)-subharmonic function, that is

$$(\omega + dd^c u)^k \wedge \omega^{n-k} \ge 0, \quad k = 1, ..., m,$$

such that

$$(\omega + dd^c u)^m \wedge \omega^{n-m} = cf\omega^n.$$

for a given function f. The existence of solutions for smooth positive f on a compact X is due D. Zhang and G. Székelyhidi (independently). The case of a continuous non-negative right hand side was solved by Nguyen and myself.

However the correct definition of the m-Hessian measure for **bounded** (ω, m)-sh functions proved to be much more difficult than in the case of plurisubharmonic ones. Recently we were able to show the following.

Theorem 1. *Let $u_1, ..., u_p$ with $1 \le p \le m-1$ be bounded $m-\omega$-sh functions. Assume $T_0 = 1$ and $T_{p-1} = dd^c u_{p-1} \wedge \cdots \wedge dd^c u_1$. Then, the current*

$$dd^c u_p \wedge \cdots \wedge dd^c u_1 := dd^c[u_p T_{p-1}]$$

is a well-defined current of order zero whose coefficients are signed Radon measures. Moreover,

$$dd^c u_p \wedge \cdots \wedge dd^c u_1 \wedge \omega^{n-m}$$

is a positive $(n-m+p, n-m+p)$-current.

For psh functions the currents above are positive, in our case we only have currents of order zero.

This allows to mimic pluripotential theory, in particular to define the capacity in an open subset of $\mathbb{C}^n$

$$cap_m(E) = \sup\left\{\int_E H_m(u) : u \text{ is } m-\omega\text{-sh in } \Omega, -1 \le u \le 0\right\}.$$

We prove quasi-continuity of $m-\omega$-sh functions with respect to this capacity. The capacity is not realized by the relative extremal function. However we have a characterization of a polar set E by $cap_m^*(E) = 0$. Using this we obtain the equivalence of polar sets and negligible sets (famous Lelong's problem for psh functions).

Among the new items in the proofs is an inequality for elementary symmetric polynomials (refining Lin-Trudinger):

Theorem 2. *There exists a uniform constant C, depending on n, m, such that the following inequalities are satisfied.*

(a) *For $1 \leq i \leq m-1$ and $\lambda \in \Gamma_m$*

$$\frac{\lambda_1 \cdots \lambda_m}{\lambda_i} \leq C(S_{m-1;i}\, S_{m-1})^{\frac{1}{2}}.$$

(b) *Generally, for $1 \leq \ell \leq n$ and increasing multi-indices $(i_1, ..., i_{m-1})$,*

$$\prod_{i_s \neq \ell; s=1}^{m-1} |\lambda_{i_s}| \leq C\,(S_{m-1;\ell} S_{m-1})^{\frac{1}{2}}\,.$$

Next we study $m-\omega$-sh functions on compact Hermitian manifolds. A key result is the uniform integrability of normalized (ω, m)-sh functions with respect to Hessian measures of bounded (ω, m)-sh functions.

Theorem 3. *Let $v_0 \in SH_m(\omega)$ be such that $\sup_X v_0 = 0$. Let $0 \leq u \leq 1$ belong to $SH_m(\omega)$. There exist uniform constants C_m and D_m depending only on n, m, such that*

$$\int_X -v_0(\omega + dd^c u)^m \wedge \omega^{n-m} \leq C_m,$$

and

$$\int_X -v_0 du \wedge d^c u \wedge (\omega + dd^c u)^{m-1} \wedge \omega^{n-m} \leq D_m.$$

This leads to a characterizations of polar sets of (ω, m)-sh functions.

Theorem 4. *Let $E \subset X$ be a Borel set. The following statements are equivalent.*

(a) *E is a globally m-polar set.*
(b) *E is a locally m-polar set.*
(c) *$cap_m^*(E) = 0$.*

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Baily–Borel compactifications and b-semiampleness

Benjamin Bakker

(joint work with Stefano Filipazzi, Mirko Mauri, Jacob Tsimerman)

The Siegel modular variety $A_g := \mathrm{Sp}_{2g}(\mathbb{Z})\backslash\mathbb{H}_g$ starts off life as a complex analytic variety and parametrizes principally polarized abelian varieties. Motivated by ideas from reduction theory, Satake [11] introduced a topological compactification

$$\overline{A}_g := A_g \sqcup A_{g-1} \sqcup \cdots A_1 \sqcup A_0$$

in 1956 and suggested it was the topological space underlying a complex analytic variety. This expectation was confirmed in 1958 by Baily [2], who furthermore showed $\overline{A}_g$ naturally underlies a projective complex algebraic variety, realized as $\operatorname{Proj} R$ for a finitely generated graded ring of automorphic forms R. Shortly thereafter, following further work of Satake [13, 12], a celebrated paper of Baily–Borel [3] constructed the corresponding projective algebraic compactifications of any locally symmetric variety in 1966. These compactifications enjoy many important properties, and have found many applications.

The general theory of variations of Hodge structures was developed at roughly the same time by Griffiths and Deligne. Roughly speaking, to any smooth projective family $f : X \to Y$, the local system $V_{\mathbb{Z}} := R^k f_* \mathbb{Z}_X$ together with the natural filtration on the associated flat vector bundle $R^k f_* \Omega^\bullet_{X/Y}$ gives such a variation. In particular, choosing a basepoint $y \in Y$ and letting Γ be the image of the monodromy representation on $V_{\mathbb{Z},y}$, there is a natural map

$$\phi : Y \to \Gamma \backslash \mathbb{D}$$

where $\mathbb{D}$ is a period domain—a certain open subset of the variety of flags on $V_{\mathbb{C},y}$ of the appropriate type. In his 1970 ICM address, Griffiths [7] suggested that the closure of the image of the period map ϕ (which he proved was a complex analytic variety) should be algebraic, and that it should admit a compactification closely analogous to the Baily–Borel compactifications described above.

The algebraicity (in fact quasiprojectivity) of the closure Z of the image of ϕ was established by the author with Y. Brunebarbe and J. Tsimerman in [4] using o-minimal GAGA. In more recent work with S. Filipazzi, M. Mauri, and J. Tsimerman [5] we confirm the remainder of Griffiths' conjecture.

Specifically, for a polarizable $\mathbb{Z}$-variation of Hodge structures V on Y with underlying local system $V_{\mathbb{Z}}$ and filtration $F^\bullet V_{\mathcal{O}}$ on the associated flat vector bundle $(V_{\mathcal{O}}, \nabla)$, let

$$L_Y := \otimes_p \det F^p V_{\mathcal{O}}$$

be the Griffiths line bundle. We define a section of L_Y^k to have *moderate growth* if its Hodge norm grows sub-polynomially, or equivalently if its pullback $L_{Y'}^k$ to a resolution Y' extends to a section of the natural extension[1] $(L_{Y'}^k)_{\overline{Y}'}$ to a log smooth compactification $\overline{Y}'$. We define the ring of moderate growth sections $B_Y := \bigoplus_k H^0_{\mathrm{mg}}(Y, L_Y^k)$.

Theorem 1 (B–Filipazzi–Mauri–Tsimerman)**.** *Let Z be the closure of the image of a period map. Then*

(1) *The ring B_Z is finitely generated, $Z^{\mathrm{BB}} := \operatorname{Proj} B_Z$ is a projective compactification of Z, and the ample bundle $\mathcal{O}_{Z^{\mathrm{BB}}}(n)$ (for sufficiently divisible n) naturally restricts to L_Z^n.*

[1]If the local monodromy is unipotent there is no ambguity; otherwise, some care must be taken in making this precise. In general one must also be careful with distinctions between stacks and coarse spaces, and one must take a sufficiently divisible power of the line bundle.

(2) *Any analytic morphism* $(\Delta^*)^\ell \to Z$ *for which the resulting morphism* $(\Delta^*)^\ell \to \Gamma\backslash\mathbb{D}$ *is locally liftable*[2] *extends to a morphism* $\Delta^\ell \to Z^{\mathrm{BB}}$. *Moreover,* $\mathcal{O}_{Z^{\mathrm{BB}}}(n)$ *(for sufficiently divisible* n*) pulls back to* $(L^n_{(\Delta^*)^\ell})_{\Delta^\ell}$.

Corollary 2. *Let* $(\overline{Y}, D)$ *be a log smooth algebraic space and* V *a polarizable integral pure variation of Hodge structures on* $\overline{Y}\backslash D$ *with unipotent local monodromy. Then the Griffiths bundle* $L_{\overline{Y}}$ *of* V *is semiample.*

Corollary 3. *Let* $\mathcal{Y}$ *be a reduced separated Deligne–Mumford stack equipped with a quasifinite period map. Then the coarse space* Y *has a compactification* Y^{BB} *for which some power* L^n_Y *of the Griffiths bundle extends to an ample bundle* $\mathcal{O}_{Y^{\mathrm{BB}}}(n)$ *and such that for any analytic morphism* $g : (\Delta^*)^\ell \to \mathcal{Y}$, *the map on coarse spaces extends to* $\bar{g} : \Delta^\ell \to Y^{\mathrm{BB}}$ *with the property that* $\bar{g}^*\mathcal{O}_{Y^{\mathrm{BB}}}(n)$ *and* $(L^n_{(\Delta^*)^\ell})_{\Delta^\ell}$ *are naturally identified.*

Corollary 3 for instance applies to any moduli stack of polarized varieties with an infinitesimal Torelli theorem, such as Calabi–Yau manifolds and most hypersurfaces.

We also prove a version of Corollary 2 for *any* line bundle arising from Hodge theory. We say a variation is a CY variation if the deepest piece of the Hodge filtration is a line bundle, and in this case we call this line bundle the Hodge bundle. The Griffiths bundle of V, for example, is the Hodge bundle of the variation $\otimes_p \wedge^{\mathrm{rk}\, F^p V_{\mathcal{O}}} V$. We then have the following:

Theorem 4 (B–Filipazzi–Mauri–Tsimerman)**.** *Let* $(\overline{Y}, D)$ *be a proper log smooth algebraic space,* V *a polarizable integral pure CY variation of Hodge structures on* $\overline{Y}\backslash D$ *with unipotent local monodromy, and* $M_{\overline{Y}}$ *the Hodge bundle on* $\overline{Y}$. *If* $M_{\overline{Y}}$ *is integrable with torsion combinatorial monodromy, then it is semiample.*

The integrability condition means that for any analytic germ $T \subset \overline{Y}$ for which the Hodge bundle is flat, it is flat on the Zariski closure T^{Zar} as well. The torsion combinatorial monodromy condition means that for any nodal curve $g\colon C \to \overline{Y}$ for which $g^*M_{\overline{Y}}$ is numerically trivial (hence torsion on each component), it is in fact torsion. Both conditions are needed in the statement of Theorem 4.

Theorem 4 is a step in the proof of Theorem 1: the Griffiths bundle has torsion combinatorial monodromy by a theorem of Green–Griffiths–Robles [6] (the integrability is standard). We also use Theorem 4 to resolve the b-semiampleness conjecture, formulated in increasing generality by Mori, Kawamata, Shokurov, Ambro, and Prokhorov–Shokurov; see [9, 1, 10] and references therein.

Theorem 5 (B–Filipazzi–Mauri–Tsimerman)**.** *Let* $f\colon (X, \Delta) \to Y$ *be an lc-trivial fibration from a pair* (X, Δ) *such that* Δ *is effective over the generic point of* Y. *Then the moduli part* $\mathbf{M}$ *is b-semiample.*

There is a version of Theorem 1 for the ring of moderate growth sections of a Hodge bundle satisfying the two conditions of Theorem 4, which in particular implies:

[2]The local liftability is equivalent to the condition that $(\Delta^*)^\ell \to \Gamma\backslash\mathbb{D}$ be the period map of a variation. If Γ is torsion-free the local liftability condition is automatic.

Corollary 6. *Let $\mathcal{Y}$ be a moduli stack of polarized smooth Calabi–Yau varieties. Then the coarse space Y has a unique normal compactification Y^{BBH} for which some power M_Y^n of the Hodge bundle of the variation of Hodge structures on middle cohomology extends to an ample bundle $\mathcal{O}_{Y^{\mathrm{BBH}}}(n)$ and such that for any family $g : (\Delta^*)^\ell \to \mathcal{Y}$, the map on coarse spaces extends to $\bar{g} : \Delta^\ell \to Y^{\mathrm{BBH}}$ with the property that $\bar{g}^*\mathcal{O}_{Y^{\mathrm{BBH}}}(n)$ pulls back to $(M^n_{(\Delta^*)^\ell})_{\Delta^\ell}$.*

In brief, for the proof of Theorem 4 we first use the result of [4] to construct a map $q : \overline{Y} \to \overline{Z}$ on the level of definable topological spaces such that $\overline{Z}$ has a locally closed definable stratification $\overline{Z}^i$ by subspaces (with decreasing dimension) for which $\overline{Y}^i := q^{-1}(\overline{Z}^i) \to \overline{Z}^i$ is algebraic and for which $M_{\overline{Y}}|_{\overline{Y}^i}$ descends to an ample bundle. Letting $\overline{Z}^{<i}$ be the union of the first i strata, we proceed by induction as follows. Assuming $\overline{Z}^{<i}$ is algebraic, we use Hodge theory to glue the definable analytic structures of $\overline{Z}^{<i}$ and $\overline{Z}^i$ into a definable analytic variety $\overline{Z}^{\leq i}$. We then use o-minimal GAGA to algebraize $\overline{Z}^{\leq i}$. For the proof of Theorem 5, we identify the moduli part with the Hodge bundle on the variation coming from middle cohomology of the minimal lc center of a sufficiently nice model. We then verify the two conditions in Theorem 4 using the immersivity of the Kodaira–Spencer map (for the integrability) and results of Kollár [8] on the $\mathbb{P}^1$-linking of minimal lc centers (for the torsion combinatorial monodromy).

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Quantitative Weyl law for Toeplitz operators

Nikhil Savale

This talk is concerned with Weyl laws for pseudodifferential and Toeplitz operators with quantified remainder estimates. We first review the pseudodifferential case.

1. Pseudodifferential operators

Let X^n be a smooth, compact and n-dimensional manifold. Let $P \in \Psi^m_{\mathrm{cl}}(X)$ be a self-adjoint, pseudodifferential operator. We say that P is polyhomogeneous if its total symbol has an asymptotic expansion

$$P = p^W, \quad \text{where} \quad p \sim p_m + p_{m-1} + \dots, \tag{1}$$

for some smooth functions $p_{m-j} \in C^\infty(T^*X)$, $j = 0, 1, 2, \dots$ on the cotangent space T^*X that are homogeneous of degree $m - j$. Assuming the principal symbol $\sigma(P) = p_m$ to be elliptic, the spectrum of P is discrete. Denote by $N[0, \lambda]$ the number of eigenvalues contained in the interval $[0, \lambda]$. We denote by $\Sigma_1 = \{(x, \xi) \in T^*X | p_m(x, \xi) = 1\} \subset T^*X$, the level set of the principal symbol and by $e^{tH_{p_0}}$ the Hamiltonian flow for the principal symbol on it. Let $T_0 > 0$ the shortest period of $e^{tH_{p_0}}$.

Next, define the recurrence set

$$S_{T,\varepsilon} = \left\{ (x, \xi) \subset \Sigma_1 | \exists t \in \left[\frac{1}{2}T_0, T\right] \text{ s.t. } d\left(e^{tH_{p_0}}(x, \xi), (x, \xi)\right) \leq \varepsilon \right\} \tag{2}$$

and its extension $S^e_{T,\varepsilon} = \{(x, \xi) \subset \Sigma_1 | d((x, \xi), S_{T,\varepsilon}) \leq \varepsilon\}$ for each $T > \frac{1}{2}T_0$, $\varepsilon > 0$. These are defined with respect to a distance function d on the phase space that is equivalent to a manifold distance, although the dependence of the above 2 on d is suppressed in the notation.

Next we further, for each $\delta \in \left[0, \frac{1}{2}\right)$ and $\ell > 0$, define the Ehrenfest time via

$$T^{\ell,\delta}_E(\lambda) = \begin{cases} \frac{\lambda^{\frac{1}{C}\left(\frac{1}{2}-\delta-\ell\right)}}{\frac{1}{2}-\delta}; & \text{if } \exists C > 0 \text{ s.t. } \left|\partial^\alpha e^{tH_{p_0}}\right| \leq 1 + |t|^{C|\alpha|}, \ \forall \alpha \in \mathbb{N}_0^{2n-1}, \\ \frac{|\ln \lambda|}{\Lambda_{\max}+\ell}; & \text{otherwise.} \end{cases} \tag{3}$$

$$\text{Here} \quad \Lambda_{\max} = \limsup_{t\to\infty} t^{-1} \sup_{x\in\Sigma_1} \ln\left|de^{tH_{p_0}}(x)\right| \tag{4}$$

is the maximum expansion rate of the Hamilton flow $e^{tH_{p_0}}(x)$ on the energy level Σ_1.

Our first result is the following general estimate on the Weyl remainder for P.

Theorem 1. [6] *Let $P \in \Psi^m_{h,cl}(X)$ be a self-adjoint, polyhomogeneous elliptic pseudodifferential operator. For each $\varepsilon = \lambda^{-\delta/2}$, $\delta \in \left[0, \frac{1}{2}\right)$, $\ell > 0$ and $T \leq$*

$\left(\frac{1}{2}-\delta\right).T_E^{\ell,\delta}(\lambda)$, *the Weyl counting function of the interval satisfies*

$$N[0,\lambda]=\lambda^{\frac{n}{2}}(2\pi)^{-n}\left[vol\ \{(x,\xi)\,|p_m(x,\xi)\leq 1\}+\lambda^{-\frac{1}{2}}R_\lambda\right] \tag{5}$$

$$\textit{with}\quad |R_\lambda|\leq(vol\ (\Sigma_1))\,T^{-1}+O\left(\nu\left(S^e_{T,\varepsilon}\right)+T^{-2}+\lambda^{-\frac{1}{2}+\delta}\right) \tag{6}$$

as $\lambda\to\infty$.

Interesting specializations of the above arise depending on estimates for volumes of the recurrence sets. Firstly, it recovers the classical results of Hörmander [5] and Duistermaat-Guillemin [4]. Namely, one obtains Hörmander's Weyl law ($R_\lambda = O(1)$) as well as Duistermaat-Guillemin's Weyl law ($R_\lambda = o(1)$ assuming the set of periodic Hamilton trajectories is of measure zero on the energy level Σ_1) from 5. Another interesting specialization is when the Hamilton flow is a contact Anosov flow. By an appropriate choice of ε, T in this case, one obtains a logarithmic improvement $R_\lambda = o\left(\frac{1}{\ln\lambda}\right)$ which is Bérard's Weyl law [1]. The general theorem 1 above is a quantified version of the Weyl's law given by Volovoy [7].

2. Toeplitz operators

A generalization of a pseudodifferential operator is given by the notion of a Toeplitz operator [2, 3]. To define it, consider a strongly pseudoconvex CR manifold $\left(X,T^{1,0}X\right)$. It's Levi distribution is defined via $HX=\mathrm{Re}\left[T^{1,0}X\oplus\overline{T^{1,0}X}\right]\subset TX$ as a subbundle of the tangent space. The annihilator of Levi distribution is defined via $\Sigma=HX^\perp\subset T^*X$ as a sub-bundle of the cotangent space. For a strongly pseudoconvex CR manifold the annihilator is a symplectic submanifold of the cotangent space. The space of CR functions is defined via $H_b^0(X)=\left\{f|Zf=0,\forall Z\in C^\infty\left(T^{1,0}X\right)\right\}$. While the Szegő projector is the orthogonal projector onto it $\Pi_b : L^2(X,\mu)\to H_b^0(X)$, that is defined with respect to some smooth measure μ on X. A Toeplitz operator is now defined as the composition

$$T_Q=\Pi_bQ\Pi_b : H_b^0(X)\to H_b^0(X) \tag{7}$$

for any pseudodifferential operator Q on the manifold X.

In [2], Boutet de Monvel showed that pseudodifferential operators on $\mathbb{R}^n$ can be conjugated, via a Hermite operator, to Toeplitz operators on the boundary of a strongly pseudoconvex domain in $\mathbb{C}^n$. Thus the above notion of a Toeplitz operator generalizes that of a pseudodifferential operator. The calculus of Toeplitz operators was developed in [3]. This includes their composition properties and symbolic calculus. In particular, the principal symbol of a Toeplitz operator as in 7 is defined via $\sigma(T_Q)=\sigma(Q)|_\Sigma\in C^\infty(\Sigma)$ as a function on the annihilator of the Levi distribution. The annihilator being symplectic, the symbol thus has a well-defined Hamiltonian vector field $H_Q=H_{\sigma(T_Q)}$ along Σ.

For Toeplitz operators, the analog of the Weyl law's of Hörmander and Duistermaat-Guillemin were proved by Boutet de Monvel-Guillemin in [3]. Following the calculus, the recurrence set 2, Ehrenfest time 3 and maximal expansion rate 4 can be similarly defined for the Hamilton vector field H_Q of a Toeplitz

operator. We can then prove a quantitative Weyl law as in 1 for Toeplitz operators generalizing the earlier results of Boutet de Monvel-Guillemin. Rather than restate the general of the same kind, we give a novel specialization. The following corresponds to Bérard's Weyl law for Toeplitz operators.

Theorem 2. *Let* $T_Q : H_b^0\left(X^{2n-1}\right) \to H_b^0\left(X^{2n-1}\right)$ *be an elliptic Toeplitz operator. If* H_Q *is a contact Anosov vector field, then the Weyl counting function satisfies the asymptotics*

$$N(\lambda) = \lambda^n (2\pi)^{-n} \, vol\{(x,\xi) \,|\sigma(T_Q) \leq 1\} + O\left(\frac{\lambda^{n-1}}{\ln \lambda}\right)$$

as $\lambda \to \infty$.

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Equidistribution results for Grassmannian embeddings

Dan Coman

(joint work with Turgay Bayraktar, Bingxiao Liu and George Marinescu)

Let (X, ω) be a (connected) compact Kähler manifold of dimension n, (L, h^L) be a positive line bundle on X, and (E, h^E) be a Hermitian holomorphic vector bundle of rank $r \leq n$ on X. For $p \geq 1$, we set

$$L^p := L^{\otimes p}, \; V_p := H^0(X, L^p \otimes E), \; d_p := \dim V_p - 1.$$

We let h^{L^p} and $h^{L^p \otimes E}$ be the Hermitian metrics induced by h^L and h^E on L^p, respectively on $L^p \otimes E$. We endow V_p with the L^2-inner product induced by this metric data, namely

$$(S, S')_p = \int_X \langle S, S' \rangle_{h^{L^p \otimes E}} \frac{\omega^n}{n!}, \; S, S' \in V_p. \tag{1}$$

We endow its dual space $V_p^\star = H^0(X, L^p \otimes E)^\star$ with the inner product induced by the one on V_p. Let $\mathbb{G}(r, V_p^\star)$ be the Grassmannian of r-dimensional subspaces of $V_p^\star$. The Kodaira map is defined by

$$\Phi_p : X \to \mathbb{G}(d_p + 1 - r, V_p) = \mathbb{G}(r, V_p^\star)\,, \;\; \Phi_p(x) = \{s \in V_p : \, s(x) = 0\}.$$

Since L is positive, the above map is a well-defined holomorphic embedding for all p sufficiently large. Let $\mathscr{T}$ be the universal holomorphic vector bundle over $\mathbb{G}(r, V_p^\star)$ and $h^{\mathscr{T}}$ be the Hermitian metric on it induced by the inner product on $V_p^\star$. Let $(\mathscr{T}^\star, h^{\mathscr{T}^\star})$ be the dual holomorphic vector bundle. We denote by $c_k(\mathscr{T}^\star, h^{\mathscr{T}^\star})$ the k-th Chern form of $(\mathscr{T}^\star, h^{\mathscr{T}^\star})$. We prove a generalization of Tian's theorem for Chern forms of any degree.

Theorem 1. [1] *Let (X, ω) be a compact Kähler manifold of dimension n, (L, h^L) be a positive line bundle on X, and (E, h^E) be a Hermitian holomorphic vector bundle on X of rank $r \leq n$. Then for every $0 \leq k \leq r$, we have in the $\mathscr{C}^\infty$ topology as $p \to \infty$,*

$$\frac{1}{p^k}\,\Phi_p^\star(c_k(\mathscr{T}^\star, h^{\mathscr{T}^\star})) = \binom{r}{k} c_1(L, h^L)^k + O\left(\frac{1}{p}\right).$$

Assume further that the bundle (L, h^L) polarizes (X, ω), that is $\omega = c_1(L, h)$. Then, for every $0 \leq k \leq r$, we have in the $\mathscr{C}^\infty$ topology as $p \to \infty$,

$$\frac{1}{p^k}\,\Phi_p^\star(c_k(\mathscr{T}^\star, h^{\mathscr{T}^\star})) = \binom{r}{k}\omega^k + \frac{1}{p}\binom{r-1}{k-1} c_1(E, h^E) \wedge \omega^{k-1} + O\left(\frac{1}{p^2}\right).$$

Tian's theorem [10] (see also [8, 3, 11]) follows from Theorem 1 by taking E to be the trivial line bundle $X \times \mathbb{C}$. In recent years, Tian's theorem was generalized to the case of singular Hermitian holomorphic line bundles, and moreover to the setting of compact normal analytic spaces X (see [4, 5, 6]).

An important application of Tian's theorem [10] is the study of the asymptotic distribution of zeros of random sequences of sections of powers of a line bundle. This started with the work of Shiffman and Zelditch [9] (see [2] and the references therein for interesting results in this direction). Theorem 1 can be applied to prove the equidistribution of zeros of random sections in V_p. By Bertini's theorem, for generic $s \in V_p$, its zero set $Z_s = \{s = 0\}$ is a complex submanifold of X of codimension r. Let $[s = 0]$ be the current of integration over Z_s and endow the projective space $\mathbb{P}V_p$ with the probability measure $\Upsilon_p = \omega_{\mathrm{FS}}^{d_p}$, where ω_{FS} denotes the Fubini-Study form on $\mathbb{P}V_p$. Let $(\mathcal{H}, \Upsilon) = \left(\prod_{p=1}^\infty \mathbb{P}V_p\,, \prod_{p=1}^\infty \Upsilon_p\right)$. Using the theory of meromorphic transforms due to Dinh-Sibony [7] we prove the following:

Theorem 2. [1] *Let (X, ω) be a compact Kähler manifold of dimension n, (L, h^L) be a positive line bundle on X, and (E, h^E) be a Hermitian holomorphic vector bundle of rank $r \leq n$ on X. Then there exist $C > 0$ and $p_0 \in \mathbb{N}$ such that the following holds:*

For any $\gamma > 1$, there exist subsets $\mathcal{E}_p = \mathcal{E}_p(\gamma) \subset \mathbb{P}V_p$ such that for $p > p_0$ we have

(i) $\Upsilon_p(\mathcal{E}_p) \leq Cp^{-\gamma}$;

(ii) if $s_p \in \mathbb{P}V_p \setminus \mathcal{E}_p$, then

$$\left|\left\langle \frac{1}{p^r}\,[s_p = 0] - c_1(L, h^L)^r, \varphi \right\rangle\right| \leq C\gamma\,\frac{\log p}{p}\,\|\varphi\|_{\mathscr{C}^2(X)}, \tag{2}$$

for any $(n-r, n-r)$ form φ of class $\mathscr{C}^2$ on X. Moreover, the estimate (2) holds for Υ-a.e. sequence $\{s_p\}_{p\geq 1} \in \mathcal{H}$ provided that p is large enough. Hence $p^{-r}\,[s_p = 0] \to c_1(L, h^L)^r$ in the weak sense of currents as $p \to \infty$, almost surely.

We also consider the expectation of the currents of integration along degeneracy sets of random holomorphic sections, in the following general setting:

(A) (X, ω) is a (connected) compact Kähler manifold of dimension n; (E, h^E) is a Hermitian holomorphic vector bundle of rank r on X, and

$$1 \leq r \leq n,\ V := H^0(X, E),\ N := \dim V - 1,\ \mathbb{P}V = \mathbb{P}H^0(X, E).$$

(B) For every section $S \in V$, $Z_S := \{x \in X :\ S(x) = 0\} \neq \varnothing$.

(C) $N \geq r$ and for every $x \in X$, V spans the fiber E_x of E over x.

We endow V with the natural L^2-inner product defined as in (1) using the Hermitian metric h^E. We consider the dual $V^\star$ with the inner product determined by that on V, and we let ω_{FS} be the induced Fubini-Study form on the projective space $\mathbb{P}V$. Let $\mathbb{G}(k, V)$ denote the Grassmannian of (complex) k-planes in V. By (C), the Kodaira map

$$\Phi_E : X \to \mathbb{G}(N+1-r, V) = \mathbb{G}(r, V^\star),\ \Phi_E(x) = \{s \in V :\ s(x) = 0\}, \tag{3}$$

is well-defined and holomorphic. Let $\mathscr{T} \to \mathbb{G}(r, V^\star)$ be the universal bundle endowed with the Hermitian metric $h^{\mathscr{T}}$ induced by the inner product on $V^\star$, and let $(\mathscr{T}^\star, h^{\mathscr{T}^\star})$ be the dual holomorphic vector bundle.

Let $1 \leq k \leq r$. Given $(s_1, \ldots, s_k) \in (\mathbb{P}V)^k$, the degeneracy set $D_k(s_1, \ldots, s_k)$ is the set of points x where $s_1(x), \ldots, s_k(x)$ are linearly dependent. We denote by $[D_k(s_1, \ldots, s_k)]$ the current of integration along $D_k(s_1, \ldots, s_k)$. Let μ_k be the product probability measure on $(\mathbb{P}V)^k$ determined by the Fubini-Study volume ω_{FS}^N on $\mathbb{P}V$. Let $\mathbb{E}_k(\mu_k)$ be the expectation current of the current valued random variable $(s_1, \ldots, s_k) \longmapsto [D_k(s_1, \ldots, s_k)]$, defined by

$$\langle \mathbb{E}_k(\mu_k), \phi\rangle = \int_{(\mathbb{P}V)^k} \langle [D_k(s_1, \ldots, s_k)], \phi\rangle\, d\mu_k\,,$$

where ϕ is an $(n+k-r-1, n+k-r-1)$ form on X. By using the fact that the Poincaré dual of $D_k(s_1, \ldots, s_k)$ is $c_{r+1-k}(E)$, we obtain a precise formula for the expectation currents.

Theorem 3. [1] *Let (X, ω), (E, h^E) verify (A)-(C), and assume that the Kodaira map $\Phi_E : X \to \mathbb{G}(r, V^\star)$ defined in (3) is an embedding. Then*

$$\mathbb{E}_k(\mu_k) = \Phi_E^\star\big(c_{r+1-k}(\mathscr{T}^\star, h^{\mathscr{T}^\star})\big),\ 1 \leq k \leq r.$$

These results are joint with Turgay Bayraktar, Bingxiao Liu and George Marinescu.

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Semi-classical Heat Kernel Asymptotics on Complex Manifolds with Boundary

Weixia Zhu

(joint work with Chin-Yu Hsiao, George Marinescu)

Let M be a relatively compact domain with smooth boundary $X = \partial M$ in an n-dimensional complex manifold M', and let $(L, h^L) \to M'$ be a Hermitian holomorphic line bundle. The aim of the talk was to describe a heat-kernel approach to semi-classical asymptotics for the $\bar{\partial}$-Neumann Laplacian on $(0,q)$-forms with values in L^k.

For compact complex manifolds, Demailly's holomorphic Morse inequalities [3] express the asymptotic growth of the cohomology groups $H^q(M, L^k)$ in terms of the curvature of L. Bismut [2] gave a heat equation proof of these inequalities. For complex manifolds with boundary, Berman [1] obtained corresponding holomorphic Morse inequalities by Bergman kernel methods. In this setting the boundary produces an additional contribution involving the Levi form. The purpose of our work is to recover these boundary Morse inequalities by heat-kernel methods and to obtain a more precise local description of the boundary contribution.

Let $\Box_k^q$ be the $\bar{\partial}$-Neumann Laplacian acting on $(0,q)$-forms with values in L^k. We assume that Kohn's condition $Z(q)$ holds on the boundary. The main difficulty, compared with the compact case, is that the natural scaling near the boundary is anisotropic: the complex tangential variables have scale $k^{-1/2}$, while the normal direction has scale k^{-1}. To encode this boundary scale at the level of the operator,

we introduce a suitable k-dependent Hermitian metric near X. After applying the corresponding anisotropic rescaling, the scaled operator has the form

$$\square^q_{(k)} = \square^q_{\phi_0} + \varepsilon_k, \qquad \varepsilon_k \to 0,$$

where $\square^q_{\phi_0}$ is a model weighted $\bar\partial$-Neumann Laplacian on a Siegel-type half-space

$$\overline{M}_0 = \Big\{(z, z_n) \in \mathbb{C}^{n-1} \times \mathbb{C};\ \operatorname{Im} z_n + \sum_{j=1}^{n-1} \lambda_j |z_j|^2 \le 0\Big\}.$$

The coefficients λ_j are allowed to have arbitrary signs, so the model is not restricted to the strongly pseudoconvex case.

Our first main result is an explicit formula for the heat kernel of the model operator $\square^q_{\phi_0}$. This formula contains the usual harmonic-oscillator contribution coming from the curvature of the line bundle L and a boundary contribution depending on the Levi form. The second main result is the convergence of the scaled heat kernels to this model heat kernel, locally uniformly with all derivatives. The uniform estimates needed for this convergence are obtained using the condition $Z(q)$.

Combining the boundary scaling asymptotics with the standard interior asymptotics gives the heat trace limit

$$\lim_{k\to\infty} k^{-n} \operatorname{Tr}_q \left(e^{-\frac{t}{k}\square^q_k}\right) = \Theta_q(t).$$

The function $\Theta_q(t)$ is the sum of an interior term, determined by the curvature of L, and a boundary term involving both the curvature and the Levi form. Letting $t \to +\infty$ gives weak and strong holomorphic Morse inequalities for complex manifolds with boundary. The interior contribution is the usual Demailly term, while the boundary term agrees with Berman's boundary contribution, written in our notation. Thus the result gives a heat-kernel proof of Berman's holomorphic Morse inequalities.

The same heat trace asymptotics also have consequences for non-compact manifolds through exhaustion. Applying the domain result to exhaustion domains yields Morse inequalities on q-convex and q-concave manifolds. In particular, in the 1-concave case one obtains lower bounds for the dimension of spaces of holomorphic sections of high tensor powers of semipositive line bundles. Under suitable boundary-curvature assumptions, these estimates imply that the line bundle is big and lead to a Moishezon-type compactification criterion.

Moreover, the heat trace asymptotics lead to a semi-classical Weyl law for the $\bar\partial$-Neumann Laplacian. Let

$$N^q_k(\lambda) = \operatorname{Tr}_q \mathbf{1}_{[0,\lambda]}\left(\frac{1}{k}\square^q_k\right), \qquad \lambda \ge 0,$$

be the spectral counting function of the semi-classically normalized operator. There exists a positive Borel measure μ^q on $[0,\infty)$ whose Laplace transform is the limiting heat trace $\Theta_q(t)$, and

$$N^q_k(\lambda) = k^n \mu^q([0,\lambda]) + o(k^n)$$

at every continuity point of the distribution function $\lambda \mapsto \mu^q([0,\lambda])$. The limiting measure is described by the spectral distributions of the interior and boundary model operators. The interior part is the familiar Demailly–Ma–Marinescu [3, 4] local spectral distribution, while the boundary part is encoded by the spectral measure of the boundary model operator.

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Higher Lelong numbers versus full Monge-Ampère mass

Duc-Viet Vu

(joint work with Do Duc Thai)

Let $n \geq 2$ be an integer. Let z be the standard coordinates on $\mathbb{C}^n$. Denote by $\|z\|$ the Euclidean norm of z. Let ψ be a plurisubharmonic function (psh for short) on an open neighborhood U around 0 in $\mathbb{C}^n$ such that ψ is locally bounded outside 0. Recall that $d^c := \frac{i}{2\pi}(\bar\partial - \partial)$, and $\mathrm{dd}^c = \frac{i}{\pi}\partial\bar\partial$. For $1 \leq k \leq n$, *the higher Lelong number* of ψ of order k is given by

$$e_k(\psi) := \int_{\{0\}} (\mathrm{dd}^c\psi)^k \wedge (\mathrm{dd}^c \log\|z\|)^{n-k}.$$

The number $e_1(\psi)$ is the Lelong number of ψ at 0 and is usually denoted by $\nu(\psi, 0)$. The higher Lelong numbers are closely related to the notion of log canonical threshold. We refer to [1] for a fundamental lower bound of the log canonical threshold in terms of higher Lelong numbers, see also [2, 3, 4] for applications to birational geometry. If $g_1, \ldots, g_N$ are holomorphic functions defined on an open neighborhood of 0 in $\mathbb{C}^n$ and $\psi = \log\sum_{j=1}^N |g_j|$, then $e_n(\psi)$ is equal to the Samuel multiplicity of the ideal generated by germs of $g_1, \ldots, g_N$ at 0; see [5]. These facts are instances illustrating the relevance of higher Lelong numbers to algebraic geometry.

A major problem in the theory of singularity of psh functions is to clarify the relation between higher Lelong numbers and the ideal multiplier sheaves associated to ψ. In this research direction, Demailly conjectured that for every psh function ψ locally bounded outside 0, the sequence of higher Lelong numbers at 0 of psh functions with analytic singularities approximating ψ in his analytic regularization of psh functions would converge to those of ψ. If this were true, it would imply the

well-known conjecture by Guedj and Rashkovskii independently (see [8, 9]) stating that $e_n(\psi) = 0$ if and only if $e_1(\psi) = 0$ (recall however that by [1], one has that $e_k(\psi) = 0$ if and only if $e_1(\psi) = 0$ for every $1 \leq k \leq n-1$). The latter conjecture has remained largely open although some important cases were confirmed such as when ψ is toric ([6]) or has sup-analytic singularities ([10, 7, 11]). We refer to [7] for a detailed summary of developments regarding the Guedj-Rashkovskii conjecture.

Recently, Chi Li [12] constructed a counter-example to the above conjecture of Demailly. The importance of such a counter-example lies in the fact that it implies that in general the higher Lelong numbers can not be recovered using the usual analytic approximation from above as in Demailly's analytic regularisation theorem. Nevertheless since the example in [12] only concerns with psh functions of positive Lelong number at 0, it gives us no clear clue how to deal with the Guedj-Rashkovskii conjecture.

To go into details, we need a few more things. We are interested in ψ such that $e_1(\psi) \geq 1$, or in other words, $\psi \leq \log \|z\| + O(1)$ near 0. In this case, by monotonicity (see [13, Lemma 4.1] or [14]), one has

$$1 \leq e_1(\psi) \leq \cdots \leq e_n(\psi). \tag{1}$$

Let $\mathbb{P}^{n-1}$ be the complex $(n-1)$-dimensional projective space, and $\pi : \mathbb{C}^n \backslash \{0\} \to \mathbb{P}^{n-1}$ be the natural projection. Let ω_{n-1} be the Fubini-Study form on $\mathbb{P}^{n-1}$, and let u be an ω_{n-1}-psh function. Hence $\mathrm{dd}^c u + \omega_{n-1} \geq 0$. Since $\pi^*\omega_{n-1} = \mathrm{dd}^c \log \|z\|$, we see that $\pi^* u + \log \|z\|$ is a psh function on $\mathbb{C}^n \backslash \{0\}$ which is bounded from above near 0. Thus it can be extended to a psh function on $\mathbb{C}^n$. Put

$$\varphi := \max\{\pi^* u + \log \|z\|, 2 \log \|z\|\}.$$

In [12], it was predicted that $e_n(\varphi) = 1$ if and only if u is of full Monge-Ampère mass, *i.e.,*

$$\int_{\mathbb{P}^{n-1}} \langle (\mathrm{dd}^c u + \omega_{n-1})^{n-1} \rangle = 1,$$

where for every closed positive $(1,1)$-current T, the expression $\langle T^n \rangle$ denotes the non-pluripolar product of $T, \ldots, T$ (n times). The conjecture was verified in [12] when u has analytic singularities, or e^u is continuous and $\{u = -\infty\}$ contained in a compact subset of an affine chart of $\mathbb{P}^{n-1}$. Let

$$\tilde{e}_k(u) := 1 - \int_{\mathbb{P}^{n-1}} \langle (\mathrm{dd}^c u + \omega_{n-1})^k \rangle \wedge \omega_{n-1}^{n-1-k}.$$

These numbers satisfy

$$0 \leq \tilde{e}_1(u) \leq \cdots \leq \tilde{e}_{n-1}(u) \leq 1 \tag{2}$$

by the monotonicity of non-pluripolar products. The following theorem is the main result in my joint-work [15] with Duc Thai Do (Hanoi) confirming in particular the above conjecture of Chi Li.

Theorem 1. *([15]) There is a constant $C > 0$ independent of u such that*

$$\tilde{e}_{k-1}(u) \leq e_k(\varphi) - 1 \leq C \tilde{e}_{k'}(u)$$

for every $1 \leq k \leq n$*, where* $k' := \min\{k, n-1\}$*. In particular,* $e_n(\varphi) = 1$ *if and only if* u *is of full Monge-Ampère mass.*

As in [12] or [15], one can deduce from Theorem 1 that even if two plurisubharmonmic germs are valuatively equivalent (at 0), then their higher Lelong numbers at 0 are not necessarily equal in general.

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On a construction of non-Kähler compact complex manifolds

Stéphanie Cupit-Foutou

Hopf manifolds are well-known examples of complex manifolds which are not Kähler whenever they are of dimension greater than 1. They can be defined as quotients of a punctured affine space acted on by an infinite cyclic group.

While considering some peculiar actions of complex abelian Lie groups on affine spaces, López de Medrano and Verjovsky constructed non-Kähler complex manifolds as quotients for such actions, generalizing Hopf manifolds ([9]). Their construction was in turn generalized first by Meersseman in [10] and even more by Bosio in [3] while considering the following actions (for $m > 2n$)

$$\begin{aligned} (\mathbb{C}^m \times \mathbb{C}^*) \times \mathbb{C}^n &\longrightarrow \mathbb{C}^n \\ (z, w, v) &\longmapsto (we^{\ell_1(z)}v_1, \ldots, we^{\ell_n(z)}v_n) \end{aligned} \tag{1}$$

where the ℓ_j's are given $\mathbb{C}$-linear forms on $\mathbb{C}^m$. Bosio characterized the open subsets U of $\mathbb{C}^n$ on which the abelian Lie group $\mathbb{C}^m$ acts freely and properly; he thus proved that the compact complex manifolds $U/\mathbb{C}^m$ he obtained were not Kähler by using a classical cohomological argument. These manifolds are now called LVMB-manifolds. See [3] for details.

As observed in [6], if the linear forms ℓ_j are all rational then the action (1) is the restriction of an algebraic action of the $2m$-dimensional algebraic torus on $\mathbb{C}^n$ to one of its closed cocompact subgroups isomorphic to $\mathbb{C}^m$. Moreover, the quotient $U/(\mathbb{C}^*)^{2m}$ is a complete toric algebraic variety over which the LVMB-manifold $U/\mathbb{C}^m$ fibers. Several other works have studied LVMB-manifolds drawing parallels with toric geometry. In particular, Battisti proved in [2] that LVMB-manifolds can be defined up to a fan supported on a *quasi-lattice* and an abelian complex Lie group. More recently, Katzarkov et al. introduced quantum toric stacks as a generalizations of simplicial complete toric varieties for fans supported on quasi-lattices; they proved that LVMB-manifolds fiber over such stacks ([8]; see also [1] for pioneering works).

The present note deals with an extension of this construction to the *spherical* setting with particular focus on the *horospherical* case. Specifically, take a connected reductive complex algebraic group G and $P \subset G$ a parabolic subgroup that is, G/P is a flag manifold. Let N be a finitely generated abelian subgroup of the $\mathbb{R}$-span of the cocharacter lattice of P, and let $\mathcal{F}$ be a *colored fan* supported on N together with an epimorphism $\rho : \mathbb{Z}^{\mathcal{F}(1)} \to N$. One can show that these data naturally yield an action (1) of some Lie group $\mathbb{C}^m$ on the G-module $V := \oplus_j H^0(G/P, L_j) \oplus \mathbb{C}^n$ for some n, the sum running over the line bundles associated to the Schubert divisors of G/P. Let $\widetilde{X} = \mathrm{Spec}(\mathrm{Cox}(G/P)) \times \mathbb{C}^n$ where $\mathrm{Cox}(G/P)$ denotes the Cox ring of the flag manifold G/P. Then the action (1) on V just mentioned restricts to an action on the subvariety $\widetilde{X}$ of V.

Theorem 1. *Suppose $\mathcal{F}$ is complete and simplicial. Then there exists an open smooth G-variety $U = U(\mathcal{F})$ of $\widetilde{X}$ on which $\mathbb{C}^m$ acts freely and properly discontinuously. The compact complex manifold $X(\mathcal{F}) = U/\mathbb{C}^m$ is non-Kähler and acted on by G. If N is a lattice, the complex manifold $X(\mathcal{F})$ fibers over a complete* horospherical *G-variety, that is a compactification of an homogeneous G-space G/H where H is an algebraic subgroup of G whose normalizer is the group P itself.*

To prove that the complex manifold $X(\mathcal{F})$ is not Kähler, it suffices to notice that its closed G-orbits are not projective. It follows readily from our construction

that $X(\mathcal{F})$ is embedded in a LVMB-manifold. When N is trivial, $X(\mathcal{F})$ is a compact non-projective homogeneous G-space. In particular, Hopf manifolds can be obtained this way with $G = SL_n$. It can be proved that none of the manifolds $X(\mathcal{F})$, except for the Hopf manifolds, are LVMB-manifolds. When the fan is *colorless*, the complex manifolds $X(\mathcal{F})$ are isomorphic to fiber products of the form $G \times^H Z$ where G/H is compact and non-projective and Z is a LVMB-manifold.

The following theorem is the horospherical analog of a theorem of Ishida proved for LVMB-manifolds ([7]).

Theorem 2. *The manifolds $X(\mathcal{F})$ constructed in Theorem 1 are transversally Kähler if and only if the fan $\mathcal{F}$ is polytopal.*

The two above theorems are stated and proved in [5]. The construction of the manifolds $X(\mathcal{F})$ is being extended to the *spherical* setting in [4]. In particular, the manifolds $X(\mathcal{F})$ we obtain therein fibers over non-algebraic stacks – an object we introduce as a generalization of spherical varieties.

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Parabolic Dynamics on K3 Surfaces: Archimedean Height Pairing and Canonical Currents

Junyu Cao

Let $\pi : X \to \mathbb{P}^1$ be an elliptic K3 surface, and let $T : X \to X$ be a fiber-preserving automorphism. For a regular value s of π, write $X_s = \pi^{-1}(s)$, and let ω_{FS} denote the Fubini–Study form on $\mathbb{P}^1$. We call T parabolic if the induced action on $\mathrm{H}^{1,1}(X, \mathbb{R})$ has infinite order. By Cantat's classification, this is the zero-entropy infinite order case for automorphisms of K3 surfaces [1]. The cohomological growth is quadratic, and the invariant nef ray is generated by the fiber class. Thus, for

a Kähler form ω on X, writing $\omega_s = \omega|_{X_s}$, the natural canonical current is the quadratically normalized limit

$$\eta_T(\omega) := \lim_{n\to\infty} \frac{(T^*)^n\omega}{n^2}.$$

The result discussed in this talk is that this limit exists as a current and has a continuous potential. More precisely, there are a constant $c > 0$ and a function $u \in C^0(\mathbb{P}^1)$, smooth away from the discriminant locus of π, such that

$$\eta_T(\omega) = \frac{1}{2}\pi^*\big(c\,\omega_{\mathrm{FS}} + \sqrt{-1}\partial\overline{\partial}u\big).$$

Thus the canonical current for parabolic dynamics is pulled back from the base of the elliptic fibration.

Set $\xi = T^*\omega - \omega$. A double telescoping identity reduces the convergence problem to understanding $T^*\xi - \xi$. The parabolic cohomological structure gives a constant $c > 0$ and a smooth real-valued function f on X such that

$$T^*\xi - \xi = c\,\pi^*\omega_{\mathrm{FS}} + \sqrt{-1}\partial\overline{\partial}f.$$

The key point is to split the function f into a base term and a coboundary. For a smooth $(1,1)$-form α with zero fiber integral, let $\phi_{\alpha,s}$ be the normalized solution of

$$\sqrt{-1}\partial\overline{\partial}\phi_{\alpha,s} + \alpha|_{X_s} = 0, \qquad \int_{X_s} \phi_{\alpha,s}\,\omega_s = 0.$$

These are the preferred potentials used by Filip–Tosatti [4]. Taking $\alpha = \xi$ and writing $\phi = \phi_\xi$, one obtains on the smooth locus

$$f + T^*\phi - \phi = \pi^*u$$

for a function u on the base. An averaging argument along the orbits of T, due to Tate [5], shows that u is bounded. Consequently, as currents on X,

$$T^*\xi - \xi = \pi^*\big(c\,\omega_{\mathrm{FS}} + \sqrt{-1}\partial\overline{\partial}u\big) + \sqrt{-1}\partial\overline{\partial}(\phi - T^*\phi)$$

where the first term is T-invariant and the second is a T-coboundary. Substitution into the telescoping formula proves the current convergence and gives the formula for $\eta_T(\omega)$ with $u \in L^\infty(\mathbb{P}^1)$.

It remains to prove continuity of u at the singular fibers. After normalizing $\int_{X_s}\omega = 1$, integration over smooth fibers gives

$$u(s) = \int_{X_s} f\,\omega + \int_{X_s} (T^*\phi - \phi)\,\omega.$$

The first term is a standard fiber integral and extends Hölder continuously by Barlet's theorem [6]. The second term is naturally expressed using an Archimedean height pairing $\langle\alpha,\beta\rangle(s) = \int_{X_s}\phi_{\alpha,s}\,\beta$; in the dynamical situation it is

$$\langle T^*\omega - \omega,\, (T^{-1})^*\omega - \omega\rangle(s).$$

The analytic input is the asymptotic behavior of this pairing in a one-parameter degeneration of Riemann surfaces [3]: for fiberwise cohomologically trivial smooth $(1,1)$-forms α, β,

$$\langle \alpha, \beta \rangle(s) - c_{\alpha,\beta} \log |s|^2 \in C^0.$$

When the central fiber is reduced, this coefficient is topological:

$$c_{\alpha,\beta} = v_\alpha^t M^+ v_\beta,$$

where M^+ is the Moore–Penrose pseudoinverse of the intersection matrix $M = (E_i \cdot E_j)$ of the irreducible components E_i of the central fiber, and $v_\alpha = (\int_{E_i} \alpha)_i$, $v_\beta = (\int_{E_i} \beta)_i$. The proof separates vertical divisor terms, handled by Poincaré–Lelong and fiber integrals, from a remainder controlled by the small-eigenvalue theorem of Dai–Yoshikawa [2] and the estimates of Filip–Tosatti [4].

Applied to the dynamical potential u, the height-pairing asymptotic shows that any singularity is logarithmic. Since the averaging argument already gives $u \in L^\infty$, the logarithmic coefficient must vanish. Therefore u extends continuously across the discriminant locus.

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A quantitative and pluripotential approach to complex hyperbolicity

Duc-Bao Nguyen

(joint work with Tien-Cuong Dinh, Duc-Viet Vu)

The main goal of this report is to introduce some numerical invariants that measured the hyperbolicity of complex manifolds. Recall that a compact complex manifold X is *Kobayashi hyperbolic* if and only if there exists no non-constant entire curve $f : \mathbb{C} \to X$.

The first motivation for us is Demailly's notion of algebraic hyperbolicity. A projective manifold X is *algebraically hyperbolic* if there exists a positive constant ϵ such that for any irreducible closed curve $C \subset X$, we have

$$-\chi(\widehat{C}) \geq \epsilon \int_C \omega,$$

where $\widehat{C}$ is a normalization of C and ω is a Hermitian metric on X. It is known that Kobayashi hyperbolicity implies algebraic hyperbolicity. However, the notion of algebraic hyperbolicity does not take account the information of entire curves.

The second motivation is Nevanlinna current associated to entire curves. Let $f : \mathbb{C} \to X$ be an entire curve. Then, we can associated to f a positive closed current $T[f]$ that capture some information of f. However, in general, such current is not unique and may possess pathological properties.

The third motivation is the theory of directed currents in holomorphic dynamics and foliation theory, which can describe the tangent structure of curves.

The main idea is to replace the curve C in the definition of algebraically hyperbolic by some class of currents that contains Nevanlinna currents. The number ϵ will serve as the invariant we want to define.

To do that, we recall the notion of Demailly Semple tower, which is a natural compactification of jet spaces (see [1]). This allows to take into account of higher derivatives.

Let (X, V) be a complex directed manifold (i.e. V is a holomorphic subbundle of T_X). We define the projectivized 1-jet of (X, V) by putting $\widetilde{X} = \mathbb{P}(V)$ and

$$\widetilde{V}_{(x,[v])} := \Big\{\xi \in T_{(x,[v])}\widetilde{X} : \pi_*(\xi) \in \mathbb{C}v\Big\} \text{ for } [v] \text{ such that } \mathbb{C}v \subset V_x \subset T_xX.$$

The Demailly-Semple tower is defined inductively by putting: $(X_0, V_0) := (X, T_X)$ and $(X_{k+1}, V_{k+1}) := \widetilde{X_k}, \widetilde{V_k}$ for $k \in \mathbb{Z}^{\geq 0}$. We denote by $\pi_{k,l}$ is the natural projection from X_k to X_l for $k > l$.

A positive closed current T of bi-dimension $(1, 1)$ is k-liftable if there exists a positive closed current $T_{[k]}$ on X_k such that $T_{[k]}$ is directed by V_k and $(\pi_{k,l})_*T_{[k]} = T$. Such $T_{[k]}$ is called an almost k-lifting of T. A current T is called ∞-liftable if it is k-liftable for all k. It is not hard to see that if T is k-liftable then T is l-liftable for all $l < k$.

For each $k \geq 2$, there is a divisor $D_k := \mathbb{P}(T_{X_{k-1}/X_{k-2}})$. We define the regular and singular locus of X_k by putting

$$X_k^{[\mathrm{sing}]} := \bigcup_{k \geq j} \pi_{k,j}^{-1}(D_j), \qquad X_k^{[\mathrm{reg}]} := X_k \setminus X_k^{[\mathrm{sing}]}.$$

If T is a k-liftable current and $T_{[k]}$ is an almost k-lifting of T, then $\mathbf{1}_{X_k^{[\mathrm{reg}]}} T_{[k]}$ is also an almost k-lifting of T (this is due to the directed structure). Such current is called a k-lifting of T.

Define $u_k := \mathcal{O}_{X_k}(1)$. We have $D_k = u_k - \pi_{k,j-1}^* u_{k-1}$. We have the following observation: for C is an irreducible closed curve inside X, $C_{[k]} \cdot u_k$ is increasing to $-\chi(\widehat{C})$ (we have the equality for k big enough). We put $\chi_k(C) := -C_{[k]} \cdot u_k$. For a k-liftable current T, we put

$$\chi_k(T) := \sup\Big\{ - T_{[k]} \cdot u_k : T_{[k]} \text{ is a } k\text{-lifing of } T\Big\},$$

and $\chi_\infty(T) := \lim_{k\to\infty} \chi_k(T)$ for ∞-liftable current T. We can prove that $\chi_k([C]) = \chi_k(T)$ for every irreducible closed curve C. Thus, $\chi_\infty([C]) = \chi(\widehat{C})$.

We define the hyperbolic indices $\mathrm{hyp}^{(k)}(X,\omega) := \inf -\chi_k(T)/\|T\|_\omega$ where infimum runs over all k-liftable current T, and $\mathrm{hyp}^{(\infty)}(X,\omega) := \sup_k \mathrm{hyp}^{(k)}(X,\omega)$.

Let $X \subset \mathbb{P}^n$ be a projective manifold and $f : \mathbb{C} \to X$ be a non-constant entire curve. Let $f_{[k]}$ denotes the k-lift of f. For each k, we can construct a k-liftable current $T[f]$ which is a Nevanlinna current of f such that $T[f_{[k]}]$ is an almost k-lifting of f. In this case, we can prove that there exists a constant $C(k)$ such that $-\chi_k(T[f])/\|T[f]\|_\omega \leq C(k)$ where $\omega := \omega_{\mathrm{FS}}|_X$. Thus, if $\mathrm{hyp}^{(k)}(X,\omega)$ big enough, then X is Kobayashi hyperbolic.

The main result is an estimates for hyperbolic indices of general hypersurfaces in $\mathbb{P}^n$, which can be view as a quantitative version of the Kobayashi conjecture.

Theorem 1. Let X_d be a general hypersurfaces of degree d in $\mathbb{P}^n$. Then

$$\mathrm{hyp}^{(n)}(X_d, \omega_{\mathrm{FS}}) \geq \delta(n)d - \lambda(n),$$

for constants δ, λ depending only on n.

The proof of this theorem is a combination of previous results on Kobayashi conjecture (see [2, 4]) and the theory of density currents (see [3]).

The introduction of liftable currents also allows us to formulate an analytic approach to the Kobayashi conjecture: The first step involves developing an analogous theory in the logarithmic setting. Next, one could employ deformation methods to construct examples of low-degree hypersurfaces with positive hyperbolic indices. The conjecture would then follow, provided that a semicontinuity theorem in the (countable) Zariski topology for the hyperbolic indices can be established. We expect the analytical softness of liftable currents to play a crucial role in this strategy.

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Plurisubharmonic upper enveloppes and nef divisors with isolated LC centres

Mihai Păun

(joint work with Junyan Cao)

Extension theorems for twisted pluricanonical forms defined on algebraic subspaces of projective manifolds are of paramount importance for applications in birational geometry. In my lecture I discussed a few results around this topic, which are part of a joint work in progress with J. Cao.

Consider the following setup: X is a projective manifold and

$$S = \sum_{i=1}^{N} S_i, \quad B := \sum_{j=1}^{M} \mu_j B_j$$

are two divisors such that $S + B$ *snc*. Moreover let D be an effective divisor such that $D \simeq K_X + S + B$ and moreover $|S| \subset |D| \subset |S + B|$, where $|D|$ denotes the support of the divisor D.

The following problem was proposed in [1].

Conjecture 1. *Let $m_0 \in \mathbb{N}$ and let $s_A \neq 0$ section of an ample line bundle. Assume that $\mu_j \in]0, 1[\cap\mathbb{Q}$ and consider $u \in H^0\big(S, m_0(K_X + S + B)|_S\big)$ such that $u^{\otimes k} \otimes s_A$ extends to X, for all k. Then so does u.*

As explained in [1], the motivation for this conjecture is to establish the equality between the numerical dimension and the Kodaira dimension of the canonical bundle of X, provided that the latter is positive or zero. In *loc. cit.* the following result is established.

Theorem 2 ([1])**.** *The conjecture has a positive answer provided that the support of S is non-singular, e.g. if $N = 1$.*

Even if this is not explicitly stated, the following notion plays a key role in the proof of the above result.

Definition 3. *Let $\{\alpha\} \in H^{1,1}(X, \mathbb{R})$ be a pseudo-effective class. We denote by V_α its psh upper envelope, defined as follows*

$$V_\alpha := \sup\big\{f \in L^1(X, \mathbb{R}) : \alpha + dd^c f \geq 0, \quad f \leq 0\big\}.$$

In case $\{\alpha\} = c_1(L)$ is the first Chern class of a line bundle, one can imagine an "algebraic" version of this notion, in which one only takes into account functions f given by log of norms of holomorphic sections of powers of L, appropriately normalised. During the lecture I reviewed the main steps in the proof of 2 from this perspective: the interplay between the upper envelope V_α for

$$\{\alpha\} = c_1(K_X + S + B + \epsilon A)$$

and its algebraic version.

The case $N \geq 2$ in which the centre S has at least two components appears to be much more subtle. Before attacking the problem of extending pluricanonical sections, the first step would be to establish an Ohsawa-Takegoshi type extension in this context. In this respect we have the following statement, due to Cao, Demailly and Matsumura.

Theorem 4. [2] *Let $(E, h_E) \to \mathcal{X}$ be a holomorphic line bundle, $\mathcal{X}$ is holomorphically convex. Let ψ be a quasi-psh function with log poles, such that $\exists\, \delta > 0$ with*

$$\sqrt{-1}\Theta(E, h_E) + (1 + t\delta)dd^c\psi \geq 0 \qquad \textit{for all } t \in [0, 1].$$

Then the restriction map below is surjective

$$H^0\big(\mathcal{X}, (K_{\mathcal{X}} + E) \otimes \mathcal{I}(h_E)\big) \to H^0\big(X, (K_{\mathcal{X}} + E) \otimes \mathcal{I}(h_E)/\mathcal{I}(e^{-\psi} h_E)\big).$$

Inspired by the very interesting method of proof in 4, we obtain in [3] the following new result.

Theorem 5. [3] *Let Δ_Y be an effective $\mathbb{Q}$-divisor on a projective manifold Y, such that the following hold.*

- *There exists $\pi : X \to Y$ birational, $\pi^\star(K_Y + \lambda_0 \Delta) = K_X + S + B$ for some $\lambda_0 \in \mathbb{Q}_+$ and moreover $\pi(S) = \{y_0\}$. We assume that $K_Y + \lambda_0 \Delta$ is nef.*
- *There exists an effective, divisor D on X, such that $D \simeq K_X + S + B$ and the support condition in the setup holds.*

Let V_k be the upper psh envelope for the class $\{\alpha_k\} := c_1(K_Y + \lambda_0 \Delta + 1/kA)$. Assume moreover that $\forall\, \epsilon > 0$ there exists $C_\epsilon \leq 0$ so that we have

$$V_k(y_0) \geq \epsilon \log \frac{1}{k} + C_\epsilon \tag{1}$$

for all k. Then u extends.

In other words, we show that *if* the ratio between value at the center y_0 of the psh upper envelopes corresponding to the classes approximating $c_1(K_Y + \lambda_0 \Delta_Y)$ and $\log 1/k$ is small enough, then there exists a section of a multiple of $K_Y + \lambda_0 \Delta_Y$ which is non-vanishing at y_0. Another result established in [3] is that the property (1) is birational, in the sense that it holds for a line bundle $L \to Y$ if and only if it holds for $\pi^\star(L)$.

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Kodaira-Iitaka dimension and multiplicity: an analytic perspective

Siarhei Finski

We report on the recent work [4] where we give analytic formulas for the Kodaira-Iitaka dimension and the multiplicity of graded linear series.

We fix a connected complex projective manifold X, $\dim_{\mathbb{C}} X = n$, and a holomorphic line bundle L over X. Consider a *(graded) linear series* $W = \oplus_{k=0}^{\infty} W_k$, $W_0 = \mathbb{C}$, defined as a subring of the associated *section ring*

$$R(X, L) := \oplus_{k=0}^{\infty} H^0(X, L^{\otimes k}).$$

To exclude some trivial cases, we make an additional assumption that $W \neq \mathbb{C}$, implying in particular that L is *effective*, i.e., $H^0(X, L^{\otimes k}) \neq \{0\}$ for $k \in \mathbb{N}^*$ large

enough. Without loosing the generality (as otherwise one could restrict to a tensor power of L), we further assume

$$W_k \neq \{0\}, \quad \text{for } k \in \mathbb{N} \text{ large enough.}$$

Kaveh-Khovanskii in [6, Theorem 3.4] established that there is $\kappa(W) \in \{0, 1, \ldots, n\}$, called the *Kodaira-Iitaka dimension* of W, so that the following limit

$$\mathrm{vol}_\kappa(W) := \lim_{k\to\infty} \frac{\dim W_k}{k^{\kappa(W)}/\kappa(W)!}$$

exists and is positive. We call the above limit the *multiplicity* of W.

Our analytic formulas for the Kodaira-Iitaka dimension and the multiplicity will be based on a notion of a plurisubharmonic (*psh*) envelope. The latter definition can be traced back to Siciak [8]; in the current context of linear series it was introduced by Hisamoto [5]. To recall it, for any norm N_k on W_k, we associate a singular metric $FS(N_k)$ on $L^{\otimes k}$ as

$$|l|_{FS(N_k),x} = \inf_{\substack{s\in W_k \\ s(x)=l}} \|s\|_{N_k}, \quad \text{for any } x \in X, l \in L_x^{\otimes k},$$

with the convention that the infimum equals $+\infty$ if there is no $s \in W_k$ such that $s(x) = l$.

It is easy to see that for any continuous metric h^L on L, the sequence of Fubini-Study (singular) metrics associated with the sup-norms $\mathrm{Ban}_k^\infty[W](h^L)$ on W_k, $k \in \mathbb{N}^*$, is submultiplicative. By the Fekete's lemma, the following limit

$$P[W, h^L] = \Big(\lim_{k\to\infty} FS(\mathrm{Ban}_k^\infty[W](h^L))^{\frac{1}{k}}\Big)_*$$

exists as a (singular) metric on L, where $(\cdot)_*$ is the lower semicontinuous regularization. Standard results imply that $P[W, h^L]$ has a psh potential.

Theorem. For any graded linear series $W \subset R(X, L)$, we have

$$\kappa(W) = \max\Big\{i \in \{0, 1, \ldots, n\} : c_1(L, P[W, h^L])^i \wedge \omega^{n-i} \neq 0\Big\},$$

where ω is any fixed Kähler form on X.

Our next result gives an analytic expression for the multiplicity of a graded linear series. It will involve another important invariant of the linear series, which we now recall.

Recall that any linear series defines the rational maps $X \dashrightarrow \mathbb{P}(W_m^*)$ that send a point $x \in X$ to the hyperplane $H_{m,x} \subset W_m$ consisting of all sections in W_m that vanish at x (and are undefined whenever all sections in W_m vanish at x). We denote by Y_m the closure of the image of this map. It is well-known that the rational maps

$$\varphi_m : X \dashrightarrow Y_m$$

are birationally equivalent for m large enough, and

$$\kappa(W) = \dim(Y_m), \quad \text{for } m \in \mathbb{N} \text{ large enough.}$$

We denote by $[\omega] \in H^{1,1}(X) \cap H^2(X, \mathbb{R})$ the cohomology class of ω, and let $\varphi_*[\omega]^{n-\kappa(W)}$ be the fiber integral of ω along a general fiber of φ_m for m large enough. It is possible to show that this quantity depends on the cohomology class $[\omega]$ of ω, on X, L, W, but not on m, as indicated by the notations.

Theorem. For any graded linear series $W \subset R(X, L)$, we have

$$\mathrm{vol}_\kappa(W) = \frac{1}{\varphi_*[\omega]^{n-\kappa(W)}} \int_X c_1(L, P[W, h^L])^{\kappa(W)} \wedge \omega^{n-\kappa(W)}.$$

We now turn our attention to a pointwise refinement of the above results, which we formulate in the language of the partial Bergman kernels. We fix a continuous metric h^L on L and a positive Borel measure μ, which – to simplify the presentation – we assume to have a full support. We denote by $\mathrm{Hilb}_k[W](h^L, \mu)$ the positive definite form on W_k defined for $s_1, s_2 \in W_k$ as follows

$$\langle s_1, s_2 \rangle_{\mathrm{Hilb}_k[W](h^L,\mu)} = \int_X \langle s_1(x), s_2(x) \rangle_{(h^L)^k} \cdot d\mu(x).$$

We say that a measure μ is *Bernstein-Markov* with respect to W and h^L, if for each $\epsilon > 0$, there is $k_0 \in \mathbb{N}$, so that

$$\mathrm{Ban}_k^\infty[W](h^L) \leq C \cdot \exp(\epsilon k) \cdot \mathrm{Hilb}_k[W](h^L, \mu), \quad \text{for any } k \geq k_0.$$

We define a sequence of measures on X as follows

$$\mu_k[W, \mu, h^L] := \frac{1}{k^{\kappa(W)}} B_k[W, \mu, h^L](x, x) \cdot d\mu(x),$$

where $B_k[W, \mu, h^L](x, x) \geq 0$ is the partial Bergman (or Christoffel-Darboux) kernel, defined as

$$B_k[W, \mu, h^L](x, x) := \sum_{i=1}^{m_k} |s_i(x)|^2_{(h^L)^k},$$

where s_i, $i = 1, \ldots, m_k$ is an orthonormal basis of $(W_k, \mathrm{Hilb}_k[W](h^L, \mu))$ and $m_k := \dim W_k$.

It was established by Berman-Boucksom-Witt Nyström [1] that when $W = R(X, L)$ and L is a big line bundle, $\mu_k[W, \mu, h^L]$ converge weakly towards the equilibrium measure of h^L; see also Bloom-Levenberg [2], [3].

When X admits an action of a compact connected Lie group G which can be lifted to a holomorphic action on L, we can consider the G-invariant sections: $W_k := H^0(X, L^{\otimes k})^G$. Under certain assumptions, Ma-Zhang in [7] established that $\mu_k[W, \mu, h^L]$ converge weakly, as $k \to \infty$, to an explicit measure.

From these two examples, together with many others, it is natural to expect that the weak convergence holds in general. However, this expectation is false: in [4], we construct an explicit (and easy) counterexample.

Nevertheless, a weaker form of the weak convergence still holds. To state this result, we fix a complex analytic space Z and a holomorphic map $\rho : X \to Z$, which we assume to factorize, for $m \in \mathbb{N}$ divisible enough, through the rational

maps φ_m and $\psi_m : Y_m \to Z$. One can show that the following measure on Z does not depend on the choice of ω:

$$\mu_{\mathrm{eq}}[W, Z, h^L] := \frac{1}{\varphi_*[\omega]^{n-\kappa(W)}} \rho_* \Big(c_1(L, P[W, h^L])^{\kappa(W)} \wedge \omega^{n-\kappa(W)} \Big).$$

Theorem. If the measure μ is Bernstein-Markov with respect to W and h^L, then, as $k \to \infty$, the sequence of measures $\rho_*(\mu_k[W, \mu, h^L])$ on Z converges weakly towards $\mu_{\mathrm{eq}}[W, Z, h^L]$.

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On the metric SYZ conjecture

Yang Li

In the 1990's, motivated by mirror symmetry, Strominger-Yau-Zaslow proposed the conjecture that polarised Calabi-Yau manifolds near the maximal degeneration should admit a special Lagrangian torus fibration at least on the generic region.

More precisely the metric SYZ conjecture says the following: Let $\pi : (X, L) \to \mathbb{D}^*$ be a polarised maximal degeneration of compact Calabi-Yau manifolds. Given any $0 < \delta \ll 1$, for sufficiently small t depending on δ, there exists a special Lagrangian T^n-fibration with respect to the Calabi-Yau structure $(X_t, \omega_{CY,t}, \Omega_t)$, on an open subset of X_t whose normalised Calabi-Yau measure is at least $1 - \delta$.

A prototypical example of maximal degeneration is the Fermat family

$$X_t = \{Z_0 Z_1 \dots Z_{n+1} + t \sum_0^{n+1} Z_i^{n+2} = 0\} \subset \mathbb{CP}^n, \quad 0 < |t| \ll 1.$$

It was well understood that the conjecture would follow if the Calabi-Yau metric on the generic region can be asymptotically approximated by the semiflat metrics:

$$\omega_{CY,t} \approx dd^c u(\frac{\log |z_1|}{\log |t|}, \dots \frac{\log |z_n|}{\log |t|})$$

where u solves the *real Monge-Ampère equation* $\det(D^2u) = \text{const}$ on the essential skeleton $Sk(X)$. However, the naïve attempt to formulate the real Monge-Ampère equation on $Sk(X)$ runs into many difficulties. I will discuss how non-archimedean pluripotential theory provides a path to go around these problems. This reduces the metric SYZ conjecture to a regularity question about the non-archimedean Monge-Ampère equation.

Let $K = \mathbb{C}(\!(t)\!)$, and consider the base changed family X_K over the formal punctured disc. For any $l \geq 0$, we say a K-basis $\{\theta_\alpha^l\} \subset H^0(X_K, lL)$ satisfies *valuative independence*, if for every $x \in Sk(X)$ viewed as the valuation val_x, we have

$$val_x(\sum_\alpha a_\alpha \theta_\alpha^l) = \min_{\alpha: a_\alpha \neq 0} (val(a_\alpha) + val_x(\theta_\alpha^l)), \quad \forall a_\alpha \in K.$$

This canonical basis is partly motivated by the Gross-Siebert theta functions in mirror symmetry, and very recently Blum-Liu gave a birational geometric proof for its existence in great generality. It turns out that the existence of a valuative independent basis for the ring of sections of the polarisation line bundle, provides the key algebro-geometric input to resolve this question, leading to a proof of the metric SYZ conjecture.

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Multiplier rigidity for complex Hénon maps

Romain Dujardin

(joint work with Serge Cantat)

1. Introduction and problem setting

The multiplier rigidity problem investigates whether a dynamical system within a specific class is completely determined by the multipliers (eigenvalues) of its periodic orbits. This problem sits within the broader domain of spectral rigidity problems, which spans geometry and dynamics.

In one-dimensional rational dynamics, multipliers are essential tools in the study of moduli spaces as they define algebraic functions there. While early work of McMullen [9] established local multiplier rigidity for one-dimensional rational functions, a complete characterization of isospectral maps remains open.

In joint work with Serge Cantat [5] (see also [4]), we address these questions in the setting of polynomial automorphisms of $\mathbb{C}^2$ with a dynamical degree $\lambda_1(f) > 1$ (known as *loxodromic* automorphisms). A fundamental structure theorem by

Friedland and Milnor [7] asserts that any such automorphism is conjugate to a finite composition of monic and centered generalized Hénon maps:

$$g = h_k \circ \cdots \circ h_1 \quad \text{where} \quad h_i(x, y) = (a_i y + p_i(x), x)$$

This composition yields two invariants: the multi-degree $\mathbf{d} = (d_1, \ldots, d_k)$ and the multi-Jacobian $\mathbf{a} = (a_1, \ldots, a_k)$, and the dynamical degree is $\lambda(f) = d = \prod_{i=1}^{k} \deg(p_i)$. The parameter space of such compositions with fixed multidegree $\mathbf{d}$ and multi-Jacobian $\mathbf{a}$ is denoted by $\mathcal{H}^k_{\mathbf{d},\mathbf{a}}$.

2. Main Rigidity Theorems

For a periodic orbit of exact period n, its multipliers λ_1, λ_2 are the eigenvalues of the differential Df^n. For orbits of saddle type, these eigenvalues are separated into stable and unstable components ($|\lambda^s| < 1 < |\lambda^u|$). They can be organized into the *trace spectrum* (all traces $\lambda_1 + \lambda_2$ across all periods) and the *unstable multiplier spectrum* (only the unstable eigenvalues λ^u of saddle cycles).

We establish several rigidity results:

Theorem A (Hénon case): *A single complex Hénon map ($k = 1$) is determined up to finitely many choices by its trace spectrum or its unstable multiplier spectrum. Locally, any Hénon map close to a base map f_0 with an identical spectrum must equal f_0.*

Theorem B (Compositions of Hénon maps): *The global conjugacy class of a Hénon composition is determined up to finitely many possibilities by its multi-degree, fixed multi-Jacobian, and either its trace or unstable multiplier spectrum.*

Theorem C (C^1 Rigidity): *The C^1 conjugacy class of a complex Hénon map or composition (with fixed multidegree and multi-Jacobian) is finite.*

3. Stable Algebraic Families and Boundedness

The proofs for Theorems A, B, and C rely on a global structural stability result inspired by McMullen's approach:

Theorem D (No Stable Algebraic Families): *Any stable irreducible algebraic family of complex Hénon maps is trivial. Likewise, any stable irreducible algebraic family of compositions of Hénon maps with a fixed multi-Jacobian is trivial.*

The stability alluded to here is the weak J^* stabilty introduced by Dujardin-Lyubich. To connect this theorem to spectral rigidity, we define $\Lambda(f_0)$ as the set of maps sharing the same spectrum as a base map f_0 and show that it is a stable algebraic family. For the trace spectrum, $\Lambda(f_0)$ is explicitly algebraic, but is is not trivial that it defines a stable family For the unstable multiplier spectrum, it is the exact opposite: stability follows from the definition, but algebraicity does not, since the notion of begin a saddle point is not an algebraic one.

4. Asymptotic Behavior of Lyapunov Exponents

The overarching mechanism behind Theorem D is that any hypothetical non-trivial stable algebraic family must be unbounded in the parameter space, giving rise to degeneration phenomena at infinity.

In one-dimensional polynomial dynamics, the degeneracy of a polynomial p can be quantified using its maximal escape rate $M(p) = \max\{G_p(c) : p'(c) = 0\}$, where G_p is the dynamical Green function. The Manning-Przytycki formula [8, 10] relates this escape rate directly to the Lyapunov exponent $\chi(\mu_p)$ of the equilibrium measure:

$$\log d + M(p) \leq \chi(\mu_p) \leq \log d + (d-1)M(p)$$

In a stable one-dimensional algebraic family, the multipliers of repelling periodic points must remain constant. Because periodic points are equidistributed towards the equilibrium measure, the Lyapunov exponents of typical periodic orbits closely approximate $\chi(\mu_p)$. The lower bound then implies that $M(p)$ must remain bounded, so the only possibility for a stable family is to be a finite set.

To replicate this logic in $\mathbb{C}^2$, we define a similar measure of degeneration for Hénon compositions:

$$M(f) = \max(M(p_i), i = 1, \ldots, k)$$

where $M(p_i)$ represents the one-dimensional escape rates of the individual polynomial factors. Every loxodromic map possesses a canonical equilibrium measure μ_f with positive and negative Lyapunov exponents $\chi^+(\mu_f)$ and $\chi^-(\mu_f)$ (Bedford-Lyubich-Smillie [1]). Then the central technical estimate of the paper is:

Theorem E (Divergence of Lyapunov Exponents): *For a given multidegree* $\mathbf{d}$ *and multi-Jacobian* $\mathbf{a}$*, there exist positive constants* C_1, C_2 *such that:*

$$C_1 M(f) \leq \chi^+(\mu_f) \leq C_2 M(f)$$

for all $f \in \mathcal{H}^k_{\mathbf{d},\mathbf{a}}$ *with* $M(f) \geq 1$.

The proof is based on an exact formula for $\chi^+(\mu_f)$ due to Bedford and Smillie [2] based on the notion of "unstable critical point". Analyzing the geometry of the unstable current T^-, alongside classical bounds one-dimensional bounds of Branner and Hubbard [3] for the geometry of polynomials in one dimension we are able to locate these unstable critical points and obtain the desired estimate.

When applied to Theorem D, if a stable algebraic family were unbounded, $M(f)$ would diverge to infinity, forcing the Lyapunov exponent $\chi^+(\mu_f)$ to diverge concurrently. However, stability forces the multipliers—and consequently the Lyapunov exponents of the dense saddle orbits—to remain entirely constant, which provides the necessary contradiction.

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Effective equidistribution of periodic points in holomorphic dynamics

Lucas Kaufmann
(joint work with Henry de Thélin, Tien-Cuong Dinh)

Equidistribution phenomena arise naturally in many areas of mathematics. The typical picture is that a sequence of points in a space X, defined in a natural way, converges to a limiting distribution, namely a probability measure on X. In the context of dynamical systems, two natural examples are given by iterated pre-images of a point and by periodic points whose period tends to infinity.

Let $f : \mathbb{P}^k \to \mathbb{P}^k$ be a non-invertible holomorphic endomorphism of complex projective space of dimension k. Such a map is given in homogeneous coordinates by homogeneous pmynomials of degree $d \geq 2$ without common factors. The equilibrium measure of f can be defined as

$$\mu = \lim_{n\to\infty} \frac{1}{d^{nk}} (f^n)^* \omega_{\mathrm{FS}}^k,$$

where ω_{FS} is the Fubini-Study form on $\mathbb{P}^k$ normalized so that ω_{FS}^k has volume one.

The equidistribution of pre-images was established by Brolin [3], Freire–Lopes–Mañé [8], and Lyubich [9] in dimension one, and by Briend–Duval [2] and Dinh–Sibony [4] in arbitrary dimensions. They showed that the iterated pre-images of a non-exceptional point converge to the equilibrium measure of f. More precisely, there exists an algebraic exceptional set $\mathscr{E} \subset \mathbb{P}^k$ such that

$$\lim_{n\to\infty} \frac{1}{d^{kn}} \sum_{x\in f^{-n}(a)} \delta_x = \mu \quad \text{if and only if} \quad a \in \mathbb{P}^k \setminus \mathscr{E}, \tag{1}$$

where the convergence is in the weak sense.

These convergence results were later quantified by Dinh–Sibony (2010), who proved that equidistribution occurs at an exponential rate with respect to the number of iterations. More precisely, we have:

Theorem 1 (Dinh-Sibony). *Let f be a holomorphic endomorphism of algebraic degree $d \geq 2$ of $\mathbb{P}^k$. Let μ be the equilibrium measure of f and $1 < \lambda < d$ a constant. There is an invariant proper analytic subset E_λ, possibly empty, of $\mathbb{P}^k$ such that if a is a point outside of E_λ and if φ is a $\mathscr{C}^\alpha$ function on $\mathbb{P}^k$ with $0 < \alpha \leq 2$, then*

$$\left|\langle d^{-kn}(f^n)^*(\delta_a) - \mu, \varphi\rangle\right| \leq A\Big[1 + \log^+ \frac{1}{\operatorname{dist}(a, E_\lambda)}\Big]^{\alpha/2} \|\varphi\|_{\mathscr{C}^\alpha} \lambda^{-\alpha n/2},$$

where $A > 0$ is a constant independent of n, a, φ and $\log^+(\cdot) := \max(0, \log(\cdot))$.

Concerning periodic points, it has been known since the above mentioned works of Lyubich in dimension one and Briend–Duval [1] in higher dimension that they also equidistribute toward the equilibrium measure. More precisely, for each $n \geq 1$, let $P_n := \{x \in \mathbb{P}^k : f^n(x) = x\}$ be the set of periodic points of period n of f. Then,

$$\lim_{n\to\infty} \frac{1}{d^{kn}} \sum_{a \in P_n} \delta_a = \mu, \tag{2}$$

where again the convergence is in the weak sense. Differently from the case of preimages, the estimation on the speed of convergence in (2) is a more challenging problem. When $k = 1$, Favre–Rivera-Letelier [6] obtained an exponential speed of convergence when f is defined over a number field as a corollary of general quantitative arithmetic equidistribution theorems, see also [10] for further refined quantitative results of arithmetic nature. In higher dimensions, Yap [11] obtained an exponential speed of convergence for endomorphisms of $\mathbb{P}^2$ defined over a number field and in [5], Dinh and Yap extended this result to higher dimensions, still for maps defined over a number field.

In this talk, I presented a result on the speed of convergence of periodic points for all endomorphisms. The result is new even in dimension one. Our proof also allows us to prescribe extra dynamical condition on the periodic points and still have exponential convergence.

Theorem 2 (de Thélin-Dinh-Kaufmann). *Let f be a holomorphic endomorphism of $\mathbb{P}^k$ of algebraic degree $d \geq 2$, μ be its equilibrium measure and $J_k := supp(\mu)$ be its small Julia set. Then, as n tends to infinity, the repelling periodic points of period n of f on J_k are equidistributed with respect to μ at an exponential rate.*

More precisely, for every constants $0 < \gamma < 1$ and $0 < \alpha \leq 1$, there exist a constant $0 < \xi < 1$ independent of α and another constant $A_\alpha > 0$ such that the following holds.

Let P_n be the set of periodic points of period n of f. Let $P_{n,\gamma}$ be the set of points $a \in P_n \cap J_k$ such that $\|Df^n(a)^{-1}\| \leq d^{-\frac{1-\gamma}{2}n}$ and Q_n be any set such that $P_{n,\gamma} \subset Q_n \subset P_n$, counting multiplicities or not. Then, we have

$$\Big|\Big\langle \frac{1}{d^{kn}} \sum_{a \in Q_n} \delta_a - \mu, \phi \Big\rangle\Big| \leq A_\alpha \xi^{\alpha n} \|\phi\|_{\mathscr{C}^\alpha},$$

for any $\mathscr{C}^\alpha$ test function ϕ on $\mathbb{P}^k$, where δ_a denotes the Dirac mass at a.

When $k = 1$ every repelling periodic point is in the Julia set. Moreover, the Fatou-Shishikura inequality asserts that there are at most $2d - 2$ nonrepelling cycles. However, when $k \geq 2$, Fornaess-Sibony showed that some endomorphisms may have infinitely many repelling periodic points outside J_k, see [7]. Our proof allows us to estimate the number of such points.

Corollary 3. *Let A_n denote the number of non-repelling periodic points of order n of f and B_n the number of periodic points of order n outside the small Julia set J_k, counting multiplicities. Then, there exists a constant $0 < \xi < 1$ such that $A_n = O(\xi^n d^{kn})$ and $B_n = O(\xi^n d^{kn})$ as n tends to infinity.*

The proof of the above theorem uses a geometric method inspired by the approach of Lyubich combined with analytic tools from pluripotential theory. The core of the proof lies in the construction of good inverse branches with controlled geometry. To achieve this, we consider a regular square tiling in affine charts of $\mathbb{P}^k$, consisting of cells of size $r = d^{-\gamma n}$ for some constant $\gamma > 0$. An important difficulty is to estimate the mass with respect to μ of the cells touching the post-criticial set PC_n of f of order n, a hypersurface outside of which local inverse branches of f^n are well defined. Since the degree of PC_n grows exponentially with n, such estimate is non-trivial and is not a simple consequence of the fact that μ is moderate (meaning that it satisfies a version of Skoda's exponential integrability theorem).

Another key part of the argument consists in identifying an appropriate notion of good inverse branches that are sufficiently contracting, and in establishing a way to count such branches that send a cell strictly inside itself, which will ultimately produce a periodic point. The direct use of the mixing properties for a large iterate f^n turns out to be insufficient. Indeed, although all the relevant terms (the proportion of bad inverse branches, the mass of small exceptional sets, etc.) are individually exponentially small, the size of the cells is also exponentially small and comparing these quantities in a way that yields effective control is delicate. To overcome this obstacle, we use a two-step argument, based on a splitting of the iterate $f^n = f^{\lfloor \zeta n \rfloor} \circ f^{\lceil (1-\zeta) n \rceil}$ where $0 < \zeta < 1$ is a parameter to be chosen appropriately. This allows us to exploit the quantitative equidistribution of preimages at both steps, ultimately yielding the desired exponential rate.

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Siu's analyticity theorem for positive pluriharmonic currents

Viet-Anh Nguyen

(joint work with Tien-Cuong Dinh)

Let X be a complex manifold of dimension n. Let T be a positive closed current of bi-dimension (q,q) on X and $\nu(T,x)$ denote the Lelong number of T at a point $x \in X$. A classical theorem by Siu says that the function $x \mapsto \nu(T,x)$ is upper semi-continuous for the analytic Zariski topology on X. More precisely, the super-level set $E_c := \{\nu(T,\cdot) \geq c\}$ is a (possibly empty) analytic subset of dimension at most q of X for every constant $c > 0$. Furthermore, we have the following Siu's decomposition

$$T = \sum_{i \in I} \lambda_i [V_i] + T_0,$$

where $\{V_i\}_{i \in I}$ is a (possibly empty) finite or countable family of analytic subsets of dimension q on X, $\lambda_i \in \mathbb{R}^+$, and T_0 is a positive pluriharmonic current of bidimension (q,q) on X whose Lelong number vanishes outside a finite or countable union of analytic subsets of X of dimension less than q, see [2, 8] for more details. In this work, we study the following long standing problem which is still open.

Problem 1. *Let T be any positive pluriharmonic current on a complex projective, or more generally, a compact Kähler manifold X of dimension n. Is Siu's theorem true for T ?*

The following is our main theorem which solves the above problem for the case of bi-dimension $(1,1)$ currents.

Theorem 2. *Let X be a complex projective manifold. Let T be a positive pluriharmonic current of bidimension $(1,1)$ on X. Then for any constant $c > 0$ the set E_c of points of X where the Lelong number $\nu(T,\cdot)$ of T is larger or equal to c is a (possibly empty) analytic subset of dimension at most 1 of X. Moreover, we have the decomposition*

$$T = \sum_{i \in I} \lambda_i [V_i] + T_0,$$

where $\{V_i\}_{i\in I}$ *is a (possibly empty) finite or countable family of compact analytic curves in* X, $\lambda_i \in \mathbb{R}^+$, *and* T_0 *is a positive pluriharmonic current of bidimension* $(1,1)$ *on* X *whose Lelong number vanishes outside a finite or countable set.*

We call $\sum_{i\in I} \lambda_i[V_i]$ *the analytic part* and T_0 *the non-analytic part* of T. The following corollary illustrates an application for foliation theory. The case of currents directed by a singular holomorphic foliation by Riemann surfaces has been obtained in [3].

Corollary 3. *Let* X *be a projective manifold. Let* A_k *with* $k \in \mathbb{N}$ *be Borel subsets of* X *of finite* 2*-dimensional Hausdorff measure. Let* T *be a positive pluriharmonic current of bidimension* $(1,1)$ *on* X. *Assume that* T *does not give mass outside the set* $\bigcup_{k\in\mathbb{N}} A_k$. *Then*

$$T = \sum_{i\in I} \lambda_i[V_i],$$

where $\{V_i\}_{i\in I}$ *is a (possibly empty) finite or countable family of compact analytic curves in* X *and* $\lambda_i \in \mathbb{R}^+$.

In the setting of singular foliations by Riemann surfaces, leaves are often Zariski dense in the phase space. By the last corollary, such leaves cannot support positive pluriharmonic currents. In other words, if a leaf supports a positive pluriharmonic current, it should be an algebraic leaf.

In [6, 7] the second author extended some results of the theory of tangent currents initiated by the first author and Sibony in [4, 5] to positive pluriharmonic currents on any complex manifolds. This, combined with our study, also allows us to obtain the following result.

Theorem 4. *Let* X *be a compact Kähler manifold. Let* $\mathcal{E} \subset H^{1,1}(X,\mathbb{R})$ *be the pseudoeffective cone, that is, the closed convex cone given by the set of pseudoeffective classes, i.e., the classes that contain a closed positive* $(1,1)$*-current. Let* T *be a positive pluriharmonic current of bidimension* $(1,1)$ *on* X *which does not give mass to any proper analytic set. Then the cohomology class* $\{T\}$ *of* T *belongs to the dual of the pseudoeffective cone* $\mathcal{E}$. *Moreover, if* $\dim X = 2$, *then* $\{T\}$ *is nef and also big unless it is closed. In particular, if* X *is a complex projective manifold and if* T *is a positive pluriharmonic current of bidimension* $(1,1)$ *on* X *which does not give mass to any complex hypersurface, then* $\{T\}$ *is movable in the sense of* [1].

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Recognition of submanifolds in homogeneous spaces via free rational curves

Sung Yeon Kim
(joint work with Yun Gao, Sui-Chung Ng)

The rigidity of proper holomorphic maps between bounded symmetric domains can date back to the classical Alexander's theorem [1], which states that every proper holomorphic self map of a complex unit ball of dimension at least 2 is an automorphism. Henkin-Tumanov [2] generalized the theorem to irreducible bounded symmetric domains of rank at least 2. Motivated by the study of Hermitian metric rigidity, Mok conjectured that every proper holomorphic map from an irreducible bounded symmetric domain with rank at least two to another bounded symmetric domain with rank at least two is always a totally geodesic isometric embedding with respect to the Bergman metrics whenever the rank of the target domain is less than or equal to the rank of the source domain. Mok's conjecture was resolved affirmatively by Tsai [3], whose proof involves a key concept called invariantly geodesic subspace, which has since become indispensable for the study of proper holomorphic maps between bounded symmetric domains of higher rank.

For a given irreducible bounded symmetric domain Ω of rank at least two, the set $\mathcal{K}$ of all maximal invariantly spaces becomes a homogeneous space on which the automorphism group of Ω acts transitively. We call $\mathcal{K}$ moduli space of maximal invariantly geodesic subspaces. Consider the universal family $\mathcal{U}$ of maximal invariantly geodesic subspaces, which is the tautological holomorphic fiber bundle over $\mathcal{K}$. Then any holomorphic map f between two bounded symmetric domains Ω and Ω' that respects maximal invariantly geodesic subspaces can be translated to a meromorphic map $f^\flat$ between the moduli spaces of maximal invariantly geodesic subspaces by pulling back and pushing down f via the canonical double fibrations $\mathcal{U} \to \Omega$, $\mathcal{U} \to \mathcal{K}$ and $\mathcal{U}' \to \Omega'$, $\mathcal{U}' \to \mathcal{K}'$, By analyzing the geometric properties of these double fibrations and the induced moduli map $f^\flat$ including the VMRT(variety of minimal rational tangents) respecting property, Ng ([4]) obtained rigidity phenomenon for proper holomorphic mappings between bounded symmetric domains of type-I. The rigidity results for type-I domains have been further advanced by the works of Kim- Zaitsev [5], Chan [6] and Huang-Lu-Tang-Xiao [7], incorporating the semi-standard embeddings. Recently, Kim-Mok-Seo [8] have made a generalization of the above results to general bounded symmetric domains with small rank differences ($\mathrm{rank}(\Omega') < 2\ \mathrm{rank}(\Omega) - 1$) by proving the rigidity

of rational maps between certain pair of rational homogeneous spaces of Picard number one that preserves minimal rational curves. If $\mathrm{rank}(\Omega') = 2\ \mathrm{rank}(\Omega) - 1$, then the mofuli map $f^\flat$ no longer respects minimal rational curves. However, the image of a general minimal rational curve under $f^\flat$ is a conic section, which is a degree two smooth rational curve. It is therefore natural to explore further the case of rational maps which send minimal rational curves to higher degree rational curves of fixed type.

Let X be a projective manifold. A nonconstant holomorphic map $f : \mathbb{P}^1 \to X$ is said to be free if and only if $f^*(TX)$ is semipositive. A projective manifold X is uniruled if and only if there exists a free parametrized rational curve in X. If X admits a free rational curve, then deformations of a free rational curve sweep out a Zariski-open subset of X.([9]). From now on, we let $X = G/P$ and $X' = G'/P'$ be rational homogeneous spaces, where G and G' are connected simple complex Lie groups, $P \subset G$ and $P' \subset G'$ are maximal parabolic subgroups. A holomorphic map $f : X \to X'$ is called a model map if there exists a group homomorphism $\phi : G \to G'$ with $\phi(P) \subset P'$ such that the induced map $\bar{\phi} : G/P \to G'/P'$ is equivalent to f. If f is a model map, then the image of a minimal rational curve is equivalent to each other. For a given model map $f : X \to X'$, we call a holomorphic map F defined on a connected open set $U \subset X$ is modeled on f of order k if for each point $x \in U$, there exists an automorphism $\gamma_x \in X'$ such that $\gamma_x \circ F$ is equal to f up to order k at x. If $f : X \to X'$ respects minimal rational curves, then F respects varieties of minimal rational tangents. In [10], Hong and Mok considered rational homogeneous manifolds $X = G/P$ of Picard number 1 associated to long simple roots of simple Lie groups G and characterize standard embeddings of certain rational homogeneous submanifolds $Z \subset X$ of Picard number 1 comming from a subdiagram of marked Dynkin diagram in terms of embeddings of varieties of minimal rational tangents.

In this talk, we will consider orbits Σ of real form of G in rational homogeneous manifolds G/P of Picard number one when Σ is saturated with minimal rational curves. Especially, we will consider rational moduli map $f^\flat : \mathbb{B}^{p,q} \to \mathbb{B}^{p',q'}$, where $\mathbb{B}^{p,q}$ and $\mathbb{B}^{p',q'}$ are respectively the moduli spaces of maximal invariantly geodesic subspaces of type-I bounded symmetric domains $D^I_{p,q}$ and $D^I_{p',q'}$. If $p' = 2p - 1$ and $p \geq 5$, then the image of a general minimal rational curve under $f^\flat$ is a degree two rational curve contained in a 2-dimensional projective space in $\mathbb{P}^{p',q'}$. We show that $f^\flat$ is determined by the projective second fundamental form. Then we classify all possible second fundamental forms of $f^\flat$. As an application, we classify proper holomorphic maps between $D^I_{p,q}$ and $D^I_{2p-1,q'}$ which respect maximal invariantly geodesic subspaces. In Harish-Chandra coordinates, we give explicit expression of such maps. This is a joint work with Y. Gao and S.-C. Ng.

Toward a generalization of such rigidity phenomonon, one can use Lie group representations. For a simple Lie group G, consider a representation V of G. Then there exists a group homomorphism $\phi : G \to GL(V)$. Suppose that the image $\phi(G)$ is contained in a simple Lie group $G' \subset GL(V)$. Suppose further that ϕ induces a holomorphic map f between G-homogeneousspace and G'-homogeneous space

of Picard number one. Then the image of a minimal rational curve under f is equivalent to each other. Therefore it would be interesting to know under which condition a holomorphic map F defined on a connected open set $U \subset X$ modeled on f of order k is equivalent to f itself. It would be interesting to know as well whether k-th fundamental form determines F.

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Perturbations of parabolic endomorphisms in dimension 2

Jasmin Raissy

(joint work with Matthieu Astorg, Lorena López-Hernanz)

In this joint work with Matthieu Astorg and Lorena López-Hernanz, we study perturbations of holomorphic endomorphisms of $\mathbb{C}^2$ tangent to the identity. Our goal is to understand how the dynamics changes when a fully parabolic fixed point is perturbed.

Parabolic implosion describes the bifurcation phenomena that occur when a parabolic fixed point is perturbed. In one complex variable, the theory was developed by Lavaurs [5].

Consider the family of quadratic polynomials in dimension 1 $P_c(z) = z^2 + c$, with $c \in \mathbb{C}$. At the parameter $c = 1/4$, the fixed point $z = 1/2$ is parabolic.

The parameter $c = 1/4$ is the cusp of the main cardioid of the Mandelbrot set (see Figure 1). When $c > 1/4$, the critical orbit escapes to infinity, but for parameters close to $1/4$ it remains trapped for a very long time near the former

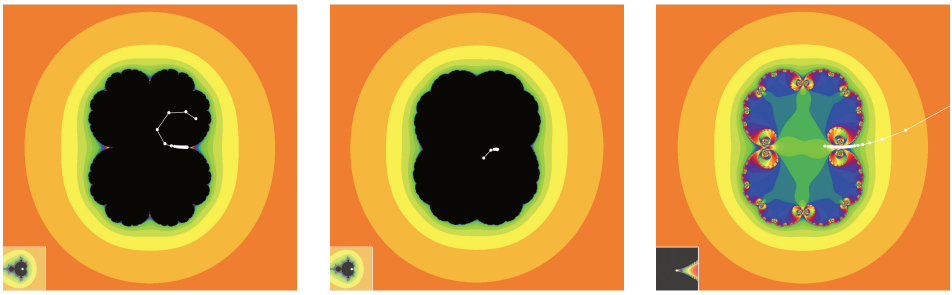

FIGURE 1. The quadratic family $P_c(z) = z^2 + c$ near the parabolic parameter. Left: $c = 1/4$. Centre: $c = 1/4 - 0.01$ (inside the Mandelbrot set). Right: $c = 1/4 + 0.01$ (outside the Mandelbrot set).

parabolic fixed point. The limiting behaviour of this transit is described by the theory of parabolic implosion.

Lavaurs proved that this passage admits a non-trivial limit. The corresponding limiting maps, now called *Lavaurs maps*, are constructed using Fatou coordinates and describe the transition through the perturbed parabolic region. They provide a refined description of the limiting dynamics and play a fundamental role in the study of parabolic bifurcations, Julia sets and parameter spaces.

The purpose of our work is to establish a *Lavaurs theorem* for perturbations of holomorphic maps tangent to the identity in dimension two. We consider holomorphic maps $F\colon \mathbb{C}^2 \to \mathbb{C}^2$ tangent to the identity at the origin. A complex direction $[v] \in \mathbb{P}^1(\mathbb{C})$ is called *characteristic* if the corresponding complex line passing through the origin and directed by $[v]$ is invariant under the first non-linear term in the homogeneous expansion of the map. Under suitable assumptions on a *non-degenerate* characteristic direction, Hakim proved the existence of parabolic basins whose orbits converge to the origin tangentially to such direction [4].

Parabolic implosion in dimension two has been investigated in several settings. Bedford, Smillie and Ueda developed a theory for semi-parabolic fixed points [3]. In the fully parabolic case, namely when the differential at the fixed point is the identity, Bianchi obtained a Lavaurs-type theorem for a class of perturbations preserving an invariant complex line [2].

We consider holomorphic maps tangent to the identity at order two, with a non-degenerate characteristic direction together with a formal invariant curve tangent to this direction. In contrast with previous works, we do not assume the existence of an invariant complex line.

Theorem 1 (Astorg, López-Hernanz, R. [1])**.** *Let $F\colon \mathbb{C}^2 \to \mathbb{C}^2$ be a holomorphic map tangent to the identity at order two and non-dicritical. Assume that F admits a non-degenerate characteristic direction $[v] \in \mathbb{P}^1(\mathbb{C})$ with director α satisfying $\Re(\alpha) > 2$ and a formal invariant curve tangent to $[v]$. Then for any $q \in \mathbb{C}$ there exists a holomorphic family of holomorphic maps $(F_\varepsilon)_{\varepsilon\in\mathbb{D}}$ with $F_0 = F$ such that*

for any $\sigma \in \mathbb{C}$ there exist a sequence $\varepsilon_n \to 0$ such that

$$F^n_{\varepsilon_n} \to \mathcal{L}_{\sigma,q} := (\Phi^o)^{-1} \circ A_{\sigma,q} \circ \Phi^\iota$$

uniformly on compact subsets of the parabolic basin $\mathcal{B}_{[v]}$ of F with respect to $[v]$, where $A_{\sigma,q}(X,Y) := (X+\sigma, e^{\pi q}Y)$.

The maps Φ^i and Φ^o denote incoming and outgoing Fatou coordinates. Thus the limiting maps $\mathcal{L}_{\sigma,q}$ play exactly the same role as the classical Lavaurs maps in one-dimensional dynamics.

Our approach relies on the construction of suitable approximate Fatou coordinates together with a detailed analysis of perturbed orbits inside the parabolic basin. The main novelty is that the construction does not rely on the presence of an invariant complex line. Replacing this geometric assumption by the existence of a formal invariant curve tangent to a non-degenerate characteristic direction, we obtain a Lavaurs theorem in a substantially more general setting.

As in the one-dimensional setting, the existence of Lavaurs maps has important consequences for the behaviour of Julia sets. As an application of our result, we obtain lower-semicontinuity results for the large Julia set J_1 and discontinuity of the small Julia set J_2 for suitable holomorphic endomorphisms of $\mathbb{P}^2(\mathbb{C})$.

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Pluripotential geometry on semi-positive effective divisors of numerical dimension one

Takayuki Koike

My research interest lies mainly in finding geometric criteria for the semi-positivity of holomorphic line bundles. Let L be a holomorphic line bundle on a complex manifold X. One says that L is *positive* (resp. *semi-positive*) if it admits a C^∞ Hermitian metric h such that $\sqrt{-1}\Theta_h > 0$ (resp. $\sqrt{-1}\Theta_h \geq 0$) holds at every point of X. Kodaira's embedding theorem may be regarded as a fundamental illustration of the philosophy that the positivity of the curvature of a metric on a line bundle controls the global geometry of the ambient compact complex manifold. From this viewpoint, it seems natural to ask what kind of geometric structure should be expected when the positivity assumption is weakened to semi-positivity.

A conjectural picture which has guided my recent work is the following. Here and below, by a *unitary flat* line bundle, I mean a holomorphic line bundle admitting a system of holomorphic local trivializations whose transition functions are locally constant valued in $\mathrm{U}(1) = \{t \in \mathbb{C} \mid |t| = 1\}$.

Conjecture ([1, 2]): Let X be a complex manifold, L a semi-positive holomorphic line bundle on X, and Y a compact reduced hypersurface of X. Assume that $L|_Y$ is unitary flat. Then there exists a neighborhood V of Y such that $L|_V$ is unitary flat.

A positive answer in the non-singular case was obtained in [3, Corollary 1.5]. So far, I have mainly studied this problem in the case where X is compact Kähler and $L = [D](:= \mathcal{O}_X(D))$ for an effective divisor D on X. If one writes $Y = \operatorname{Supp} D$, then the above conjecture predicts that the semi-positivity of $[D]$ should force the unitary flatness of $[D]$ on a neighborhood of Y. Based on the recent preprint [5], I reported the result in which we proved that this is indeed the case. As the converse also holds under some natural assumptions (when the numerical dimension $\mathrm{nd}([D])$ of $[D]$ is one), we have the following: Let X be a compact Kähler manifold and D an effective nef divisor on X with $\mathrm{nd}([D]) = 1$, and denote by Y the support of D. Then the following conditions are equivalent [5, Theorem 1.1]:
(i) the line bundle $[D]$ is semi-positive;
(ii) there exists a neighborhood V of Y such that $[D]|_V$ is unitary flat;
(iii) there exist a covering $\{V_j\}$ of Y by open subsets of X and holomorphic functions $s_j \colon V_j \to \mathbb{C}$ whose zero sets coincide with $V_j \cap Y$, such that $|s_j/s_k| \equiv 1$ on each intersection $V_j \cap V_k$. Thus, in this case, the semi-positivity of $[D]$ can be characterized by the existence of a certain holomorphic foliation / Levi-flat neighborhood geometry around the support Y. Note that, under an additional torsion-type assumption, one can go further and relate semi-positivity to semi-ampleness [5, Theorem 1.2].

This naturally raises the following rough question: when does such a neighborhood of Y, together with the associated pseudoflat geometry or foliation, actually exist? When X is a surface and Y is a smooth compact complex curve with topologically trivial normal bundle, this problem was studied in depth by Ueda [6]. *Ueda theory* describes the neighborhood structure of Y in terms of the linearizability of transition functions of local defining functions, and introduces the Ueda classes as the corresponding obstruction classes (defined by extracting each n-th jet obstruction for the linearization), which has been an important background for my work [1, 2, 3].

Based on another preprint [4], I also discussed this neighborhood problem from a more differential-geometric and cohomological point of view through the *formal principle for line bundles* (FPLB). Let Y be an analytic subset of a complex manifold X. One says that the FPLB holds on neighborhoods of Y in X if, for any neighborhood V of Y and any holomorphic line bundle L on V, the formal triviality of L along Y implies the holomorphic triviality of L on a (smaller) neighborhood of Y. When Y is a non-singular hypersurface with unitary flat normal bundle and

$L = [Y] \otimes \widetilde{N_{Y/X}}^{-1}$, the obstruction classes introduced in [4] coincide, up to a non-zero constant multiple, with the Ueda classes [4, Remark 3.10], where $\widetilde{N_{Y/X}}$ is the flat line bundle on a tubular neighborhood of Y whose restriction to Y coincides with the normal bundle. In this sense, the paper gives a reinterpretation of Ueda's obstruction theory in a more general framework. The main result of [4] gives practical sufficient conditions for the validity of the FPLB. Let X be a compact Kähler manifold, Y a non-empty analytic subset of X, and put $M := X \setminus Y$. Then the FPLB holds on neighborhoods of Y in X if either
(i) the natural map $H^2_c(M, \mathcal{O}_M) \to H^2(M, \mathcal{O}_M)$ is injective, or
(ii) there exists an effective divisor D on X such that $\operatorname{Supp} D = Y$ and the natural map

$$H^2_c(M, \mathcal{O}_M) \to \varprojlim_{\mu} H^2(X, \mathcal{O}_X(-\mu D))$$

is injective [4, Theorem 1.2]. The proof is based on the construction of an obstruction class

$$v(\mathcal{I}_Y, L) \in \operatorname{Ker}\big(H^2_c(M, \mathcal{O}_M) \to H^2(M, \mathcal{O}_M)\big)$$

for a holomorphic line bundle L on a neighborhood of Y, and on reducing the problem to a $\partial\overline{\partial}$-equation near the boundary of the complement.

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Completed volumes and the DR-cycle

Martin Möller

(joint work with Miguel Prado)

The moduli space of projectivized k-differentials $\mathbb{P}\Omega^{\otimes k}\mathcal{M}_{g,n}(\mu)$ with zeros and poles of orders prescribed by $\mu = (m_1, \ldots, m_n)$ comes with a natural finite volume form. In particular for the case of Abelian differentials $k = 1$ and quadratic differentials $k = 2$ these Masur-Veech volumes of these moduli spaces have important implicitations for the dynamics on translation and half-translation surfaces.

A challenging problem is to find formulas for the computation of these volumes, that can be efficiently evaluated and such that the large genus limits can be derived. For the case of abelian differentials, the large limit conjecture [3] has been proven

in recent years [5, 2]. For quadratic differentials the principal strata (all $m_i = \pm 1$) was shown in [1] while the general case is open.

There are three strategies to compute Masur-Veech volumes. First, the initial strategy of counting square-tiled surfaces of Zorich and Eskin-Okounkov and relying on the quasimodularity of the generating function has sucessfully been used for abelian differentials [5] and for a low genus data base by Goujard, but it will play no role in the sequel of this paper. Second, one may decompose the (half-) translation surfaces into ribbon graphs with glued-in cylinders and count ribbon graphs. Key to make this strategy usefuly is to relate the count of integer metrics on ribbon graphs to intersection numbers of Witten-Kontsevich's combinatorial classes on the moduli space of stable curves $\overline{\mathcal{M}}_{g,n}$. This strategy was implemented for abelian differentials and principal strata of quadratic differentials in a series of papers by Delecroix, Goujard, Zograf and Zorich. Third, Masur-Veech volumes can be computed as intersection numbers of tautological classes on the multi-scale compactification of the moduli spaces $\mathbb{P}\Omega^{\otimes k}\mathcal{M}_{g,n}(\mu)$ themselves, proven for abelian differentials [5] and quadratic differentials without relative periods and conjecturally in general [6]. In the case without relative periods the class that computes the Masur-Veech volume is simply the top power ξ^d of the tautological line bundle where $d = \dim \mathbb{P}\Omega^{\otimes k}\mathcal{M}_{g,n}(\mu)$.

Ribbon graph completed volumes. To adapt the ribbon graph strategy for quadratic differentials with general odd entries m_i, Duriev–Goujard-Yakovlev introduced in [8] the notion of *completed volumes* $\overline{\mathrm{vol}}(\mu)$ of moduli spaces of quadratic differentials: Instead of counting metrics on Ribbon graphs, they evaluate Kontsevich's polynomials in terms of combinatorial classes, even though they overcount by contributions of degenerate ribbon graphs. Their main theorem [8, Theorem 2.6] is (for the case of $n \geq 4$ singularities) a recursive formula for completed volume in terms of the Masur-Veech volume and lower complexity volumes.

Double ramification (DR)-cycle completed volumes. To evaluate intersection numbers on the multi-scale compactification one may also consider using a completion. The double ramification (DR)-cycle can be considered as an alternative 'compactification' of $\mathbb{P}\Omega^{\otimes k}\mathcal{M}_{g,n}(\mu)$ in an appropriate blowup of $\overline{\mathcal{M}}_{g,n}$ having additional components that lie entirely over the boundary of $\overline{\mathcal{M}}_{g,n}$. Chiodo–Holmes [7] recently proved a closed formula for intersection numbers of arbitrary powers ξ^r of the tautological line bundle on the DR-cycle, which was conjectured in [4]. It is thus natural to define the *DR-completed volume* $\overline{\mathrm{vol}}^{\mathrm{DR}}(\mu)$ of any of the moduli spaces $\mathbb{P}\Omega^{\otimes k}\mathcal{M}_{g,n}(\mu)$ to be the appropriate intersection number, that is, ξ^d in the case without relative periods, against the DR-cycle, and to aim for a recursive formula in terms of the true Masur-Veech volume and of volumes of lower complexity moduli spaces.

Refering to [9] for the precise volume definitions we can state the main result of this paper as:

Theorem 1. *The DR-completed volume and the (Ribbon graph) completed volume agree, i.e.,*

$$\overline{\mathrm{vol}}^{\mathrm{DR}}(\mu) = \overline{\mathrm{vol}}(\mu)$$

for all signatures of quadratic differentials with $m_i > -2$ odd for $i = 1, \ldots, n$.

In fact, the theorem is not just and equality of two numbers, but a termwise equality of two recursions that form the two constituents of the proof.

The theorem now opens a two-fold way to approach the large genus asymptotic conjectures: Use either of the two tautological class expression for the completed volume to compute its large genus limit and deduce the conjecture in [3] by inverting these recusions.

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Dynamics of Hénon-like maps

Fabrizio Bianchi

(joint work with Tien-Cuong Dinh, Karim Rakhimov)

Complex Hénon maps form one of the central classes of holomorphic dynamical systems. They provide a natural invertible generalization of polynomial dynamics from complex dimension one to two. Their study goes back to the work of Hubbard and Oberste-Vorth, Bedford–Lyubich–Smillie, and Fornæss–Sibony, among others. A fundamental feature of this theory is that their dynamics can be described by two Green functions, associated respectively with forward and backward iteration. Their complex Laplacians define two positive closed currents T^+ and T^-, and the wedge product $\mu = T^+ \wedge T^-$ is the distinguished invariant measure of the system.

These objects encode complementary aspects of the dynamics. The measure μ is the measure of maximal entropy and governs, for instance, the asymptotic distribution of saddle periodic points. The currents T^+ and T^-, on the other hand, describe the asymptotic distribution of images and preimages of analytic curves. This interplay between geometric and measure-theoretic properties is one of the useful features of the pluripotential approach.

In higher dimension, one natural algebraic analogue is given by Hénon-Sibony maps, that is, polynomial automorphisms of $\mathbb{C}^k$ whose extensions to projective space have disjoint indeterminacy sets. Their pluripotential theory was developed in particular by Dinh–Sibony. In this setting the Green objects are no longer merely $(1,1)$-currents, but positive closed currents of intermediate bidegree. Nevertheless, many features of the two-dimensional theory persist: one obtains invariant Green currents, an equilibrium measure of maximal entropy, and a rich geometric structure for the currents.

The aim of the talk was to discuss to what extent this picture survives outside the algebraic framework. The class of Hénon-like maps provides a natural setting for this question. Roughly speaking, these are invertible holomorphic maps on a bounded convex domain $D = M \times N \subset \mathbb{C}^p \times \mathbb{C}^{k-p}$, which geometrically expand in p directions and contract in the remaining $k-p$ directions. More precisely, such a map sends a vertical open subset of D properly and invertibly onto a horizontal open subset of D. This is a local, geometric notion of hyperbolicity, not tied to an algebraic compactification.

This class is quite flexible. It contains perturbations of algebraic examples, but it is infinite-dimensional and includes genuinely non-algebraic systems. In dimension two, Hénon-like maps were introduced by Hubbard and Oberste-Vorth and further studied by Dujardin. In higher dimension, the theory was developed by Dinh, Nguyên and Sibony. One can construct Green currents and an invariant equilibrium measure in this setting, but the absence of a compact algebraic background creates new difficulties: there is no ambient cohomology group on which the map acts, and many standard cohomological arguments are unavailable.

The first issue concerns dynamical degrees. In compact Kähler dynamics, the dynamical degree of order s measures the exponential growth of the action on $H^{s,s}$, or equivalently the growth of volumes of analytic sets of dimension s. By Gromov's theory and the Khovanskii–Teissier inequalities, the sequence of these degrees is log-concave. In particular, they increase up to a dominant degree and then decrease. For Hénon-like maps one can still define dynamical degrees by measuring the growth of masses of horizontal and vertical currents. More precisely, d_s^+, for $0 \leq s \leq p$, measures the exponential growth under forward iteration of horizontal positive closed currents of dimension s, while d_s^-, for $0 \leq s \leq k-p$, is the analogous quantity for vertical positive closed currents under backward iteration. The main dynamical degree is $d = d_p^+ = d_{k-p}^-$. However, the usual cohomological proof of log-concavity is no longer available.

The first main result, obtained jointly with Tien-Cuong Dinh and Karim Rakhimov, shows that the expected monotonicity nevertheless holds.

Theorem 1 [1]. Let f be a Hénon-like map of type $(p, k-p)$, and let d be its main dynamical degree. Then the forward and backward dynamical degrees are monotone on the two sides of the main degree. More precisely,

$$1 = d_0^+ \leq d_1^+ \leq \cdots \leq d_p^+ = d = d_{k-p}^- \geq \cdots \geq d_1^- \geq d_0^- = 1.$$

Thus the main dynamical degree is indeed the dominant one, even in this non-algebraic setting. This is important because it identifies the correct analogue of the dominant-degree condition familiar from algebraic dynamics. In particular, if d is strictly larger than the two adjacent degrees d_{p-1}^+ and d_{k-p-1}^-, then it strictly dominates all the remaining dynamical degrees.

Let us briefly indicate the idea of the proof. In the algebraic case, the comparison between consecutive dynamical degrees is encoded in cohomology. Here it has to be recovered directly at the level of currents. Starting from a horizontal positive closed current, one constructs a holomorphic family deforming it and realizes this family as slices of a current in a larger space. A projection then produces a current of higher dimension whose iterates control the original one. The main technical input is a set of quantitative estimates for solving d, $\bar{\partial}$, and dd^c equations on convex domains, combined with Skoda-type pluripotential estimates.

The second result concerns the regularity and geometry of the Green currents. A Hénon-like map has two canonical Green currents T^+ and T^-, obtained as normalized limits of suitable pull-backs and push-forwards. Their intersection $\mu = T^+ \wedge T^-$ is the equilibrium measure. In the algebraic Hénon-Sibony setting, the Green currents have Hölder continuous super-potentials and are woven; in dimension two, the corresponding $(1,1)$-currents are laminar. The question is whether the same conclusions remain true for general Hénon-like maps.

The answer is affirmative under the natural gap condition singled out by the monotonicity theorem.

Theorem 2 [2]. Assume that the main dynamical degree d satisfies

$$d > \max(d_{p-1}^+, d_{k-p-1}^-).$$

Then the Green currents T^+ and T^- have Hölder continuous super-potentials. They are woven currents. Moreover, the equilibrium measure $\mu = T^+ \wedge T^-$ is a Monge-Ampère measure with Hölder continuous potential, and in particular has positive Hausdorff dimension. In the extremal cases $p = 1$ or $p = k-1$, the corresponding Green current is laminar.

The Hölder continuity of the super-potentials gives a strong regularity property for the Green currents and allows one to define their intersection in a robust way. The woven and laminar conclusions give geometric information: the currents can be approximated, or represented, by averages of integration currents over analytic submanifolds, and in the laminar case by compatible families of holomorphic disks.

The proof combines quantitative pluripotential estimates with geometric control of analytic lifts. In algebraic settings one often lifts analytic sets to a Grassmannian bundle and controls the volume of the lift cohomologically. Here this is replaced by a notion of shadow for currents in a product with a Grassmannian.

If too much volume were created in the lift, one would produce lower-dimensional positive closed currents whose growth contradicts the domination of the main dynamical degree. This gives the required woven, and in the extremal cases laminar, structure.

These results show that many central features of classical Hénon dynamics persist in the broader Hénon-like setting. Although the algebraic compactification and cohomological action are missing, pluripotential methods still recover the correct hierarchy of dynamical degrees, the Hölder regularity of Green currents, and their geometric decomposition. In this sense, Hénon-like maps provide a flexible non-algebraic framework in which one can study invertible holomorphic dynamics with non-uniform expanding and contracting directions.

Let us finally mention one related development. Recent work of Luo–Zhou proves the equidistribution of saddle periodic points towards the equilibrium measure for Hénon-like maps, completing another important part of the analogy with classical Hénon maps.

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The Yau–Tian–Donaldson conjecture for constant scalar curvature Kähler metrics

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(joint work with Sébastien Boucksom)

1. Calabi's question

Let (X, ω_0) be a compact Kähler manifold of dimension n. By the dd^c-lemma, every Kähler form ω cohomologous to ω_0 is of the form $\omega = \omega_0 + dd^c\varphi$ for a function $\varphi \in C^\infty(X)$, unique up to an additive constant.

In the 1950s Calabi asked whether a given Kähler class on X admits a canonical Kähler form ω. For example, ω is a *Kähler–Einstein* metric/form if its Ricci curvature is proportional to ω, $\mathrm{Ric}\,\omega = \lambda\omega$, where $\lambda \in \mathbb{R}$ is a constant. As $\mathrm{Ric}\omega \in c_1(X)$, this is only possible if $c_1(X)$ has a sign.

More generally, ω is a *constant scalar curvature Kähler* metric, or cscK metric, if the scalar curvature $S(\omega) = \frac{n\mathrm{Ric}\,\omega\wedge\omega^{n-1}}{\omega^n}$ is constant. Yet more general notions of canonical metrics exist, notably extremal metrics.

Here I will stick to cscK metrics, and focus on the algebraic case, in which the given Kähler class is the first Chern class of an ample line bundle L on X. In this case, ω will be the curvature of a smooth positive metric on L.

2. The Yau–Tian–Donaldson conjecture

It was noticed early on that there are obstructions to the existence of a cscK metric. For example, it follows from results by Matsushima and Lichnerowicz proved in the 1950s that the existence of a cscK metric in $c_1(L)$ implies that the automorphism group $\mathrm{Aut}(X, L)$ is reductive. This can be used to prove that the blowup of the projective plane at a single point does not admit a cscK metric (in any Kähler class).

After further work, a conjecture by Yau [13], Tian [12], and Donaldson [10], emerged. In a vague form, this YTD conjecture states that the existence of a cscK metric in $c_1(L)$ should be an algebraic condition on (X, L).

Famously, the YTD conjecture was first proved for Fano manifolds X (with $L = -K_X$ the anticanonical bundle) by Chen–Donaldson–Sun [6], the algebraic condition being *K-polystability*, defined in terms of *test configurations* for (X, L).

3. Main result

Boucksom and I proved in [4] that the existence of a cscK metric in $c_1(L)$ is equivalent to (X, L) satisfying an algebraic condition that we call $\hat{\mathrm{K}}$-stability. (A different YTD correspondence was independently proved by Darvas and Zhang in [8].) I will discuss our result only briefly here, referring to [4] as well as the survey [5] for more details and references.

As the name is supposed to suggest, $\hat{\mathrm{K}}$-stability is a condition on a metric completion of the space of test configurations for (X, L). A test configuration, as introduced in [10], is a type of one-parameter degeneration of (X, L), but it can also be seen as a metric on the analytification (in the sense of Berkovich [1]) of L with respect to the *trivial* absolute value on the ground field $\mathbb{C}$. Notions involving this absolute value are necessarily algebraic in nature.

From this point of view, the space $\mathcal{H}_{\mathrm{na}}$ of test configurations for (X, L) can be viewed as a non-Archimedean version of the space $\mathcal{H}$ of smooth positive metrics on L. The latter admits a natural *Darvas metric* [7], with metric completion $\widehat{\mathcal{H}}$ being the space of psh metrics on L of finite energy. Similarly, $\mathcal{H}_{\mathrm{na}}$ can be equipped with a metric, with completion $\widehat{\mathcal{H}}_{\mathrm{na}}$.

The space $\widehat{\mathcal{H}}_{\mathrm{na}}$ is needed to define the very concept of $\hat{\mathrm{K}}$-polystability, whereas $\widehat{\mathcal{H}}$ is crucial to running the variational method for finding cscK metrics. Moreover, the elements of $\widehat{\mathcal{H}}_{\mathrm{na}}$ can be viewed as geodesic rays in the space $\widehat{\mathcal{H}}$. The proof in [4] relies on non-Archimedean pluripotential theory as developed in [3], crucial work by Chi Li [11], and a complex-analytic regularity theorem by Berman–Demailly [2] and Di Nezza–Trapani [9].

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