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Geometry and Combinatorics: The Model Theoretic Perspective

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Abstract. Our meeting focused mainly on geometric and combinatorial aspects in model theory and their interactions with geometric group theory, algebraic geometry, valued fields, and combinatorics. The model-theoretic perspective provided a unifying point of view for applications in seemingly different mathematical areas. At the same time, phenomena and problems encountered in applications provided fresh impetus for further develoments in abstract model theory. It also provided a welcome meeting point for reasearchers from different areas of mathematics.

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Introduction by the Organizers

The common theme of this conference was the model theoretic perspective on geometries arising in algebraic geometry, geometric group theory and combinatorics. Recent progress on these topics has brought model theoretic methods into the view of an ever wider range of core areas of mathematics, in turn driving the search for new model theoretic tools. These links were made visible during the conference.

The model theoretic focus on geometries can be traced back to Zilber's conjecture stating that any strongly minimal structure should either be trivial, vector space like or an algebraic curve over an algebraically closed field. While this conjecture has long been refuted by Hrushovski, it inspired a lot of work on model theoretic characterizations of algebraic curves. Hrushovski and Zilber gave one

such characterisation in terms of abstract *Zariski geometries*, which in turn was one of the crucial tools in Hrushovski's proof of the Manin-Mumford conjecture. Very recently, several long standing important cases of the restricted trichotomy conjecture were obtained: for strongly minimal structures definable in algebraically closed fields (by Castle in characteristic 0, and by Castle, Hasson, Ye in positive characteristic); and the higher dimensional case for strongly minimal structures definable in o-minimal structures (by Castle).

Aiming for another abstract characterisation of algebraic curves among strongly minimal structures, the model theoretic concept of ampleness was introduced by Pillay. It aims to give an abstract characterisation of projective spaces over fields in terms of the forking geometry. In this way, an n-ample theory should correspond to a projective space of dimension $n+1$ and it was shown by Pillay that stable fields are indeed n-ample for all n. Recent work has produced many examples of ample theories, brining together the group theoretic and the field theoretic point of view: namely both the free groups and right-angled buildings provide natural examples of complexes with stable and ample theory. However, in the strongly minimal context few examples exist apart from projective spaces over algebraically closed fields.

While algebraic geometry has long been in the focus of model theoretic developments, recent progress on the geometries arising from (hyperbolic) groups has lead to fruitful interactions with model theory. In both branches, associated complexes like the curve complex, the free factor complex or projective geometry give rise to combinatorial structures, which are then studied from a model-theoretic point of view.

1. Algebraic Geometry

Hrushovski's proof of the Manin-Mumford conjecture was pivotal in the further development and interactions between algebraic geometry and model theory. It has lead to numerous fruitful collaborations not only in the context of non-archimedian geometry and Berkovich spaces, but also with respect to new resolutions of singularities as recently described by Halupczok and Bradley-William.

A particular highlight in the conference was the talk by Ben Castle about topological reconstruction results in the spirit of the celebrated Borel-Tits result on algebraic groups. The recent book of Kollar-Lieblich-Olsson-Sawin (KLOS) showed that many varieties are determined by their Zariski topologies. More precisely, similar to the algebraic group case, a homeomorphism between absolutely irreducible, normal, projective varieties of dimension at least 2 over uncountable fields of characteristic zero always decomposes into a field isomorphism and an isomorphism of varieties in a precise sense. The authors of the book leave open several questions, essentially asking what happens in positive characteristic and for arbitrary varieties (quasi-projective, maybe reducible, maybe non-normal). In joint with Ronan O'Gorman, Ben Castle explained how they use model theory to give the best possible answer to all of these questions in the special case of algebraically closed fields.

Another exciting talk was given by Thomas Scanlon, who gave an excellent exposition of the history of tilting and explained how to use a slight extension of first order logic to a version incorporating predicates from continuous logic in order to recover the concept of tilting.

2. Geometric group theory

In 2006 Sela and independently Kharlapovich and Myasnikov answered and old question of Tarski and showed that the non-abelian free groups are elementarily equivalent, using quantifier elimination down to boolean combinations of inductive formulas. Sela then generalized the methods to torsion-free hyperbolic groups and showed that they are stable in the sense of Shelah. This opened the way for a model-theoretic study of free and hyperbolic groups.

One of the highlights of the conference was a talk by Sela on extensions of his work on the free group to algebraic geometry over free associative algebras. Even though the theory is only at its beginnings, this seems to open fascinating new perspectives.

The investigation of complexes arising from hyperbolic groups started with the proofs of ampleness of the free group (Sklinos, Ould-Houcine, Tent) using the curve complex as an example. Then it was shown by Tent that right angled buildings of rank $n+1$ are ω-stable and strictly n-ample. In her talk, Zahra Mohammadi explained joint results with Tent about the ω-stability of the free factor complexes. Since the theory of the free group is stable but not superstable, this result is quite surprising and already led to new topics and questions about the model theory of hyperbolic spaces in general, in particular regarding the ampleness hierarchy in these complexes.

Recently, Disarlo, Koberda and de la Nuez Gonzales started developing bridges between model theory, geometric topology, and geometric group theory starting from the mapping class groups. In particular, they showed that the curve graph is ω-stable and they give a model theoretic interpretation of Ivanov's metaconjecture.

3. Combinatorics

Model theory provides general tools for converting asymptotic quantitative questions about properties of families of finite (hyper-)graphs or groups into qualitative questions about the structure of the limiting infinite objects, to which the infinitary model-theoretic machinery can be applied. There have been several exciting developments in recent years illustrating the fruitful cross-interaction between model theory and additive and extremal combinatorics.

One important line of work starts with Hrushovski's 2012 paper on approximate groups, that is finite symmetric subsets A of a group G which are almost closed under product in the sense that all products of two elements of A are contained in finitely many left translates of A. Using ultraproducts and stability-theoretic techniques, he showed that up to commensurability, a pseudofinite approximate subgroup maps homomorphically into a Lie group (the so-called Lie Model Theorem). This to make a link to Hilbert's 5th Problem. They vastly extend and

generalize Gromov's celebrated result on groups with polynomial growth, leading to the asymptotic classification of finite approximate subgroups by Breuillard, Green and Tao.

In this area of research, Krupinski's talk about approximate subrings was particularly remarkable. In joint work with Machado they prove a structure theorem for finite approximate subrings and they aim is to develop a general framework for the sum-product phenomenon that applies uniformly across arbitrary rings. The main result identifies nilpotent quotients as the fundamental obstruction to growth under both addition and multiplication. The main application is a ring-theoretic counterpart of Gromov's theorem on groups of polynomial growth and the results on approximate subgroups by Breuillard, Green and Tao on the one hand and Hrushovski on the other hand. In this ring-theoretic problem the principal tool in the proof is the existence of definable locally compact models for arbitrary approximate subrings.

The Elekes-Rónyai Theorem states that, given a polynomial P(x,y) over the complex numbers that mixes addition and multiplication non-trivially, it exhibits expansion: namely, for any finite sets A, B of complex numbers of size n, the set $P(A, B)$ has size approximately $n^{1+\eta}$ for some positive constant η independent of n. In her talk, Zou presented an analogous result in the context of polynomial automorphisms of $\mathbb{C}^2$ for the (non-algebraic) group $\mathrm{Aut}(\mathbb{C}^2)$. Namely, a finite subset of $\mathrm{Aut}(\mathbb{C}^2)$ of bounded degree exhibits expansion unless it is concentrated on a nilpotent algebraic subgroup of $\mathrm{Aut}(\mathbb{C}^2)$.

4. Neostability and generalized tameness conditions

Stability theory was developped in the late 60s and early 70s by Shelah and others. Generalizing linear independence in vector spaces and algebraic independence in fields, he defined the notion of *forking independence* by a purely combinatorial condition; in the class of *stable* theories this independence is very well behaved. In certain cases it gives rise to a notion of dimension called *rank*; in this case independence is characterized by the preservation of rank: $\bar{a}$ and $\bar{b}$ are independent over C if $\dim(\bar{a}/C\bar{b}) = \dim(\bar{a}/C)$. In recent years there have been far reaching generalizations the notion of dimension which allow us to extend methods from stable structures to a wider class of universes. Hrushovski noticed that the logarithm of the pseudofinite counting measure (modulo the convex hull of the integers) yields a kind of dimension with values in some ordered abelian group. In the presence of a measure (as in pseudofinite structures) this leads to stability-like phenomena, most notably an amalgamation and a stabilizer theorem, which is the main ingredient for Hrushovski's Lie Model Theorem, in the context of approximate subgroups as discussed above.

In the presence of a group structure, even just the existence of a well-behaved integer-valued dimension function for definable (interpretable) sets has important algebraic consequences. In particular, finite-dimensional structures form a common unifying setting for groups of finite Morley rank and groups definable in an o-minimal theory. Karhumäki and Invitti both presented results from joint work

with Wagner on groups with such a notion of dimension. Karhumäki focusing on Sylow-2-subgroups of such groups showed in particular that in pseudo-finite groups Sylow-2-subgroups behave well while Invitti presented far-reaching generalizations from known results of groups of finite Morley rank, which is a very special case of dimension, to general groups of small dimension.

Another highlight was the talk by Ramsey on a variant of Mekler's construction for fields. Mekler's construction is a method to encode an arbitrary structure in a certain graph, and then the graph in a group of nilpotency class 2, preserving many model-theoretic tameness conditions and yielding group examples with prescribed properties. Ramsey first interprets a graph Γ in a certain a projective profinite group G, and then realizes G as the absolute Galois group of a pseudo-algebraically closed field K. He shows that Γ has SOP_n if and only if G has SOP_n if and only if K has SOP_n, thus obtaining the first examples of strictly NSOP_n fields.

Workshop: Geometry and Combinatorics: The Model Theoretic Perspective

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Abstracts

Model-Theoretic Tameness in Finite Extension of Groups

Yatir Halevi
(joint work with Saharon Shelah)

A basic and recurring structural question in group theory is: Which properties are inherited or transferred in finite group extensions?

In the case of model-theoretic stability, the question is:

Question 1. *Let G be an infinite group, and let $H \leq G$ be a subgroup of finite index. Is stability preserved when passing between G and H? More precisely, if G is stable, must H be stable? Conversely, if H is stable, must G be stable?*

One can, of course, ask this for any model-theoretic tameness notion.

The direction of transferability of stability up in finite extensions is known for abelian groups: abelian-by-finite groups are stable (folklore, essentially due to [1]). The general question in this direction goes back at least to [2]. A counterexample was given by Simon Thomas and was communicated by Baldwin in [3, Example 3.2]: The semidirect product of $\mathrm{SL}_2(\mathbb{C})$ and $\mathbb{Z}/2\mathbb{Z}$ under the action of complex conjugation.

We answer the general question negatively. The situation can be made as bad as possible with respect to model-theoretic tameness.

Theorem 1. *Let p be an odd prime. There exists an ω-stable group H of nilpotency class 2 and exponent p such that:*

1. *There is a supergroup $G \supseteq H$ satisfying that $[G : H] = 2$ and that $(G, \cdot)$ interprets ZFC.*
2. *There is a normal subgroup $K \trianglelefteq H$ of index p such that $(K, \cdot)$ interprets ZFC.*

Remark 1. *The group is produced by applying Mekler's construction* [4]. *The desired supergroup is obtained by producing an automorphism, and the subgroup is obtained with the aid of an appropriate linear functional on the $\mathbb{F}_p$-vector space $H/Z(H)$.*

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On ω-categorical k-strong Lie algebras

Christian d'Elbée

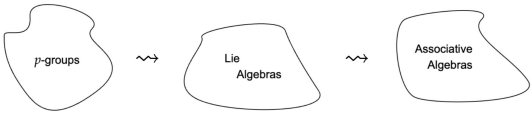
The Wilson Conjecture on ω-categorical groups is a classical question on the behaviour of locally nilpotent ω-categorical groups which can be stated as follows. Recall that a group G is n-Engel if it satisfies the identity $[...[x, \underbrace{y], y], ..., y}_{n \text{ times}}] = 1$.

Conjecture 1 (Wilson, 1981, [1]). *Every n-Engel ω-categorical group is nilpotent.*

The analogous question for Lie algebras (*every ω-categorical n-Engel Lie algebra is nilpotent*, where n-Engel means satisfying the identity $[...[x, y], y], ..., y] = 0$) is also an open statement. On the other hand, the corresponding question for associative algebra was answered a long time ago, in two beautiful papers of Cherlin [2, 3]. In this talk, we will focus on the question of nilpotency for n-Engel Lie algebras over $\mathbb{F}_p$, in particular in the case where $p > n$.

The three questions above relate to classical problems of Burnside-type (the restricted Burnside Problem, the Kostrikin Problem, the Kurosh problem), at several levels. In terms of quality, Burnside-like statements take the form: a structure satisfying a condition of the form "bounded exponent" (bounded exponent, n-Engel, bounded nilexponent) is locally nilpotent. With the extra assumption of ω-categoricity, we expect not *local* but *global* nilpotency[1].

In terms of methods, the study of Burnside-like problems often involves the following maps.



The left arrow embodies what is generally known as *Lie methods in group theory*, or how to reduce group-theoretic questions to Lie algebra questions. The right arrow merely represents how one effectively computes in a Lie algebra: via its enveloping (associative) algebra.

Let us consider the particular case of the Burnside problem for groups of prime exponent p which were proven by Kostrikin in the 1960s: given $r \in \mathbb{N}$ there are only finitely many *finite* r-generated groups of exponent p. In fact, this statement reduces to proving that, for all $p > n$, every n-Engel Lie algebra over $\mathbb{F}_p$ is locally nilpotent, which is what Kostrikin. It took 30 years to generalize the arguments of Kostrikin in order to remove the condition $p > n$, which Zelmanov did in the early 1990s, as part of his solution to the restricted Burnside problem.

In this talk, we will present some results in the study of ω-categorical n-Engel Lie algebras of characteristic $p > n$, focusing on the right arrow of the map above

[1] In fact, this intuition can be pushed further. In [1], Wilson also conjectured that ω-categorical groups generate a cross variety, that is, a variety generated by a finite group. This were proved for metanilpotent-by-finite ω-categorical groups. This hints toward the intuition that, on some level, ω-categorical groups carry features of finite groups.

and revisiting the work of Kostrikin. This study will involve the use of Lie algebras which are *k-strong*, a generalisation of the k-Engel condition, tracing back to the work of Higman on the solution to the restricted Burnside problem for groups of exponent 5, and connected to the asymptotic behaviour of the *nilpotency function* of locally nilpotent Lie algebras.

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Definable groups and fields in differentially closed fields of positive characteristic

Charlotte Bartnick

Definable groups in theories of fields can often be characterised in terms of algebraic groups. For example, by a result of Pillay [4], every definable group in a differentially closed field of characteristic 0 embeds into an algebraic group. The proof relies on a generalization of Weil's theorem [8] that an algebraic group can be obtained, up to bi-rational equivalence, from an irreducible variety with a function that generically resembles a group operation (i.e. from a pre-group).

In this talk, we will show that the analogue of Pillay's result holds true for definable groups in differentially closed fields of positive characteristic. Differentially closed fields are the existentially closed models of the theory of differential fields. In positive characteristic, their theory DCF_p was first investigated by Shelah and Wood ([5], [9, 10]). Due to the presence of pth roots, the theory DCF_p is not ω-stable and does not eliminate quantifiers in the language of differential rings (unlike its characteristic 0 equivalent DCF_0). Yet, it still lies in the tame class of stable theories.

To characterize (type-)definable groups in DCF_p, we follow the strategy of Bouscaren and Delon in [2] for the theory $SCF_{p,e}$ of separably closed fields of finite degree of imperfection: They proved that every definable group is isomorphic to the field-rational points of an algebraic group, so every type-definable group embeds into an algebraic group. In fact, the proof of their embedding result for type-definable groups can be generalized to more stable theories T of fields with additional structure. Instead of an explicit description of the definable closure (as it is used for $SCF_{p,e}$, given by the iterated λ-functions), it is enough to assume that the following property holds in a sufficiently saturated model M of the $\mathcal{L}$-theory T:

Property ($\star$). *Let N be a small elementary substructure of M and consider definably closed sets $A = \mathrm{dcl}(A)$ as well as $B = \mathrm{dcl}(B)$ with $N \subset A, B$. If A and B are non-forking independent over N, then the definable closure $\mathrm{dcl}(A, B)$ is contained in the subfield $A \cdot B$ generated by A and B.*

This property is satisfied by the theory DCF_p: A differential subfield A of the ambient model M is definably closed if and only if it is differentially perfect. By results of Shelah [5], the product of two differentially perfect fields in a model of DCF_p, which are non-forking independent over an elementary substructure, is again differentially perfect.

Using only Property $(\star)$ and the fact that T is a stable theory of fields, we obtain the following theorem:

Theorem 1. *Suppose that T is a stable theory of fields such that Property $(\star)$ holds. Every type-definable group $(G, \cdot)$, which is definable over a small model N inside the sufficiently saturated model M of T, is definably isomorphic over N to a subgroup of the M-points $H(M)$ of an algebraic group H, defined over N.*

The strategy of the proof follows [2, Proposition 4.2]. We first find an isomorphism of the group to a generically rational group and then apply Weil's theorem on pre-groups [8].

Besides DCF_p, the theorem also applies to the theory $SCF_{p,\infty}$ of separably closed fields of infinite degree of imperfection or the theory of proper pairs of algebraically closed fields.

Moreover, we will discuss how the result can be used to characterise definable fields: For differentially closed fields in characteristic 0, Sokolovic [6] showed that every definable field of finite Morley rank is definably isomorphic to the field of constants. Moreover, every definable field of infinite rank is isomorphic to the model M itself. In the proof, the embedding result on groups is applied to a simple group obtained from the definable field F and then a finer structural analysis is carried out.

Recently, Wang [7] provided a general result on definable fields assuming the existence of an embedding of the semi-direct product $F_+ \rtimes F^\times$ into an algebraic group. Using his result, the study of definable fields in our examples reduces to the study of definable subfields of the model M. To describe the latter, the strategy employed by Messmer [3] for separably closed fields of finite degree of imperfection generalizes and yields the following:

Lemma 1. *Suppose that T is a stable theory of separably closed fields of positive characteristic p satisfying Property $(\star)$ and such that for a sufficiently saturated model M of T the full induced structure on the subfield M^{p^∞} is strongly minimal. Every definable field F in the model M is separably closed of the same degree of imperfection as M.*

In particular, every definable field in a model of DCF_p is again separably closed of infinite degree of imperfection.

In contrast to the characteristic 0 case, it can also be shown that there are definable subfields of the model M of DCF_p not definably isomorphic to M. The proof of the lemma proceeds by showing that for every definable subfield F of M there is a natural number n such that $M^{p^n} \subset F$. It remains open whether F is a finite extension of some M^{p^n}.

The results presented in this talk are partially available in the preprint [1].

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Topological reconstruction theorems

Ben Castle

(joint work with Ronan O'Gorman)

1. The KLOS theorem

This talk is about an application of model theory to algebraic geometry. We start with a background story. Let K_1, K_2 be fields. The classical fundamental theorem of projective geometry asserts that any bijection $\phi : \mathbb{P}^2(K_1) \to \mathbb{P}^2(K_2)$ sending lines to lines must arise from a field isomorphism followed by an algebraic isomorphism. Concretely, given such ϕ, we can find a factorization $\phi = f \circ \sigma$ where $\sigma : \mathbb{P}^2(K_1) \to \mathbb{P}^2(K_2)$ is the induced map of a field isomorphism $K_1 \to K_2$ and $f \in \mathrm{PGL}_3(K_2)$.

It is natural to consider an analogous problem for maps sending *curves* to *curves*. For simplicity, take K_1, K_2 algebraically closed, and now consider a bijection $\phi : \mathbb{P}^2(K_1) \to \mathbb{P}^2(K_2)$ sending curves to curves in both directions. Is there a similar factorization $\phi = f \circ \sigma$? This time, the answer depends on the fields:

Fact 1 (Wiegand-Krauter, 1981 [1])**.** *For any two primes p and q, there is a curve-preserving bijection $\mathbb{P}^2(\overline{\mathbb{F}_p}) \to \mathbb{P}^2(\overline{\mathbb{F}_q})$.*[1]

In particular, such a bijection cannot involve an isomorphism of fields if $p \neq q$. However, in the 'opposite' setting, Kollár-Lieblich-Olsson-Sawin (KLOS) showed:

[1]Side-note: this is a very short paper giving essentially a model-theoretic argument; it axiomatizes a weak second order theory of projective planes over algebraic closures of finite fields, then shows categoricity via a back and forth construction, enumerating the curves over each field and matching them one at a time.

Fact 2. [2] *If K_1, K_2 are uncountable algebraically closed fields of characteristic zero, then any curve-preserving bijection $\mathbb{P}^2(K_1) \to \mathbb{P}^2(K_2)$ factors as above into a field isomorphism $K_1 \to K_2$ followed by an element of $\mathrm{PGL}_3(K_2)$.*

Why should 'uncountable' matter? Note that if ϕ were degree-preserving, Fact 2 would reduce to the original fundamental theorem. The hard case is when e.g. the degrees $\deg(\phi(L))$ vary as L runs over all lines in K_1. Now the idea is that if K_1, K_2 are uncountable, then by a pigeonhole argument, the value $\deg(\phi(L))$ is at least 'constant on a large set' (whatever that means), and this together with an intersection counting argument shows that ϕ is actually degree-preserving.

Fact 2 is part of a larger book of KLOS, which deals with recovering varieties from their scheme topologies by developing generalized versions of the fundamental theorem of projective geometry (the idea is that 'curve-preserving bijections' are just 'Zariski homeomorphisms'). One version of their main result is:

Fact 3. [2] *Let K_1, K_2 be uncountable fields of characteristic zero, and $\phi : X_1 \to X_2$ a scheme homeomorphism, where X_i is an absolutely irreducible, normal, projective variety of dimension at least 2 over K_i. Then we can factor $\phi = f \circ \sigma$, where σ is (induced by) a field isomorphism $K_1 \to K_2$ and ϕ is an isomorphism of K_2-varieties. In particular, ϕ underlies a scheme isomorphism $X_1 \to X_2$.*

2. What more can we expect?

Fact 3 was a big surprise in algebraic geometry, since it says the whole algebraic structure of the varieties X_i above is determined by their Zariski topologies. On the other hand, the statement has notable limitations. For instance, their method does not work even for smooth affine varieties over $\mathbb{C}$ (let alone varieties in positive characteristic). KLOS *do*, however, give several 'speculations' about what should be true in general. So let us ask the following:

Question 1. *Let K_1, K_2 be uncountable fields, and let X_i be an arbitrary quasi-projective variety over K_i, all of whose irreducible components have dimension at least 2. What can we say about the homeomorphisms $X_1 \to X_2$?*

Example 1. *In positive characteristic, there are new homeomorphisms coming from inseparable maps. For example, over $\overline{\mathbb{F}_p}$, the map $(x, y) \mapsto (x, y^p)$ on $\mathbb{A}^2$ does not factor into a field isomorphism and an isomorphism of varieties.*

Example 2. *For non-normal varieties, we get new homeomorphisms coming from bijective normalizations. E.g. let C be the curve $y^2 = x^3$. Then C is not isomorphic to $\mathbb{A}^1$, but the map $t \mapsto (t^2, t^3)$ from $\mathbb{A}^1$ to C is a homeomorphism (it is the normalization map $\widetilde{C} \to C$). One can also make this example higher-dimensional by mapping $\mathbb{A}^2$ to $\mathbb{A}^1 \times C$ in the obvious way.*

Example 3. *For disconnected varieties, we can get maps of shemes that don't arise from maps of fields. For example, over a given K, suppose $X = C \cup D$ with C and D closed and disjoint. Now take a self-homeomorphism $\phi : X \to X$ acting as the identity on C and as a random field automorphism (fixing the definition of D) on D. Then ϕ is a scheme morphism, but with a little work, one can show*

that ϕ typically fails to factor as $f \circ \sigma$ like above. A similar argument works even if $C \cap D$ is finite.

Given all this, KLOS propose the following informal picture:

(1) Fact 3 should hold verbatim if we only change 'projective' to 'quasi-projective' (e.g. this includes 'affine').
(2) If we take away the characteristic zero assumption, the same should work 'up to purely inseparable morphisms and field extensions' (the latter can be ignored if we assume algebraically closed).
(3) If we take away normality, then a homeomorphism $\phi : X_1 \to X_2$ should lift to a homeomorphism of normalizations $\widetilde{\phi} : \widetilde{X_1} \to \widetilde{X_2}$, where everything now works as above.
(4) If we don't assume irreducibility, we should still get a *scheme morphism of normalizations*, and if there are no 'finite intersections' (Example 3), this scheme map should factor in the desired way.

3. New Results

Our result, informally, is that (1)-(4) above hold over algebraically closed fields, by replacing the variant fundamental theorem of projective geometry from [2] with the Zilber trichotomy for ACF-relics [3]. We start with the irreducible case:

Theorem 1. *Let K_1, K_2 be uncountable algebraically closed fields, and let X_i be an irreducible quasi-projective variety of dimension ≥ 2 over K_i. Let $\phi : X_1 \to X_2$ be a homeomorphism. Then ϕ lifts to a homeomorphism $\widetilde{\phi} : \widetilde{X_1} \to \widetilde{X_2}$ of normalizations, and $\widetilde{\phi}$ factors as $f \circ \sigma$ where σ is induced by a field isomorphism $K_1 \to K_2$ and f is a universal homeomorphism of K_2-varieties.*

Why does model theory help here? The original KLOS proof showed that (in their setting) homeomorphisms send *linear systems* to *linear systems*. This is the correct analog of detecting which curves in $\mathbb{P}^2$ are lines. In our general setting, the geometry is not as well-behaved, and linear systems are not the right invariant (we need something coarser). We propose *definable families* as the correct coarsening. Precisely, embed X_1 into $\mathbb{P}^n$ where n is minimal, and let $\mathcal{H}$ be the family of irreducible components of hyperplane sections in X_1. The majority of the paper is a proof that ϕ must send $\mathcal{H}$ to a *definable family* of varieties in X_2 (this is really a quite involved proof of the definability of a single set; note that we only get definability, not anything stronger like closedness). Once this is done, we view ϕ as an isomorphism of certain structures $\mathcal{X}_i$ definable in the K_i, and apply the trichotomy. With a bit of work, the $\mathcal{X}_i$ are shown to be bi-interpretable with K_i, and this is shown to imply the theorem.

Now for the reducible case:

Theorem 2. *Let K_i be uncountable algebraically closed fields, and X_i a quasi-projective variety over K_i, all of whose irreducible components have dimension at least 2. Let $\phi : X_1 \to X_2$ be a homeomorphism. Then:*

(1) *ϕ lifts to a universal homeomorphism of normalized shemes. $\widetilde{\phi} : \widetilde{X_1} \to \widetilde{X_2}$.*

(2) *Assume X_1 cannot be written as the union of two proper closed subsets with finite intersection. Then $\widetilde{\phi}$ factors as $f \circ \sigma$, where σ is induced by a field isomorphism and f is a universal homeomorphism of K_2-varieties.*

(1) is a corollary of Theorem 1 (applied separately to each irreducible component and noting that the normalizations separate the components). (2) adds an *internality* argument. We again put a structure on X_i, this time by naming separate structures on each component in disjoint languages (so the trichotomy applies on each component). Then using the 'no finite intersections' property above, we show that all of all the components are internal to each other, and then show this implies bi-interpretability with the field as in the irreducible case.

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A "Mekler Construction" for fields?

Nicholas Ramsey

One important thread within model theory focuses on whether the various regions of the model-theoretic map of the universe have (theories of) fields in them and, if so, what those fields might look like. Concerning the analogous question for groups, a construction of Mekler [7] has been an important engine for producing algebraic examples across the various dividing lines in model theory. More precisely, Mekler showed that for each 'nice graph' Γ, there is a (nil-2, exponent p, for an odd prime p) group M_Γ such that the type counting function for the theory of M_Γ is the same as that of the theory of Γ; it follows immediately that stability and NIP are preserved. Many papers since then have elaborated on this theme, showing that, in many situations, model-theoretic properties of the graph Γ are preserved under passage to the group M_Γ. We are interested in addressing whether a similar sort of construction might be possible for fields.

The good news is that there is already a candidate construction for coding graphs into fields. In connection with questions concerning decidability, Cherlin, van den Dries, and Macintyre [5] showed how to build, from a graph Γ, a projective profinite group $\widetilde{G_\Gamma}$ whose inverse system codes Γ. Realizing $\widetilde{G_\Gamma}$ as the absolute Galois group of a PAC field, then, produces a field that interprets Γ. For this to serve as an analogue of the Mekler construction, though, one would need to know that the model-theoretic complexity of the field corresponds to that of Γ. The bad news is that for many model-theoretic properties, this is outright false: if Γ is an infinite graph, then any PAC field with absolute Galois group $\widetilde{G_\Gamma}$ will be

unbounded and thus will have TP_2 and IP_k for all k [4, 6]. We show, however, that there are important dividing lines that are preserved by this construction.

We focus on a test question concerning whether there is, for all $n \geq 3$, a field whose theory has Shelah's property SOP_n and does not have SOP_{n+1}. Chatzidakis [3] showed that a PAC field K has NSOP_n theory if and only if the theory of the inverse system of its absolute Galois group $S(\mathcal{G}(K))$ has NSOP_n theory. Thus, the analysis reduces to showing that, for an arbitrary graph Γ, Γ has SOP_n theory if and only if $S(\widetilde{G_\Gamma})$ has SOP_n theory. This is shown by a detailed analysis of the model theory of the theories $\mathrm{Th}(S(\widetilde{G_\Gamma}))$, for which we were able to describe the full elementary class, as well as describing types and algebraic closure. Building on this, we prove a 'relative stationarity' theorem, which shows that amalgamation problems in the inverse system have solutions if the corresponding problem has a solution in the graph. This analysis relies on Chatzidakis's detailed treatment of the model theory of free pro-p groups [2] as well as more recent results on H-structures, due to Berenstein, Carmona, and Vassiliev [1].

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Sylow p-subgroups in finite-dimensional groups

Ulla Karhumäki

(joint work with Frank O. Wagner)

In [Wag20], Wagner defined a general notion of dimension, thus aiming to unify results from stability and simplicity theory, o-minimality, and the study of pseudo-finite structures. In his paper, he studied the basic properties of dimensional groups. When studying the finer structure of finite-dimensional groups, one aims for results similar to the finite Morley rank and o-minimal case: a group G is called *finite-dimensional* if there is a dimension function dim from the collection of all interpretable sets in models of $\mathrm{Th}(G)$ to $\mathbb{N} \cup \{-\infty\}$ such that, for any formula $\phi(x, y)$ and interpretable sets X and Y, the following hold:

(1) *Invariance*: If $a \equiv a'$ then $\dim(\phi(x, a)) = \dim(\phi(x, a'))$.

(2) *Algebraicity*: If $X \neq \emptyset$ is finite then $\dim(X) = 0$, and $\dim(\emptyset) = -\infty$.

(3) *Union*: $\dim(X \cup Y) = \max\{\dim(X), \dim(Y)\}$.
(4) *Fibration*: If $f : X \to Y$ is an interpretable map such that $\dim(f^{-1}(y)) \geq d$ for all $y \in Y$ then $\dim(X) \geq \dim(Y) + d$.

The dimension of a tuple a of elements over a set B is defined as

$$\dim(a/B) := \inf\{\dim(\phi(x)) : \phi \in \mathrm{tp}(a/B)\},$$

and the dimension is called

- *additive* if $\dim(a, b/C) = \dim(a/b, C) + \dim(b/C)$ holds for any tuples a and b and for any set C; and
- *fine* if $\dim(X) = 0$ implies that X is finite.

One of the Sylow Theorems for finite groups states that the Sylow p-subgroups of a finite group are conjugate. Conjugacy results for Sylow p-subgroups in different classes of infinite groups also exist. For example, it is known that in any linear algebraic group over an algebraically closed field, the Sylow p-subgroups are conjugate. In the wider class of groups of finite Morley rank, this is an open conjecture. For $p = 2$ the conjecture holds [BP90]. For $p \neq 2$ the conjugacy of Sylow p-subgroups is known for the restricted class of soluble groups of finite Morley rank [ACCN98]. The well understood Sylow 2-theory plays a significant role in many deep results on groups of finite Morley rank, such as a) the classification of simple groups of finite Morley rank of *even* type [ABC08] b) the identification of a simple group of Morley rank 3 with $\mathrm{PSL}_2(K)$ [Fr18], and c) in definably primitive permutation groups (G, X) of finite Morley rank, bounding $RM(G)$ in terms of $RM(X)$ [BC08]. Motivated by this, we aim towards a good Sylow p-theory in the finite-dimensional context.

In a recent work with Wagner [KW26], we first generalise a structural result of p-subgroups from the finite Morley rank setting to the finite-dimensional setting:

Theorem 1 (with Wagner)**.** *Let G be a finite-dimensional group with fine and additive dimension. Suppose that P is either a 2-subgroup of G, or a locally finite p-subgroup of G. Then P is nilpotent-by-finite.*

In [Ug25] Uğurlu Kowalski shows that there is a pseudo-finite field F so that the simple pseudo-finite group $\mathrm{PSL}_2(F)$ has non-conjugate Sylow 2-subgroups; in this example the Sylow 2-subgroups are necessarily infinite. As simple pseudo-finite groups are finite-dimensional with fine and additive dimension, this means that in general one cannot expect to prove a conjugacy result for Sylow p-subgroups, even in the case $p = 2$, for finite-dimensional groups with fine and additive dimension. However, it is easy to see that in this example, $\mathrm{PSL}_2(F)$, the Sylow 2-subgroups are almost conjugate:

Definition 1. *Let G be a group. We say that the Sylow p-subgroups of G are* almost conjugate *in G if for any two Sylow p-subgroups S, T of G, there is $g \in G$ so that S is commensurable with T^g (i.e. $[S : S \cap T^g]$ and $[T^g : T^g \cap S]$ are both finite).*

Note also that in a pseudo-finite group any definable abelian normal subgroup which is 2-torsion free is also 2-divisible; we call groups in which definable abelian normal subgroups behave this way *AD_2-groups*.

Our main result for $p = 2$ is the following:

Theorem 2 (with Wagner)**.** *Suppose that G is a finite-dimensional group with fine and additive dimension. If definable sections of G are AD_2-groups then the Sylow 2-subgroups of G are almost conjugate.*

By the descending chain condition on definable subgroups, in a group of finite Morley rank and in an *o*-minimal group, any definable abelian subgroup splits to a sum of a definable bounded exponent characteristic subgroup and a definable divisible characteristic subgroup. We call groups in which definable abelian normal subgroups behave this way *ABD-groups*. As a corollary of Theorem 2 we observe that in any finite-dimensional group with fine and additive dimension whose definable sections are ABD-groups the Sylow 2-subgroups are conjugate. In particular, we recover the conjugacy of Sylow 2-subgroups in groups of finite Morley rank and prove the conjugacy of Sylow 2-subgroups in *o*-minimal groups (as far as we know this is a new result).

For arbitrary prime we prove the following.

Theorem 3 (with Wagner)**.** *Let G be a soluble finite-dimensional group with fine and additive dimension. Assume that definable sections of G are ABD-groups. Then the Sylow p-subgroups of G are conjugate.*

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Pseudofinite fields with an additive and a multiplicative character

Stefan Marian Ludwig

In [1] Ax gave a natural axiomatisation of the common theory of all finite fields. An infinite model of this theory is called a pseudofinite field. Naturally one might wonder whether the model-theoretic treatment can be extended to richer structures (of number-theoretic interest) than the pure field structure. Important objects to consider, ubiquitous in number theory, are character sums. A general **character sum** is an expression of the form

$$\mathrm{S}(V,f,g,q) := \sum_{\bar{x}\in V(\mathbb{F}_q)} \Psi(f(\bar{x}))\chi(g(\bar{x}))$$

where Ψ, χ are additive and multiplicative characters, V a variety or (more generally) a definable set, and f, g rational functions. Here, we assume that the data of (V, f, g) is given over $\mathbb{Z}$, and that for any $\mathbb{F}_q$ non-trivial Ψ and χ are chosen. Kowalski observed [4] that character sums are well behaved with respect to definable sets in finite fields by obtaining, under mild assumptions, the bound

$$|\mathrm{S}(V,f,g,q)| \leq Cq^{\dim(V)-\frac{1}{2}}$$

for $C > 0$ a constant independent of q.

More recently, Hrushovski gave a model-theoretic account thereof [3], which generalises Ax's work, introducing the theory PF^+ of pseudofinite fields (in characteristic 0) equipped with a generic additive character. This is done working in a mild version of continuous logic allowing for a predicate to take values in a compact complex set (the unit circle).

In my talk I presented results from the preprint [5]. It considers a generalisation of Hrushovski's work to the theory $\mathrm{PF}^{+,\times}$ of pseudofinite fields equipped with both a generic additive and a generic multiplicative character. Here, genericity just means that the characters are not of finite order in a model of $\mathrm{PF}^{+,\times}$ (as this otherwise can be treated on the level of (pure) pseudofinite fields already. The language $\mathcal{L}_{+,\times}$ is an expansion of $\mathcal{L}_{\mathrm{ring}}$ by two unary complex-valued continuous logic predicates Ψ and χ. The first main result is the following generalisation of Ax's Theorem to the theory of all finite fields with a generic additive and a generic multiplicative character $\mathrm{Th}^{\lim}_{\mathcal{L}_{+,\times}}((\mathbb{F}_q, \Psi_q, \chi_q)_q)$.

Theorem 1. $(F, \Psi, \chi) \models \mathrm{Th}^{\lim}_{\mathcal{L}_{+,\times}}((\mathbb{F}_q, \Psi_q, \chi_q)_q)$ *if and only if*

- F *is a field of characteristic* 0 *with Galois group* $Gal(F) = \hat{\mathbb{Z}}$.
- $\chi : (F^\times, \cdot) \to S^1$ *is a group homomorphism and* $\chi(0) = 0$.
- $\Psi : (F, +) \to S^1$ *is a group homomorphism.*
- *Given any absolutely irreducible curve* $C \subset \mathbb{A}^n$ *defined over* F, $\alpha : \mathbb{A}^n \to \mathbb{A}^k$ *an integral linear map and* $\beta : \mathbb{G}_m^n \to \mathbb{G}_m^l$ *an integral multiplicative map such that (with* $C' = \mathbb{G}_m^n \cap C$*)*
  - $C_1 := \alpha(C)$ *is not contained in any rational hyperplane over* F,
  - $C_2 := \beta(C')$ *is not contained in any multiplicative coset over* F.

Then,

$$(\Psi^{(k)} \times \chi^{(l)})(\Delta(F)) \subseteq_{\text{dense}} \mathbb{T}^{k+l}$$

where $\Delta(F) := \{(\alpha(\bar{z}), \beta(\bar{z})) \mid \bar{z} \in C'(F)\}$.

The main ingredients of the proof are for one direction the Weil bounds on character sums over absolutely irreducible curves, and for the other direction, apart from Ax's proof, a quantative version of Weyl's equidistribution criterion (the Erdős-Turán-Koksma inequality).

We have quantifier elimination in a natural expansion of the language. In particular, two models are elementary equivalent over any common relatively algebraically closed substructure. In general, $\mathrm{PF}^{+,\times}$ turns out model-theoretically tame in the following sense.

Theorem 2. $\mathrm{PF}^{+,\times}$ *is model-complete (after naming some constants). Moreover, it is simple and decidable (in the sense of continuous logic).*

An important result in the model theory of finite fields is the generalisation of the Lang-Weil bounds to definable sets obtained by Chatzidakis-van den Dries-Macintyre in [2]. Their theorem in particular includes the definability in the parameters of the (normalised) counting measure of a definable set in a (pseudo)finite field. Adapting their result and using ideas from [4], as well as the quantifier elimination statement for $\mathrm{PF}^{+,\times}$, we obtain the following in our context.

Theorem 3. *Let* $P(\bar{x}, \bar{y})$ *be an* $\mathcal{L}_{+,\times}$*-formula and* $B_{\bar{y}}$ *an* $\mathcal{L}_{\text{ring}}$*-definable set. For any finite field* $\mathbb{F}_q$ *where* $q = p^e$ *consider the function* $f : \mathbb{F}_q^{|\bar{y}|} \to \mathbb{C}$ *given by* $f(\bar{y}) = 0$*, if* $B_{\bar{y}} = \emptyset$*, and otherwise by*

$$f(\bar{y}) := \frac{\sum_{\bar{x} \in B_{\bar{y}}(\mathbb{F}_q)} P(\bar{x}, \bar{y})}{|B_{\bar{y}}(\mathbb{F}_q)|}.$$

Now for any $\epsilon > 0$*, there exists an* $\mathcal{L}_{+,\times}$*-formula* $\tilde{P}(\bar{y})$ *such that for any triple* $(\mathbb{F}_q, \Psi_q, \chi_q)$ *(consisting of a finite field with additive and multiplicative character) and any* $\bar{a} \in \mathbb{F}_q^{|\bar{y}|}$*,*

$$|\tilde{P}(\bar{a}) - f(\bar{a})| \leq \epsilon.$$

The above entails character sums over definable sets as consider by Kowalski in [4]. In contrast to [4], the proof only uses the Weil bounds and moreover the result includes the purely multiplicative case (i.e., definable expressions only involving χ).

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On the superstable part of the free group

Rizos Sklinos
(joint work with Javier de la Nuez González, Chloé Perin)

Let F be a nonabelian free group. By the work of Sela [6], and independently of Kharlampovich and Myasnikov [2], all nonabelian free groups have the same first-order theory. This theory, usually denoted T_{fg}, is stable by work of Sela [7], but it is not superstable; the latter was first shown by Gibone, see Poizat [4]. Thus, from the point of view of stability theory, the free group occupies an interesting position: it is stable enough to admit a meaningful independence calculus, but its global rank-theoretic behaviour is far from that of a superstable theory.

The aim of this talk is to discuss the problem of isolating the *superstable part* of the free group. More precisely, given a first-order formula $\varphi(\bar{x}, \bar{a})$ over F, one would like to understand when the definable set $\varphi(x, \bar{a})$ has ordinal-valued Shelah rank. The guiding conjecture is the following.

Conjecture: A first-order formula $\varphi(\bar{x}, \bar{a})$ over F is superstable, equivalently has ordinal-valued Shelah rank, if and only if its solution set does not change upon passing from F to any elementary extension which is again a free group.

The conjecture says that a formula $\varphi(\bar{x}, \bar{a})$ over F_n is superstable precisely when

$$\varphi(F_n, \bar{a}) = \varphi(F_\omega, \bar{a}).$$

The left-to-right direction of this conjecture was proved in joint work with Perin [3]: if a definable set over F_n is superstable, then it does not acquire new solutions in F_ω.

The first part of the talk will be devoted to this necessity direction. Using tools from geometric stability theory, together with the definable simplicity of the free group in the group-theoretic sense, we explain why superstable definable sets are rigid under passage to elementary extensions which are again free groups.

The converse direction requires a deeper use of the machinery developed by Sela in his solution of the Tarski problem, especially the analysis of definable sets through Diophantine envelopes [5, 8]. I will recall some of the relevant terminology and explain how definable sets over free groups can be analysed through the geometry of limit groups, that is, finitely generated models of the universal theory of a nonabelian free group. In this framework, the behaviour of a definable set is reflected in the structure of the corresponding resolutions and in the way these resolutions interact with graded limit groups.

The second part of the talk will focus on the connection between this geometric structure and Shelah rank. In joint work with de la Nuez González and Perin [1], we initiated the study of Shelah rank for definable sets in the theory of the free

group, and obtained lower bounds for the Shelah rank of varieties whose coordinate groups are hyperbolic towers. The broader aim is to develop a geometric calculus for Shelah rank in the free group, analogous to the role played by Zariski dimension in the calculation of Morley rank for varieties over algebraically closed fields. In this analogy, the geometry of Diophantine envelopes and resolutions is expected to provide the relevant dimension theory for definable sets in free groups. I will explain how such geometric features of Diophantine envelopes give rise to rank computations for definable sets in free groups, and how this provides evidence for the sufficiency direction of the conjecture.

Overall, the talk will describe a program relating two different kinds of structure: the model-theoretic rank of definable sets, and the geometry of Diophantine envelopes arising from Sela's analysis of imaginaries.

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Pseudofinite dimensions, entropy, and internal imaginaries

Martin Bays

(joint work with Itay Kaplan, Amador Martin-Pizarro, Geva Yashfe)

Shannon entropy can be seen as a real-valued dimension function; one incarnation of this is that if $\overline{a}$ is a finitely supported random n-tuple and $\overline{a}_I$ denotes the subtuple in co-ordinates $I \subseteq n$, then $I \mapsto H(\overline{a}_I)$ is a polymatroid. A polymatroid is **almost entropic** if it lies in the closure in $\mathbb{R}^{\mathcal{P}(n)}$ of such entropic polymatroids. We draw some connections with another real-valued dimension function arising in the work of Hrushovski [Hru12], coarse pseudo-finite dimension. We first observe, as a consequence of the representability of almost entropic polymatroids by cosets [CY02], that the almost entropic polymatroids are precisely those of the form $I \mapsto \boldsymbol{\delta}(\overline{a}_I)$ where $\overline{a}$ is a tuple in a pseudofinite structure and $\boldsymbol{\delta}$ is a continuous coarse pseudofinite dimension. Applying this to pseudofinite fields recovers the result of [Mat24] that algebraic matroids are almost entropic.

We proceed to adapt the proof of an information theoretic result of Ahlswede and Körner [AK75] to the $\boldsymbol{\delta}$ setting, and use this to obtain existence of weak canonical bases. The precise statement requires a little setup, since the canonical base may live in a quotient by an internal equivalence relation which need not be definable. Consider a non-principal ultraproduct M over ω of finite structures in a countable language $\mathcal{L}$, and $\boldsymbol{\delta}$ the coarse pseudofinite dimension with $\boldsymbol{\delta}(X_0) = 1$ where $X_0 \subseteq M$ is $\mathcal{L}$-definable. Let Int be the structure with a sort for each $n \in \mathbb{N}$ consisting of the internal subsets of M^n, with the subset relation on each sort and Cartesian product as binary functions. Say M is **adequate** if the substructure L of Int given by $\mathcal{L}$-definable sets is elementary. This is a drastic, but for our purposes harmless, assumption, which in particular makes $\boldsymbol{\delta}$ continuous. Given a countable $A \subseteq M$, say an expansion of M by internal sets is **A-independent** if $(L, \vec{A}) \preceq (L', \vec{A})$, where L' denotes the definable sets in the expansion, and we expand by predicates for the intersections with A-definable sets in Int; this implies in particular that $\boldsymbol{\delta}^{\mathcal{L}'} = \boldsymbol{\delta}^L$ on A. Finally, given $a, b \in A^{\mathrm{eq}}$ with $\boldsymbol{\delta}(ab) < \infty$, say $e \in A^{\mathrm{eq}}$ is a **weak canonical base** for a over b if $\boldsymbol{\delta}(e/b) = 0$ and $\boldsymbol{\delta}(a/eb) = \boldsymbol{\delta}(a/e)$, and if $e' \in A^{\mathrm{eq}}$ also satisfies this then $\boldsymbol{\delta}(e/e') = 0$.

Theorem 1. *Given a, b as above, after passing to an A-independent expansion there exists a weak canonical base for a over b, which moreover remains a weak canonical base in any further A-independent expansion.*

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Varieties over free associative algebras

Zlil Sela

(joint work with Agatha Atkarskaya)

In the 1960's and 70's ring theorists (P. M. Cohn, G. M. Bergman and others) tried to study the structure of sets of solutions to systems of equations (varieties) over free associative (non-commutative) algebras.

They tackled some fundamental pathologies that prevented any attempt to conjecture what can be the structure of these varieties.

The main difficulty that Cohn and Bergman observed was the lack of unique factorization. In fact every finite distributive lattice is the lattice of divisors of an element in a non-commutative free associative algebra. The major positive result that was known about these varieties was Bergman's centralizer theorem,

that proves that the centralizer of a non-trivial element is isomorphic to the ring of polynomials in one variable over the defining field.

The centralizer theorem implies that the Diophantine problem over free associative algebras reduces to Hilbert 10th problem, and hence it is, in particular, undecidable.

We managed to study binomial varieties, i.e., varieties over a free associative algebra in which every polynomial in the system that defines the variety has exactly two monomials (where each monomial is in terms of the variable and the defining coefficients).

To do that we first analyze homogeneous varieties, i.e., the set of homogeneous solutions to a homogeneous binomial system of equations. We reduce the structure theory of these homogeneous varieties to the study of varieties over a countably generated free semigroup. We further prove that given an arbitrary set of n-tuples in a free group or semigroup it is possible to associate with the set a Makanin-Razborov diagram that encodes the set, and if the diagram can be chosen to be of minimal rank, then the least complexity one is canonical (up to an appropriate isomorphism).

Given a general binomial variety, we start by analyzing the top homogeneous parts of the solution set. These are, in particular, homogeneous solutions to a homogeneous system of equations. Hence, with the top homogeneous parts of the solutions we can associate a Makanin-Razborov diagram.

Then we apply a (non-commutative) Hensel type principle, and prove that we can use the diagram that is associated with the top homogeneous parts to analyze the codegree d parts of the solutions for any natural number d.

On the structure of approximate rings

Krzysztof Krupiński
(joint work with Simon Machado)

This talk is based on my joint paper with Simon Machado [10].

The sum-product phenomenon, first studied by Erdős and Szemerédi [6], is a fundamental phenomenon in additive combinatorics. The sum-product phenomenon proved by Erdős and Szemerédi expresses the principle that the additive and multiplicative structures of the ring of integers, $\mathbb{Z}$, are incompatible: there is a constant $\epsilon > 0$ such that for any finite subset $A \subseteq \mathbb{Z}$, either the sumset $A + A = \{a + b : a, b \in A\}$ or the product set $A \cdot A = \{ab : a, b \in A\}$ must be large, specifically of size at least $|A|^{1+\epsilon}$. Since its introduction, the sum-product phenomenon has found numerous striking applications. For instance, in results by Helfgott [7] and Bourgain–Gamburd [2], establishing the sum-product phenomenon in various rings was pivotal in order to prove spectral gap for Markov operators. It has also striking applications in number theory via exponential sums [1].

Foundational works by Bourgain–Katz–Tao [3], and Tao [12] have further explored how the phenomenon manifests in rings satisfying a "few zero divisors"

assumption. These works distinguish between rings with few zero divisors, which are amenable to elementary methods, and rings with many scales (such as $\mathbb{Z}/p^n\mathbb{Z}$), which require advanced techniques like Bourgain's multiscale analysis.

In our work, we argue that the key distinction lies not in the presence or absence of zero divisors, but rather in the structural properties of nilpotency. We show that the sum-product phenomenon is fundamentally tied to the nilpotency of certain substructures within the ring, offering a unified approach to understanding this phenomenon across *all* rings.

We present a simplified version of our sum-product phenomenon. For a subset Z of a ring R and $n \in \mathbb{N}_{>0}$, by nZ we denote the set of sums of n elements of Z.

Theorem 1 (Sum-product phenomenon in all rings)**.** *Let $\epsilon > 0$. There exists a non-decreasing unbounded function $f\colon \mathbb{N} \to \mathbb{N}$ such that the following holds.*

Let R be a ring and $X \subseteq R$ be a finite subset. Define the set $X' := 4(X-X) + (X-X)\cdot 4(X-X)$. Then:

- *either $|X+X+X\cdot X| \geq f(|X|)|X|$,*
- *or there exists a subring $R' \subseteq R$ and an ideal $I \subseteq R' \cap X'$ such that R'/I is nilpotent and $|X' \cap R'| \geq |X|^{1-\epsilon}$.*

While this result provides a unified framework, it comes at the cost of quantitative precision, i.e. the function f is unspecified. Unlike the classical sum-product phenomenon in $\mathbb{Z}$, where the growth is known to be a power function, the precise growth rate here remains unknown. This limitation reflects the nature of our methods, which rely on ideas developed in the proof of the structure theorem for finite approximate subgroups by Breuillard, Green, and Tao [5].

A more precise statement, relating structural data to the ratio $\frac{|X+X+XX|}{|X|}$, requires the notion of *approximate rings*. Recently introduced by the first author in [9], this concept captures the lack of growth under addition and multiplication quantitatively. A subset X of a ring is called a *K-approximate subring* if it is additively symmetric (i.e., $0 \in X$ and $X = -X$) and the set $X \cdot X \cup (X+X)$ can be covered by K additive translates of X. Variants of this definition have appeared previously in the literature, often in finitary contexts (e.g., [8, 4]).

It is straightforward to verify that a *finite* K-approximate subring satisfies $|X+X+XX| \leq K^3|X|$. Conversely, a covering argument shows that if a subset X satisfies $\frac{|X+X+XX|}{|X|} \leq K$, then the difference set $X-X$ is a $K^{O(1)}$-approximate subring. Similarly, if the ambient ring is unital and $1 \in X$ with $|XX - XX| \leq K|X|$, then $X-X$ forms a $K^{O(1)}$-approximate subring (see [8, Lemma 3.3.1]).

Let X be an approximate subring. We define recursively

$$X_0 = X \text{ and } X_{n+1} := X_n + X_n + X_n \cdot X_n.$$

Then the union $\bigcup_{n\in\mathbb{N}} X_n$ coincides with the subring $\langle X \rangle$ generated by X. We state now our main structural result for *finite* approximate subrings.

Theorem 2. *For any $K \in \mathbb{N}$, there exist constants $N_1(K), \ldots, N_4(K) \in \mathbb{N}$ such that for every finite K-approximate subring X, there exists an $N_1(K)$-approximate*

subring $Y \subseteq 4X + X \cdot 4X$ which is $N_2(K)$-commensurable to X, and an ideal $I \lhd \langle Y \rangle$ contained in $Y_{N_3(K)}$ such that $\langle Y \rangle / I$ is nilpotent of class at most $N_4(K)$.

Furthermore, we establish two regimes for these constants:

- *One may choose $N_3(K) = 1$ and $N_1(K) = K^{510} + K^{22}$;*
- *Alternatively, one may choose $N_4(K) = \lfloor 4 \log_2(K) \rfloor$ and $N_1(K) = K^{510} + K^{22}$.*

A direct application of Theorem 2 is the aforementioned sum-product phenomenon, i.e. Theorem 1. Another application is a ring-theoretic analogue of Gromov's theorem on groups of polynomial growth. Below the definition, we state a simplified version of our theorem. For a subset X of a ring R by $X^{\leq n}$ we denote the subset of R consisting of the elements obtained from X using $+$ and $\cdot$ so that the elements of X are used at most n times (counting repetitions). A ring R generated by a finite additively symmetric set X has *polynomial growth* if there exists $d \in \mathbb{N}$ and constant C such that $|X^{\leq n}| \leq Cn^d$ for all positive integers n.

Theorem 3. *A finitely generated torsion-free ring has polynomial growth if and only if it is virtually nilpotent.*

Question 1. *Can $N_2(K)$ in Theorem 2 be polynomially bounded?*

A positive answer to this and related questions would have inportant further applications, e.g. we would get an explicit power function f in Theorem 1.

A fundamental tool in the proof of Theorem 2 is the existence of locally compact models for arbitrary approximate subrings which was proved in [9] via model-theoretic methods involving model-theoretic connected components of definable groups and rings. The fact that locally compact models exist for arbitrary (not necessarily finite or pseudofinite) approximate subrings enabled us to generalize a sum-product phenomenon due to Meyer [11] beyond the scope of classical additive combinatorics. Meyer's theorem essentially asserts that infinite, uniformly discrete approximate subrings of $\mathbb{R}$ are commensurable with $P(K)$ for some number field $K \subseteq \mathbb{R}$, where $P(K)$ is defined as the collection of all Pisot–Salem numbers that generate K (recall that Pisot–Salem numbers are real algebraic integers $\alpha > 1$ such that all other Galois conjugates of α have modulus less than or equal to 1).

A consequence of our main result concerning approximate subrings of semi-simple, finite-dimensional real algebras (which we do not state here) is the following generalization of Meyer's theorem, which for simplicity we state here for simple real algebras. Recall that simple, finite-dimensional real algebras are precisely the matrix algebras $M_n(D)$, where D is $\mathbb{R}$, $\mathbb{C}$, or the quaternions $\mathbb{H}$.

Theorem 4. *Let X be an infinite, uniformly discrete approximate subring in a finite-dimensional simple real algebra A, and assume X spans A. There exists a number field K, an archimedean place v, a K_v-algebra structure on A extending the $\mathbb{R}$-algebra structure, and a K_v-basis $(e_1, \ldots, e_d)$ of A such that X is commensurable*

with $\sum_{i=1}^{d} \mathcal{O}_{K,v} \cdot e_i$[1]*, where* $\mathcal{O}_{K,v} := \{\alpha \in K : \forall w \in S_K \setminus \{v\}, |\alpha|_w \leq 1\}$ *and* S_K *denotes the set of all places of* K*.*

In the specific case where $A = M_n(\mathbb{R})$*, we have* $K_v = \mathbb{R}$*,* $\mathcal{O}_{K,v}$ *corresponds to the set of Pisot–Salem numbers* $P(K)$*, and* X *is commensurable with* $\sum_{i=1}^{d} P(K) \cdot e_i$*.*[2]

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Sentences over Random Groups

Sobhi Massalha

Julia Knight formulated a natural conjecture predicting that, given a first-order sentence, its truth value over a random group in the few-relators model coincides with its truth value over non-abelian free groups. Our work extends the scope of this conjecture to random groups in the Gromov density model at the optimal density range $d < 1/2$.

We prove the conjecture for sentences belonging to the Boolean algebra of universal sentences, as well as for sentences of minimal rank with an arbitrary number of quantifiers, over random groups of density $d < 1/2$. In this extended abstract, we present our results and outline the main ideas and strategies underlying the proofs.

[1] In fact, $\mathcal{O}_{K,v}$ is identified here with $\sigma_v[\mathcal{O}_{K,v}]$ for an embedding σ_v of K in $\mathbb{C}$ such that the place v corresponds to the absolute value $x \mapsto |\sigma_v(x)|$.

[2] Here, K is identified with $\sigma_v[K]$.

Around 1945, Alfred Tarski posed a well-known question concerning the first-order theory of free groups. He asked whether any two non-abelian finitely generated free groups are elementarily equivalent. This question was answered affirmatively by Z. Sela through his seminal work, published in a series of seven papers entitled *Diophantine Geometry over Groups.*

Theorem 1. *(Z. Sela,* [Se1, Se2, Se3, Se4, Se5, Se6, Se7]*) Every two non-abelian free groups share the same first order theory.*

For a different approach to Tarski's problem, see also [Kh-My].

One may restate this result as follows. Let ψ be a first order sentence. Then one non-abelian free group satisfies ψ if and only if all non-abelian free groups satisfy ψ. In light of this observation, it is natural to fix a first-order sentence ψ that is true in non-abelian free groups and seek groups Γ that do not satisfy ψ. It turns out that this phenomenon is negligible. Namely, if a sentence ψ is true in non-abelian free groups, then it is also true in almost all groups, in an appropriate probabilistic sense.

Note the distinction between this approach and the study of groups that are elementarily equivalent to non-abelian free groups. In the setting described above, one focuses on a single sentence, whereas in the latter setting the entire first-order theory is involved. These two approaches are not equivalent. Indeed, almost all groups are not elementarily equivalent to non-abelian free groups.

Let us now make the mathematical framework precise. To this end, we must define what we mean by "almost all groups"; that is, we need to introduce a probabilistic model of groups from which a group can be chosen at random. (Gromov density model). Fix a natural number $k \geq 2$, a free group F_k with a fixed basis $a = (a_1, ..., a_k)$, and fix a real number $0 \leq d \leq 1$. The number d is called the *density* of the model. Given an integer l, let B_l denote the set of cyclically reduced words of length l in F_k

$$B_l := \{r \in F_k : |r| = l,\ r \text{ is cyclically reduced}\}.$$

We also define the set $\mathcal{M}_l$, called *the set of group presentations of level* l, to be the collection of all the tuples $T = \left(r_1, ..., r_{s(l)}\right)$ consisting of $s(l) = |B_l|^d$ elements $r_i \in B_l$, $i = 1, ..., [s(l)]$.

Given a property P, we call the number

$$p_l := \frac{|\{T \in \mathcal{M}_l : \text{The group presentation } \langle a : T\rangle \text{ satisfies the property } P\}|}{|\mathcal{M}_l|},$$

the probability that *the random group of level* l *and density* d satisfies the property P. If the limit $p_0 = \lim_{l\to\infty} p_l$ exists, then we call it *the probability that the random group of density* d *satisfies the property* P.

If $p_0 = 1$, we say that the random group of density d satisfies the property P with *overwhelming probability.* In this case we say also that *almost all groups of density* d *satisfy the property* P. In contrast, if $p_0 = 0$, we say that the property P is *negligible in density* d.

Note that if the probability that a random group of density d satisfies the property P is p_0, then the probability that it satisfies the property $\neg P$ is $1 - p_0$. In particular, a random group of density d satisfies the property P if and only if the property $\neg P$ is negligible in density d. Note that, for a fixed d, the number of relators in a presentation of level l grows exponentially with l. This approach to random groups is motivated by the philosophy of small cancellation theory, namely, the study of group-theoretic structure through the combinatorial properties of defining relations. Indeed, we have the following facts.

Theorem 2. *(*[Gr], [Ol]*).*

(1) *When $d > 1/2$, the random group of density d contains at most two elements.*
(2) *When $d < r/2$, for r a real number, the random group of density d is small cancellation $C'(r)$.*
(3) *When $d < 1/2$, the random group of density d is torsion-free hyperbolic. Moreover, there exists a real number $\alpha_0 = \alpha_0(d)$, that depends on d only, so that the hyperbolicity constant of a random group of level l and density d is at most $\alpha_0 l$.*

Returning to our main theme, we can formulate Knight's question in the setting of the Gromov density model as follows.

Conjecture 1. *Let $0 \leq d < \frac{1}{2}$, let $k \geq 2$ be an integer, and let ψ be a first order sentence in the language of groups. The random group of density d satisfies ψ (with overwhelming probability) if and only if the free group F_k satisfies ψ.*

We begin our investigation of this problem with sentences belonging to the Boolean algebra of universal sentences.

Theorem 3. *(*[Ma1]*, Theorem 10.3). Let $0 \leq d < \frac{1}{2}$, and let $k \geq 2$ be an integer. Let ψ be a first order sentence in the language of groups, and assume that ψ belongs to the Boolean algebra of universal sentences. Then the random group of density d satisfies ψ (with overwhelming probability) if and only if the free group F_k satisfies ψ.*

The statement of Theorem 3 was proved for densities $d < \frac{1}{16}$ in [Kh-Sk], using a different argument from the one we use in [Ma1].

In [Se1, Se2, Se3, Se4, Se5, Se6, Se7], Sela introduced a collection of powerful tools that can be applied to a given sentence ψ over a non-abelian finitely generated free group. Roughly speaking, when a sentence ψ is true in a non-abelian finitely generated free group, these tools give a syntactic "proof" of its validity. Moreover, these tools, together with the "proof" they produce, are shared uniformly by all non-abelian finitely generated free groups. In particular, the "proof" of ψ obtained over a non-abelian finitely generated free group can be used to establish the validity of ψ over every other non-abelian finitely generated free group.

Our strategy toward Conjecture 1 is to prove that all of these tools, originally developed over free groups, can be applied to random groups without altering either their structure or their functionality.

The most basic among these tools is the Makanin-Razborov diagram, introduced in [Se1]. The Makanin-Razborov diagram associated with a system of equations over a free group is a finite diagram that describes the structure of its solution set. The more advanced tools introduced by Sela rely heavily on the theory of Makanin-Razborov diagrams.

Consequently, the first step toward proving Conjecture 1 is to show that the Makanin-Razborov diagram of a given system of equations can be used uniformly to describe the solution sets of that system over random groups.

To this end, we prove the following statement, which is of independent interest.

Theorem 4. *(*[Ma1]*, Theorem 10.1). Let $0 \leq d < \frac{1}{2}$, let $k \geq 2$ be an integer, and consider the density model with k generators and density d. Let Σ_0 be a system of equations. Then the random presentation $\pi_\Gamma : F_k \to \Gamma$ of density d satisfies the following property with overwhelming probability.*

The set of solutions of the system Σ_0 in Γ equals the projection to Γ of the set of solutions of Σ_0 in F_k.

In the second stage of our approach to Conjecture 1, we consider general sentences of minimal rank. Our goal at this stage is to prove that a minimal rank sentence is true in free groups if and only if it is true in a random group of density $d < \frac{1}{2}$.

Theorem 5. *(*[Ma2]*, Theorem 10.8). Let $0 \leq d < \frac{1}{2}$, let $k \geq 2$ be an integer, and let ψ be a first order sentence of minimal rank. Let Γ be a random group of density d. Then ψ is true in the free group F_k if and only if ψ is true in Γ (with overwhelming probability).*

In [Se5, Se6], Sela established a quantifier elimination procedure for arbitrary formulas over free groups. He proved that every definable set belongs to the Boolean algebra of AE-sets (All-Exist) over free groups. The tools required for this quantifier elimination procedure are developed in [Se1, Se2, Se3, Se4].

For the purposes of our problem in the second stage, we establish a uniform quantifier elimination procedure for formulas of minimal rank. More precisely, given a minimal rank formula $V(p)$, we prove the existence of a formula $\varphi(p)$ belonging to the Boolean algebra of AE-formulas such that the formulas $V(p)$ and $\varphi(p)$ are equivalent (that is, they define the same set) over the free group F_k, which serves as the ambient free group in the density model, and are equivalent over almost all the groups of density $d < \frac{1}{2}$.

Theorem 6. *(*[Ma2]*, Theorem 9.12). Let $0 \leq d < \frac{1}{2}$, let $k \geq 2$ be an integer, and let $V(w, p)$ be a minimal rank formula in the Boolean algebra of AE-formulas. Let $\pi V(p)$ be the projection of $V(w, p)$. There exists a formula $R(p)$ in the Boolean algebra of AE-formulas so that:*

(1) *The formulas $\pi V(p)$ and $R(p)$ define the same set over F_k.*
(2) *Let Γ be a random group of density d. With overwhelming probability, the formulas $\pi V(p)$ and $R(p)$ define the same set over Γ.*

The uniform quantifier elimination procedure presented in [Ma2] is essentially the quantifier elimination procedure for formulas over free groups developed in [Se5]. In other words, for every minimal rank formula fixed in advance, the quantifier elimination procedure for that formula presented in [Se5] remains valid over almost all groups of density $d < \frac{1}{2}$.

The minimal rank assumption imposed on the formula does not restrict the number of quantifiers. Rather, it highlights the geometric methods involved in analyzing the structure of formulas over free groups and random groups, while simplifying some of the technical arguments that arise in the study of general formulas. This assumption was originally introduced in [Se4, Se5] in the analysis of AE-sentences and AE-formulas over free groups.

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On the Model Theory of the Free Factor Complex

Zahra Mohammadi Khangheshlaghi
(joint work with Katrin Tent)

The free factor complex, $\mathcal{AF}_n$, is a simplicial complex naturally associated with the free group $\mathcal{F}_n$ of rank n, and the complex of conjugacy classes of free factors, $\mathcal{OF}_n$, is the quotient of the free factor complex by the action of $Inn(\mathcal{F}_n)$: namely, the vertices of $\mathcal{AF}_n$ are the proper free factors of $\mathcal{F}_n$, and edges correspond to containment. The complex $\mathcal{OF}_n$ plays a central role in geometric group theory, serving as an analogue of buildings for linear algebraic groups and of the curve complex for free groups.

From the perspective of model theory, it offers a natural setting for exploring interactions between stability theory and geometric structures. In [1], Bestvina and Bridson proved that for $n \geq 3$,

$$Aut(\mathcal{AF}_n) \cong Aut(\mathcal{F}_n) \quad \text{and} \quad Aut(\mathcal{OF}_n) \cong Out(\mathcal{F}_n) \cong Aut(\mathcal{F}_n)/Inn(\mathcal{F}_n).$$

A key ingredient in their argument is the definability of certain subsets of these graphs in the language of graphs. In particular, they showed that the rank of every vertex in $\mathcal{AF}_n$ and $\mathcal{OF}_n$ is definable in this language. This observation provides a starting point for studying these complexes from a model-theoretic perspective.

In my talk, I presented the current state of our results for $\mathcal{AF}_n$ and $\mathcal{OF}_n$ from the perspective of model theory.

For a vertex x, let $LK^-(x)$ (respectively $LK^+(x)$) denote the set of vertices in the neighborhood of x whose rank is strictly smaller (respectively greater) than that of x. This set is definable for every x. Moreover, if x has rank k in $\mathcal{AF}_n$, then

$$LK^-_{\mathcal{AF}_n}(x) \cong \mathcal{AF}_k \qquad \text{and} \qquad LK^-_{\mathcal{OF}_n}(x) \cong \mathcal{OF}_k.$$

In particular, for every $m \leq n$, the structure $\mathcal{AF}_m$ is definable inside $\mathcal{AF}_n$.

However, rank 2 is exceptional. All proper free factors of $\mathcal{F}_2$ have rank 1, so there is no inclusion relation between distinct vertices. Instead, one considers the graph defined by the complement relation: two vertices are adjacent if their free product is the entire free group. It was shown in [1] that this relation is definable in free factor complexes of higher rank.

By analysing subgraphs of these two graphs, we obtained axiomatizations of $\mathcal{OF}_2$ (which is isomorphic to the Farey graph) and $\mathcal{AF}_2$, and proved that both structures are ω-stable.

Definition 1. *The Farey graph $F_1 = F_1(e_0)$ of level 1 consists of two triangles sharing the edge e_0. Inductively, the Farey graph $F_{n+1}(e_0)$ of level $n+1$ is the graph obtained from F_n by adding a new vertex v_e for every boundary edge e of F_n, with new edges joining v_e precisely to the vertices of e.*

*Finally, the **Farey graph** is the union of these graphs*

$$G_F = G_F(e_0) := \bigcup_{n \in \mathbb{N}} F_n.$$

Our main results are the following.

Theorem 1 ([2])**.** *The theory of the Farey Graph is axiomatized by*

(1) *every vertex is contained in an edge;*
(2) *every edge is contained in two triangles; and*
(3) *every finite subgraph of cardinality at least two has two removable vertices*

and is ω-stable of Morley rank ω.

Theorem 2 ([3])**.** *The theory* $\mathrm{Th}(\mathcal{AF}_2)$ *of the free factor complex of rank* 2 *is ω-stable, and $\mathcal{AF}_2$ is its prime model.*

Our current goal is to establish that the free factor complex $\mathcal{AF}_n$ and the complex of conjugacy classes of free factors $\mathcal{OF}_n$ are stable for every $n \geq 3$.

Definition 2. *A* standard apartment *in $\mathcal{AF}_n$ (and in $\mathcal{OF}_n$) is the full subcomplex spanned by the free factors generated by the non-empty proper subsets of a basis of $\mathcal{F}_n$.*

To approach the stability problem, we make use of the following properties.

- By [1], the property of being a standard apartment is first-order. Moreover, if X is a standard apartment in $\mathcal{AF}_n$, then $\mathrm{acl}(X) = \mathcal{AF}_n$ (and similarly for $\mathcal{OF}_n$). This implies that $\mathcal{OF}_n$ is ω-homogeneous, that it is the prime model of $\mathrm{Th}(\mathcal{OF}_n)$, and that every model of $\mathrm{Th}(\mathcal{OF}_n)$ can be written as a union of copies of $\mathcal{OF}_n$.
- For every vertex x, $LK^+_{\mathcal{AF}_n}(x) \cong LK^+_{\mathcal{OF}_n}([x])$, where $[x]$ denotes the conjugacy class of x.
- If x is a vertex of rank m, then $LK^+_{\mathcal{AF}_n}(x)$ is a union of definable copies of $LK^+_{\mathcal{AF}_{n-m}}(x)$ as a substructure.

These observations suggest an inductive approach to stability via the structure of links and the model-theoretic control provided by standard apartments and we conjecture that $\mathcal{AF}_n$ and $\mathcal{OF}_n$ are ω-stable for every $n \geq 3$.

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Generic (non-)differentiability in weakly o-minimal and C-minimal theories

Will Johnson

In o-minimal and P-minimal theories of fields, definable functions are generically differentiable, i.e., differentiable off a nowhere dense set. What about weakly o-minimal and C-minimal fields? (See [13, 6, 10, 5] for background on o-minimal, P-minimal, weakly o-minimal, and C-minimal theories of fields.)

I am fairly confident that one can build an example of a weakly o-minimal field in which generic differentiability fails. Here's how.

Let $\alpha \in \mathbb{R}$ be differentially transcendental, i.e., not 0-definable in the structure $(\mathbb{R}, +, \cdot, \exp)$. Consider the structure $(\mathbb{R}, +, \cdot, x^\alpha)$. This structure is o-minimal [15], model complete after adding a constant symbol for α [11, Theorem 3.4], and has a decidable first-order theory relative to an oracle for α [9] in large part because an analogue of Schanuel's conjecture holds [2]. Let $K = \mathrm{dcl}(\varnothing) \prec (\mathbb{R}, +, \cdot, x^\alpha)$. I can probably prove the following:

Likely Theorem 1.

(1) *There is a unique derivation $\delta : K \to K$ satisfying the following conditions:*

$$\delta(x^\alpha) = \alpha x^{\alpha-1}\delta x$$
$$\delta\alpha = 1.$$

(2) *If $f : K^n \to K$ is a definable function in the structure $(K, +, \cdot, x^\alpha)$, then there is another function $g : K^{2n} \to K$ definable in the same structure, such that*

$$\delta f(x_1, \ldots, x_n) = g(x_1, \ldots, x_n, \delta x_1, \ldots, \delta x_n) \text{ for } x_1, \ldots, x_n \in K.$$

For part (1), the strategy for proving *uniqueness* is as follows:

- Use the model completeness of $(\mathbb{R}, +, \cdot, \leq, \alpha, x^\alpha)$ [11, Theorem 3.4], and the fact that $K = \mathrm{dcl}(\varnothing)$, to show that every element of K appears as a coordinate of a tuple $\bar{x} = (x_1, \ldots, x_n)$ which is an isolated solution of a system of equations of the form

$$P_1(x_1, \ldots, x_n) = 0$$
$$P_2(x_1, \ldots, x_n) = 0$$
$$\cdots$$
$$P_m(x_1, \ldots, x_n) = 0,$$

  where each function P_i is a polynomial in $x_1, \ldots, x_n, x_1^\alpha, \ldots, x_n^\alpha$ with coefficients in the ring $\mathbb{Q}[\alpha]$.
- "Take derivatives" to ensure that the polynomial equations are transverse, in the sense that $m = n$ and the matrix $(\partial P_i/\partial x_j)_{i,j}$ is invertible at the isolated solution.
- Formally apply δ to the system of equations, and solve a system of linear equations to read off the values of $\delta x_1, \delta x_2, \ldots, \delta x_3$.

The *existence* in part (1) is harder. The uniqueness proof gives a recipe for calculating δx for any $x \in K$, but the problem is to show that this recipe is consistent. In its present form, the proof depends on the Ax-Schanuel theorem [1], though I expect the proof can be simplified using the Schanuel property for x^α already proven in [2]. The proof of (2) is similar to the uniqueness proof of (1), using model completeness to find a system of equations which defines f.

Once Theorem 1 is established, the following is an easy consequence:

Likely Theorem 2. *Let C be the field of constants $C = \{x \in K : \delta x = 0\}$.*

(1) *If $x \in C$ then $x^\alpha \in C$.*
(2) *$\alpha \notin C$.*
(3) *In the structure $(C, +, \cdot, \leq, x^\alpha)$, the function x^α is nowhere differentiable.*
(4) *The structure $(C, +, \cdot, \leq, x^\alpha)$ has a weakly o-minimal theory.*

In particular, generic differentiability can fail for a one-variable definable function in a weakly o-minimal theory of fields.

I expect a similar failure of generic differentiability to exist in the setting of C-minimal fields, though I have checked none of the relevant details. For C-minimal fields of positive characteristic, generic differentiability trivially fails (consider $f(x) = x^{1/p}$), so we are primarily interested in the case of characteristic 0. In characteristic 0, generic differentiability *does* hold if one assumes the property of "definable completeness" [7, Theorem 9.5].

All this discussion has implications for the study of definable groups and definable fields in weakly o-minimal and C-minimal theories of fields. There is a well-known strategy for using generic continuity and generic differentiability of definable functions to classify definable fields and prove non-trivial facts about definable groups. For example, generic differentiability is used by Pillay [12] to prove that if K is an infinite definable field in a p-adically closed field, then K is a finite extension of the home sort. A much stronger result was later obtained by Halevi, Hasson, and Peterzil, who show that in any P-minimal expansion of a p-adically closed field, any infinite *interpretable* field is a finite extension of the home sort [4, Theorem 1(3)]. Again, the proof depends on generic differentiability in P-minimal theories, which was established in [8].

Since generic differentiability fails in weakly o-minimal fields, and probably also C-minimal fields, what can we say about (say) definable fields? The weakly o-minimal case is hard, and is currently being investigated by Hasson and his collaborators, if I understand correctly. For the C-minimal case, I strongly suspect one can prove the following:

Conjecture 1. *Let $(K, +, \cdot, \ldots)$ be a C-minimal expansion of a field of characteristic 0.*

(1) *If F is an infinite definable field in the structure K, then F is definably isomorphic to K.*
(2) *If F is an infinite interpretable field in the structure K, then one of three things happens:*
   - *F is definably isomorphic to K.*
   - *F is internal to the residue field.*
   - *F is internal to the value group.*

The strategy for proving this would be to build a "definable completion" similar to what Wencel does for nonvaluational weakly o-minimal fields [14], and then transfer statements about F to the definable completion, similar to what Eleftheriou does when studying definable groups in nonvaluational weakly o-minimal

fields [3]. For the case of interpretable fields, one would use the Halevi-Hasson-Peterzil machinery [4] to split into two cases:

- $\dim(F) > 0$. In this case, F looks "locally" like a definable field, and one can run the same argument as for definable fields.
- $\dim(F) = 0$. In this case, F should be connected to the residue field or value group or to the sort $K/\mathcal{O}$ (where $\mathcal{O}$ is the valuation ring), though some "local linearity" property in $K/\mathcal{O}$ ought to exclude the third case.

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The Sum-Product Phenomenon via Continuous Real Dimension

Sergei Starchenko
(joint work with Yuyan He)

1. Introduction and Historical Context

The sum-product phenomenon is a core theme in additive combinatorics, asserting that a subset A of a ring or field cannot be simultaneously close to being an additive subgroup and a multiplicative subgroup. In other words, both the sumset $A + A$

and the productset AA cannot remain small unless A itself structurally mirrors an existing algebraic subfield.

This behavior was first classically formalized for finite sets of integers, establishing that there exist constants $C, \delta > 0$ such that:

$$\max\{|A + A|, |AA|\} \geq C|A|^{1+\delta} \tag{1}$$

In fields of finite characteristic, the landscape is inherently more intricate due to the presence of actual subfields. Classical extensions of this phenomenon to finite fields demonstrate that if a subset A does not structurally mimic a subfield, it must exhibit significant expansion under either addition or multiplication.

1.1. **The Model-Theoretic Approach.** This talk introduces a model theoretic generalization of the sum-product phenomenon. Instead of quantifying the "size" of a set using traditional counting measures or discrete cardinalities, we transition to a model-theoretic framework where size is governed by a **continuous real dimension** encoded within first-order logic. We demonstrate that small expansion under addition and multiplication implies the presence of an explicitly **definable subfield** of comparable dimension.

2. Axiomatizing Continuous Real Dimension

Let T be a complete first-order theory and $M \models T$ a model. A *continuous real dimension* is a mapping $\delta : Def(M) \to \mathbb{R}_{\geq 0} \cup \{+\infty\}$ on the collection of parameter-definable sets satisfying the following axioms for any $X, Y \in Def(M)$:

- **Union:** $\delta(X \cup Y) = \max\{\delta(X), \delta(Y)\}$
- **Product:** $\delta(X \times Y) = \delta(X) + \delta(Y)$
- **Finiteness:** $\delta(X) = 0$ if X is a finite set
- **Intersection:** $\delta(X \cap Y) \leq \min\{\delta(X), \delta(Y)\}$
- **Invariance:** $\delta(X) = \delta(Y)$ if there exists a definable bijection $f : X \to Y$
- **Subadditivity** For any definable surjection $f : X \to Y$,

  $$\delta(X) = \sup\{\alpha + \beta : \alpha \in \mathbb{R} \cup \{\infty\}, \beta \leq \{z \in Y : \delta(f^{-1}(z)) \geq \alpha\}\}.$$

- **Continuity:** for any formula $\phi(\bar{x}, \bar{y})$ and $s < t \in \mathbb{R}$, there is a $\emptyset-$definable D such that

  $$\{a : \delta(\phi(\bar{x}, a)) \leq s\} \subseteq D \subseteq \{b : \delta(\phi(\bar{x}, b)) < t\}.$$

This dimension generalizes classical geometric dimensions; for instance, o-minimal and geometric theories using algebraic closure (acl) dimension, alongside some theories of finite Morley rank. Importunately, it captures coarse pseudo-finite dimension, which treats asymptotic bounds of finite sets within an ultraproduct as a real-valued dimension.

Using this framework, we extend δ to partial types by taking the infimum over defining formulas, which allows us to naturally recover robust model-theoretic definitions for **genericity** and **independence** ($\vec{a} \perp_A \vec{b}$).

3. Combinatorial Machinery in the Dimensional Setting

We adopt some tools of classical additive combinatorics into continuous dimensions.

3.1. **Generalized Ruzsa Triangle Inequality.** We prove that for type-definable sets X, Y, Z of finite dimension in a dimensional group, the following holds:

$$\delta(X) + \delta(Y - Z) \leq \delta(X - Y) + \delta(X - Z) \tag{2}$$

The proof relies on generic independent realizations, the underlying symmetry and additivity of the dimension function.

3.2. **Plünnecke-Ruzsa Sumset Estimates.** We adapt Plünnecke's original strategy of utilizing commutative layered graphs. By tracking injections within a definably commutative 3-layered graph $\Gamma = (U_0 \sqcup U_1 \sqcup U_2, E)$, we establish that if a type-definable set expands minimally at the first layer, its higher expansions are bounded. This yields the Sumset Estimate: if $\delta(X + Y) = \delta(X) < \infty$, then for any $m, n \in \mathbb{N}$, $\delta(mY - nY) \leq \delta(X)$.

3.3. **Balog-Szemerédi-Gowers (BSG) Lemma & The Katz-Tao Theorem.** Using n-gons and triangles we proof a weak variant of the BSG lemma. This serves as the bridge to proving the dimensional version of the **Katz-Tao Lemma**. We show that if a type-definable set X has identical sum and product dimensions $(0 < \delta(X) = \delta(X + X) = \delta(XX) < \infty)$, then for any generic types p, q on X, the difference of their products expands minimally:

$$\delta(pq - pq) = \delta(X) \tag{3}$$

4. The Main Theorem: Constructing the Definable Field

Theorem 1. *Let F be a sufficiently saturated model of a theory expanding the theory of fields, equipped with a continuous real dimension δ. Let X be a complete type over a small parameter set A satisfying $0 < \delta(X+X) = \delta(XX) = \delta(X) < \infty$. Let p be a generic type on X over A, and $Y = a^{-1}p$ for some $a \models p$.*

*Then $\frac{Y-Y}{Y-Y}$ is a **definable field**, and $\delta\left(\frac{Y-Y}{Y-Y}\right) = \delta(X)$.*

4.1. **Outline of Proof Strategy.**

(1) **The $(\star)$ Property:** We define a stabilizing property for any element $x \in F$: x satisfies $(\star)$ if $\delta(xY + Y) = \delta(Y)$. Using the dimensonal Ruzsa triangle inequality, we prove that the collection of elements satisfying $(\star)$ forms a field.

(2) **Equivalence to the Quotient Set:** We show via a non-injectivity argument that an element x satisfies $(\star)$ if and only if it can be written as a quotient of differences: $x \in \frac{Y-Y}{Y-Y}$.

(3) **Definability via Subfield Stabilization Theorems:** While the quotient set is initially type-definable, model-theoretic structural properties of type-definable subfields of finite dimension inside a dimensional field

ensure that it is automatically first-order definable over the same parameters.

5. Finitary Applications and Conclusion

The advantage of this model-theoretic approach is its generalizability across arbitrary fields. By applying our main theorem to coarse pseudo-finite dimension and invoking ultraproduct structures, we immediately recover a clean, general finitary statement across all fields.

Corollary 1. *For any $\delta \in (0,1)$, there is $\epsilon(\delta) \in (0,1)$ and $N(\delta) \in \mathbb{N}$ such that for any finite subset $A \subseteq F$ of a field, if $|A+A|, |AA| \leq |A|^{1+\epsilon(\delta)}$ and $|A| \geq N(\delta)$, then there is a subfield $E \subseteq F$ satisfying:*

$$|A|^{1-\delta} \leq |E| \leq |A|^{1+\delta} \tag{4}$$

Expanding polynomial automorphisms of $\mathbb{C}^2$

Tingxiang Zou
(joint work with Martin Bays)

The Elekes–Rónyai Theorem [1] states that if a polynomial $f(x,y) \in \mathbb{R}[x,y]$ depends non-trivially on both variables, then for any finite subsets $A, B \subseteq \mathbb{R}$ of size n, the image set

$$f(A,B) := \{f(a,b) : a \in A,\ b \in B\}$$

must exhibit expansion: there exists a constant $\eta > 0$ such that $|f(A,B)| \geq n^{1+\eta}$ for all sufficiently large n, unless f is essentially induced by addition or multiplication.

More precisely, we say that $f(x,y) \in \mathbb{R}[x,y]$ is *additive* (respectively *multiplicative*) if there exist univariate polynomials $g, h, t \in \mathbb{R}[z]$ such that

$$f(x,y) = g(h(x) + t(y)) \text{ (respectively } f(x,y) = g(h(x) \cdot t(y)).)$$

The Elekes–Rónyai Theorem asserts that if f is neither additive nor multiplicative, then there exist constants $c, \eta > 0$ such that for all finite sets $A, B \subseteq \mathbb{R}$ with $|A| = |B| = n$,

$$|f(A,B)| \geq cn^{1+\eta}.$$

Raz, Sharir, and Solymosi later showed in [2] that one may take $\eta = \frac{1}{3}$.

A key reason for this dichotomy is the Elekes–Szabó Theorem [3]. Roughly speaking, it implies that a polynomial $f(x,y)$ exhibits expansion unless it is essentially governed by a one-dimensional algebraic group law. Since f is polynomial, the only one-dimensional algebraic groups that can account for non-expansion are the additive and multiplicative groups.

Together with Martin Bays, we generalised the Elekes–Rónyai Theorem to a two-dimensional setting. More specifically, we consider pairs of polynomials

$$(F(x,y;\bar{z}), G(x,y;\bar{z}))$$

over $\mathbb{C}$, under the assumption that for the parameters $b \in \mathbb{C}^{|\bar{z}|}$ under consideration, the specialised map

$$(F(x, y; b), G(x, y; b))$$

defines a polynomial automorphism of $\mathbb{C}^2$. In contrast to the one-dimensional case, where only one-dimensional (and hence abelian) algebraic groups arise, nilpotent algebraic subgroups of $\mathrm{Aut}(\mathbb{C}^2)$ can account for non-expansion as well. By classifying the nilpotent algebraic subgroups of $\mathrm{Aut}(\mathbb{C}^2)$, we obtained the following analogue of the Elekes–Rónyai Theorem [4].

Let $F(x, y, \bar{z}), G(x, y, \bar{z})$ be polynomials over $\mathbb{C}$. We say that the pair (F, G) is *coordinate separable* if there exist automorphisms $g, h \in \mathrm{Aut}(\mathbb{C}^2)$ and polynomials $t_0(\bar{z})$, $t_1(\bar{z})$, and $s(y, \bar{z})$ such that

$$g \circ (F(x, y, \bar{z}), G(x, y, \bar{z})) \circ h$$

is one of the following:

- $(t_0(\bar{z})x,\ t_1(\bar{z})y)$;
- $(t_0(\bar{z})x,\ y + t_1(\bar{z}))$;
- $(t_1(\bar{z})^\ell x + t_0(\bar{z})y^\ell,\ t_1(\bar{z})y)$ for some $\ell \in \mathbb{N}$;
- $(x + s(y, \bar{z}),\ y + t_1(\bar{z}))$.

A coordinate separable pair gives rise, after specialising the parameters $\bar{z}$, to elements of a nilpotent algebraic subgroup of $\mathrm{Aut}(\mathbb{C}^2)$. Conversely, if (F, G) is not coordinate separable and the finite sets involved do not concentrate on proper subvarieties, then expansion must occur.

Let

$$F(x, y, \bar{z}), G(x, y, \bar{z}) \in \mathbb{C}[x, y, \bar{z}] \setminus \mathbb{C}[x, y].$$

Suppose that (F, G) is not coordinate separable. Let $W \subseteq \mathbb{C}^{|\bar{z}|}$ be a subvariety. Then for every $\varepsilon > 0$ there exists $\eta > 0$ such that for all finite sets $A \subseteq \mathbb{C}^2$ and $B \subseteq W(\mathbb{C})$ satisfying

(1) $|A|^{1/\varepsilon} \geq |B| \geq |A|^{\varepsilon} \geq \frac{1}{\eta}$;
(2) $(F(x, y, b), G(x, y, b)) \in \mathrm{Aut}(\mathbb{C}^2)$ for all $b \in B$;
(3) for every proper subvarieties $V \subseteq \mathbb{C}^2$ and $W' \subseteq W$ of complexity $< \frac{1}{\eta}$,

$$|A \cap V| \leq |A|^{1-\varepsilon} \qquad \text{and} \qquad |B \cap W'| \leq |B|^{1-\varepsilon},$$

we have

$$|(F, G)(B) * A| := \left|\{(F(a, b), G(a, b)) : a \in A,\ b \in B\}\right| \geq |A|^{1+\eta}.$$

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Lie rings in finite-dimensional theories

Moreno Invitti

Lie rings are a natural generalisation of Lie algebras in which one does not assume the existence of an underlying vector space structure. More precisely, a Lie ring $\mathfrak{g}$ is an abelian group equipped with a bilinear map $[\cdot,\cdot]$ that is antisymmetric and satisfies the Jacobi identity. Lie rings played a fundamental role in Zelmanov's solution of the restricted Burnside problem [4, 5].

Lie rings have recently attracted considerable attention in model theory, following the work of Rosengarten, Deloro, and Tindzogho Ntsiri. A natural question in this context is an analogue of the Algebraicity Conjecture for Lie rings: if $\mathfrak{g}$ is a simple Lie ring of finite Morley rank, is $\mathfrak{g}$ definably isomorphic to a Lie algebra over an algebraically closed field? Although the conjecture remains open, Rosengarten [2] classified connected Lie rings of Morley rank at most 3, revealing a striking similarity with the corresponding classification for groups. More than twenty years later, Deloro and Tindzogho Ntsiri [1] classified simple connected Lie rings of Morley rank 4.

Finite-dimensional theories, introduced by Frank Wagner [3], generalise both theories of finite Lascar rank and o-minimal theories, thereby providing a broader framework in which to study the previous conjecture. One may ask whether the results known for Lie rings of finite Morley rank extend to finite-dimensional Lie rings. While the general case remains unresolved, we show that such extensions hold for connected finite-dimensional Lie rings and for NIP finite-dimensional Lie rings. Specifically, we prove that a connected Lie ring of dimension 1 is abelian, and that a connected Lie ring of dimension 2 is solvable. Moreover, we show that a simple connected Lie ring of finite dimension satisfies the DCC. This result is fundamental in proving that a connected Lie ring of dimension 3 is either isomorphic to $\mathfrak{sl}_2(K)$, where K is a definable field of dimension 1, or is "anisotropic". Anisotropic Lie rings constitute the Lie ring analogue of bad groups, and they arise naturally in finite-dimensional theories, as illustrated by the Lie ring $\mathfrak{so}_3(\mathbb{R})$. Similarly, we prove that a simple connected Lie ring of dimension 4 is anisotropic.

In the NIP setting, we prove that a Lie ring of dimension 1 is virtually abelian and that a Lie ring of dimension 2 is virtually solvable. We then introduce a new notion of simplicity, namely strong absolute simplicity, which plays a central role in the analysis of NIP Lie rings of finite dimension. We further show that a strongly absolutely simple NIP Lie ring of finite dimension satisfies the DCC. This allows us to classify NIP Lie rings of dimensions 3 and 4 by reducing the problem to the above classification of connected Lie rings of finite dimension.

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The tilting equivalence as a bi-interpretation

Thomas Scanlon
(joint work with Sivain Rideau-Kikuchi and Pierre Simon)

This is a report on joint work with Sivain Rideau-Kikuchi and Pierre Simon appearing in the preprint `arXiv:2505.01321`.

A perfectoid field is a complete non-trivially normed field $(K, |\cdot|)$ of residue characteristic p for some prime number p for which the set of norms $|K^\times|$ is dense in $\mathbb{R}_+$ and the map $x \mapsto x^p$ is surjective as a self-map on the ring $\mathcal{O}(K)/p\mathcal{O}(K)$ where $\mathcal{O}(K) = \{x \in K : |x| \leq 1\}$ is the ring of integers of K. Given a perfectoid field K, the tilt $K^\flat$ is obtained, as a multiplicative monoid, as the projective limit of the system in which K maps to itself repeatedly via $x \mapsto x^p$:

$$K^\flat := \varprojlim(\cdots \longrightarrow K \xrightarrow{x \mapsto x^p} K \xrightarrow{x \mapsto x^p} K)$$

The tilt $K^\flat$ carries a natural structure of a perfect complete normed field of characteristic p.

It can happen that two different perfectoid fields have isomorphic tilts. For a fixed perfect complete nontrivially normed field L of characteristic p the family of untilts, that is, characteristic zero perfectoid fields K with $K^\flat \cong L$, is parameterized by the Fargues-Fontaine curve [3]. That is, there is a natural correspondence between perfectoid fields and perfect non-trivially normed fields fields of positive characteristic given with a point on the Fargues-Fontaine curve.

This tilting correspondence has been used to transfer properties between mixed and positive characteristic. For example, in [6], Scholze extends the tilting operation to adic spaces, establishes an equivalence of categories of adic spaces over K and its tilt $K^\flat$, and uses this equivalence to transfer the truth of the weight monodromy conjecture for complete intersections in positive characteristic to mixed characteristic. For another celebrated use of perfectoids, consider André's proof of the direct summand conjecture in [1] in which he ports positive characteristic proof techniques to mixed characteristic.

Unlike the case of the Ax-Kochen-Ershov-type theorems in which theories of henselian fields of mixed characteristic and of positive characteristic converge as the residue characteristic grows, the tilt/untilt correspondence is not asymptotic; it allows for direct comparisons between mixed characteristic perfectoid fields and positive characteristic perfect fields with exactly the same residue fields. This is curious from the standpoint of mathematical logic as the correspondence could not possibly reflect an equality of theories, since, for instance, the characteristic of the field is captured by the theory. We were motivated by the problem of

answering the riddle of how the tilt/untilt correspondence might be explained through mathematical logic.

An answer is given by Jahnke and Kartas in [5]. Their main theorem is that if K is a perfectoid field, then for any non-principal ultrapower $K^{\mathfrak{U}}$, the tilt $K^\flat$ of K may be realized as an elementary substructure of a residue field of $K^{\mathfrak{U}}$. (See [5, Theorem 6.2.3] for details.) Their proof gives an equivalence of categories between the category of finite étale algebras over K and the corresponding category over $K^\flat$ and the theorem itself allows them to replace parts of almost mathematics [4] with ordinary commutative algebra.

Our answer is that the tilt/untilt correspondence is a quantifier-free bi-interpretation in continuous logic, largely based on a reformulation of Fargues and Fontaine's account of the tilting equivalence [3]. We note that the existence of this bi-interpretation implies that various important features of perfectoid fields are preserved by the tilt/untilt correspondence. These consequences include the Fontaine-Wintenberger theorem that a perfectoid field and its tilt have canonically isomorphic Galois groups, identifications between adic spaces over a perfectoid field and adic space over its tilt, and an approximation lemma whereby the tilt of an algebraic variety over a perfectoid field may be approximated by sets defined by quantifier-free formulas in the ordinary (first-order discrete) language of valued fields.

Our approach leverages the pioneering work of Ben Yaacov on metric valued fields [2]. We encounter a crucial technical issue that in Ben Yaacov's theory the valuation ring of the valued field in question is not definable. We deal with this by working with a modified theory of metric valued fields in which the universe is taken to be the valuation ring and the valuation is permitted to properly refine the valuation corresponding to the structural norm. We find an intermediate third theory of truncated perfectoid fields, where the intended structures take the form $\mathcal{O}/(\varpi)$ where $\mathcal{O}$ is the valuation ring of a perfectoid field and ϖ is a "pseudouniformizer": an element of the valuation ring with $0 < |\varpi| \leq |p|$. Even though these truncated perfectoid rings come equipped with a metric, they are essentially discrete as the norm of a nonzero element is bounded below by $|\varpi|$. The Frargue-Fontaine parameters give the data necessary to encode all perfectoid fields with and their tilts having residue ring $\mathcal{O}/(\varpi)$ in this truncated perfectoid ring.

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Nonabelian Brunn–Minkowski inequalities and model theory

Chieu-Minh Tran
(joint work with Yifan Jing, Simon Machado, and Ruixiang Zhang)

The purpose of the talk was to explain a connection between three themes which, at first sight, belong to rather different parts of mathematics: additive combinatorics, convex geometry, and model theory. The starting point is the classical small doubling problem. If A is a finite subset of an abelian group, then the size of

$$A + A = \{a_1 + a_2 : a_1, a_2 \in A\}$$

measures, in a crude but useful way, how much additive structure A has. For instance, arithmetic progressions have small doubling, whereas very sparse sets such as $\{2^1, \dots, 2^n\}$ have essentially quadratic growth under addition. Freiman's theorem says that subsets of $\mathbb{Z}$ with small doubling are controlled by generalized arithmetic progressions. A sharper dream, expressed by the Polynomial Freiman–Ruzsa philosophy, asks for polynomial bounds and for models which are not merely algebraic but convex: one expects a finite set with $|A+A| \leq K|A|$ to be efficiently modelled by a convex portion of a generalized arithmetic progression.

In the nonabelian setting one replaces sums by products. If G is a group and $A, B \subseteq G$, then

$$AB = \{ab : a \in A,\ b \in B\}.$$

The basic questions become: how small can A^2 or A^3 be, and what does small product growth imply about the structure of A? This point of view has been central in the development of nonabelian additive combinatorics. A paradigmatic result is Helfgott's product theorem for $\mathrm{SL}_2(\mathbb{F}_p)$, which became one of the inputs in the Bourgain–Gamburd method for constructing expander graphs. Later work of Hrushovski, Breuillard–Green–Tao, and Pyber–Szabó showed that, in broad generality, approximate groups are governed by nilpotent or Lie-theoretic objects.

Hrushovski's contribution introduced a model-theoretic viewpoint into the subject. Very roughly, one forms an ultraproduct of a sequence of approximate groups, obtains a limiting object equipped with a finitely additive or Loeb measure, and then extracts from it a locally compact, and eventually Lie, model. In an informal form, Hrushovski's Lie model theorem says that an ultra-approximate group is virtually modelled by a precompact neighbourhood of the identity in a Lie group. The power of this statement is that it converts combinatorial small doubling hypotheses into geometric information. Its weakness, for the purposes of quantitative additive combinatorics, is that the ultraproduct and Gleason–Yamabe ingredients are qualitative. Thus, while the theorem identifies the correct geometric universe, it does not by itself give the polynomial bounds one would want in a nonabelian analogue of the Polynomial Freiman–Ruzsa conjecture.

This suggests the following guiding problem. Can Hrushovski's Lie model theorem be refined into a quantitative "convex model theorem"? Such a result should say, schematically, that if A is a small-doubling set in a group equipped with a reasonable notion of size, then A is efficiently controlled by a convex object in a Lie group of bounded complexity. Here "reasonable size" may mean counting measure, Haar measure, metric entropy, or another compatible size notion. The word "convex" is the delicate part: in a nonabelian group there is no linear structure, and hence no immediate analogue of a convex body or a convex progression. One therefore needs a convexity theory intrinsic to Lie groups and compatible with product growth.

This is where Brunn–Minkowski theory enters. In Euclidean space, the Brunn–Minkowski inequality states that for suitable measurable sets $A, B \subseteq \mathbb{R}^d$,

$$\mu(A+B)^{1/d} \geq \mu(A)^{1/d} + \mu(B)^{1/d}.$$

It is one of the basic theorems of convex geometry, and its equality and stability phenomena single out convex sets. In additive combinatorics, it explains why convex progressions are the natural extremal objects. A nonabelian convex model theorem should therefore be accompanied by nonabelian Brunn–Minkowski inequalities: inequalities for $\mu(AB)$ in Lie groups which identify the correct dimension parameter and the correct near-extremal sets.

One representative result, joint with Yifan Jing and Ruixiang Zhang, gives such an inequality for a large class of noncompact groups. In a simplified form, let G be a connected noncompact unimodular real matrix group, let μ be a Haar measure, and let

$$n = \dim(G) - h(G),$$

where $h(G)$ is the maximal dimension of a compact subgroup of G. Then for nonempty compact subsets $A, B \subseteq G$, one has

$$\mu(AB)^{1/n} \geq \mu(A)^{1/n} + \mu(B)^{1/n}.$$

For $G = \mathbb{R}^d$, this recovers the classical exponent d. For nonabelian Lie groups, the subtraction of $h(G)$ reflects the fact that compact directions do not contribute to expansion in the same way as noncompact directions. The proof proceeds by an induction on dimension, using the structure theory of Lie groups. One new feature is that nonunimodular groups cannot simply be ignored: even when the final theorem is stated in the unimodular case, the induction naturally passes through nonunimodular intermediate groups.

Compact groups require a different analysis, since small sets may concentrate near proper subgroups. The motivating example is a problem of Breuillard and Green concerning small open sets in $\mathrm{SO}(3,\mathbb{R})$. The question asks whether every sufficiently small open set $A \subseteq \mathrm{SO}(3,\mathbb{R})$ satisfies

$$\mu(A^2) > 3.99\,\mu(A).$$

The constant is not accidental. In $\mathrm{SO}(3,\mathbb{R})$, the largest proper closed subgroups have dimension 1, while the ambient group has dimension 3. Thus the relevant codimension is 2, and the expected minimal doubling constant is $2^2 = 4$. Jointly

with Yifan Jing and Ruixiang Zhang, we proved this lower bound in the asymptotic form above, and also proved a corresponding product-set version.

The proof can be interpreted as a model-theoretic compactness argument. If the desired inequality failed, one could take a sequence of smaller and smaller counterexamples and pass to an ultralimit. A coarse Lie model theorem then turns the limiting approximate group into a locally compact Lie-theoretic object. Since the original sets in $\mathrm{SO}(3,\mathbb{R})$ shrink to measure zero, the limiting object is noncompact. The noncompact Brunn–Minkowski inequality can then be applied to this model, giving the forbidden expansion lower bound. Thus model theory acts as a bridge: it transports a compact, small-scale problem into a noncompact Lie group, where Brunn–Minkowski geometry is available.

Subsequent work of Simon Machado gives a sharp general version for compact connected Lie groups. If G is compact connected, $d_G = \dim G$, and d_H is the maximal dimension of a proper closed subgroup, then small measurable subsets satisfy an asymptotic lower bound of the form

$$\mu_G(A^2) \geq \big(2^{d_G - d_H} - o(1)\big)\mu_G(A), \qquad \mu_G(A) \to 0.$$

More generally, there is a Brunn–Minkowski-type inequality for products AB, and a stability statement: near-minimal examples are essentially neighbourhoods of maximal proper subgroups. This confirms the expectation that the correct nonabelian convex objects are not arbitrary approximate groups, but geometric neighbourhoods organized by the subgroup structure of the ambient Lie group.

The broader message of the talk was that model theory should not be viewed merely as a black-box compactness device. It provides a language in which one can formulate limiting versions of additive-combinatorial problems, identify the Lie groups which control those limits, and separate qualitative structure from quantitative geometry. The proposed convex model theorem would be a quantitative refinement of this philosophy. It would combine the model-theoretic extraction of Lie models with Brunn–Minkowski inequalities and stability results inside those Lie models. Such a theorem would be a step toward a genuinely nonabelian Polynomial Freiman–Ruzsa theory, where small product growth is controlled not only by nilpotent algebraic structure, but also by convex geometric structure.

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Definably Simple Groups in Valued Fields

Immanuel Halupczok
(joint work with Jakub Gismatullin and Dugald Macpherson)

It is a recurring theme in model theory (and specifically in this conference) to classify groups definable in certain nice structures. More precisely, it often turns out that every non-abelian definably simple group is essentially already a linear algebraic group. Such a result has been obtained around 2000 by Peterzil–Pillay–Starchenko [4] for groups definable in o-minimal structures. The goal of this talk is to present an analogous result for groups definable in "nice" valued fields.

The assumptions are the following:

- Let K be a henselian valued field of characteristic 0 (but arbitrary residue characteristic).
- Let L be a 1-h-minimal expansion of the valued field language on K. (The notion of 1-h-minimality is an analogue of o-minimality for valued fields, introduced in [1, 2].)
- Let G be an L-definable group with the following properties:
  (1) The universe of G is a subset of K^m, for some m (and not of some other sorts).
  (2) G is infinite.
  (3) G is non-abelian.
  (4) G is definably almost simple, meaning that every definable normal subgroup $N \triangleleft G$ is either equal to G or finite.

The result we obtain in [3] is that such a group G is very close to being a linear algebraic group, namely:

Theorem 1. *Under the above assumptions, there exists a definable homomorphism $f\colon G \to H(K)$, for some linear algebraic K-group H, whose kernel is $Z(G)$, whose image contains $H(K)^+$, and where moreover*

(*) *H is semisimple, almost K-simple and K-isotropic.*

Here, $H(K)^+$ is the subgroup of $H(K)$ generated by all unipotent elements. It is known that (*) implies that $H(K)^+$ is "almost equal to $H(K)$". Note that $H(K)^+$ is a normal subgroup of $H(K)$, so if it is definable, definable simplicity of G implies that the image of f is equal to $H(K)^+$. (However, we do not know whether $H(K)^+$ is always definable.)

In the talk, I explained the result in some more detail, and I gave a sketch of the proof.

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