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Random Matrices

Organized by
Paul Bourgade, New York
Friedrich Götze, Bielefeld
Alice Guionnet, Lyon

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Abstract. Random matrix theory (RMT) has emerged as a central object of study across physics, computer science, and mathematics. Originally introduced by Wigner to describe energy levels in heavy nuclei, RMT reveals universal statistical properties of eigenvalues, dependent solely on the matrix symmetry class. Over the past three decades, it has found applications in diverse fields, from growth models and number theory to multivariate statistics and data science.

Research in RMT follows two interconnected paths: integrability, which seeks explicit distributions for specific models, and universality, which extends these properties to broader classes of probabilistic systems. Recent breakthroughs include the proof of the Anderson transition for band matrices in any dimension, universality for non-Hermitian matrices and d-regular graphs, and the resolution of long-standing conjectures in number theory.

Key open challenges persist, particularly in random Schrödinger operators (e.g., proving delocalization in physical dimension 3), Coulomb gases (crystallization phenomena), large deviations, and random tensors. The interplay between universality and integrability continues to drive innovation, with new connections to machine learning, extreme value theory, and quantum field theory.

This workshop brought together outstanding researchers from a variety of mathematical backgrounds whose areas of research are linked to random matrices. While there are strong links between their motivations, the techniques used by these researchers span a large swath of mathematics, ranging from purely algebraic techniques to stochastic analysis, classical probability theory, operator algebra, supersymmetry, orthogonal polynomials, etc.

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Introduction by the Organizers

The workshop *Random matrices* focused on several recent breakthroughs in random matrix theory, including the resolution of long-standing conjectures.

One of them concerns the Anderson transition conjecture for one-dimensional random band matrices, which predicts that the eigenvectors of a band matrix are localized if the band width is much smaller than the square root of the dimension, but delocalized if it is much larger. This long-standing conjecture was finally resolved in 2025, in independent works presented during the Monday afternoon session. Volodymyr Riabov outlined the Zig-Zag strategy, developed with Laszlo Erdös and others. Jun Yin introduced generalized random band matrices in any dimension, and explained the proof of delocalization through a loop hierarchy method and its tree approximation, elaborated with S. Dubova, F. Yang, and H. T. Yau. The critical regime, where the band has width of the order of the square root of the dimension, was tackled by Marya and Tatyana Shcherbina by supersymmetry techniques. For full Wigner matrices, Mohammed Osman discussed precise delocalization properties for Wigner matrices and the Gumbel law for the largest entry of individual eigenvectors. Lucas Benigni provided an alternative perspective on the quantitative universality of the eigenvectors of Wigner matrices, based on a flow regularization approach.

Another breakthrough was the proof of edge universality for d-regular graphs down to $d = 3$, achieved by T. McKenzie, J. Huang and H.-T. Yau. Jiaoyang Huang presented a novel approach to these questions, leveraging loop equations to characterize local distributions. Theo McKenzie proposed a general criterion for sparse random graphs, guaranteeing that the distribution of local neighborhoods of eigenvectors is given by a Gaussian wave. Johannes Alt considered both undirected and directed critical Erdős-Rényi graphs through the time evolution of a free quantum particle in the Hermitian case, and the emergence of localized states in the non-Hermitian case.

Important advances were also made to analyze the spectrum of non-Hermitian matrices, which is particularly difficult because of its instability. To deal with this instability, a usual path is to use Green's formula and to estimate the smallest singular value of related matrices to study the large dimension limits of the spectrum. Julian Sahasrabudhe addressed the Spielman-Teng conjecture, focusing on deriving the optimal upper bound on the probability that the smallest singular value of a matrix with independent Bernoulli entries is small. Gernot Akemann explored the universality classes for non-Hermitian matrices and discussed the conjecture that there are only 3 generic local bulk and edge statistics.

The study of the asymptotics of characteristic polynomials and their convergence to Gaussian multiplicative chaos has also seen remarkable progress recently, motivated by its connection to number theory. Giorgio Cipolloni and Tatyana Shcherbina discussed these developments in the context of non-Hermitian matrices. Djalil Chafai presented singular central limit theorems for the logarithm of characteristic polynomials in the context of the fermionic Coulomb gas on the

2-sphere. Fanny Augeri focused on characteristic polynomials in random Jacobi matrices, while Gaultier Lambert examined the so-called Gaussian beta ensembles.

Another impressive development in random matrix theory over the last few years concerns the concept of strong asymptotic freeness. The notion of asymptotic freeness of several random matrices simply says the trace of polynomials in these matrices converges towards the trace of free operators, as defined in free probability. Strong asymptotic freeness additionally requires convergence of the operator norm of these polynomials to its free counterpart. Haagerup and Thorbjornsen proved that independent Gaussian matrices are asymptotically strongly free in 2005. This result has since been generalized in several directions, including high dimensional tensors (implying the Peterson-Thom conjecture) and to permutation matrices. Benoit Collins proposed a new, simplified proof of strong convergence for tensored matrices. Ramon Van Handel described how strong asymptotic freeness applies to random covers of hyperbolic surfaces. Mireille Capitaine showed that strong asymptotic freeness also extends to multiplicative Brownian motions on the general linear group, while Nick Cook discussed the convergence and the evolution of the spectrum of these non-Hermitian multiplicative Brownian motions on the general linear group.

Yet another highlight was the deeper understanding of the Toda chain, motivated by the connection of its generalized Gibbs measures with Beta-ensembles introduced by Herbert Spohn. Tamara Grava presented a direct approach to study these generalized Gibbs measures through a fine analysis of their density computed by Van Moerbeke. Herbert Spohn discussed recent progress in understanding the diffusion of a tagged Toda fluid quasiparticle, which was made rigorous by Amol Aggarwal very recently.

The conference also explored applications and generalizations. Two talks focused on generalizations to tensors, motivated for instance by quantum mechanics. Razvan Gurau highlighted the challenges in developing a theory analogous to free probability, where invariants would play the role of trace of polynomials, particularly due to the lack of concentration of part of the invariants. Stéphane Dartois presented two recent strategies for bounding the injective norms of tensors by using complexity or a moment method. Gérard Ben Arous showed how random matrices can be used to study the complexity of random functions, detect hidden signals, and design algorithms for signal recovery. Afonso Bandeira discussed new progress on a conjecture by Spencer concerning the matrix discrepancy of group representations. Finally, Mark Rudelson showed how to bound the condition number of a matrix by adding a sparse matrix (depending on a number of parameters similar to the dimension) rather than a Gaussian matrix as was already known after the result by Sankar-Spielman-Teng. Jana Reker discussed large deviations for the largest eigenvalue of Kronecker matrices, an important step to understanding the entropy of structured matrices and polynomials in several matrices.

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Workshop: Random Matrices

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Abstracts

Three universality classes of non-Hermitian random matrices

Gernot Akemann
(joint work with G. Angermann, N. Aygün, A. Mielke, P. Päßler, C. Raitzig and T. Winkler, and with Y. V. Fyodorov and D. Savin)

Non-Hermitian random matrices enjoy important applications, including open, dissipative systems in Quantum Physics and many others [1]. Based on symmetries alone, random matrices serve as the simplest model - often analytically tractable - to predict universal aspects of spectral statistics in non-Hermitian Hamiltonians of physical systems. To date, the complex eigenvalue and eigenvector statistic of non-Hermitian random matrices is far from being understood and a very active field of current research.

The simplest ensembles introduced by Jean Ginibre in 1965 [2] consist of real non-symmetric, complex non-self adjoint and quaternion non-self dual non-Hermitian random matrices, with independent and identically distributed (i.i.d.) Gaussian elements. By now a comprehensive understanding of the complex eigenvalue and corresponding eigenvector statistics exists for all 3 ensembles, as reviewed in [3]. However, these 3 ensembles do not cover all universal statistics observed in physical systems, e.g. for complex symmetric Hamiltonians [4].

How many different non-Hermitian random matrix ensembles are there? In analogy to the celebrated Altland-Zirnbauer classification of symmetry classes of Hermitian random matrices, an analogous classification was proposed by Bernard and Le Clair, which has been refined recently by Kawabata et al. [5]. It is common to all symmetry classes that the spectrum becomes compactly supported in the plane in the limit of large matrix size $N \to \infty$. Shortly after [5], it has been conjectured based on numerics [6], that amongst the 38 symmetry classes there are only 3 generic universality classes of local statistics in the bulk of the support of the spectrum. Only at special symmetry points in the spectrum, i.e. at the origin, along the real line (e.g. for the real Ginibre ensemble) or the imaginary axis, the spectral statistics of the 38 classes may differ in the large-N limit.

The simplest representatives for the 3 conjectured generic universality classes are the ensemble of complex Ginibre matrices - called class A following the Cartan classification of [5] - complex symmetric (class AI†), and complex self-dual (class AI†) Gaussian random matrices. More precisely, let $J \neq J^\dagger$ be a non-self adjoint $N \times N$ matrix with complex matrix elements distributed as

$$P_N(J) \sim \exp\left[-N\mathrm{Tr}[JJ^\dagger]\right], \text{ with } \quad J_{ij} \in \mathbb{C}. \tag{1}$$

The known normalisation constant suppressed here depends on the ensemble. They are characterised by the following conditions, where $J^\dagger = \overline{J}^T$ is the complex conjugate transposed:

- class A: $J \neq J^\dagger$,
- class AI†: $J = J^T \neq J^\dagger$,
- class AII†: $J = \Sigma J^T \Sigma \neq \Sigma \overline{J} \Sigma$, where $\Sigma = \begin{pmatrix} 0 & -i\mathbf{1}_{N\times N} \\ i\mathbf{1}_{N\times N} & 0 \end{pmatrix}$.

In [6] it is proposed to parametrise matrix J of class AII†, which is of size $2N \times 2N$ instead, in terms of 3 non-Hermitian $N \times N$ random matrices A, B, C,

$$J = \begin{pmatrix} A & B \\ C & A^T \end{pmatrix}, \tag{2}$$

where $B = -B^T$, $C = -C^T$, and $A \neq A^\dagger$ has no symmetry.

In this talk we first review the conjecture of [6]. It is based on heuristic arguments and the numerical evaluation of the nearest neighbour (NN) spacing distribution $p_{\mathrm{NN}}(s)$ in the bulk of the spectrum, for 9 of the 38 classes and i.i.d. matrix elements with Gaussian or Bernoulli distribution. Then, recent progress is reported on our analytical and numerical findings in classes AI† and AII† for $p_{\mathrm{NN}}(s)$, the spectral density of complex eigenvalues $R_1(z)$, and the complex spacing ratio $\varrho(z)$ to be defined below, both for bulk and edge statistics.

An additional motivation to study the local statistics of classes AI† and AII† is that an approximate effective description for the NN and next-to-nearest neighbour (NNN) spacing distribution has been found numerically in [4]. They follow closely the same distributions as in a two-dimensional Coulomb gas (2DCG), see (3), also determined numerically, at different values of inverse temperature $\beta = 1.4$ and 2.6, respectively. Thus understanding the local statistics of the ensembles AI† and AII† would give us insights into the local statistics of the 2DCG, which is a difficult problem for general values of $\beta > 0$.

The 3 above mentioned quantities are defined as follows. Let $\mathcal{P}_N(z_1, \ldots, z_N)$ be the joint complex eigenvalue distribution function of the respective ensemble. While unknown for classes AI† and AII†, in class A it is given by that of a 2DCG,

$$\mathcal{P}_N^{\mathrm{A}}(z_1, \ldots, z_N) \sim \prod_{i<j}^{N} |z_j - z_i|^\beta \exp\Big[-\sum_{j=1}^{N} |z_j|^2 \Big], \tag{3}$$

in a Gaussian potential at $\beta = 2$, representing a determinantal point process. The spectral density normalised to N is then defined in all 3 ensembles as

$$R_1(z_1) := N \int_{\mathbb{C}^{N-1}} d^2 z_2 \ldots d^2 z_N \mathcal{P}_N(z_1, \ldots, z_N). \tag{4}$$

Assuming permutation invariance for $\mathcal{P}_N(z_1, \ldots, z_N)$ for all 3 ensembles, we have the following definitions for point z_1 with NN z_2 and NNN z_3:[1]

[1] In the complex plane these may not be unique.

$$
\begin{aligned}
p_{\mathrm{NN}}(s) &:= \int_{\mathbb{C}^N} d^2z_1 \dots d^2z_N \mathcal{P}_N(z_1,\dots,z_N)\delta(|z_2-z_1|-s)\prod_{j=3}^{N}\Theta\left(|z_j-z_1|-s\right), \\
\varrho(z) &:= c_N \int_{\mathbb{C}^N} d^2z_1 \dots d^2z_N\ \mathcal{P}_N(z_1,\dots,z_N)\delta^{(2)}\left(z-\frac{z_2-z_1}{z_3-z_1}\right) \\
&\quad \times\Theta\left(|z_3-z_1|-|z_2-z_1|\right)\prod_{j=4}^{N}\Theta\left(|z_j-z_1|-|z_3-z_1|\right), \qquad (5)
\end{aligned}
$$

with the normalisation c_N to be determined, and $\Theta(s)$ the Heaviside function.

Our results in [7] and [8] can be summarised as follows. In [7] we derive an analytical expression for $R_1^{\mathrm{AI}^\dagger}(z_1)$ at finite-N. This proves the circular law for class $\mathrm{AI}^\dagger$ and leads to a new edge-scaling limit which we conjecture to be universal. It is different from the known erfc-law at the edge for class A.

In [8] we introduce a conditional point process by setting $z_1 = 0$ in (5), following the idea of Grobe, Haake and Sommers in computing $p_{\mathrm{NN}}^{\mathrm{A}}(s)$. Given that the origin is a generic bulk point, and the spectrum is translationally invariant in the bulk limit, as well known in class A [3], this leads to the correct limiting $p_{\mathrm{NN}}^{\mathrm{A}}(s)$. Applied to the complex spacing ratio $\varrho^{\mathrm{A}}(z)$ in class A, the expression in (5) simplifies drastically. This justifies the uncontrolled approximation made in [9] for $\varrho^{\mathrm{A}}(z)$. The resulting radial integral over a five-diagonal determinat of size N becomes exact when conditioning. Thus the limiting conditional point process explains the excellent numerical results in [9] for $\varrho^{\mathrm{A}}(z)$ and its moments. On the numerical side, we show in [8] that $p_{\mathrm{NN}}(s)$ and $\varrho(z)$ differ for all 3 classes between the edge and the bulk, thus establishing 3 different conjectured edge universality classes. In particular, we explain quantitatively the difference in repulsive behaviour of these quantities through the effective description as a 2DCG at $\beta = 1.4,\ 2$ and 2.6 for classes $\mathrm{AI}^\dagger$, A and $\mathrm{AII}^\dagger$, respectively. Last but not least we confirm the conjecture, that for all 3 classes, both at edge and bulk the small argument expansion, $p_{\mathrm{NN}}(s) \sim s^3$, is cubic. For class A this was known in the bulk and we prove it at the edge.

To summarise, it would be highly desirable to, first, improve our mathematical understanding of the symmetry classes $\mathrm{AI}^\dagger$ and $\mathrm{AII}^\dagger$, and second, to show that together with class A they exhaust all 38 symmetry classes for the generic local bulk and edge statistics. This establishes a challenging research programme.

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Spectra of critical Erdős–Rényi graphs

Johannes Alt
(joint work with Raphael Ducatez, Antti Knowles, Sarah Timhadjelt)

Matrices associated with random graphs form a natural class of random matrix models. Their spectral properties exhibit rich phenomena with relevance to spectral graph theory and quantum mechanics, among others. The hopping of a free quantum particle on a graph is governed by the adjacency matrix, which plays the role of the quantum Hamiltonian.

For a random graph, this Hamiltonian becomes disordered. In the Erdős-Rényi random graph $\mathbb{G} = \mathbb{G}(N, d/N)$ with vertex set $[N] := \{1, \dots, N\}$, each edge of the complete graph on $[N]$ is kept independently with probability d/N for some N-dependent parameter $d \in (0, N)$. Despite its simple definition, $\mathbb{G}$ exhibits a rich range of behaviours. For large d, the graph $\mathbb{G}$ is connected, whereas isolated vertices appear when $d < \log N$. On the same critical scale, namely when $d/\log N$ is small enough, the adjacency matrix A of $\mathbb{G}$ develops large eigenvalues[1] with localized eigenvectors [4, 3]. Each such eigenvector is exponentially localized around a unique high-degree vertex. When $d/\log N$ is large enough, all eigenvectors of A are completely delocalized [7, 9, 1] – the typical behaviour for random matrices [8]. For smaller d, all eigenvalues of the bulk spectrum away from zero have delocalized eigenvectors [1]. For a more detailed account about the eigenvector structure of A, along with complete references to the literature, we refer to the introduction of [4]. The corresponding phase diagram is shown in [4, Figure 1.1].

The time evolution of a quantum mechanical system is governed by its Hamiltonian through the Schrödinger equation. We consider a free quantum particle hopping on the Erdős-Rényi graph $\mathbb{G}$ with initial location $x \in [N]$ at time $t = 0$. The evolution of such particle is described by the Schrödinger equation

$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{v}(t) = H\mathbf{v}(t), \qquad \mathbf{v}(0) = \mathbf{1}_x \tag{1}$$

with $\mathbf{v}(t) \in \mathbb{C}^N$, where $H := A/\sqrt{d}$ and $\mathbf{1}_x$ denotes the standard basis vector. We work with H for convenience, as apart from at most N^ε many, all eigenvalues of H remain of order 1. The solution to (1) is given by the matrix exponential

[1]The matrix A possesses an outlying eigenvalue near d. For simplicity, we neglect it and the associated eigenvector from this overview.

$\mathbf{v}(t) = \mathrm{e}^{-\mathrm{i}tH}\mathbf{1}_x$ for all $t \geq 0$. At time t, the probability to find the particle at vertex $y \in [N]$ is $|\langle \mathbf{1}_y, \mathbf{v}(t)\rangle|^2$, the squared modulus of the scalar product.

To determine $\mathrm{e}^{-\mathrm{i}tH}$ more explicitly, we approximate $\mathbb{G}$ in a neighbourhood of x with a tree $\mathbb{T}$ rooted at x such that the degrees of x in $\mathbb{G}$ and $\mathbb{T}$ coincide and all other vertices in $\mathbb{T}$ have d children. To state the tree approximation with regard to the time evolution $\mathrm{e}^{-\mathrm{i}tH}$, we introduce some notation. The sphere in $\mathbb{G}$ of radius n in the graph distance $\mathrm{dist}_{\mathbb{G}}$ around x is denoted by $S_n(x) := \{y \in [N] : \mathrm{dist}_{\mathbb{G}}(x,y) = n\}$. By extending $\mathbb{T}$ to an infinite tree and using tridiagonalisation around x, the adjacency matrix of $\mathbb{T}$ rescaled by $d^{-1/2}$ is unitarily equivalent to $Z(\alpha_x)$, where

$$Z(\alpha) := \begin{pmatrix} 0 & \sqrt{\alpha} & & \\ \sqrt{\alpha} & 0 & 1 & \\ & 1 & 0 & \ddots \\ & & \ddots & \ddots \end{pmatrix}, \qquad \alpha_x := \frac{|S_1(x)|}{d}.$$

The infinite self-adjoint matrix $Z(\alpha)$ acts on $\ell^2(\mathbb{N})$ as a bounded operator and α_x is the normalized degree of the vertex x. For $S \subset [N]$, we define $\mathbf{1}_S := \sum_{x \in S} \mathbf{1}_x$.

In [6], R. Ducatez, A. Knowles and I prove the following theorem, which approximates the solution $\mathrm{e}^{-\mathrm{i}tH}\mathbf{1}_x$ with the time evolution on $\mathbb{T}$ via $Z(\alpha_x)$, together with the spheres in $\mathbb{G}$ around x.

Theorem 1. *If* $(\log N)^{1/3} \leq d \leq N^{1/4}$, $0 < t \leq \sqrt{d} \wedge \mathrm{diam}(\mathbb{G})/4$ *and* α_x *is sufficiently large then for any constant* $\nu > 0$,

$$\mathrm{e}^{-\mathrm{i}tH}\mathbf{1}_x = \sum_{n=0}^{\infty} u_n(t,\alpha_x)\mathbf{s}_n + o(1), \qquad u_n(t,\alpha) := (\mathrm{e}^{-\mathrm{i}tZ(\alpha)})_{n0}, \quad \mathbf{s}_n := \frac{\mathbf{1}_{S_n(x)}}{\|\mathbf{1}_{S_n(x)}\|}$$

in the ℓ^2*-norm on* $\mathbb{C}^N$ *with probability at least* $1 - O(N^{-\nu})$. *We use the convention* $\mathbf{s}_n := 0$ *if* $S_n(x) = \varnothing$.

Some upper bound on t is necessary due to a simple version of Poincaré recurrence. The diameter $\mathrm{diam}(\mathbb{G}) = \frac{\log N}{\log d}(1 + o(1))$ with probability at least $1 - o(1)$.

Theorem 1 is useful since $u_n(t,\alpha)$ can be represented as a contour integral in the complex plane, which is amenable to asymptotic analysis. This is the second main result of [6], which we present informally to keep this exposition concise. In fact, $t \mapsto u_n(t,\alpha)$ describes a radially symmetric wave travelling through $\mathbb{T}$, with a sharp wave front at $n = 2t$. Depending on the value of n relative to the wave front and the root, the behaviour of u_n falls into one of four regimes.

- *Beyond the wave front*, $n > 2t$: $u_n(t,\alpha)$ decays exponentially in t and n.
- *Around the wave front*, $n \approx 2t$: $u_n(t,\alpha)$ is described by an Airy function on the scale $t^{-2/3}$ with a decay of $t^{-1/3}$.
- *Behind the wave front and away from the root*, $0 < n < 2t$: the contour integral is well approximated by a Gaussian integral and $u_n(t,\alpha)$ decays like $t^{-1/2}$.
- *Near the root*, $n \approx 0$: $u_n(t,\alpha)$ is governed by a Faddeeva or Kramp function, which emerges as the critical point of the phase of the integrand approaches the pole of the integrand. The Faddeeva function is essentially the Stieltjes transform of the Gaussian density.

For the precise statements, we refer to [6].

For the directed counterpart, i.e. the Erdős-Rényi digraph, S. Timhadjelt and I study the eigenvector structure in the recent preprint [5]. If X is the adjacency matrix of the Erdős-Rényi digraph $\mathbb{H}$ with vertex set $[N]$ and edge probability d/N then all entries of X are independent $\mathrm{Ber}(d/N)$-distributed. Thus, (most) eigenvalues of X are complex. A detailed account about the eigenvalues of X can be found in [5]. The next theorem is a simplified version of the main result of [5].

Theorem 2. *For any constant $\delta > 0$, the following holds with probability at least $1 - o(1)$.*

(1) If $d \geq (1+\delta)\log N$ then all left and right eigenvectors[2] of X are completely delocalized.

(2) If $d \gg (\log N)^{24/25}$ then all left and right eigenvectors of X with eigenvalue larger than $\delta\sqrt{d}$ are completely delocalized.

We note that the lower bound $d > \log N$ is optimal for the delocalization of all eigenvectors, even when considering only the giant component of $\mathbb{H}$ and omitting isolated vertices. When $d < \log N$, with high probability, there are leaves attached to the giant component. Depending on the edge's orientation, each leaf induces a fully localized left or right eigenvector of X with eigenvalue 0.

We highlight the contrast between Theorem 2 and the corresponding undirected case with Hermitian adjacency matrix whose large eigenvalues have localized eigenvectors if $d \leq (b_* - o(1))\log N$ with $b_* := (2\log 2 - 1)^{-1}$, see [4]. This is another instance where phenomena characteristic of random matrices are more *robust* in the non-Hermitian setting than in the Hermitian one. However, the delocalization proof is more complicated than in the Hermitian case, as explained next.

The proof of Theorem 2 is based on applying Girko's Hermitization trick to X and establishing a local law for the resolvent of the Hermitization, which involves an interesting family of approximants for the resolvent entries. Some resolvent entries diverge causing extra difficulties compared to the Hermitian setting. Further proof details and additional spectral properties of X are presented in [5].

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Maximum of the characteristic polynomial of random Jacobi matrices

Fanny Augeri
(joint work with Ofer Zeitouni)

In the present work, we study the maximum of the characteristic polynomial of the following random Jacobi matrix

$$J_n = \begin{pmatrix} b_n & a_{n-1} & 0 & \cdots & 0 \\ a_{n-1} & b_{n-1} & a_{n-2} & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & a_2 & b_2 & a_1 \\ 0 & \cdots & \cdots & a_1 & b_1 \end{pmatrix}. \tag{1}$$

where $(a_k)_{k\geq 1}$ and $(b_k)_{k\geq 1}$ are two independent sequences of independent random variables such that

$$\mathbb{E}(a_k^2) = k + O(1),\ \mathrm{Var}(a_k^2) = kv + O(1),\ \mathbb{E}(b_k) = 0,\ \mathrm{Var}(b_k) = v + O\Big(\frac{1}{k}\Big), \tag{2}$$

for any $k \geq 1$, where v is some positive constant, and such that

$$\sup_{k\geq 1} \mathbb{E}\big(e^{\mathfrak{h}_0|b_k|}\big) < +\infty,\ \sup_{k\geq 1} \mathbb{E}\big(e^{\frac{\mathfrak{h}_0}{\sqrt{k}}|a_k^2 - \mathbb{E}(a_k^2)|}\big) < +\infty, \tag{3}$$

for some $\mathfrak{h}_0 > 0$.

We denote by p_n the characteristic polynomial of the scaled Jacobi matrix $J_n/\sqrt{n}$ defined by $p_n(z) = \det(z\mathrm{I}_n - J_n/\sqrt{n})$ for any $z \in \mathbb{R}$, where I_n denotes the identity matrix of size $n \times n$. Our main result reads as follows.

Theorem 1. *Let $\eta > 0$ and denote by $I_\eta := \{z \in \mathbb{R} : \eta \leq |z| \leq 2 - \eta\}$.*

$$\frac{\max_{z\in I_\eta}\big(\log|p_n(z)| - n\big(\frac{z^2}{4} - \frac{1}{2}\big)\big) - \sqrt{v}\log n}{\log\log n} \underset{n\to+\infty}{\longrightarrow} -\frac{3\sqrt{v}}{4}, \quad \text{in probability.}$$

We believe similar results hold for the imaginary part of $\log p_n(z)$, which is an eigenvalue-counting function. However, we chose not to present those here in order to keep this already long paper at a reasonable length. Similarly, the restriction to $z \in I_\eta$ is done for convenience and probably can be extended to the full interval $[-2, 2]$.

As a particular case of Theorem 1, we obtain the second order asymptotics of the maximum of the absolute value of the logarithmic potential of *Gaussian β Ensembles*. The *Gaussian β Ensemble*, where $\beta > 0$, is the law on real n-particles configurations given by

$$\mathbb{P}_{n,\beta} = C_{n,\beta} \prod_{1\le i<j\le n} |\lambda_i - \lambda_j|^\beta e^{-\frac{\beta}{4}\sum_{i=1}^n \lambda_i^2} \prod_{i=1}^n d\lambda_i,$$

where $d\lambda_i$ denotes the Lebesgue measure on $\mathbb{R}$ and $C_{n,\beta} > 0$ is a normalisation constant. When $\beta = 1$, respectively $\beta = 2$, $\beta = 4$, the Gaussian β Ensemble coincides with the joint law of the eigenvalues of the Gaussian Orthogonal Ensemble (GOE), respectively the Gaussian Unitary Ensemble (GUE) and the Gaussian Symplectic ensemble (GSE). For general $\beta > 0$, there is no natural dense random matrix model whose spectrum is distributed according to $\mathbb{P}_{n,\beta}$, but Edelman and Dimitriu [6] found that $\mathbb{P}_{n,\beta}$ can be interpreted as the joint density of the eigenvalues of a certain tridiagonal model. More precisely, if b_k is a centered Gaussian with variance $2/\beta$ and $\sqrt{\beta}a_k$ has a χ-distribution of degree βk, then the corresponding random Jacobi matrix has a spectrum distributed as $\mathbb{P}_{n,\beta}$. One can easily check that such coefficients b_k and a_k satisfy the moments and integrability assumptions (2)-(3) with $v = 2/\beta$. Hence, we get as an immediate consequence of Theorem 1 the following corollary. Denote by $p_\lambda(z)$ the logarithmic potential of the n-particles configuration $\lambda \in \mathbb{R}^n$, that is,

$$p_\lambda(z) := \frac{1}{n}\sum_{i=1}^n \log(z - \lambda_i), \ z \in \mathbb{R},$$

with the convention that $\log(0) = -\infty$.

Theorem 2. *Let $\beta, \eta > 0$. Denote by $I_\eta := \{z \in \mathbb{R} : \eta \le |z| \le 2 - \eta\}$. Under $\mathbb{P}_{n,\beta}$,*

$$\frac{\max_{z\in I_\eta}\left(\log|p_\lambda(z)| - n\left(\frac{z^2}{4} - \frac{1}{2}\right)\right) - \sqrt{\frac{2}{\beta}}\log n}{\log\log n} \underset{n\to+\infty}{\longrightarrow} -\frac{3}{2\sqrt{2\beta}}, \quad \text{in probability.}$$

Theorem 2 confirms the first two leading terms of the Fyodorov and Simm conjecture, stated for the GUE in [8]. It is worthwhile to note that given Theorem 2, one obtains the same result (corresponding to $\beta = 1, 2$) for Wigner type matrices with Gaussian-divisible entries, see [2, Theorem 1.4].

This work contributes to the body of research motivated by the study of logarithmically correlated fields in the context of random matrices, and the seminal conjectures of Fyodorov–Hiary–Keating [7] concerning the characteristic polynomial of random unitary matrices and their links with the maxima of the Riemann zeta function over short intervals (see [8] for the Hermitian counterpart). We refer to the literature review in [16] for a historical account and pointers to the literature for both the circular ensemble results and the logarithmically correlated fields angle.

We next discuss ensembles with real eigenvalues. The (logarithm of the) characteristic polynomial of random matrices is an example of a linear statistics of the spectrum, albeit with a family of (singular) test functions. Central limit theorems

for smooth test functions were discussed (for general β) in [9], while the CLT for the characteristic polynomials of such matrices in the case $\beta = 2$ follow from the analysis in [11], see also [4]; we refer to [1] and [10] for a historical review. The CLT for the determinant of such matrices (and of general Wigner matrices) was derived by [17], who also used the basic recursions we use, with $z = 0$ (where only the elliptic regime survives). More recently, we mention the derivation of multi-points and central limit theorems for the characteristic polynomial of β ensembles in [3], and the refinements developed in [2], where the leading order in Theorem 1 for β-ensembles is derived. For the Gaussian β ensembles, a detailed study was undertaken in [14, 13, 15] (see also [10] for the hyperbolic and parabolic regimes, motivated by the study of spherical spin-glasses). In the determinantal ($\beta = 2$) case, where analytical techniques are available, we also refer to [12] and [5]. Our work here, while self contained, builds on ideas in [1], where a (pointwise) central limit theorem for the logarithm of the characteristic polynomial of random Jacobi matrices was derived.

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Matrix Discrepancy for Representations of Finite Groups – Extended Abstract of talk based on [1]

Afonso S. Bandeira
(joint work with Helmut Bölcskei)

The matrix Spencer conjecture is an intriguing matrix generalization of Spencer's Six Deviations Suffice [6].

Conjecture 1. [2, 4, 5] *There exists a universal constant C such that for any integer n and any set of $n \times n$ matrices $A_1, \ldots, A_n$ each with spectral norm at most one,*

$$\min_{\varepsilon \in \{\pm 1\}^n} \left\| \sum_{k=1}^{n} \varepsilon_k A_k \right\| \leq C\sqrt{n}.$$

In this talk we show a proof of this conjecture for the case when the matrices are representations of a finite group. More precisely, given a finite group G, we show that there exist signs $\varepsilon \in \{\pm 1\}^G$ such that

$$\left\| \sum_{g \in G} \varepsilon_g \rho(g) \right\| \leq C\sqrt{|G|},$$

where ρ is the left regular representation of G, and C is a universal constant.

This special case of the Matrix Spencer conjecture was posed in [3], where it was established for simple groups.

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Law of Fractional Logarithm and numerical radius of random matrices

Zhigang Bao
(joint work with Giorgio Cipolloni, László Erdős, Joscha Henheik, Oleksii Kolupaiev)

In this report, we consider the standard Wigner matrices and Ginibre matrices, with notations and scalings specified as follows.

- *Wigner matrices:* Let $X = (x_{ij})_{i,j\in\mathbb{N}}$ be a doubly infinite array of independent, up to symmetry $x_{ij} = \overline{x}_{ji}$, real ($\beta = 1$) or complex ($\beta = 2$) random variables. We assume them to have mean 0 and variances given by $\mathbb{E}x_{ij}^2 = 1 + \delta_{ij}$ for $\beta = 1$ and $\mathbb{E}|x_{ij}|^2 = 1$, $\mathbb{E}x_{ij}^2 = \delta_{ij}$ for $\beta = 2$. For simplicity, we further assume $\mathbb{E}|x_{ij}|^k \le C_k < \infty$ for some positive constant C_k, for all k. For any $N \in \mathbb{N}$, we define the N-dimensional Wigner matrix as $W^{(N)} := N^{-1/2}X^{(N)}$, where $X^{(N)}$ is the $N \times N$ upper-left corner of X.
- *Ginibre matrices:* Let $Y = (y_{ij})_{i,j\in\mathbb{N}}$ be a doubly infinite array of independent real ($\beta = 1$) or complex ($\beta = 2$) random variables. We assume them to have mean 0 and variances given by $\mathbb{E}y_{ij}^2 = 1$ for $\beta = 1$ and $\mathbb{E}|y_{ij}|^2 = 1$, $\mathbb{E}y_{ij}^2 = 0$ for $\beta = 2$. For simplicity, we further assume $\mathbb{E}|y_{ij}|^k \le C_k < \infty$ for some positive constant C_k, for all k. For any $N \in \mathbb{N}$, we define the N-dimensional Ginibre matrix as $A^{(N)} := N^{-1/2}Y^{(N)}$, where $Y^{(N)}$ is the $N \times N$ upper-left corner of Y.

Let $\lambda_1(W^{(N)})$ be the largest eigenvalue of $W^{(N)}$, and define

$$\widetilde{\lambda}_1(W^{(N)}) \equiv N^{2/3}(\lambda_1(W^{(N)}) - 2).$$

The Bai–Yin law for $\lambda_1(W^{(N)})$ and Tracy–Widom law for $\widetilde{\lambda}_1(W^{(N)})$ can be regarded as the analogues of the Law of Large Numbers and the Central Limit Theorem for classical iid sum. It is then natural to ask for a logarithmic law for $\lambda_1(W^{(N)})$, analogous to the Law of the Iterated Logarithm. This question was first posed by Kalai [4] and studied in the pioneering work of Paquette and Zeitouni [5], which derived the precise limsup of $\widetilde{\lambda}_1(W^{(N)})$ for GUE. They termed this result the Law of the Fractional Logarithm (LFL). The authors of [5] also provided lower and upper bounds for the liminf of $\widetilde{\lambda}_1(W^{(N)})$ and conjectured the true limit, a conjecture only very recently resolved in [3], again, for GUE. In our recent work [2], we established the following LFL for all Wigner matrices (both real and complex):

Theorem 1. *For both real ($\beta = 1$) and complex ($\beta = 2$) Wigner minor process, we have*

$$\limsup_{N\to\infty} \frac{\widetilde{\lambda}_1(W^{(N)})}{(\log N)^{2/3}} = \left(\frac{1}{2\beta}\right)^{2/3}, \qquad \liminf_{N\to\infty} \frac{\widetilde{\lambda}_1(W^{(N)})}{(\log N)^{1/3}} = -\left(\frac{8}{\beta}\right)^{1/3}.$$

We then turn to another fundamental but much less explored spectral statistics of random matrices. It is the numerical radius of the Ginibre matrices. The numerical range, or field of values, of a square matrix $B \in \mathbb{C}^{N\times N}$ is defined as the

range of its Rayleigh quotient, i.e.,

$$\mathcal{R}(B) = \{x^* Bx : x \in \mathbb{C}^N,\ \|x\|_2 = 1\},$$

which is a convex set by Toeplitz–Hausdorff theorem. The numerical radius of B is defined as $r(B) = \max\{|z| : z \in \mathcal{R}(B)\}$. It is well-known that $\rho(B) \le r(B)$ and $\frac{1}{2}\|B\|_{\mathrm{op}} \le r(B) \le \|B\|_{\mathrm{op}}$. Further one also has $r(B^m) \le (r(B))^m$ for all $m \in \mathbb{N}$. Here we used $\rho(B)$ and $\|B\|_{\mathrm{op}}$ to denote the spectral radius and operator norm of B, respectively. The last two inequalities immediately imply the fundamental inequality $\|B^m\|_{\mathrm{op}}^{1/m} \le 2^{1/m} r(B)$. In contrast, by Gelfand's formula we know $\lim_{m\to\infty} \|B^m\|_{\mathrm{op}}^{1/m} = \rho(B)$, which holds only in the limiting sense. Actually, $\rho(B)$ or a simple multiple of it cannot be used to bound $\|B^m\|_{\mathrm{op}}^{1/m}$ for fixed m and instead one can use $2^{1/m} r(B)$ as stated above. Since the quantity $\|B^m\|_{\mathrm{op}}^{1/m}$ is of great importance in many iterative algorithms and in real applications m is always finite, numerical radius is then widely used since it is a better indicator of the rate of convergence of iterative methods than the spectral radius due to this basic fact. Mathematically, a remarkable fact is

$$r(B) = \max_{\theta \in [0,2\pi)} \lambda_1(H(e^{\mathrm{i}\theta} B)),$$

where $H(D) = (D + D^*)/2$ is the Hermitian part of a square matrix D, and $\lambda_1(\cdot)$ is again the largest eigenvalue. The above representation allows one to study the largest eigenvalue of a parametric Hermitian matrix instead. Especially, when we choose $B = A^{(N)}$, $H(e^{\mathrm{i}\theta} A^{(N)})$ becomes a Wigner matrix (further scaled by $1/\sqrt{2}$), for any fixed parameter $\theta \in [0, 2\pi)$. Hence, $\lambda_1(H(e^{\mathrm{i}\theta} A^{(N)}))$ is a stochastic process with time domain $\theta \in [0, 2\pi)$, and marginal distribution given by Tracy–Widom law asymptotically. Then the study of $r(A^{(N)})$ boils down to the study of the extrema of such process over $\theta \in [0, 2\pi)$. On a high level, the process $\{\lambda_1(H(e^{\mathrm{i}\theta} A^{(N)}))\}_{\theta \in [0,2\pi)}$ can be viewed as a continuous analogue of the sequence $\{\lambda_1(W^{(N)})\}_{N \in \mathbb{N}}$ considered in the LFL problem.

In a recent preprint [1], we derived the following first two orders of the numerical radius for complex Ginibre matrices.

Theorem 2. *For the complex* $(\beta = 2)$ *Ginibre matrices, we have*

$$r(A^{(N)}) = \sqrt{2} + \frac{(\log N)^{2/3}}{4\sqrt{2} N^{2/3}}(1 + o(1)).$$

As both problems concern the extrema of Airy-like processes over a long time period, we needed to first prove a correlation-decorrelation transition for the process from small to large time scales. For the proof of LFL, we developed a martingale approach for the correlation part and applied a dynamic approach for the decorrelation part. For the problem of the numerical radius, we embedded the two-point marginals of the process into DBM and applied the correlation-decorrelation transition for the DBM.

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The role of Random Matrix Theory for the optimization algorithms of machine learning

Gerard Ben Arous

(joint work with R.Gheissari, J. Huang, A. Jagannath)

The loss functions of modern Data Science and Machine Learning provide a very rich and important family of random functions of many variables. Understanding the performance of optimization algorithms on these loss functions, in the presence of over-parametrization, is a central question of Machine Learning.

It has been proposed in recent work, (with Reza Gheissari (Northwestern) and Aukosh Jagannath (Waterloo)), that these optimization dynamics, when successful, are eventually taking place in fact in much lower dimensional sub-manifolds (the "summary statistics"). The current works (with Reza Gheissari (Northwestern), Jiayoang Huang (Wharton) and Aukosh Jagannath (Waterloo)), show how RMT can help find these low dimensional structures, using a dynamical version of the BBP transition along the path of the optimization algorithm, typically defined by a Stochastic Gradient Descent.

Indeed the Hessian of the loss along the optimization path defines a dynamical large random matrix. One can also use the Gram matrix of the gradients of the loss along the SGD path. When the spectrum of these random matrices manages to expel outliers, one can naturally expect that the projection on the eigenspace of these outliers could be good "summary statistics", and the autonomous effective dynamics could allow to understand convergence of the algorithm.

So we study here the relation between the training dynamics via stochastic gradient descent (SGD) and the spectra of empirical Hessian and G-matrices over the course of training. We prove that, in two canonical classification tasks for multiclass high-dimensional mixtures and either 1 or 2-layer neural networks, both the SGD trajectory and emergent outlier eigenspaces of the Hessian and gradient matrices align with a common low-dimensional subspace.

Moreover, in multilayer settings this alignment occurs per layer, with the final layer's outlier eigenspace evolving over the course of training, and exhibiting rank deficiency when the SGD converges to sub-optimal classifiers. This establishes some of the rich predictions that have arisen from extensive numerical studies in

the last decade about the spectra of Hessian and information matrices over the course of training in over-parametrized networks.

We rigorously show the following phenomenology in two canonical high-dimensional classification tasks with k "hidden" classes:

(1) Shortly into training, the empirical Hessian and empirical G-matrices have $C(k)$ many outlier eigenvalues. Their corresponding eigenvectors, along with the SGD trajectory, align with a common latent $C(k)$-dimensional subspace. In particular, the SGD and outlier eigenspaces align well with one another. Here $C(k)$ is explicit and depends on the model and the performance of the classifier to which the SGD converges.
(2) In multi-layer settings, this alignment happens within each layer, i.e., the first layer parameters align with the outlier eigenspaces of the corresponding blocks of the empirical Hessian and G-matrices, and likewise for the second layer parameters.
(3) This alignment is not predicated on success at the classification task: when the SGD converges to a sub-optimal classifier, the empirical Hessian and G-matrices have lower rank outlier eigenspaces, and the SGD aligns with those rank deficient spaces.

The first model we consider is the basic example of supervised classification of general k-component Gaussian mixture models with k linearly independent classes by a single-layer neural network. We show that the matrices have an outlier-minibulk-bulk structure throughout the parameter space, and derive limiting dynamical equations for the trajectory of appropriate summary statistics of the SGD trajectory. Importantly, the same low-dimensional subspace is at the heart of both the outlier eigenspaces and the summary statistics.

At this level of generality, the SGD can behave very differently within the outlier eigenspace. As an example of the refined phenomenology that can arise, we further investigate the special case where the means are orthogonal; here the SGD aligns specifically with the single largest outlier eigenvalue, which itself has separated from the other $k-1$ outliers along training.

To demonstrate our results in more complex multi-layer settings, we consider supervised classification of a GMM version of the famous XOR problem of Minsky and Papert [5]. This is one of the simplest models that requires a two-layer neural network to solve. We use a two-layer architecture with a second layer of width k.

The alignment of the SGD with the matrices' outlier eigenspaces occurs within each layer, the first layer having an outlier space of rank two, and the second layer having an outlier space of rank 4 when the dynamics converge to an optimal classifier.

This second layer's alignment is especially rich, as when the model is over-parametrized (k large), its outlier space of rank 4 is not present at initialization, and only separates from its rank-k bulk over the course of training. Moreover, the SGD for this problem is known to converge to sub-optimal classifiers with probability bounded away from zero, and we find that in these situations, the

alignment still occurs but the outlier eigenspaces into which the SGD moves are rank-deficient compared to the number of hidden classes.

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Quantitative eigenvector universality for generalized Wigner matrices

Lucas Benigni

The guiding question of [1] is the following: to what extent do eigenvectors of a Wigner matrix behave like the eigenvectors of the Gaussian Orthogonal Ensemble, whose eigenbasis is Haar-distributed? For the GOE, $o(N)$ renormalized entries of a fixed eigenvector are essentially independent Gaussian random variables. The goal is to prove analogous statements for generalized Wigner matrices without relying on exact orthogonal invariance.

Let W be a real symmetric generalized Wigner matrix, with independent centered entries satisfying

$$\sum_{i=1}^{N} \mathbb{E}|w_{ij}|^2 = 1, \qquad \frac{c}{N} \le \mathbb{E}|w_{ij}|^2 \le \frac{C}{N},$$

and with uniformly bounded moments. Denote its eigenvalues by $\lambda_1 \le \cdots \le \lambda_N$ and its eigenvectors by $\mathbf{u}_1, \dots, \mathbf{u}_N$. Since real eigenvectors are only defined up to sign, we introduce independent Rademacher variables ε_i and set

$$\mathbf{v}_i = \sqrt{N}\,\varepsilon_i \mathbf{u}_i.$$

The first result is joint universality of arbitrary deterministic projections of finitely many eigenvectors. Let $k_1, \dots, k_p$ be distinct eigenvector indices, let $a_1, \dots, a_p$ be fixed, and set $Q = \sum_i a_i$. For deterministic unit vectors $\mathbf{q}_{i\alpha} \in \mathbb{R}^N$, define

$$X_N = \big(\langle \mathbf{q}_{i\alpha}, \mathbf{v}_{k_i}\rangle\big)_{1\le i\le p,\, 1\le \alpha\le a_i} \in \mathbb{R}^Q.$$

Then X_N converges, in the sense of smooth bounded test functions, to the centered Gaussian vector G_N with covariance

$$\mathbb{E}[G_{i\alpha} G_{j\beta}] = \delta_{ij}\langle \mathbf{q}_{i\alpha}, \mathbf{q}_{i\beta}\rangle.$$

This gives joint eigenvector universality for arbitrary deterministic projections and arbitrary eigenvector indices, including the edge, without a four-moment matching assumption extending the results of [3, 5, 2, 4].

The method also gives information on extreme eigenvector statistics. For a fixed eigenvector index k, and every $\varepsilon > 0$, there exists $a = a(\varepsilon) > 0$ such that

$$\mathbb{P}\left(\sqrt{\frac{N}{\log N}}\|\mathbf{u}_k\|_\infty \in \left[\frac{1}{\sqrt{2}} - \varepsilon,\ \sqrt{2} + \varepsilon\right]\right) \geq 1 - N^{-a}.$$

The upper bound is the optimal first-order delocalization estimate of Benigni–Lopatto, while the new contribution is the lower bound. The constant $1/\sqrt{2}$ is not expected to be optimal; the GOE prediction is convergence in probability to $\sqrt{2}$. The limitation comes from the strength of the available two-point Gaussianization estimates in the Paley–Zygmund argument for the number of large coordinates.

Under an additional smoothness assumption on the entry distributions, the dynamical estimates are quantitative enough to treat a growing number of projections. Let

$$R = \dim \operatorname{Span}\{\mathbf{q}_{i\alpha} : 1 \leq i \leq p,\ 1 \leq \alpha \leq a_i\}.$$

If

$$Q \leq RQ \leq CN^{1/2-b}$$

for some $b \in (0, 1/2]$, then for smooth h satisfying suitable polynomial bounds and $\|\nabla^2 h\|_\infty \leq N^{\delta_2}$ with $\delta_2 < b$, one has

$$|\mathbb{E}h(X_N) - \mathbb{E}h(G_N)| \leq C_h N^{-\varepsilon}.$$

This yields, for example, joint Gaussianity of $N^{1/2-\varepsilon}$ entries in a fixed row of the eigenbasis, of $N^{1/4-\varepsilon}$ arbitrary entries of a fixed eigenvector, and of $m \times m$ eigenbasis minors with $m \leq N^{1/6-\varepsilon}$. The same approach gives an explicit Kolmogorov bound:

$$\mathrm{d_K}(X_N, G_N) \leq N^{-b/3+\kappa}$$

for every $\kappa > 0$. In particular, for a fixed number of projections this gives $\mathrm{d_K}(X_N, G_N) \leq N^{-1/6+\kappa}$. These exponents are not expected to be optimal even in comparison with the Haar case, but they are explicit and hold beyond invariant ensembles.

We now explain the main idea of the proof. Start from Dyson Brownian motion

$$H_t = H_0 + \frac{1}{\sqrt{N}} B_t,$$

where B_t is a real symmetric Brownian motion. Conditionally on the eigenvalue path, the eigenvectors evolve according to the Dyson vector flow with generator

$$\mathcal{L}_t = \frac{1}{2}\sum_{i \neq j} c_{ij}(t)\mathcal{X}_{ij}^2, \qquad c_{ij}(t) = \frac{1}{N(\lambda_i(t) - \lambda_j(t))^2},$$

where

$$\mathcal{X}_{ij} = \mathbf{u}_i \partial_{\mathbf{u}_j} - \mathbf{u}_j \partial_{\mathbf{u}_i}.$$

The classical eigenvector moment flow [2] analyzes the evolution of moments of this process. Here the dynamics is studied directly at the level of test functions.

Fix a mesoscopic range ℓ. The generator is split into a short-range part, where $|i-j| < \ell$, and a long-range part, where $|i-j| \geq \ell$. The short-range part is left unchanged, while the long-range part is replaced by an Ornstein–Uhlenbeck mechanism. For one coordinate variable $v_i = \sqrt{N}\, u_i(\alpha)$, this leads to the reference generator

$$\widehat{\mathcal{L}}_t = \frac{1}{2} \sum_{|i-j|<\ell} c_{ij}(t) \mathcal{X}_{ij}^2 + \frac{1}{2} \sum_i \beta_i(t)(\partial_i^2 - v_i \partial_i), \qquad \beta_i(t) = \sum_{|i-j|\geq \ell} c_{ij}(t).$$

The reference flow no longer lives on the sphere, but its invariant measure is the standard Gaussian product measure. Using eigenvalue rigidity and the logarithmic Sobolev inequality for the Ornstein–Uhlenbeck part, one obtains quantitative relaxation of the backward reference semigroup toward Gaussian equilibrium:

$$\left|\widehat{\mathcal{P}}_{0,t} h(\mathbf{v}) - \mathbb{E} h(G)\right| \leq C \|h\|_\infty \sqrt{N} \exp\left[-c\left(\frac{Nt^3}{\ell}\right)^{1/3}\right].$$

It remains to compare the true Dyson vector flow with the reference flow. The difference of the two generators is supported only on the long-range part and can be written schematically as a diagonal fluctuation term plus an off-diagonal term:

$$(\mathcal{L}_t - \widehat{\mathcal{L}}_t) g = \sum_{|i-j|\geq \ell} c_{ij}(t)(v_j^2 - 1)\partial_i^2 g - \sum_{|i-j|\geq \ell} c_{ij}(t) v_i v_j \partial_{ij}^2 g.$$

A Duhamel formula expresses the discrepancy between the two expectations as the time integral of this error along the actual eigenvector flow. The resulting terms are controlled using eigenvalue rigidity, isotropic delocalization, eigenvector fluctuation estimates on intervals, and L^2-contractivity of the reference dynamics. The key point is that the long-range interactions are individually weak, while their replacement by the Ornstein–Uhlenbeck component retains enough coercivity to force Gaussian relaxation.

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Semilocalization for inhomogeneous random graphs

Thomas Buc—d'Alché
(joint work with Antti Knowles)

We consider the localization of eigenvector phenomenon for the adjacency matrix of a random inhomogeneous graph. The random graph is sampled using the *Generalized random graph* (GRG) model (see [1]), in which each vertex x is equipped with a weight w_x. The probability that an edge between two vertices x and y appears in the graph is then proportional to $w_x w_y$. Hence, by specifying the weight sequence, we determine the expected degree sequence. Notice that by choosing a constant sequence of weights, we recover the Erdős-Rényi random graph model.

We consider the case of sparse random graphs. We assume that the sequence of weights has bounded mean and variance, but allow for somewhat heavy tails – a behavior very different from the one of the Erdős-Rényi model studied in [2, 3, 4, 5]. We show that near the edges of the spectrum, the eigenvectors are semilocalized in the sense that their mass concentrates around a small set of resonant vertices. For the extremal eigenvalues, we establish localization around a single vertex.

The proof is similar in spirit to the one used in [2] for showing (semi-)localization in the Erdős-Rényi case. However, the neighborhoods of vertices of high degree in the sparse Erdös-Rényi case and in the GRG case with a heavy-tailed distribution of weights are very different: while in the first case the small neighborhoods of vertices of high degrees are disconnected with high probability, in the second case those big vertices tend to cluster together. In order to obtain effective estimates in the presence of highly inhomogeneous degrees, we introduce a new economical pruning procedure that carefully extracts a forest from the original graph, whose adjacency matrix is compared to that of the original graph using a suitably constructed local coupling to random trees with independent edges. This forest approximation of our random graph allows a very explicit construction of approximate eigenvectors and description of the localization phenomenon.

We expect that when the distribution of weights has a too heavy tail, we should observe a qualitatively different localization/delocalization phenomenon: in such regime, some regions of the random graph may look like a complete graph, which would impact drastically the localization phenomenon. This is left to future investigation.

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Strong convergence of multiplicative brownian motions on the general linear group

Mireille Capitaine
(joint work with Marwa Banna, Guillaume Cébron)

The *Ginibre ensemble* consists of random matrices with independent and identically distributed complex Gaussian entries, each with variance $1/N$, and the *Gaussian Unitary Ensemble* (GUE) is obtained by taking the Hermitian part of the Ginibre ensemble. Now, if we endow the set of Hermitian matrices with the inner product $\langle A, B\rangle_N = N\Re(Tr(A^*B))$, a matrix H from the GUE can be seen as the end point H_1 of a Hermitian Brownian motion $(H_t)_{t\geq 0}$ starting at $H_0 = 0$. This stochastic process has additive increments $H_t - H_s$ (for $s < t$) which are stationary and stochastically independent. Now, let $(U_t)_{t\geq 0}$ be the stochastic process defined via the Stratonovitch stochastic differential equation:

$$dU_t = U_t \circ i\, dH_t, \qquad U_0 = I_N.$$

This procedure is the usual one to define a Brownian motion in a Lie group from a Brownian motion in its Lie algebra, $(iH_t)_{t\geq 0}$ being a Brownian motion in the Lie algebra $\mathfrak{u}(N)$ of skew-Hermitian matrices. Here, it yields a Brownian motion $(U_t)_{t\geq 0}$ in the unitary group $U(N)$ with stationary and independent multiplicative increments $U_s^{-1}U_t$ (for $s < t$). Similarly, if we endow the set $M_N(\mathbb{C})$ of all matrices with the inner product $\langle A, B\rangle_N = N\Re(Tr(A^*B))$, a matrix Z from the Ginibre ensemble can be seen as the end point Z_1 of a Brownian motion $(Z_t)_{t\geq 0}$ on $M_N(\mathbb{C})$ starting at $Z_0 = 0$. The stochastic exponential map yields the stochastic process $(G_t)_{t\geq 0}$ defined from $(Z_t)_{t\geq 0}$ via the Stratonovitch stochastic differential equation:

$$dG_t = G_t \circ i\, dZ_t, \qquad G_0 = I_N.$$

The process $(G_t)_{t\geq 0}$ is a standard Brownian motion in the general linear group $GL_N(\mathbb{C})$, with stationary and independent multiplicative increments $G_s^{-1}G_t$ (for $s < t$).

In the early 90's, a deep relation between *random matrices* and *free probability* was found by Voiculescu : a sequence of tuples of independent random matrices $H^{(1)}, \dots, H^{(p)}$ sampled from the GUE is *asymptotically free.* Note that asymptotic freeness holds more generally for independent random matrices which are unitarily invariant. Let $(H_t; t \in [0;1])$ be a Brownian motion taking values in the space of Hermitian $N \times N$ matrices normalized such that $\mathbb{E}\left((H_t)_{ij}(H_t)_{kl}\right) = \frac{t}{N}\delta_{il}\delta_{jk}$. Asymptotic freeness yields that a large-N limit of this Brownian motion is provided by a free Brownian motion. Namely, let (A, φ) be a noncommutative probability space and $(s_t; t \in [0;1])$ be a free Brownian motion in (A, φ), then for all choices of $t_1, \cdots, t_n \in [0, 1]$, one has

$$tr_N\left(H_{t_1} \dots H_{t_n}\right) \xrightarrow[N\to+\infty]{} \varphi(s_{t_1} \dots s_{t_n}),$$

almost surely, where tr_N denotes the normalized trace $\frac{1}{N}Tr$. In 1997, Biane [2] suggested to study the limit of the processes $(U_t)_{t\geq 0}$ and $(G_t)_{t\geq 0}$ as $N \to \infty$. He proved that $(U_t)_{t\geq 0}$ converges in $*$-distribution to a limiting process $(u_t)_{t\geq 0}$

called the *free unitary Brownian motion*, and conjectured that $(G_t)_{t\geq 0}$ converges as well to a certain process called the *free multiplicative Brownian motion*. This last conjecture was proved in 2013 by Cébron in [3]. In 2020, Driver, Hall and Kemp [4] introduced the following three-parameters family of Brownian motions in $GL_N(\mathbb{C})$ that has been shown [4, Theorem 3.2] to be, up to multiplication by a scalar process, the most general class of Brownian motions in $GL_N(\mathbb{C})$ that preserves the unitary invariance of increments, a property that ensures asymptotic freeness as $N \to \infty$. Let $\theta \in \mathbb{R}$, $a \geq 0$ and $b \geq 0$ that are not both zero, and set $\lambda = a^2 + b^2$ and $\tau = \lambda - e^{2i\theta}(a^2 - b^2)$. We denote by $(H_t)_{t\geq 0}$ and $(\tilde{H}_t)_{t\geq 0}$ two independent Brownian motions on the set of Hermitian $N \times N$ matrices endowed with the inner product $\langle \cdot, \cdot \rangle_N$. The *rotated elliptic Brownian motion* $(Z_{\lambda,\tau}(t))_{t\geq 0}$ on $M_N(\mathbb{C})$ is then defined by

$$Z_{\lambda,\tau}(t) = e^{i\theta}(aH_t + ib\tilde{H}_t), \tag{1}$$

whose law is uniquely defined by the parameters $\lambda \geq 0$ and $\tau \in \mathbb{C}$. The multiplicative (λ, τ)-Brownian motion $(G_{\lambda,\tau}(t))_{t\geq 0}$ is the diffusion process on $GL_N(\mathbb{C})$ which is the solution of the Stratonovich stochastic differential equation

$$dG_{\lambda,\tau}(t) = G_{\lambda,\tau}(t) \circ i\, dZ_{\lambda,\tau}(t) \quad \text{with} \quad G_{\lambda,\tau}(0) = I_N. \tag{2}$$

It is well-defined for all $\lambda \geq 0$ and $\tau \in \mathbb{C}$ such that $|\tau - \lambda| \leq \lambda$; $(G_{\lambda,\tau}(t))_{t\geq 0}$ is invertible for all $t \geq 0$, has independent stationary multiplicative increments. . The case $(\lambda, \tau) = (1, 0)$ yields the Brownian motion $(U_t)_{t\geq 0}$, while the case $(\lambda, \tau) = (1, 1)$ corresponds to the Brownian motion $(G_t)_{t\geq 0}$. The convergence in $*$-distribution of $(G_{\lambda,\tau}(t))_{t\geq 0}$ is proved in [5] when τ is real and can be extended in a straightforward way for $\tau \in \mathbb{C}$, yielding the convergence in $*$-distribution of $(G_{\lambda,\tau}(t))_{t\geq 0}$ towards a process $(g_{\lambda,\tau}(t))_{t\geq 0}$, known as the *free multiplicative* (λ, τ)*-Brownian motion* defined as follows. Let $(\mathcal{A}, \varphi)$ be a W^*-probability space, that is $\mathcal{A}$ is a von Neumann algebra and φ is a faithful, normal, tracial state. For any $t \geq 0$, we construct the rotated elliptic Brownian motion $z_{\lambda,\tau}(t)$ as

$$z_{\lambda,\tau}(t) = e^{i\theta}(ax_t + iby_t),$$

where a, b and θ as chosen above and where x_t and y_t are freely independent semicircular Brownian motions. Finally, we consider the free multiplicative (λ, τ)-Brownian motion $g_{\lambda,\tau}$ defined for any $t \geq 0$ as the solution of the free stochastic differential equation

$$dg_{\lambda,\tau}(t) = g_{\lambda,\tau}(t)\Big(i\, dz_{\lambda,\tau}(t) - \frac{1}{2}(\lambda - \tau)\, dt\Big) \quad \text{with} \quad g_{\lambda,\tau}(0) = I_{\mathcal{A}}. \tag{3}$$

For any $\lambda \geq 0$ and $\tau \in \mathbb{C}$ such that $|\tau - \lambda| \leq \lambda$, $(g_{\lambda,\tau}(t))_{t\geq 0}$ is invertible for all $t \geq 0$, has freely independent stationary multiplicative increments.

In 2005, Haagerup and Thorbjørnsen strengthened the asymptotic freeness of GUE matrices to a stronger form. They showed that, almost surely, as $N \to \infty$, for any noncommutative polynomial P in p variables, the operator norm $\|P(H^{(1)}, \ldots, H^{(p)})\|$ converges to $\|P(s^{(1)}, \ldots, s^{(p)})\|$, where $(s^{(1)}, \ldots, s^{(p)})$ are freely independent semicircular elements. This property is referred to as strong

convergence, and we say that $(H^{(1)},\dots,H^{(p)})$ *strongly converges* to $(s^{(1)},\dots,s^{(p)})$ as $N\to\infty$. Concerning the multiplicative (λ,τ)-Brownian motion $(G_{\lambda,\tau}(t))_{t\geq 0}$, the strong convergence has been shown by Collins-Dahlqvist-Kemp in the particular case $(\lambda,\tau)=(1,0)$, which corresponds to the Brownian motion on the unitary group. Their proof uses the convergence of the support of the empirical law of eigenvalues together with the fact that $G_{1,0}(t)$ is a normal matrix. For general parameters (λ,τ), $G_{\lambda,\tau}(t)$ is not necessarily normal. The main result of our paper [1] is the following strong convergence involving (λ,τ)-Brownian motion on $GL_N(\mathbb{C})$ for any allowed parameters (λ,τ) and deterministic matrices.

Theorem 1. *For $\lambda\geq 0$, $\tau\in\mathbb{C}$ such that $|\tau-\lambda|\leq\lambda$, let, for all $N\geq 1$, $(G_t)_{t\geq 0}:=(G_{\lambda,\tau}(t))_{t\geq 0}$ be a multiplicative (λ,τ)-Brownian motion on $GL_N(\mathbb{C})$ and let $(g_t)_{t\geq 0}:=(g_{\lambda,\tau}(t))_{t\geq 0}$ be a free multiplicative (λ,τ)-Brownian motion in $(\mathcal{A},\varphi)$. Let $A^N:=(A_1^N,\dots,A_r^N,A_1^{N*},\dots,A_r^{N*})$ be a $2r$-tuple of $N\times N$ deterministic matrices that are uniformly bounded over N such that that A^N converges strongly to $\mathbf{a}$ in $(\mathcal{A},\varphi)$. Then for any $p\geq 0$, and times $t_1,\dots,t_p\geq 0$, almost surely for any noncommutative polynomial P in $4p+2r$ indeterminates,*

$$\lim_{N\to\infty} tr_N\left(P(G_{t_1},G_{t_1}^*,G_{t_1}^{-1},(G_{t_1}^{-1})^*,\dots,G_{t_p},G_{t_p}^*,G_{t_p}^{-1},(G_{t_p}^{-1})^*,A^N)\right)$$
$$=\varphi\left(P(g_{t_1},g_{t_1}^*,g_{t_1}^{-1},(g_{t_1}^{-1})^*,\dots,g_{t_p},g_{t_p}^*,g_{t_p}^{-1},(g_{t_p}^{-1})^*,\mathbf{a})\right)$$

and

$$\lim_{N\to\infty}\left\|P(G_{t_1},G_{t_1}^*,G_{t_1}^{-1},(G_{t_1}^{-1})^*,\dots,G_{t_p},G_{t_p}^*,G_{t_p}^{-1},(G_{t_p}^{-1})^*,A^N)\right\|$$
$$=\left\|P(g_{t_1},g_{t_1}^*,g_{t_1}^{-1},(g_{t_1}^{-1})^*,\dots,g_{t_p},g_{t_p}^*,g_{t_p}^{-1},(g_{t_p}^{-1})^*,\mathbf{a})\right\|$$

with $\mathbf{a}$ being freely independent of g_t.

In the proof, we adopt an interpolation-based approach, drawing on ideas developed by Collins, Guionnet, and Parraud [6]. Our method is *multiplicative* rather than additive; multiplicative interpolation requires careful choices at each step of the proof to interpolate with either a left- or right-invariant multiplicative Brownian motion, to ensure obtaining symmetric terms that can be combined and effectively controlled.

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Aspects of the Spherical Ensemble

Djalil Chafaï
(joint work with David García-Zelada, Yuan Yuan Xu)

The spherical ensemble is a unitary invariant random matrix model obtained by taking the ratio of two inde- pendent complex Ginibre matrices. Its spectrum is a heavy-tailed determinantal Coulomb gas on the complex plane. Its equilibrium measure is the complex Cauchy distribution. This planar gas is the image by the stereographic projection of a chordal gas on the two-sphere. A crucial observation is the invariance in law of the model under inversion, related to its spherical symmetry, and that makes, in a sense, edge and bulk equivalent.

Our first result is that divided by the square root of the dimension, the spectral radius of the ratio converges in distribution, when the dimension tends to infinity, to a universal heavy-tailed distribution. We provide a mathematical proof of this high-dimensional phenomenon, under a fourth moment matching with a Gaussian case known as the complex Ginibre ensemble. Our approach involves Girko Hermitization, local law estimates for Wigner matrices, lower bound estimates on the smallest singular value, and convergence of kernels of determinantal point processes. The universality of the high-dimensional fluctuation of the spectral radius of the ratio of Girko matrices turns out to be remarkably more accessible mathematically than for a single Girko matrix!

Our second main result is a Central Limit Theorem (CLT) for linear statistics for test functions with an arbitrary finite number of logarithmic singularities. The normalization is at the order of the square root of the logarithm of the dimension. The contributions of the singularities are asymptotically independent, and the asymptotic variance depends only on the weights of the singularities. We deduce a multipoint CLT for the logarithm of the modulus of the characteristic polynomial, which is the logarithmic potential of the empirical spectral distribution. We also deduce a multipoint CLT for the linear statistics tested against the Green function of the two-sphere. As a third main result, we compute covariance asymptotics showing that the associated unnormalised fields are log-correlated. The approach relies on an interplay between probabilistic, analytic, and geometric arguments. Furthermore, we also establish, under suitable assumptions, a matrix-universality extension when the Ginibre matrices are replaced by Girko matrices. Finally, we discuss possible geometric universality extensions in which the sphere is replaced by a more general compact Riemann surface, as well as natural open problems related to Fisher–Hartwig asymptotics and Gaussian multiplicative chaos.

In short, we study the high-dimensional spectral analysis of random matrices from the spherical ensemble, a famous model of mathematical physics. We obtain CLTs for test functions with logarithmic singularities. This covers in particular the asymptotic analysis of the logarithm of the modulus of the characteristic polynomial. Another way to state what we do is to say that we study the fluctuations of logarithmic Green singularities in the spherical ensemble, viewed as a random discretization of the two-sphere. Smooth observables exhibit the usual Sobolev or

Gaussian free field fluctuations, whereas logarithmic singularities live on a larger logarithmic scale and asymptotically decouple in high-dimension, producing an explicit white-noise limit. The result gives precise asymptotics for logarithmic potentials and characteristic polynomials, with constants expressed through chordal geometry on the sphere.

All this is based on [CGZX1] and [CGZX2].

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Gaussian Multiplicative Chaos from non-Hermitian Random Matrices

Giorgio Cipolloni

(joint work with Benjamin Landon)

We consider *i.i.d. matrices*, namely $N \times N$ matrices X with real or complex independent, identically distributed (i.i.d.) centered entries, normalized so that $\mathbb{E}|X_{ij}|^2 = N^{-1}$. We also assume that all moments of X_{ij} exist. Let $\sigma_i \in \mathrm{Spec}(X)$ denote the eigenvalues of X. The empirical measure $\frac{1}{N}\sum_i \delta_{\sigma_i}$ converges weakly a.s. to a deterministic limiting measure, the *circular law* (see e.g. [4, 15]):

$$\lim_{N\to+\infty} \frac{1}{N}\sum_i f(\sigma_i) = \frac{1}{\pi}\int_{\mathbb{D}} f(z)\,\mathrm{d}z\mathrm{d}\overline{z}.$$

To study the fluctuations around the circular law, one is naturally led – via the identity $\sum_i f(\sigma_i) = \frac{1}{2\pi}\int_{\mathbb{C}} \Delta f(z)\log|\det(X-z)|\,\mathrm{d}z\mathrm{d}\overline{z}$ – to analyze the (asymptotically centered) log-characteristic polynomial,

$$\Phi_N(z) := \log|\det(X-z)| + N\frac{(1-|z|^2)_+}{2}. \tag{1}$$

For simplicity, we henceforth restrict attention to the complex case. The seminal work of Rider and Virág [21] first revealed the remarkable fact that the field $\Phi_N(z)$ has asymptotic Gaussian fluctuations with covariance structure

$$\mathrm{Cov}\big(\Phi_N(z), \Phi_N(w)\big) \approx -\frac{1}{2}\log\big[|z-w| + N^{-\frac{1}{2}}\big], \qquad |z|, |w| < 1, \tag{2}$$

by proving that $\Phi_N(z)$ converges weakly to the Gaussian Free Field (GFF). Their results was established for the complex Ginibre ensemble, where the X_{ij}'s are complex Gaussians. This result was later extended to normal matrices [1] and to general i.i.d. matrices [10].

Logarithmically correlated (or log-correlated in short) fields, such as the 2d GFF, arise in a wide range of areas of mathematics, including probability theory, statistical mechanics, SPDEs, and analytic number theory. Since the foundational works of Bramson and Kahane, considerable attention has been devoted to understanding the size and spatial distribution of the extreme values of such fields.

More precisely, a Gaussian field $\psi(x)$ on $\mathbb{R}^d$ is said to be log-correlated if $\mathrm{Cov}\big(\psi(x), \psi(y)\big) \approx -\log|x-y|$ in the sense of distributions, as $|x-y| \to 0$. Motivated in part by questions in turbulence theory, Kahane [16] in the 80's developed the theory of Gaussian multiplicative chaos (GMC). The GMC provides a rigorous framework for describing the geometry and distribution of the high points of a log-correlated field ψ through the formal random measure with "density" $e^{\gamma\psi(x)}$. Since ψ is typically only a random distribution, the expression $e^{\gamma\psi(x)}$ is not well defined pointwise. Nevertheless, the GMC can be constructed as a random fractal measure on $\mathbb{R}^d$ as the weak limit of

$$\lim_{\epsilon\to 0} e^{\gamma\psi_\epsilon(x) - \frac{\gamma^2}{2}\mathbb{E}\psi_\epsilon(x)}\mathrm{d}x, \tag{3}$$

where ψ_ϵ denotes a mollification of ψ. The limit does not depend on the choice of the mollifier and is non-trivial only for $\gamma \in (0, \sqrt{2d})$.

In 2012, Fyodorov, Hiary, and Keating (FHK) [14] initiated a new line of research in random matrix theory by uncovering connections between the extreme values of the logarithm of characteristic polynomials of random matrices and the theory of log-correlated fields. Their work led to a series of influential conjectures and developments relating random matrix statistics to the behavior of high points of log-correlated Gaussian fields; see [2, 3, 20] for recent breakthroughs on the original questions raised by FHK.

A few years later, Webb [22] established a new remarkable connection between powers of characteristic polynomials of Haar-distributed unitary random matrices and the GMC measure. This provided the first appearance of GMC in random matrix theory and has since been followed by a number of fundamental works proving convergence to GMC measures in both one- and two-dimensional settings, see e.g. [5, 7, 8, 9, 12, 13, 17, 18, 19]. Despite these advances, all existing results on convergence to GMC measures have been restricted to *invariant ensembles*. These ensembles possess a high degree of symmetry: their distributions are invariant under rotations (or unitary conjugation), and they admit explicit formulas for the joint distribution of their eigenvalues. In our recent work [11] we showed:

Theorem. *Let X be an i.i.d. matrix. For $0 \le \gamma < 2\sqrt{2}$ we have that the random measures defined on the unit disk in $\mathbb{C}$ by*

$$\frac{|\det(X-z)|^\gamma}{\mathbb{E}[|\det(X-z)|^\gamma]}\mathrm{d}z\mathrm{d}\bar{z}$$

converge in distribution (with respect to the topology of weak convergence) to the measure $\mu^{(\beta),\gamma}_{\mathrm{GMC},\kappa_4}$. *Here* $\mu^{(\beta),\gamma}_{\mathrm{GMC},\kappa_4}$ *is the GMC-measure defined as in* (3) *for the Gaussian field* $\psi(z)$, *with covariance*[1]

$$\mathrm{Cov}\big(\psi(z),\psi(w)\big) = -\frac{1}{2}\log|z-w| + \frac{\kappa_4}{4}(1-|z|^2)(1-|w|^2),$$

where $\kappa_4 := \mathbb{E}|X_{ij}|^4 - 2$ *denotes the fourth cumulant of the entries of* X.

This is the first convergence result for any non-invariant random matrix ensemble, regardless of the symmetry class. In particular, our result includes the case of the invariant real Ginibre ensemble (where the matrix entries are real Gaussians), a model that has thus far resisted even integrable approaches. Our result also provides a new proof of the complex Ginibre case established in [7] (we note that this work further treats general normal matrix ensembles). In addition, we also derive precise asymptotic for moments of products of characteristic polynomials.

A feature of our approach is that it uses very little input from any form of integrability, requiring only the asymptotic of $\mathbb{E}|\det(X)|^\gamma$ for the complex Ginibre ensemble, which can be obtained by an elementary application of Kostlan's theorem. By contrast, all previous works mentioned above make substantial use of integrability. Depending on the model, these proofs rely either on the Riemann-Hilbert method, use orthogonal polynomials on the unit circle (OPUC) in the CβE ensembles, or explicit determinantal formulas for eigenvalue correlation functions. None of these tools are available for the ensembles considered here. Instead, our analysis is based on a dynamical approach using the Dyson Brownian motion and the so-called method of characteristics. A further key ingredient in our proof of the GMC convergence is the black-box result of [12].

We conclude by mentioning two related open problems. First, one may investigate the behavior of $|\det(X-z)|^\gamma$ when $\gamma \to \gamma_c := 2\sqrt{2}$, and its connection with critical Gaussian multiplicative chaos. Second, it remains open to understand the fluctuations of the so-called thick points of this field. Both problems are unresolved even in the integrable setting of the complex Ginibre ensemble.

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A Sudakov–Fernique proof of Lehner-type edge bounds for matrix-valued GUE sums

Benoît Collins
(joint work with Yuta Yamagishi)

Let $A_0, A_1, \dots, A_n \in M_N(\mathbb{C})$ be Hermitian matrices and let $G_1, \dots, G_n$ be independent $M \times M$ GUE matrices, normalized so that $\|M^{-1/2} G_i\|_{\mathrm{op}} \to 2$ almost surely as $M \to \infty$. We consider the linear matrix-coefficient model

$$H_M = A_0 \otimes I_M + \frac{1}{\sqrt{M}} \sum_{i=1}^{n} A_i \otimes G_i.$$

In the free limit, $M^{-1/2} G_i$ is replaced by a free semicircular family (s_i), and the norm of $A_0 \otimes 1 + \sum_i A_i \otimes s_i$ is computed by Lehner's variational formula [9]. Extracting the two spectral edges from Lehner's formula, the free right edge is

$$\rho_+ = \inf_{Z>0} \lambda_{\max}\Big(A_0 + Z + \sum_{i=1}^{n} A_i Z^{-1} A_i\Big),$$

the free left edge ρ_- is given by the symmetric formula ($\sup_{Z<0}$ of $\lambda_{\min}$), and the free norm is $\rho_* = \max\{\rho_+, -\rho_-\}$.

That $\|H_M\|_{\mathrm{op}} \to \rho_*$ when the coefficients are fixed is an instance of the strong convergence phenomenon, going back to Haagerup and Thorbjørnsen [7]; see [10] for a recent survey. The known proofs rely on rather heavy machinery: master equations for the Stieltjes transform [7], free stochastic interpolation [5], matrix concentration and intrinsic freeness [2], or the recent polynomial method of Chen–Garza-Vargas–Tropp–van Handel [3, 4], which covers matrix coefficients of dimension $N = \exp(o(M))$ and signed coefficients, well beyond the regime discussed here.

The goal of the talk was different, and in the spirit of the classical concentration-and-comparison proofs of $\mathbb{E}\lambda_{\max}(M^{-1/2}G) \le 2+O(M^{-1/2})$ for a single GUE matrix: how far can one get toward the free edge constants for the model H_M using *only* Gaussian concentration and Gaussian comparison inequalities, with explicit non-asymptotic error terms? The answer is that, for positive semidefinite coefficients, this elementary route recovers Lehner's constants exactly. The main result of [6] is the following.

Theorem 1. *Assume $A_i \ge 0$ for $i = 1, \dots, n$, and $M \ge N$. Then*

$$\mathbb{E}\lambda_{\max}(H_M) \le \rho_+ + 6\sqrt{\frac{nN}{M}}\,\Big\|\sum_{i=1}^n A_i^2\Big\|_{\mathrm{op}}^{1/2},$$

the symmetric lower bound holds for $\mathbb{E}\lambda_{\min}(H_M)$ with ρ_- in place of ρ_+, and consequently

$$\mathbb{E}\|H_M\|_{\mathrm{op}} \le \rho_* + 9\sqrt{\frac{nN}{M}}\,\Big\|\sum_{i=1}^n A_i^2\Big\|_{\mathrm{op}}^{1/2}.$$

The constants 6 *and* 9 *are universal.*

The square-function factor is the familiar quantity from the endpoint noncommutative Khintchine inequality. For n fixed, uniformly bounded coefficients and $N = N_M = o(M)$, the error term is $o(1)$; the theorem thus gives a short, self-contained quantitative proof of the sharp upper edge bound in this regime, without master equations or resolvent expansions.

Ingredients of the proof. The proof has three main steps.

1. A dual variational formula over density matrices. Let $\mathcal{D}_N$ denote the density matrices ($C \ge 0$, $\operatorname{Tr} C = 1$) and, for $C \in \mathcal{D}_N$, set $B_i(C) = C^{1/2}A_iC^{1/2}$ and $E_C = \sum_{i=1}^n B_i(C)^2$. Writing $\lambda_{\max}(B(Z))$ as a supremum over $\mathcal{D}_N$ and exchanging sup and inf by Sion's minimax theorem, followed by a matrix trace AM–GM identity, yields

$$\rho_+ = \max_{C\in\mathcal{D}_N}\Big\{\operatorname{Tr}(A_0C) + 2\operatorname{Tr}(E_C^{1/2})\Big\}, \qquad \rho_- = \min_{C\in\mathcal{D}_N}\Big\{\operatorname{Tr}(A_0C) - 2\operatorname{Tr}(E_C^{1/2})\Big\}.$$

This duality, which was pointed out to us by R. van Handel, holds for arbitrary Hermitian coefficients and can be viewed as a matrix-coefficient version of [2, Lemma 3.2]. After completion of this work we learned of independent work of Kunisky [8] on semidefinite-programming formulations of Lehner's formulas, which also discusses such dual forms.

2. A full-process Sudakov–Fernique comparison. Identifying $\mathbb{C}^N \otimes \mathbb{C}^M$ with $N \times M$ matrices, every unit vector factors as $C^{1/2}W$ with $C \in \mathcal{D}_N$ and W a coisometry ($WW^* = I_N$), whence

$$\lambda_{\max}(H_M) = \sup_{C,W}\Big\{\operatorname{Tr}(A_0C) + \frac{1}{\sqrt{M}}\sum_{i=1}^{n}\operatorname{Re}\operatorname{Tr}\big(G_i\,W^*B_i(C)W\big)\Big\}.$$

We compare this process with the rectangular Gaussian process

$$Z_{C,W} = \frac{2}{\sqrt{M}}\sum_{i=1}^{n}\operatorname{Re}\operatorname{Tr}\big(\Gamma_i^*B_i(C)W\big),$$

where the Γ_i are independent $N \times M$ standard complex Gaussian matrices, using Vitale's extension [11] of the Sudakov–Fernique inequality allowing a deterministic drift (here $\operatorname{Tr}(A_0C)$). The required increment domination, across *different* fibers (C,W) and (C',V), reduces to the inequality

$$\|W^*PW - V^*QV\|_F^2 \le 2\,\|PW - QV\|_F^2, \qquad P,Q \ge 0,$$

a direct consequence of the Araki–Yamagami inequality $\||X|-|Y|\|_F \le \sqrt{2}\|X-Y\|_F$ [1]. This is the only place where positivity of the A_i is used, and it is used sharply.

3. A block-Wishart estimate. For fixed C, the supremum of the comparison process over W equals $\frac{2}{\sqrt{M}}\|\sum_i B_i(C)\Gamma_i\|_1$, by Schatten duality. Stacking the Γ_i into a single $nN \times M$ Gaussian matrix Γ bounds this, uniformly in C, by $2\,\|\Gamma\Gamma^*/M\|_{\mathrm{op}}^{1/2}\operatorname{Tr}(E_C^{1/2})$. The standard Gaussian singular-value tail estimate

$$\mathbb{P}\big\{\|\Gamma\|_{\mathrm{op}} \ge \sqrt{M} + \sqrt{nN} + t\big\} \le e^{-t^2/2},$$

produces the error term, using $\max_C \operatorname{Tr}(E_C^{1/2}) \le \|\sum_i A_i^2\|_{\mathrm{op}}^{1/2}$. The dual formula of Step 1 identifies the resulting deterministic constant with ρ_+. The left edge follows by applying the argument to $-H_M$, and the norm bound then follows by Gaussian concentration for the Lipschitz map $(G_1,\dots,G_n) \mapsto \|H_M\|_{\mathrm{op}}$.

Obstruction for signed coefficients, and open questions. The positivity assumption is essential for the sharp constant. For signed Hermitian B, the two-fiber increment inequality above fails already for 2×2 matrices: with $B = \operatorname{diag}(1,-1)$, $W = I_2$ and V a rotation by a small angle θ, it would require $\sin^2\theta \le 1 - \cos\theta$. The elementary substitute $\|W^*BW - V^*BV\|_F \le 2\|B(W-V)\|_F$ yields a valid comparison, but with $2\sqrt{2}\operatorname{Tr}(E_C^{1/2})$ in place of $2\operatorname{Tr}(E_C^{1/2})$, which no longer matches Lehner's edge. This seems to delimit rather precisely the scope of the plain Sudakov–Fernique method. Several questions remain open. (1) Can a Gaussian comparison argument (e.g. a Gordon-type minimax comparison) recover the sharp constant ρ_+ for signed Hermitian coefficients? (2) Is there a comparison-based proof of the matching bound $\mathbb{E}\lambda_{\max}(H_M) \ge \rho_+ - o(1)$ when $N = o(M)$? (3) What is the optimal dependence of the error term on n, N and M? Already for $n = N = 1$ the error $O(M^{-1/2})$ is not optimal, the sharp fluctuation scale being $M^{-2/3}$. (4) The dual formulas for $\rho_\pm$ over density matrices are efficiently computable and closely related to the semidefinite programs of [8]; can they be exploited further, e.g. for spiked or deformed models?

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Limiting spectral distribution for Brownian motions in the general linear group

Nicholas A. Cook

(joint work with Tatiana Brailovskaya, Todd Kemp and Félix Parraud)

A Brownian motion $B = (B_t)_{t\geq 0}$ in $\mathrm{GL}_N(\mathbb{C})$ is the unique strong solution to the matrix stochastic differential equation $dB_t = B_t dW_t$ with $B_0 = I$, where W is a standard Brownian motion in $\mathrm{M}_N(\mathbb{C})$ scaled to have spectral radius ~ 1 at time $t = 1$. (Note that W_1 is a normalized Ginbre matrix.) The process has independent stationary multiplicative increments $B_s^{-1}B_t \sim B_{t-s}$.

In [1], Biane introduced a process $b = (b_t)_{t\geq 0}$ of operators b_t in a non-commutative probability space satisfying the free stochastic differential equation $db_t = b_t dw_t$ with b_0 the identity. He conjectured that this process is the large-N limit (in various senses) of the matrix process B. Such convergence was confirmed in the sense of finite-dimensional $*$-distributions by Cébron [4]. An analogous result had been shown by Biane [1, 2] for unitary Brownian motion $dU_t = iU_t dX_t - \frac{1}{2}U_t dt$, where X is a standard Hermitian Brownian motion. Since U_t and the limiting unitary operator u_t are normal, their spectral distributions are determined by their $*$-moments, so Biane could deduce the almost-sure weak convergence of empirical spectral distributions $\mu_{U_t} = \frac{1}{N}\sum_{i=1}^N \delta_{\lambda_i(U_t)}$ to the spectral measure μ_{u_t}.

The problem of convergence of the empirical spectral distribution of B_t is harder as B_t, b_t are (almost-surely) non-normal: one has to deal with potential spectral

instability of B_t, and the limiting object b_t does not possess a spectral measure. A natural candidate for the limit of μ_{B_t} is the Brown measure

$$\mu_{b_t} := \frac{1}{2\pi}\Delta_z \int_0^\infty \log(x)\mu_{|b_t - z|}(dx)$$

(in the distributional sense) defined in terms of the spectral measures of self-adjoint operators $|b_t - z| = \sqrt{(b_t - z)^*(b_t - z)}$ for $z \in \mathbb{C}$.

Our main result is the following:

Theorem 1. *For any fixed $t > 0$, almost surely μ_{B_t} converges weakly to μ_{b_t}.*

In fact we prove a more general result:

- We allow for a general initial condition A, showing $\mu_{AB_t} \to \mu_{ab_t}$, assuming $A = A^N \to a$ in $*$-distribution.

  In particular, we do not require A to be normal, nor that $\mu_A \to \mu_a$. Thus, Theorem 1 implies that multiplying a matrix by a multiplicative Brownian (at any time $t > 0$) has a regularizing effect on its empirical measure, and hence provides a multiplicative analogue of a well-known result of Sniady [8] for additive Brownian motion $A + W_t$.
- We cover a more general class of processes $B_t = B_t^{\rho,\zeta} \in \mathrm{GL}(N, \mathbb{C})$ driven by Brownian motions $W = W^{\rho,\zeta}$ with entries having (possibly degenerate) "elliptical" correlation structure, where $\rho := \mathbb{E}\frac{1}{N}\mathrm{Tr}W_1^* W_1$ and $\zeta := \mathbb{E}\frac{1}{N}\mathrm{Tr}W_1^2 \in \mathbb{C}$, for which B satisfies $dB_t = B_t dW_t + \frac{\zeta}{2}B_t dt$. The "vanilla" Brownian motion considered above is the case $(\rho, \zeta) = (1, 0)$, while with $(\rho, \zeta) = (1, -1)$, $W_t = iX_t$ is skew-Hermitian, and we recover the unitary Brownian motion $(U_t)_{t\geq 0}$. The convergence in $*$-distributional sense for this general class was proved in [6].

Proof strategy. As in other works on the spectrum of non-normal random matrices, most notably the circular law of matrices with iid entries (see the survey [3]), the main problem for the proof of Theorem 1 is to establish quantitative upper bounds on large singular values of the resolvent $(B_t - z)^{-1}$ for a.e. $z \in \mathbb{C}$. This is a novel problem, as the entries of B_t have a dependency structure coming from a stochastic differential equation. For this we develop a flexible approach based on a quantitative comparison of the process over a small time increment with the driving process W in the Lie algebra:

Theorem 2. *For any $t \leq 1$, $\|B_t - I - W_t\|_{op} = O(t)$ with probability $\geq 1 - e^{-N^{2/3}}$.*

(Note that the spectrum of W_t is at scale $\sqrt{t}$.)

By the independence and stationarity of increments this allows us to replace B_t with $B_{t-\delta}\widetilde{B}_\delta \approx B_{t-\delta}(1+\widetilde{W}_\delta)$ for small δ and $(\widetilde{W}, \widetilde{B})$ an independent copy of (W, B), and condition on $B_{t-\delta}$. We then obtain satisfactory control on singular values of the resolvent using the more tractable randomness of the Gaussian elliptical matrix $\widetilde{W}_\delta$.

Theorem 2 is proved by a general procedure for controlling the coupled evolution of (B_t, W_t) that we expect can be adapted to a broad class of matrix SDE:

(1) We first show that B_t is well approximated in L^p by a multiplicative random walk $B_t \approx B_t^{(n)} := \prod_{k \le t2^n} (I + W_{k2^{-n}} - W_{(k-1)2^{-n}})$, uniformly in n (with p eventually taken of order $N^{2/3}$).

This allows us to approximate a high spectral moment $\mathbb{E}\frac{1}{N}\mathrm{Tr}|B_t - I - W_t|^{2k}$ by $\mathbb{E}\frac{1}{N}\mathrm{Tr}Q(X_1, \dots, X_{\lfloor t2^n \rfloor}, Y_1, \dots, Y_{\lfloor t2^n \rfloor})$ for a non-commutative polyomial Q evaluated at iid GUE matrices X_j, Y_j, where X_j, iY_j are the scaled Hermitian and anti-Hermitian parts of the increments of W.

(2) We apply a result from [7] that provides a $\frac{1}{N^2}$ expansion

$$\mathbb{E}\frac{1}{N}\mathrm{Tr}Q(X_1, \dots, X_{\lfloor t2^n \rfloor}, Y_1, \dots, Y_{\lfloor t2^n \rfloor}) = \sum_{q=0}^{r-1} \frac{\alpha_q}{N^{2q}} + \frac{\tilde{\alpha}_r}{N^{2r}}$$

to arbitrary order r, where the coefficients α_q have explicit (though complicated) expressions in terms of iterated applications of a non-commutative differential operator L to Q, evaluated at free semicircular variables, and the remainder coefficient $\tilde{\alpha}_r$ is similar, but involves an extended system of GUEs.

While the precise formulas for the coefficients is complicated, the differential structure allows us to effectively estimate them to arbitrary order, resulting in a sharp bound on $\mathbb{E}\frac{1}{N}\mathrm{Tr}|B_t - I - W_t|^{2k}$ for $k \sim N^{2/3}$, and hence a strong upper tail bound for $\|B_t - I - W_t\|_{op}$ at the correct scale t.

Furture directions. Apart from applications to other non-normal matrix SDE, we expect this procedure and Theorem 2 to be useful for the analysis of local eigenvalue statistics, which should exhibit universal determinantal structure.

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On two approaches to the injective norm of random tensors

Stéphane Dartois
(joint work with Benjamin M^{C}Kenna)

We summarize here two approaches to the injective norm of random tensors, based on joint work with Benjamin M^{C}Kenna. The injective norm is the tensor analogue of the operator norm of a matrix. It appears as the ground-state energy of multi-species spherical spin glasses and, in quantum information, as the maximal overlap of a pure multipartite state with product states. The first approach uses Kac-Rice to probe for the highest critical point of a relevant Gaussian process. It is sharp, but essentially limited to Gaussian random tensors due to technical limitations. The second approach is a large-moment method based on new deterministic inequalities for deterministic tensors. It is not as sharp as Kac-Rice in some Gaussian regimes, but matches Kac-Rice in others. It is also non-asymptotic and applies to several non-Gaussian and structured models.

1. Set-up

Let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, and let $T \in \mathbb{K}^{d_1} \otimes \cdots \otimes \mathbb{K}^{d_p}$. The injective norm is

$$\|T\|_{\text{inj},\mathbb{K}} := \max_{\|x^{(1)}\|_2 = \cdots = \|x^{(p)}\|_2 = 1} \left|\left\langle T, x^{(1)} \otimes \cdots \otimes x^{(p)} \right\rangle\right|.$$

For $p = 2$, this is the operator norm of a matrix. For $p \geq 3$, there is no spectral decomposition, and computing the norm is hard in general. A natural question is therefore whether random tensors admit analogues of the classical theory of the largest singular value of random matrices.

There are several motivations for this question: If T is a normalized complex tensor (meaning its Frobenius norm is set to 1), viewed as a pure multipartite quantum state $|\psi\rangle$, then $\||\psi\rangle\|_{\text{inj}}$ is its largest overlap with a product state; equivalently, $-\log \||\psi\rangle\|_{\text{inj}}^2$ is its geometric entanglement. If T is real and symmetric, $\max_{\|x\|=1} |\langle T, x^{\otimes p}\rangle|$ is the ground-state energy of the pure spherical p–spin model. Non-symmetric tensors lead instead to multispecies spin glass ground state energies.

Kac-Rice

Assume that $T \in (\mathbb{R}^d)^{\otimes p}$ has independent standard Gaussian entries (for this Kac-Rice part we limit ourselves to real random tensors). Up to normalization, the injective norm is the maximum of the Gaussian process

$$f_T(x^{(1)}, \ldots, x^{(p)}) = \frac{1}{\sqrt{p(d-1)}} \left\langle T, x^{(1)} \otimes \cdots \otimes x^{(p)} \right\rangle \tag{1}$$

on the Cartesian product of p $(d-1)$–dimensional spheres $(\mathbb{S}_{\mathbb{R}}^{d-1})^p$. If f_T is above a level u, then there is a critical point above level u. Hence, by Markov's inequality,

the probability of the event $\{\max f_T \geq u\}$ can be bounded by the expected number of critical points above level u. A one-sided Kac-Rice formula gives

$$\mathbb{E}\,\mathrm{Crt}_{f_T}([u,\infty)) \leq F(p,d)\int_u^\infty e^{-Nt^2/2}\mathbb{E}\,|\det(W_N - t)|\,dt, \qquad N = p(d-1), \tag{2}$$

where W_N is a block-hollow Gaussian orthogonal ensemble: it is a GOE-type matrix with vanishing diagonal blocks (it is the projection of a GOE matrix on the space of matrices with vanishing diagonal $(d-1)\times(d-1)$ blocks). These zero blocks come from the multilinearity of the Hamiltonian in each $x^{(i)}$. In the complex case one first quotients out $U(1)$ symmetry, and the same argument leads to a related random matrix ensemble.

The problem is turned into a random-matrix question: estimate the average determinant. In the regime where p is fixed and $d \to \infty$, we obtain, for standard Gaussian entries,

$$\forall \varepsilon > 0,\ \limsup_{d\to\infty} \frac{1}{p(d-1)} \log \mathbb{P}\left(\frac{1}{\sqrt{d}}\|T\|_{\mathrm{inj}} > \sqrt{p}\,E_0(p) + \varepsilon\right) < 0, \tag{3}$$

where $E_0(p)$ is the ground-state constant for the pure spherical p–spin model. In the second regime, with d fixed and $p \to \infty$, we prove

$$\|T\|_{\mathrm{inj}} \leq \sqrt{p(d-1)}\left(\sqrt{\log p} + \frac{\log\log p}{2\sqrt{\log p}} + O\left(\frac{1}{\sqrt{\log p}}\right)\right) \tag{4}$$

with high probability.

This method gives sharp one-sided constants in Gaussian models (only conjecturally in the large p regime). It is limited to Gaussian tensors and not well suited to non-asymptotic estimates. It is also less robust for rectangular tensors, non-Gaussian entries, or models with additional algebraic structure.

Large moments

The second approach relies on deterministic bounds. Let u_i be independent uniform vectors on the complex unit spheres of dimensions d_i. For every complex tensor T and every $k \geq 1$,

$$\|T\|_{\mathrm{inj},\mathbb{C}}^{2k} \leq \left(\prod_{i=1}^{p}\binom{d_i+k-1}{k}\right)\mathbb{E}_u\left(|\langle T, u_1\otimes\cdots\otimes u_p\rangle|^{2k}\right). \tag{5}$$

There is a real analogue with explicit Gamma function constants, and a complex symmetric version. In the complex case, the proof relies on the observation that the average of $|u^{\otimes k}\rangle\langle u^{\otimes k}|$ is proportional to the orthogonal projector onto the space of symmetric tensors. In the real case the proof uses an induction on the number of tensor indices.

Averaging these deterministic inequalities over a random tensor reduces the problem to estimating the high moments of one-dimensional projections $\langle T, u_1 \otimes \cdots \otimes u_p\rangle$, and then optimizing over k. For tensors with independent *rigidly sub-Gaussian entries*, a class introduced in our work, these moments are dominated

by the corresponding Gaussian moments. Thus, normalizing the entries to have variance $1/d$, the method leads to

$$\limsup_{d\to\infty} \mathbb{E}\|T\|_{\mathrm{inj},\mathbb{K}} \le \inf_{\alpha>0} \psi_p^{\mathbb{K}}(\alpha) \le \sqrt{p\log p}\,(1+o_p(1)), \tag{6}$$

where $\psi_p^{\mathbb{K}}(\alpha)$ is fully explicit, has a unique minimum and can be minimized numerically at each fixed p. For standard Gaussian entries this estimate is multiplied by $\sqrt{d}$. Compared with Kac-Rice, the fixed$-p$, large$-d$ constant is slightly worse, but the scaling $\sqrt{p\log p}$ is the same and the method is more robust.

We also study some structured model. Consider

$$T = \sum_{r=1}^{R} x_1^{(r)} \otimes \cdots \otimes x_p^{(r)}, \tag{7}$$

where the vectors $x_i^{(r)}$ have independent complex rigidly sub-Gaussian entries of variance $1/d$. Such tensors model random states with bounded multipartite Schmidt rank. At fixed p and R, the moment method together with a lower bound gives,

$$\lim_{d\to\infty} \mathbb{E}\|T\|_{\mathrm{inj},\mathbb{C}} = 1.$$

This illustrates a situation where the moment method extends beyond the Gaussian Kac-Rice framework and where the technique allows for dependency between the entries of T.

Outlook

The two methods provide different research avenues. Kac-Rice is sharper and is not limited to the highest critical point. The moment method is relatively elementary, non-asymptotic, and works for several non-Gaussian or structured ensembles. Open directions include: universality for tensors with independent entries, extensions to random tensor networks and other physically relevant structured random tensors, and a better understanding of fluctuations and edge-type universality for tensor injective norms.

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Large deviation principle for the Toda lattice

Tamara Grava
(joint work with Alice Guionnet, Karol K. Kozlowski, Alex Little)

The Toda lattice is an integrable system of N particles on the line whose evolution equations can be written in the so called Lax pair form where the Lax operator L is a Jacobi matrix with periodic boundary conditions. This representation immediately produces N *local* conserved quantities that are given by the powers of the trace of L. The existence of a large number of local conserved quantities suggests that the statistical behaviour of the system is captured by a generalisation of the standard Gibbs measure. Indeed, physical intuition suggests that while local parts of the system "thermalise" due to interactions with the rest of the system, this local equilibrium must respect all conservation laws. These considerations motivated the introduction of Generalised Gibbs Ensembles (GGEs) which are believed to characterise this local equilibrium. The study GGEs was pioneered by Spohn [1], who derived the convergence of the eigenvalue distribution for polynomial potentials using transfer matrix approaches and comparisons with Dumitriu–Edelman β-ensembles [2]. Subsequently, Guionnet and Memin [3] established a LDP for the empirical measure of eigenvalues for more general potentials, though the rate function remained non-explicit.

In the present work [4] we derive a large deviation principle for the spectral measure of the Lax matrix, with an explicit rate function which can be regarded as a generalisation of the free energy of the system. Moreover, the large deviation principle is proven directly at the level of the representation of the generalised Gibbs partition function given in terms of the variables realising the classical separation of variables, i.e. rectify the equations of motion. As such, this work paves the way towards the computation of the thermodynamic limit of dynamical correlation functions in the Toda chain subject to generalised Gibbs ensemble statistics.

Further developments concerns the derivation of a large deviation principle for Jacobi matrices with more general class of random entries.

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Large N (non-)factorization of the expectations of multi-trace observables

Razvan Gurau
(joint work with Felix Joos and Benny Sudakov)

For a $N \times N$ Hermitian matrix H, the traces of the powers of the matrix $\mathrm{Tr}(H^p)$ yield a class of polynomials in the matrix entries invariant under the adjoint action of the unitary group $H \to H' = UHU^\dagger$. For a generic single trace invariant probability measure

$$d\mu(H) = e^{-N\,\mathrm{Tr}[S(H)]} \prod_{a,b} dH_{ab} \ ,$$

the expectations of a product of invariants factor in the large N limit:

$$\mathbb{E}\left[\mathrm{Tr}(H^{p_1})\dots\mathrm{Tr}(H^{p_b})\right] = \mathbb{E}\left[\mathrm{Tr}(H^{p_1})\right]\dots\mathbb{E}\left[\mathrm{Tr}(H^{p_b})\right] + O(N^{-2}) \ .$$

This is a very consequential feature of random matrices. For the free probabilist this means that the large N limit is entirely captured by the cumulants $\mathbb{E}\left[\mathrm{Tr}(H^p)\right]$. For the physicist this has tremendous implications in the context of the AdS/CFT correspondence: the large N factorization implies that the CFT is classical (weakly coupled), while on the AdS side the theory is strongly coupled yielding a bridge between weakly interacting gauge theories and strongly interacting gravity theories.

Tensors are multi-linear forms on $D \geq 3$ copies of a Hilbert space of dimension N. Real tensors can be seen as multidimensional arrays $T_{a^1\dots a^D}$ of N^D real numbers (the components of the tensor in a tensor product basis) which change under local orthogonal transformations $T \to T' = O \triangleright T$, $O \in O(N)^{\otimes D}$ with $O(N)$ the orthogonal group, as

$$T'_{b^1\dots b^D} = \sum_{a^1\dots a^D} O^{(1)}_{b^1a^1}\dots O^{(D)}_{b^Da^D} T_{a^1\dots a^D} \ , \quad O^{(c)} \in O(N) \ .$$

The traces of the powers of a matrix generalize to tensor trace invariants, that is polynomials in the tensor entries indexed by D-uples of perfect matchings $\mathbf{M} = (M^1, \dots, M^D)$ over $2n$ vertices:

$$\mathrm{Tr}_{\mathbf{M}}(T) = \sum_{\text{all } a} \Big(\prod_{k=1}^{2n} T_{a^1_k\dots a^D_k} \Big) \prod_{c=1}^{D} \prod_{\{k,l\}\in M^c} \delta_{a^c_k a^c_l} \ ,$$

which in turn can be represented as edge D-colored graphs. The connected graphs correspond to the single trace invariants $\mathrm{Tr}(H^p)$ while the disconnected ones correspond to multi-traces $\mathrm{Tr}(H^{p_1})\dots\mathrm{Tr}(H^{p_b})$.

Random tensors [1] generalize random matrices: as in the matrix case, one can show that for a large class of tensor probability measures the expectations of the trace invariants exhibit a $1/N$ expansion. This motivates recent efforts to define a free probability theory adapted to tensors [2, 3].

A natural question arises: does the large N factorization of expectations also generalizes to random tensors? One would expect this from a free probability perspective and in this context it has been conjectured to be true [2]. To be precise,

given b connected edge D-colored graphs $\mathbf{M}_1, \dots, \mathbf{M}_b$, is it true that for any (or at least some) probability measure invariant under $T \to T' = O \triangleright T$, $O \in O(N)^{\otimes D}$ we have

$$\mathbb{E}\left[\mathrm{Tr}_{\mathbf{M}_1}(T) \dots \mathrm{Tr}_{\mathbf{M}_b}(T)\right] = \mathbb{E}\left[\mathrm{Tr}_{\mathbf{M}_1}(T)\right] \dots \mathbb{E}\left[\mathrm{Tr}_{\mathbf{M}_b}(T)\right] + o(N) \ ?$$

In order to answer this question we considered in [4] the simplest invariant tensor measure, the Gaussian:

$$d\mu(T) = e^{-\frac{1}{2}\sum_{a^1 \dots a^D} (T_{a^1 \dots a^D})^2} \prod_{a^1 \dots a^D} dT_{a^1 \dots a^D} \ .$$

We have shown that:

- there exist infinite families of observables for which factorization holds at all orders: an example is the so called *melonic* family,
- (surprisingly from a free probability perspective, but straightforwardly from a probabilistic combinatorics point of view) despite this and the fact that the low order examples seem to confirm it, factorization fails for almost all observables with high enough degree in the tensor entries n.

Following our work, subsequent studies focused on two main directions. First, using a combination of analytic and numerical methods the lowest order counter examples to factorization have been identified [5]. Second, more general infinite families of observables for which factorization holds have been studied [6].

So where does the lack of factorization in random tensors leave us? One needs to explore the implications of this phenomenon for tensor freeness [2, 3] and especially the role of the higher order free cumulants in this context. The implications of non factorization in physics are also quite intriguing: what exactly happens to the large N tensor CFTs? The large N limit simplifies the theories, but less that in the matrix case so what does this mean for the dual gravity theories?

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Bernoulli flow for Erdős–Rényi graphs

Joscha Henheik
(joint work with Antti Knowles)

Let $A = A(N,p)$ be the adjacency matrix of an Erdős–Rényi graph $\mathbb{G}(N,p)$ on N vertices with edge probability p. We denote its normalized version by

$$H := \sigma A\,, \qquad \sigma := \frac{1}{\sqrt{Np(1-p)}}\,. \tag{1}$$

Denote the ℓ^2-normalized eigenvectors of H by $(\boldsymbol{u}_i)_{i\in[N]}$, i.e. $H\boldsymbol{u}_i = \lambda_i \boldsymbol{u}_i$. Moreover, for vectors $\boldsymbol{v} \in \mathbb{C}^N$, we define the *non-isotropic norm*

$$\|\boldsymbol{v}\|_{\boldsymbol{e}} := \|(\mathbf{1} - \Pi_{\boldsymbol{e}})\boldsymbol{v}\| + (Np)^{-1/2}\,\|\Pi_{\boldsymbol{e}}\boldsymbol{v}\|$$

where $\Pi_{\boldsymbol{e}}$ denotes the orthogonal projection to $\boldsymbol{e} := N^{-1/2}(1,\dots,1) \in \mathbb{C}^N$.

Our first main result shows optimal delocalization in the bulk for $Np \geq C(\log N)^2$.

Theorem 1. *Fix $\kappa > 0$. Then, for any $D > 0$, there exists a constant $C = C(D,\kappa) > 0$ such that*

$$\mathbb{P}\Big[\exists i \in [N] : |\lambda_i| \leq 2-\kappa\,,\ \big|\langle \boldsymbol{u}_i, \boldsymbol{v}\rangle\big|^2 \geq \frac{C\log N}{N}\|\boldsymbol{v}\|_{\boldsymbol{e}}^2\Big] \leq N^{-D} \tag{2}$$

whenever $C(\log N)^2 \leq Np \leq N^{1-\kappa}$, uniformly in $\boldsymbol{v} \in \mathbb{C}^N$.

An analogous statement holds for the left- and right-eigenvectors associated with bulk eigenvalues of directed *Erdős–Rényi graphs.*

Previous optimal delocalization results were restricted to dense matrices (see, e.g., [2, 7]), or, in the sparse case, proved (2) with $\log N$ replaced by N^κ [12, 1].

Our second main result proves bulk universality in the regime $Np \gg (\log N)^2$. This was previously only known in the polynomially sparse regime, $Np \geq N^\kappa$ [14].

Theorem 2. *Fix $\kappa > 0$ and $E \in [-2+\kappa, 2-\kappa]$. Then, around E, the local eigenvalue statistics of H in* (1) *converge, as $N \to \infty$, in the vague topology, to the* Sine_1 *process, whenever* $(\log N)^{2+\kappa} \leq Np \leq N^{1-\kappa}$.

Our main technical result underlying both Theorems 1–2 is a *local law* for the Green function $G(z) := (H-z)^{-1}$ of H, valid for spectral parameters $z \in \mathbb{C}\setminus\mathbb{R}$ whose imaginary part is at least $(\log N)/N$. Our local law is optimal (up to constants), in terms of both the lower bound on $\operatorname{Im} z$ and the precise bounds on the Green function entries.

We prove such a local law using a novel dynamical technique, the *Bernoulli flow method* [13]. At a high level, our proof is inspired by the seminal work of Pastur [16], who combined a matrix-valued Ornstein-Uhlenbeck process with characteristics flow of the spectral parameter to access small spectral scales. This formidable idea was rediscovered by Lee-Schnelli [15] and von Soosten-Warzel [17], and proved to be extremely powerful in various contexts. We highlight recent breakthrough works on *random band matrices* [18, 11, 8, 9] and the development

of the robust *zigzag strategy* [4, 5, 6, 11], where the flow argument is complemented by a perturbative comparison step. For sparse matrices, however, such perturbative arguments, including the Green function comparison idea or cumulant expansions, are not affordable. Indeed, the Gaussian component arising from the Ornstein-Uhlenbeck process can only be removed by comparison techniques if the distribution of the matrix entries is not too far away from a Gaussian; this is no longer true for sparse matrices.

In contrast, the Bernoulli flow is a dynamical process whose endpoint is precisely the adjacency matrix of an Erdős–Rényi graph, which means that no perturbative comparison step is required. It consists of two ingredients:

(1) The *Bernoulli process* $(A_t)_{0\le t\le T}$, a Markovian jump process on $\{0,1\}^{N\times N}$.
(2) The *Bernoulli characteristic flow* $(z_t)_{0\le t\le T}$, a deterministic matrix-valued flow for the spectral parameter z.

The *Bernoulli process* has generator acting on observables $\phi : \{0,1\}^{N\times N} \to \mathbb{C}$ defined through

$$\mathcal{L}\phi(A) := \sum_{x<y} \big(\phi\big(A^{xy}\big) - \phi\big(A\big)\big)$$

where A^{xy} is obtained from A by adding the edge xy (if it is not already there). That is, the associated Markov process adds edges independently at rate one and can thus be thought of as a continuous-time analogue of the classical dynamical construction of random graphs by Erdős and Rényi in [10]. The process is designed in such a way that, if it is initialised at the empty graph, we have that

$$A_t \quad \text{is the adjacency matrix of} \quad \mathbb{G}(N,p_t) \quad \text{with} \quad p_t = 1-\mathrm{e}^{-t} \quad \text{for} \quad t\ge 0\,.$$

Given a target sparsity parameter p, the terminal time of the Markov process is given by $T := -\log(1-p)$. The matrix dynamics are combined with the matrix-valued *Bernoulli characteristic flow* given by

$$\partial_t z_t = -z_t + \sigma E - \frac{1-p_t}{1-p}\frac{\langle M_t(z_t)\rangle}{p}\,,$$

where $E_{xy} = 1-\delta_{xy}$, $\langle M\rangle := \frac{1}{N}\operatorname{Tr} M$, and M_t, the deterministic approximation in the local law, solves the *time-dependent matrix Dyson equation*

$$-\frac{1}{M_t(z_t)} = z_t - p_t\sigma E + \frac{p_t(1-p_t)}{p(1-p)}\langle M_t(z_t)\rangle.$$

As observables we choose the isotropic or average quantities

$$\phi_t(A_t) = \langle \boldsymbol{v}, \big(\sigma A_t - z_t\big)^{-1}\boldsymbol{w}\rangle \quad \text{or} \quad \phi_t(A_t) = \langle \big(\sigma A_t - z_t\big)^{-1}\rangle\,.$$

In order to show that $\phi_T(A_T) \approx \phi_0(A_0 = 0)$ for $T \sim p$, unlike for Brownian motion, the Itô formula is not useful. Instead, we use the naturally associated Dynkin martingale

$$\mathcal{M}_t := \phi_t(A_t) - \phi_0(A_0) - \int_0^t \mathrm{d}s\,\big(\mathcal{L}\phi_s(A_s) + \partial_s\phi_s(A_s)\big).$$

Roughly speaking, the proof has three main ingredients:

1. Estimate the generator terms, exploiting a cancellation with the z-flow.
2. Estimate the martingale jump sizes by one-step resolvent expansions.
3. Estimate the predictable quadratic variation.

Items 2 and 3 are the key inputs for controlling the martingale using the Burkholder-Davis-Gundy inequality with jumps.

Remark 3. *Besides the optimal local law, the second key input to proving Theorem 2 is the very recent characterization of the* Sine_1 *process via loop equations by Bourgade and Huang* [3]*, which we learned of at the Workshop at Oberwolfach.*

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A conjecture about large deviations of the spectrum of perturbed Gaussian matrices

Ella Hiesmayr
(joint work with Thomas Buc—d'Alché)

A symmetric matrix can always be described by its set of eigenvalues and eigenvectors. This still holds true when one considers random symmetric matrices, i.e. those where each entry is a random variable. One way to study such random matrices is thus to consider the empirical distribution of its eigenvalues. This is then a random probability measure and one can wonder how this measure depends on the distribution of the entries of the matrix.

One classical random matrix model are Wigner matrices, where all entries, up to the symmetry restriction, are independent, and there are only two different entry distributions: one for those on the diagonal and one for those off the diagonal. One of the first major results of random matrix theory, called Wigner's theorem [4, 5], showed that the limit of the empirical eigenvalue distribution does not depend on the precise entry distributions, as long as some weak moment conditions are satisfied.

In contrast to this universality result about the typical behaviour, we expect that the probability of rare events, in other words, large deviations, depends on the precise entry distributions. This can be seen in analogy to results about the empirical mean of a sequence of random variables, which converges to the mean for a large class of distributions, but whose probability to be far away from this typical limit depends on all moments.

Unlike the typical behaviour, large deviations of the empirical distribution are only understood for a small class of distributions. The first result in this line of inquiry was about Gaussian matrices [1], and leveraged the fact that the density of eigenvalues of Gaussian matrices is explicit. Another major result concerned matrices with heavy-tailed entries [2], in which case large deviations can be traced back to very few large entries in the matrix.

Our work has the goal to understand how different the large deviation behaviour of light-tailed matrices is from the one of Gaussian matrices. We do not expect to be able to decompose the matrix into a large negligible part and a very sparse part to which the large deviation can be traced back to. Moreover we already know that certain subgaussian matrices behave differently from Gaussian ones, as matrices where the entries can only take a finite number of values are much more likely to converge to a Dirac mass at zero than Gaussian matrices. Thus a natural next step is to consider Gaussian matrices with a small light-tailed perturbation. Providing some insights into this question is the goal of our work, which we plan to post on arXiv soon.

The advantage of considering perturbed Gaussian matrices is that we can express the density of their eigenvalues in terms of an explicit term that resembles the density of eigenvalues of Gaussian matrices, times another term that is an expected value over Haar-distributed unitary matrices. This latter term depends

on the Laplace transform of the entries of the perturbation. It can be analysed by writing it as an infinite sum of cumulants of traces of products of unitary matrices, to which we can apply a tool called Weingarten calculus. To find the leading order of these individual terms, we then use combinatorial arguments, and different ways of representing the permutations that appear in cumulants by graphs.

This analysis allows us to find the leading contribution of each term up to a small error, and thus get a formal expansion of the entire infinite sum. Based on this formal expansion, we obtain a conjecture for the rate function of large deviations of the empirical distribution of Gaussian matrices. We believe that this conjecture could be rendered analytical through an approach as the one presented in [3]. This requires showing that the expected value we analysed has a limiting behaviour that can be expressed using the conjectured rate function, a question we hope to tackle in a follow-up project.

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Random matrix statistics and loop equations

Jiaoyang Huang

(joint work with Theo McKenzie, Horng-Tzer Yau, and Paul Bourgade)

This talk concerns a mechanism behind random matrix universality which can be formulated in terms of loop, or Dyson–Schwinger, equations. I will first discuss edge universality for random regular graphs. Let A be the adjacency matrix of a uniformly random d-regular graph on N vertices, with fixed $d \geq 3$, so that, after normalization by $\sqrt{d-1}$, the nontrivial spectrum is asymptotically supported on $[-2, 2]$. In joint work with Theo McKenzie and Horng-Tzer Yau [1, 2], we prove that, after the usual $N^{2/3}$ edge scaling, the upper and lower nontrivial spectral edges converge to the GOE soft-edge statistics. In particular, the fluctuations of λ_2 and $-\lambda_N$ are governed by the Tracy–Widom$_1$ law, and this yields an asymptotic probability of about 0.69 that a random d-regular graph is Ramanujan.

The proof follows a three-step strategy. First one proves a sharp local law and optimal rigidity for the eigenvalues. Next one adds a small Gaussian component,

$$H_t = \frac{A}{\sqrt{d-1}} + \sqrt{t}\, G,$$

where G is a GOE matrix, and uses the short-time relaxation of Dyson Brownian motion to identify the edge statistics of H_t for times $t \gg N^{-1/3}$. The main comparison step is to remove almost all of this Gaussian component. This is achieved by proving a microscopic version of the loop equation at the spectral edge for the resolvent of H_t down to times $t = N^{-1/3+\varepsilon}$. The derivation uses the invariance of the random regular graph measure under local switchings and plays, for random regular graphs, the role that integration by parts plays for Wigner matrices.

To indicate the form of the limiting equations, let $\Lambda_1 \geq \Lambda_2 \geq \cdots$ denote the Airy_1 point process [6] and let $a_1 > a_2 > \cdots$ be the zeros of the Airy function. The regularized Stieltjes transform

$$S(w) = \sum_{k\geq 1}\left(\frac{1}{\Lambda_k - w} - \frac{1}{a_k}\right) - \frac{\mathrm{Ai}'(0)}{\mathrm{Ai}(0)}, \qquad w \in \mathbb{C} \setminus \mathbb{R},$$

is a random Nevanlinna function with square-root behavior at infinity. The microscopic loop equations are a hierarchy of identities for the correlation functions of S. In the GOE case, the rank-p identity has the schematic form

$$\mathbb{E}\left[\left((S(w) - \sqrt{w})^2 + 2\sqrt{w}\,(S(w) - \sqrt{w}) + \partial_w S(w)\right)\prod_{j=1}^{p-1} S(w_j)\right]$$
$$+ 2\sum_{i=1}^{p-1}\mathbb{E}\left[\partial_{w_i}\left(\frac{S(w) - S(w_i)}{w - w_i}\right)\prod_{\substack{j=1\\ j\neq i}}^{p-1} S(w_j)\right] = 0.$$

The key point is that the corresponding prelimit identities for the perturbed random regular graph are accurate at the microscopic scale, so that the Green function comparison can be closed at the edge.

In the second part of the talk I will discuss a direct characterization of random matrix statistics through loop equations, based on joint work with Paul Bourgade [3]. The guiding question is whether one can recognize the universal local point processes of random matrix theory from the equations they satisfy, without first coupling the model to a Gaussian ensemble. The answer is affirmative: for rational $\beta > 0$, the bulk loop equation hierarchy characterizes the Sine_β process [5], while the soft-edge hierarchy characterizes the Airy_β process [4]. Thus loop equations are not merely necessary identities inherited from a model; together with the correct boundary behavior they become a uniqueness principle for random matrix statistics.

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Eigenvalues of random matrices via coefficients of polynomials

Jonas Jalowy
(joint work with Sungsoo Byun, Zakhar Kabluchko, Yong-Woo Lee, Alexander Marynych, Grégory Schehr)

Consider a polynomial $P_n(z) = \sum_{k=0}^n a_{k;n} z^k$ with only real zeros $z_j \in \mathbb{R}$. As in random matrix theory, we are interested in the weak empirical limit of zeros

$$[\![P_n]\!] := \frac{1}{n} \sum_{z:P_n(z)=0} \delta_z \Rightarrow \mu, \text{ as } n \to \infty.$$

The main tool. This general question naturally has a long history, but here we focus on the direct relation between coefficients and the limiting zero distribution. In particular, we applied the following main tool, see also [15, 7, 2].

Theorem 1 ([8, 12])**.** *If $a_{k;n} \geq 0$, i.e. $z_j \leq 0$, and $P_n(1) = 1$, then,*

- $[\![P_n]\!] \Rightarrow \mu$ *on* $[-\infty, 0]$ *with Stieltjes transform* $m_\mu(z) = \int \frac{1}{z-w} d\mu(w)$
- $\frac{1}{n} \log a_{k;n} \underset{k/n \to \alpha}{\overset{n\to\infty}{\longrightarrow}} \begin{cases} g(\alpha), & \text{if } \alpha \in \mathcal{I} \subseteq [0,1] \\ -\infty, & \text{else} \end{cases}$ *for some concave* $g : \mathcal{I} \to (-\infty, 0]$

are equivalent. Moreover, $\mathbb{R}_+ \ni z \mapsto z m_\mu(z)$ *is the inverse of* $\mathcal{I} \ni \alpha \mapsto e^{-g'(\alpha)}$.

The interval $\mathcal{I} \Subset (\mu(0), \mu(\mathbb{R}))$ is explicit and the so called *exponential profile* $g \in \mathcal{C}^\infty$ is automatically 1–strongly concave. Our proof in [12] is probabilistic, by writing P_n as a generating function of sums of Bernoulli random variables, for which an LDP argument reveals that $-g$ is Legendre transform of log-potential $U_\mu(e^z)$. Apart from many classical examples of polynomials in [13], this tool gives rise to the following three applications on random matrices and free probability.

1) Coefficients of characteristic polynomials of sample covariance matrices. Let $X := (X_{i,j})_{1\leq i\leq n, 1\leq j\leq p}$ be a random matrix with iid entries satisfying $\mathbb{E}[X_{i,j}] = 0$ and $\mathbb{V}\mathrm{ar}[X_{i,j}] = \sigma^2 < \infty$. Consider the sample covariance matrix $Y = \frac{1}{p} X X^*$ with $\frac{n}{p} \to \lambda \in (0, \infty)$ and its characteristic polynomial $P_n(z) = \det(Y - z)$. It is well known that its coefficients can be represented as sums of minors, that its expected polynomial $\mathbb{E}[P_n(z)] = p^{-n} n! L_n^{(p-n)}(pz)$ is a rescaled Laguerre polynomial, and that the Marčenko-Pastur law holds, i.e. $[\![P_n]\!] \Rightarrow \mathsf{MP}_{\sigma^2,\lambda}$. An immediate consequence of Theorem 1 is:

Theorem 2 ([13])**.** *The coefficients $a_{k;n}$ of $P_n(-x)/P_n(-1)$ satisfy*

$$a_{\alpha n;n} = e^{ng(\alpha)+o(n)}, \quad \mathbb{P}\text{-}a.s.$$

for any $0 \vee (1 - \lambda^{-1}) + \varepsilon < \alpha < 1 - \varepsilon$ and any $\varepsilon > 0$ small enough. We have $e^{-g'(\alpha)} = \frac{\sigma^2 \alpha(\lambda\alpha + 1 - \lambda)}{1-\alpha}$, coinciding with Laguerre polynomials.

For fixed k, the asymptotics of $a_{n;n-k}$ are polynomials in trace powers $\mathrm{Tr}(Y^j)_{j\le k}$, which can be understood using results on smooth linear statistics.

2) LDP for real zeros of ellGinOE. Let $X \in \mathbb{R}^{n\times n}$ be a real elliptic Ginibre matrix with parameter $t \in [0,1]$, i.e. $X_{ij} \sim \mathcal{N}_{\mathbb{R}}(0,1)$ with $\mathrm{Cov}(X_{ij}, X_{ji}) = t$. Then, we have weak convergence $[\![\det(X^{(t)}/\sqrt{n} - z)]\!] \Rightarrow \mathrm{ell}_t$ to the uniform measure on the ellipse with real axis $1+t$ and imaginary axis $1-t$. Due to conjugate pairs of eigenvalues, the number of real eigenvalues $N_n := \#\{\lambda_j \in \mathbb{R}\}$ is a non-trivial random variable, whose study goes back to [6]. Its asymptotic behavior splits into the following regimes.

Strong Asymmetry: If $t \in [0,1)$ is fixed, then $\mathbb{E}N_n \sim c_t\sqrt{n}$ for some constant $c_t > 0$ and N_n satisfies a CLT with order $\sqrt{n}$ variance.

Weak Asymmetry: If $t = 1 - \beta^2/n$ converges to 1, then $\mathbb{E}N_n \sim c_\beta n$ for some constant $c_\beta > 0$ and N_n satisfies a CLT with order n variance.

We show local large deviation principles for the number of real eigenvalues.

Theorem 3 ([3]). *In the strong asymmetry regime, choose $\frac{k}{\sqrt{n}} \to \alpha > 0$, then*

$$\frac{1}{\sqrt{n}} \log \mathbb{P}[N_{2n} = 2k] \to -\sup_{u\in\mathbb{R}} \Big\{\alpha u + \frac{c_t}{\sqrt{2}} \mathrm{Li}_{3/2}(1-e^u)\Big\} \text{ as } n \to \infty,$$

where $\mathrm{Li}_s(z) = \sum_k z^k k^{-s}$. In the weak regime, choose $\frac{k}{n} \to \alpha \in (0,1)$, then

$$\frac{1}{n} \log \mathbb{P}[N_{2n} = 2k] \to -\sup_{u\in\mathbb{R}} \Big\{\alpha u - \frac{4}{\pi}\int_0^1 \log(1-(1-e^u)e^{-\beta^2 s^2})\sqrt{1-s^2}\,ds\Big\}.$$

The right tail of Theorem 3 asymptotically matches [1] for $k = n - \mathcal{O}(1)$ and its left tail matches [4] for $k = n/\log n$ or $n^{1/2}/\log n$ respectively. For real spherical Ginibre matrices, the same rate function has been (conditionally) derived in [9].

The heart of the proof relies on the following observation: In the weak regime, consider the probability generating function $P_n(z) := \sum_{k=0}^n z^k \mathbb{P}[N_{2n} = 2k]$. Using the Pfaffian structure of the eigenvalues of X, it follows $P_n(z) = \det(\mathrm{I}_n + (z-1)M_n)$ for some Hermitian M_n. After finding the limit $\frac{1}{n}\log P_n(z) \to U_\mu(z)$, we essentially identified the limiting root distribution μ such that a variant of Theorem 1 proves the existence of an exponential profile g for the coefficients of P_n. In the strong case, we extended the necessity of Theorem 1 to more general speeds $c_n = \sqrt{n}$.

3) Finite free convolutions and the heat flow. Let $P_n(z) = \sum_k a_{k;n} z^k$ and $Q_n(z) = \sum_k b_{k;n} z^k$ be two polynomials of degree $\le n$. Define the bilinear operation $\boxplus_n$ with unit z^n that preserves real rootedness

$$P_n \boxplus_n Q_n(z) := \sum_{k=0}^n \frac{1}{n!k!}\Big(\sum_{i+j=n+k} a_{i;n} b_{j;n} i!j!\Big) z^k.$$

Its connection to free probability has been revealed in [14]: If $P_n(z) = \det(X - z)$ and $Q_n(z) = \det(Y - z)$ are characteristic functions, then

$$P_n \boxplus_n Q_n(z) = \mathbb{E}_{U\sim\mathcal{U}_n}[\det(X + UYU^* - z)],$$

explaining the name *finite free convolution.* Assuming that there exists exponential profiles of P_n and Q_n, it is possible to calculate the exponential profile g of $P_n \boxplus Q_n$, and relate it via R-transforms to the free convolution $\mu \boxplus \nu$ as follows.

Theorem 4 ([12]). *If P_n, Q_n have roots in $[c,\infty]$ such that $[\![P_n]\!] \Rightarrow \mu$ and $[\![Q_n]\!] \Rightarrow \nu$, and $\mu(\infty)+\nu(\infty)<1$, then*

$$[\![P_n \boxplus_n Q_n]\!] \Rightarrow \mu \boxplus \nu.$$

Analogous results hold for finite free multiplicative convolution $\boxtimes_n \to \boxtimes$.

For compactly supported μ, ν, see [2] using finite free cumulants.

Finally, observe that any polynomial $Q_n(z)$ can be expressed as $f_n(\partial_z)z^n$ for some polynomial f_n and the holomorphic derivative ∂_z. By linearity, it follows $P_n \boxplus_n Q_n = f_n(\partial_z)P_n(z)$, leading to the actively studied question

How do (real) zeros evolve under differential operators? Specifically, for the heat flow $f_n(\partial_z) = e^{-\frac{t}{2n}\partial_z^2}$, how can we identify the weak limit $[\![e^{-\frac{t}{2n}\partial_z^2}P_n]\!] \Rightarrow \mu_t$?

Since monomials evolve into Hermite polynomials $\mathrm{He}_n(z) = e^{-\frac{1}{2}\partial_z^2}z^n$, which satisfy $[\![e^{-\frac{t}{2n}\partial_z^2}z^n]\!] \Rightarrow \mathrm{sc}_t$ as $n \to \infty$, Theorem 4 implies the following.

Theorem 5. *If $[\![P_n]\!] \Rightarrow \mu$ on $[c,\infty]$ and $t \in \mathbb{C}$, then $[\![e^{-\frac{t}{2n}\partial_z^2}P_n]\!] \Rightarrow \mu \boxplus \mathrm{sc}_t$.*

In the remainder of the talk, we briefly discussed the generalization to infinitely divisible laws as shown in [5], and the case of complex roots under heat evolution of the polynomial studied in [10, 11].

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The Eigenfunctions of Šeba's Billiard

Jon Keating
(joint work with Henrik Ueberschär)

The central aim of the field of Quantum Chaos is to characterise the influence the statistical properties of the classical dynamics of a system have on its corresponding quantum statistical properties, in the semiclassical (short wavelength) asymptotic limit. It is expected that for systems in which the classical dynamics is integrable, the eigenfunctions of the Schrödinger operator behave locally like a finite sum of plane waves, in the semiclassical limit, and that the statistical distribution of the eigenvalues coincides with that of independent, real random variables, i.e. with a Poisson distribution. In systems with chaotic classical dynamics, on the other hand, it is expected that the eigenfunctions behave locally like a large number of randomly directed plane waves, and so satisfy a central limit theorem, and that the eigenvalues are distributed statistically like those of certain classes of random matrices. Examples of integrable systems include geodesic motion on flat tori with an irrational aspect ratio. Examples of chaotic systems include Sinai-type billiards, in which a circular obstacle of radius $r > 0$ is placed on the torus, at the boundary of which classical trajectories make specular reflections and quantum eigenfunctions vanish. Establishing rigorous results consistent with these expectations has been a major goal for the past 50 years.

One also expects to find both Poisson and random matrix eigenvalue distributions in the Anderson model on a lattice; the former case corresponding to states in which the quantum eigenfunctions are localised, and the latter case to states in which the quantum eigenfunctions are delocalised. At the border between localised and delocalised states, it is believed that the eigenvalue statistics are intermediate between Poisson and random-matrix-like, exhibiting characteristics of both, and that the eigenfunctions are multifractal, i.e. with moments that scale nontrivially, with a spectrum of fractal exponents. Proving multifractality in the Anderson model is an important open problem.

In 1990, Šeba introduced a class of billiards motivated by the formal idea of taking the limit $r \to 0$ in Sinai billiards. Specifically, Šeba billiards correspond to adding a Dirac delta-function potential to a flat torus, or, equivalently, taking the Schrödinger operator to be a self-adjoint extension of the Laplacian acting on the torus with a point removed. These systems have been studied extensively in recent years, especially in the case of the square torus, where, in the absence of the delta-function potential, the eigenvalues take the form n^2+m^2, where n and m are integers. The addition of the delta-function potential corresponds to a rank-one perturbation, and so one can establish formulae for the Šeba eigenvalues and

eigenfunctions in terms of sums over n and m, i.e. in terms of sums over lattice points.

In [1], we prove that, in the momentum representation (i.e. in a Fourier basis) the eigenfunctions of the Šeba billiard corresponding to the square torus are multifractal in the semiclassical limit. We establish an explicit formula for the scaling exponents of their moments in terms of the moment parameter. Our proof relies heavily on results from number theory concerning the asymptotic number of lattice points in certain annuli. We also prove that the scaling exponents satisfy a symmetry relation that is consistent with conjectures in the Physics literature.

In [2] we prove similar results for metric star graphs. The proof in this case relies only slightly on number theory.

We believe mulifractality to be a robust phenomenon in systems that are intermediate between Poisson and random matrix theory, and in particular in singular perturbations of integrable systems.

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Two-resolvent local laws with decorrelation in energy space

Oleksii Kolupaiev

(joint work with Z. Bao, G. Cipolloni, L. Erdős, J. Henheik)

Two-resolvent local laws are the concentration bounds for the quantities of the form $G(z_1)B_1G(z_2)B_2$, where $G(z) := (H - z)^{-1}$ is the resolvent of an $N \times N$ Hermitian random matrix H, and B_1, B_2 are deterministic matrices of the same size. We assume that $\|B_j\| \lesssim 1$ in order not to carry the dependence on B_j in the estimates. For a Wigner matrix H the following averaged two-resolvent local law in the bulk regime is known from [4, Theorem 2.5]:

$$\left|\left\langle\left(G(z_1)B_1G(z_2) - M_{12}^{B_1}\right)B_2\right\rangle\right| \lesssim \frac{N^{\xi}}{N\eta^2}, \quad \eta := \min\{|\Im z_1|, |\Im z_2|\} \gg N^{-1}, \tag{1}$$

with very high probability, for any fixed $\xi > 0$. Here $\langle\cdot\rangle := N^{-1}\mathrm{Tr}$ and $M_{12}^{B_1}$ is the deterministic approximation to $G(z_1)B_1G(z_2)$ which is explicitly constructed from the solution of the *matrix Dyson equation* [5]. For more details, see, e.g., [4, Eq.(2.4)], [2, Eq.(3.8)]. Since $|\langle M_{12}^{B_1}B_2\rangle| \lesssim \eta^{-1}$, and this bound is saturated for $z_2 = \overline{z}_1$ and typical B_1, B_2, (1) is a concentration bound. In other words, the fluctuation of the two-resolvent chain is much smaller than its typical size.

In the talk I will consider a more general model, where $G_j = G_j(z_j)$ is the resolvent of a deformed Wigner matrix $H_j := W + D_j$ for $j = 1, 2$. Note that the

source of randomness in H_1, H_2 is the same, but the deformations are different. In [6] we established that

$$\left|\langle M_{12}^{B_1} B_2\rangle\right| \lesssim \left(\langle (D_1 - D_2)^2\rangle + \eta\right)^{-1}, \tag{2}$$

i.e. for $D_1 \neq D_2$ the upper bound on the lhs. of (2) improves compared to the Wigner case. This improvement carries over to the local law estimate, which is the statement of the two-resolvent local law with decorrelation in energy space [2, Theorem 3.2]. In the bulk regime we have

$$\left|\left\langle \left(G_1 B_1 G_2 - M_{12}^{B_1}\right) B_2\right\rangle\right| \lesssim \frac{N^{\xi}}{\sqrt{N\eta}\gamma}, \quad \eta := \min\{|\Im z_1|, |\Im z_2|\} \gg N^{-1}. \tag{3}$$

Here $\gamma \gtrsim \langle (D_1 - D_2)^2\rangle + \eta$ is an explicit control parameter such that γ^{-1} is an optimal upper bound on the deterministic approximation.

I will also demonstrate the analogue of (3) in the non-Hermitian set-up [3, Proposition 4.2] and discuss the following applications.

(i) Optimal eigenvector overlap bound for a differently deformed Wigner matrix [2].
(ii) Eigenvector decorrelation transition in the Wigner minor process [1].
(iii) Optimal singular vector overlap for i.i.d. matrices [3].
(iv) *Hyperuniformity* of the eigenvalue process of (real or complex) i.i.d. matrices [3].

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On characteristic polynomials of Gaussian β-ensembles

Gaultier Lambert

(joint work with Elliot Paquette)

We consider the Gaussian β-ensemble (GβE), that is the probability measure on $\mathbb{R}^N$

$$\frac{1}{Z_{N,\beta}} \prod_{i<j} |x_i - x_j|^{\beta} \exp\left\{-\beta N \sum_{i=1}^{N} x_i^2\right\} dx_1 \cdots dx_N, \tag{1}$$

for fixed $\beta > 0$. The normalization is such that the empirical measure converges to the semicircle law with density $\rho(x) = \frac{2}{\pi}\sqrt{1-x^2}\,\mathbf{1}_{|x|\le 1}$. This talk aims to report on recent works with Elliot Paquette [2, 3, 4] on *strong asymptotics* for the GβE characteristic polynomial as $N \to \infty$. In particular, we are interested in giving asymptotics which explains both the global Gaussian log-correlated structure of Φ_N and the local Sine$_\beta$ fluctuations of the eigenvalues.

For a random configuration distributed according to (1), we define

$$\Phi_N(z) = \prod_{j=1}^{N}(z-x_j)\sqrt{w_N(z)}, \qquad \sqrt{w_N(z)} = \frac{(2\sqrt{N})^N}{\sqrt{N!}}\left(\frac{2N}{\pi}\right)^{1/4} e^{-Nz^2}. \tag{2}$$

With this convention, $\mathbb{E}\Phi_N$ is the normalized Hermite function associated with the Gaussian weight e^{-2Nx^2} on $\mathbb{R}$. In particular, the asymptotics of $\mathbb{E}\Phi_N$, which also corresponds to the deterministic limit $\beta = \infty$, in the bulk and at the edge of the spectrum are given by the classical Plancherel–Rotach asymptotics. By the work of Dumitriu–Edelman [1], the sequence $\{\Phi_N\}$ can be realized on the same probability space as the characteristic polynomials of minors of an infinite tridiagonal matrix with independent entries and this leads to a second order recurrence for $\{\Phi_N\}$. This device turns the analysis of $\Phi_N(z)$ into to a problem of studying the Hermite polynomial recurrence perturbed by noise. For $z \in [-1,1]$, a key feature of this dynamics is the presence of a *turning point* at $n \simeq Nz^2$ where the behavior of the solution Φ_n changes from exponentially growing to oscillatory. Thus, for the proofs, we naturally splits the study into three differnet regimes:

- In the first part $n \ll Nz^2$, one can *linearize* the dynamics perturbatively and identify a global Gaussian behavior. This is called the *hyperbolic regime*, [2].
- The *transition window at the turning point* $n \approx Nz^2$ is of size $N^{1/3}$ and the asymptotics of the characteristic polynomial are described by the *stochastic Airy function*. This is called the *parabolic regime*, [3].
- After the turning point, $n \gg Nz^2$, the behavior of the solution is oscillator and we introduce a *Prüfer phase* to encode the solution. This is called the *elliptic regime* and it is responsible for the zeros of Φ_N in the bulk of the spectrum, [4].

We will now informally present the strong asymptotic approximation for the characteristic polynomial at the edges and in the bulk of the spectrum. These approximations are formulated within the tridiagonal model coupling on an event of *overwhelming probability*.

Edge results. We prove in [3] that uniformly for $\lambda \in \mathbb{R}$ in compact sets: as $N \to \infty$,

$$\Phi_N\left(1 + \frac{\lambda}{2N^{2/3}}\right) = N^{1/6}\frac{\exp(\sqrt{2/\beta}\,G_N)}{\mathbb{E}\exp(\sqrt{2/\beta}\,G_N)}\big(\mathrm{SAi}_0^\beta(\lambda) + o(1)\big), \tag{3}$$

where G_N is a mean-zero Gaussian with variance $\log N^{1/3} + O(1)$ and SAi^β is the *stochastic Airy function*. This function is defined in [3] as the special L^2 eigenfunction of the stochastic Airy operator from Ramírez–Rider–Virág [5], $\mathcal{H}_\beta = -\frac{d^2}{dt^2} + t + \frac{2}{\sqrt{\beta}}\dot{B}_t$. In particular, in the deterministic limit, $\mathrm{SAi}_0^\infty(\lambda) = \mathbb{E}\,\mathrm{SAi}_0^\infty(\lambda) =$

$\mathrm{Ai}(\lambda)$ is the standard Airy function which appears in the Hermite function asymptotics at the edge. As corollaries, we obtain a central limit theorem for $\log|\Phi_N|$ at the edge and the weak convergence of the rescaled edge eigenvalues to the Airy_β point process in the form of the zeros of stochastic Airy function, namely $\{2N^{2/3}(x_j-1)\}_{j=1}^N \Rightarrow \mathrm{Airy}_\beta = \{\lambda \in \mathbb{R} : \mathrm{SAi}_0^\beta(\lambda) = 0\}$ as $N \to \infty$.

Bulk results. In [4], we introduce a complex *Prüfer phase* $\varphi_N : (-1,1) \to \mathbb{C}$ to describe the bulk asymptotics of characteristic polynomial: for $z \in (-1,1)$,

$$\Phi_N(z) = m(z)^{c_\beta}\,\Re\exp\varphi_N(z), \qquad c_\beta = \tfrac{1}{2} - \tfrac{1}{\beta}, \qquad m(z) = \tfrac{2}{\pi\sqrt{1-z^2}}. \tag{4}$$

In particular $\Re\varphi_N$ governs the log-amplitude of Φ_N and $\omega_N(z) = \frac{\Im\varphi_N(z)}{\pi}$ its oscillations: one has

$$\{z \in [-1,1] : \Phi_N(z) = 0\} = \{z \in [-1,1] : \omega_N(z) \in \mathbb{Z} + \tfrac{1}{2}\}. \tag{5}$$

Our asymptotic results describe both the *macroscopic* log-correlated behavior and *microscopic* Sine_β point process behavior of the phase. Informally, for $z \in (-1,1)$ with $z \neq 0$ (say $Nz^2 \gg 1$), uniformly for $\lambda \in \mathbb{R}$ in compact sets: as $N \to \infty$,

$$\varphi_N\left(z + \frac{\lambda}{N\rho(z)}\right) = iN\pi F(z) + \sqrt{2/\beta}\,\big(M_N(z) + \Omega_N(z)\big) + \pi\Psi_N(\lambda; z) + o(1) \tag{6}$$

where

- $F(z) = \int_z^1 \rho(x)\,dx$ is the semicircle counting function, it gives the leading deterministic part of the phase.
- $\{M_N\}$ is a martingale sequence and it is an approximately Gaussian log-correlated field in the sense that

$$\left\{\frac{M_N(z_j)}{\sqrt{\log N}}\right\}_{j=1}^k \Rightarrow \mathcal{N}_\Sigma$$

  and the bracket

$$\big[M_N(x), \overline{M_N(y)}\big] \simeq \log(|x-y| \vee N^{-1})^{-1}.$$

- $\Omega_N(z) \Rightarrow \Omega$ in distribution where Ω is a non-trivial (non-Gaussian) random variable coming from the turning point; *the stochastic Airy regime.*
- $\Psi_N(\lambda; z)$ is the microscopic relative phase: $\Psi_N(\lambda; z) \Rightarrow \psi_1^\beta(\lambda)$ as a process in $\lambda \in \mathbb{R}$. The process $\{\psi_t^\beta : t \in [0,1]\}$ solves the Sine SDE introduced by Killip–Stoiciu [8] and one has

$$\mathrm{Sine}_\beta = \{\lambda \in \mathbb{R} : \Im\psi_1^\beta(\lambda) \in \mathbb{Z} + \boldsymbol{u}\}, \quad \boldsymbol{u} \sim \mathrm{Uni}[0,1] \text{ independent.}$$

  As corollaries, by (5), this implies that $\big\{(x_j - z)\frac{N\rho(z)}{\pi}\big\}_{j=1}^N \Rightarrow \mathrm{Sine}_\beta$ as established in [6] and also the convergence of ratios of characteristic polynomials converge to the stochastic zeta function of Valkó–Virág [7]:

$$\frac{\Phi_N(z + \lambda/(N\rho(z)))}{\Phi_N(z)} \Rightarrow \zeta_\beta(\lambda). \tag{7}$$

Discussion. The asymptotics (3) and (6) reveal both the macroscopic Gaussian log-correlated structure, through the martingale part M_N, and the Airy$_\beta$ and Sine$_\beta$ objects which govern the microscopic fluctuations at the edge and in bulk of the spectrum. It is also surprising that a stochastic Airy random variable Ω arises in the bulk asymptotics. It is not independent of M_N and it reflects the non-Gaussian behavior. For future research, this brings a connection to Gaussian multiplicative chaos, by considering the renormalized power of the characteristic polynomial, for a fixed $z \in (-1,1)$ and for $\gamma > 0$,

$$\frac{|\Phi_N(z)|^\gamma}{\mathbb{E}|\Phi_N(z)|^\gamma} \approx \frac{e^{\gamma\sqrt{\alpha}\,\Re(M_N(z)+\Omega_N(z))}}{\mathbb{E}e^{\gamma\sqrt{\alpha}\,\Re(M_N(z)+\Omega_N(z))}}, \qquad \alpha = 2/\beta. \tag{8}$$

In [4], we also compute the normalization in (8) as $N \to \infty$

$$\mathbb{E}|\Phi_N(z)|^\gamma \simeq N^{\frac{\gamma^2}{2\beta}} \rho(z)^{\frac{\gamma^2}{2\beta}} m(z)^{c_\beta \gamma} \Upsilon_\alpha(\gamma). \tag{9}$$

The non-elementary factor $\Upsilon_\alpha(\gamma)$ is independent of z, it is not explicit in our analysis, but it can be related to the stochastic Airy function. Finally, we conjecture that

$$\Upsilon_\alpha(\gamma) = \frac{G_\alpha(1+\gamma/2)^2}{\pi^{\gamma/2} G_\alpha(1+\gamma)} \tag{10}$$

where we call G_α the Barnes-α function; it is the appropriate solution of the equation $G_\alpha(u+1) = \Gamma(1+\alpha u) G_\alpha(u)$ with $G_\alpha(1) = 1$. The asymptotics (9) and the conjecture (10) are known for the GUE ($\beta = 2$) since Krasovsky [9]. This work relies on the determinantal structure of GUE and Riemann–Hilbert asymptotics, and our goal is to generalize these Fisher–Hartwig asymptotics for GβE for a general $\beta > 0$ with a Probabilistic proof. The conjecture (10) is supported by strong numerical evidence.

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Gaussian behavior of eigenvectors in sparse random graphs

Theo McKenzie
(joint work with Amir Dembo)

Berry's random wave conjecture is a foundational prediction in the study of quantum chaotic systems. It theorizes that the local statistics of eigenfunctions converge to those of a Gaussian random wave. The discrete version of this question is well studied, and large sparse graphs have long been used as a model of quantum chaos. Whereas a general heuristic is that these graphs have spectral measures converging to those of their local weak limits, Berry's conjecture predicts the stronger phenomenon that the local statistics of eigenvectors themselves converge to those of a Gaussian process.

The d-regular graph has been a central model for this question. The impressive work of Backhausz and Szegedy [3] proved that every eigenvector, except for the trivial eigenvector with eigenvalue d, has local statistics given by a Gaussian process on the infinite regular tree $\mathcal{T}_d$. The d-regular graph benefits from a key simplifying feature: its local weak limit is the infinite regular tree, which has a large automorphism group. Indeed, the tree is distance regular, so any limiting wave must respect this symmetry, and this invariance plays a decisive role in identifying and solving the associated formulas.

Nevertheless, there is little reason to believe the symmetry of the regular tree to be essential to chaotic behavior, and we prove a result toward Berry's conjecture for a broader class of random graph models generalizing the random regular graph. Specifically, for a large family of configuration models—random graphs with a prescribed degree distribution—we show that eigenvectors exhibit Gaussian-type statistics under a condition on the Green's function that holds for Lebesgue-almost-every energy in the spectrum, with the phenomenon driven solely by local expansion properties.

A principal innovation is a structural decomposition of the Gaussian process on the infinite tree via the Green's function, which makes this mechanism transparent. The Gaussian wave can be realized as a Gaussian process obtained by pushing forward i.i.d. Gaussian noise through the resolvent. To make this precise, let

$$G(z,\mathcal{G}) := [A_{\mathcal{G}} - zI^{-1}$$

denote the Green's function of a graph $\mathcal{G}$. For $z = \lambda + i\eta \in \overline{\mathbb{C}^+}$, graph $\mathcal{T}$, and a family $\{\xi_x\}_{x\in V(\mathcal{T})}$ of i.i.d. random variables with $\xi_x \sim \mathcal{N}(0,1)$, the Gaussian wave $\Psi^z(\mathcal{T})$ admits the representation

$$\Psi^z(\mathcal{T}) \overset{d}{=} \sqrt{\eta} \sum_{x\in V(\mathcal{T})} \xi_x \, G(z,\mathcal{T})_x, \tag{1}$$

where $G(z,\mathcal{T})_x(\cdot) := G(z,\mathcal{T})_{\cdot x}$ denotes the column of the kernel of $G(z,\mathcal{T})$ corresponding to $x \in V(\mathcal{T})$. To show any candidate distribution must be Gaussian,

we examine its entropy, and one key insight is that to compute the entropy of the Gaussian wave, it is sufficient to consider moments of the form

$$\{\langle \Psi^z(\mathcal{T}), G(z,\mathcal{T})_x \rangle\}_{x \in V(\mathcal{T})},$$

which are straightforward to calculate using the above decomposition.

While the Green's function representation of covariance is known (see [4, Section 1.1]), treating this decomposition as a primary analytical tool allows for a direct comparison between the cases $z \in \mathbb{C}^+$ and $z \in \mathbb{R}$ and provides a transparent framework for analyzing the entropy of the Gaussian wave. Equation (1) can be compared to the heuristic related to Berry's conjecture that on a hyperbolic manifold, the Fourier coefficients of the Poisson kernel should be i.i.d. Gaussian [1, 2]. Indeed, we expect this type of decomposition to hold in many models and to be useful in future work. For example, it is of interest to show a similar result for the Erdős–Rényi random graph.

The key observation in [3] is that the empirical distribution of an eigenvector in the configuration model is a *typical process*, meaning that it defines a labeling which, in the weak topology, must be close to a labeling realized by a randomly selected graph, with high probability. We show that for any infinite tree of finite cone type satisfying a mild expansion condition, the only typical process on its vertices with covariance induced by the Green's function is the Gaussian wave. This generalizes the result of Backhausz and Szegedy from the infinite regular tree to a broad class of trees. As a consequence, for random bipartite biregular graphs, the distribution of local neighborhoods of eigenvectors must approximate the Gaussian wave. Moreover, for generic configuration models, including random lifts, the local distribution of a uniformly chosen eigenvector from any arbitrarily small spectral window likewise converges to the Gaussian wave.

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Precise delocalisation and Gumbel laws for eigenvectors of Wigner matrices

Mohammed Osman

Let W_N be an $N \times N$ Wigner matrix with eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N$ and eigenvectors $\{\mathbf{u}_i\}$. We let $\beta = 1$ and $\beta = 2$ refer to real and complex matrices respectively. The eigenvectors are expected to behave similarly to columns of a

uniformly random orthogonal/unitary matrix. One manifestation of this is delocalisation. Let

$$\ell_N^{(\beta)}(D) := \begin{cases} (D+1)\log N - \log(4\pi(D+1)\log N) & \beta = 1 \\ 2(D+1)\log N & \beta = 2 \end{cases}. \tag{1}$$

For any $D > 0$, we expect that

$$\mathbb{P}\Big(\|\mathbf{u}_i\|_\infty > \sqrt{\frac{\ell_N^{(\beta)}(D)}{N}}\Big) \lesssim N^{-D}. \tag{2}$$

Another is Gumbel fluctuations for $\|\mathbf{u}_i\|$ and $\max_i \|\mathbf{u}_i\|$, the latter being a result of Jiang [7]. More precisely, we expect that

$$\lim_{N\to\infty} \mathbb{P}\Big(N\|\mathbf{u}_i\|_\infty^2 - \ell_N^{(\beta)} \le x\Big) = e^{-e^{-\frac{\beta x}{2}}}, \tag{3}$$

and

$$\lim_{N\to\infty} \mathbb{P}\Big(N \max_i \|\mathbf{u}_i\|_\infty^2 - \ell_{2N}^{(\beta)} \le x\Big) = e^{-e^{-\frac{\beta x}{2}}}, \tag{4}$$

where $\ell_N^{(\beta)} \equiv \ell_N^{(\beta)}(0)$.

Erdős–Schlein–Yau [5] prove (2) with $\ell_N^{(\beta)}(D)$ replaced by $N^\xi \log N$ for any $\xi > 0$, while Benigni–Lopatto [2] improve this to some $C(D)\log N$ growing exponentially in D. Knowles–Yin [8], Tao–Vu [9] and Bourgade–Yau [3] prove joint Gaussian fluctuations for finite subsets of components of eigenvectors, while Benigni [1] allows for growing subsets. These results are not enough to deduce convergence in distribution of the maximum entry, although the comparison arguments in [2] can be used to prove (3) for edge eigenvectors, i.e. those with index satisfying $i \wedge |N-i| < N^\epsilon$. This is done explicitly with a different approach by Bao–Bucht–Schnelli [4], who also compute order statistics beyond the maximum entry.

In this talk we present a nearly optimal version of (2) for Wigner matrices:

Theorem 1. *Let $\alpha \in (0,1)$. Then there is a constant $C > 0$ such that*

$$\max_{i\in[\alpha N,(1-\alpha)N]} \mathbb{P}\left(\|\mathbf{u}_i\|_\infty > \sqrt{\frac{\ell_N^{(\beta)}(D)}{N}}\right) \le CN^{-D}\log N. \tag{5}$$

The restriction to bulk indices $[\alpha N, (1-\alpha)N]$ is for technical convenience but the method applies to all indices. The right-hand side is optimal up to the $\log N$ factor.

For complex Wigner matrices, we can also prove Gumbel fluctuations.

Theorem 2. *Fix $E \in [-2,2]$ and $x \in \mathbb{R}$. Let $i \equiv i(E)$ be the random index of the eigenvalue closest to E. Then*

$$\lim_{N\to\infty} \mathbb{P}\Big(N\|\mathbf{u}_i\|_\infty^2 - \ell_N^{(2)} \le x\Big) = e^{-e^{-x}}. \tag{6}$$

Moreover,

$$\lim_{N\to\infty} \mathbb{P}\Big(N \max_{i\in[N]} \|\mathbf{u}_i\|_\infty^2 - \ell_{2N}^{(2)} \le x\Big) = e^{-e^{-x}}. \tag{7}$$

The single-eigenvector fluctuations are described in terms of a fixed energy and random index, as opposed to (3), which corresponds to a deterministic index and random energy. This is because our method to treat Gaussian-divisible matrices can only work with unlabelled eigenvalues/eigenvectors. More precisely, we obtain (6) by first showing that the point process

$$\Big\{\big(N(\lambda_i - E), N\|\mathbf{v}_i\|_\infty^2 - \ell_N^{(2)}\big) : i \in [N]\Big\}, \tag{8}$$

converges to a point process whose correlation functions factorise into an eigenvalue part (sine kernel) and an eigenvector part (Poisson).

The proof naturally splits into two parts: a comparison of two ensembles and a computation for Gaussian-divisible matrices. Our approach to the comparison step is based on the inclusion–exclusion principle and the idea of Erdős–Xu [6] for comparing small-probability events. In brief, by inclusion–exclusion, it is enough to compare probabilities of the form

$$\mathbb{P}\Big(N|\mathbf{u}_{i,j}|^2 - \ell_{2N}^{(\beta)} > x,\ i \in I,\ j \in J\Big), \tag{9}$$

for any fixed subsets $I, J \subset [N]$. Since I is fixed, we expect that we can exclude eigenvalues in small neighbourhoods $\mathcal{I}_i$ of λ_i, simultaneously for all $i \in I$. On this good event, $|\mathbf{u}_{i,j}|^2$ is well-approximated by an explicit function of the resolvent. The bad event should be negligible since we expect the eigenvectors and eigenvalues to be approximately independent. Since (9) is $O(N^{-|I|-|J|})$, we need to compare probabilities up to any polynomial order in N^{-1}, which is possible using the idea of Erdős–Xu.

For the Gaussian-divisible computations, we use the method of partial diagonalisation to obtain explicit integral formulas for probabilities of the form in (9), which can be analysed asymptotically. Roughly speaking, we use the Fourier representation of indicators:

$$\mathbf{1}_{(x,\infty)}(y) = \lim_{\epsilon\to 0} \int_{-\infty}^{\infty} \frac{e^{ik(y-x)}}{k - i\epsilon} \frac{\mathrm{d}k}{2\pi i}, \tag{10}$$

to reduce to expectation values of exponentials in quadratic forms in $\mathbf{u}_i$, which are the kinds of expectation values for which partial diagonalisation is well-suited.

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The canonical systems approach to limiting β-ensembles

Elliot Paquette

(joint work with Vincent Painchaud; and, for the work in progress on the quartic ensemble, Elizabeth Collins-Woodfin, Vincent Painchaud, and Brian Rider)

The local limits of one-dimensional β-ensembles are usually described as point processes, but for general $\beta > 0$ they also arise as spectra of random differential operators. At the soft edge one has the stochastic Airy operator, at the hard edge the stochastic Bessel operator, and in the bulk the stochastic sine operator. A basic difficulty is that these operators do not belong to the same elementary class: the edge operators are random Schrödinger or Sturm–Liouville type operators, while the bulk operator is a random Dirac type operator. The point of view discussed in the talk is that canonical systems give a common language for these objects, and for the finite tridiagonal and CMV models which approximate them.

A canonical system is an equation

$$Ju'(t) = -zH(t)u(t), \qquad J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

where $H(t)$ is a locally integrable, non-negative 2×2 matrix. The matrix H is the Hamiltonian or coefficient matrix. If H is invertible, this is the eigenvalue equation for the first order operator $-H^{-1}J\,d/dt$ on the Hilbert space L^2_H. If H is singular, as happens naturally for Sturm–Liouville embeddings, the right object is a self-adjoint relation rather than an operator. This mild enlargement is useful: it allows rank-one Hamiltonians and Dirac Hamiltonians to be treated by the same spectral theory.

The relevant spectral data are encoded by the transfer matrix $T_H(t,z)$, the solution of

$$J\partial_t T_H(t,z) = -zH(t)T_H(t,z), \qquad T_H(a,z) = I,$$

and by the Weyl–Titchmarsh function m_H. The latter is defined by the solution which satisfies the right boundary condition, or by the unique L^2_H direction if the right endpoint is limit point. It is a Herglotz function, hence has a representing measure. In the present examples this measure is pure point, with atoms at the eigenvalues and masses equal to the spectral weights. A useful topology on Hamiltonians is obtained by viewing $H(t)\,dt$ as a matrix-valued measure and testing

against compactly supported continuous vector functions. Under natural trace bounds, vague convergence of Hamiltonians implies convergence of transfer matrices; with the correct control of the right boundary condition, one also obtains convergence of Weyl–Titchmarsh functions and spectral measures.

The sine operator has a particularly direct canonical-system form. Let $\mathcal{B}_s = x_s + iy_s$ be hyperbolic Brownian motion in the upper half-plane, started at i, with variance parameter $4/\beta$. Then

$$R_\beta(s) = \frac{1}{2y_s}\begin{pmatrix} 1 & -x_s \\ -x_s & x_s^2 + y_s^2 \end{pmatrix}$$

is the Hamiltonian for the sine system after the logarithmic time change $s = -\log(1-t)$. For $\beta \le 2$ the endpoint $t = 1$ is limit point, while for $\beta > 2$ it is limit circle and the boundary direction is determined by $\operatorname{Re}\mathcal{B}_\infty$.

By contrast, the stochastic Airy operator

$$\mathcal{H}_\beta f = -f'' + \left(t + \frac{2}{\sqrt{\beta}}B'(t)\right) f$$

is a random Schrödinger operator. After replacing this formal expression by the standard generalized Sturm–Liouville formulation, one may embed the shifted operator $2\sqrt{E}(\mathcal{H}_\beta - E)$ into a canonical system. If f_E, g_E are the Dirichlet and Neumann solutions of the zero-energy equation for this shifted operator, then the corresponding Hamiltonian is

$$H_{\beta,E}(t) = \frac{1}{2\sqrt{E}}\begin{pmatrix} \sqrt{E}\, g_E(t)^2 & f_E(t)g_E(t) \\ f_E(t)g_E(t) & E^{-1/2} f_E(t)^2 \end{pmatrix}.$$

This matrix has rank one. Thus the Airy and sine systems look quite different as differential operators, but they are close objects in the canonical-system topology after the correct high-energy rescaling.

The main result with Vincent Painchaud is that, as $E \to \infty$, there is a time change η_E such that

$$\eta_E'(t) H_{\beta,E}(\eta_E(t)) \Longrightarrow R_\beta(-\log(1-t))$$

in the vague topology of Hamiltonians. Moreover the corresponding transfer matrices converge, locally uniformly in the spectral parameter, and the Weyl–Titchmarsh functions converge compactly on the upper half-plane. Hence the associated spectral measures converge vaguely in distribution. This is an operator-level version of the soft-edge to bulk transition, and gives an intrinsic canonical-system proof of the Airy-to-sine limit, rather than passing through a comparison with finite Gaussian β-ensembles. The same framework also gives the hard-edge to bulk transition: Painchaud has shown that the high-energy stochastic Bessel canonical system converges to the sine canonical system at the level of Hamiltonians, transfer matrices, Weyl–Titchmarsh functions, and spectral measures.

I also described work in progress with Collins-Woodfin, Painchaud and Rider on a critical quartic model. The ensemble is

$$dP_{\beta,n}(\lambda) \propto \exp\left\{-n\frac{\beta}{2}\sum_i\left(\frac{\lambda_i^4}{4} - c\lambda_i^2\right)\right\}\prod_{i<j}|\lambda_i - \lambda_j|^\beta\, d\lambda, \qquad c = 1 + \frac{s}{2}n^{-2/3}.$$

At $c = 1$ the limiting density has a quadratic zero at the origin. The first step is an exact tridiagonal Markov-chain representation: the entries (a_k, b_k) have transition densities built from local potentials $V_k(a_k, b_k \mid a_{k-1}, b_{k-1})$, and the spectrum has the desired β-ensemble law. Around a two-periodic deterministic background $\widehat{b}_k$, the critical fluctuations live on the $n^{1/3}$ scale,

$$n^{1/3}(-1)^k(b_k^2 - \widehat{b}_k^2), \qquad n^{1/3}(-1)^k a_k.$$

Under a Gaussian background these fields converge to a pair of linear diffusions driven by independent Brownian motions. The non-Gaussian part of the model remains visible through a fine-potential tilt, involving quartic terms and a stochastic integral driven by an additional Brownian motion.

The expected limiting spectral object is again a canonical system, now of Dirac type. If ϕ is the deterministic transition profile and $\mathfrak{b}, \mathfrak{a}$ are the limiting fluctuation fields, then the candidate first-order system is

$$\partial_t\psi = \frac{1}{2}\begin{pmatrix}\phi(t) + \mathfrak{b}(t) & \mathfrak{a}(t) \\ -\mathfrak{a}(t) & -\phi(t) - \mathfrak{b}(t)\end{pmatrix}\psi.$$

Taking a fundamental matrix Ψ for this equation, the conjectural Hamiltonian is $H(t) = \frac{1}{2}\Psi(t)^*\Psi(t)$. This system should be singular and limit point at both endpoints. Several natural questions remain open: to identify the canonical choice of transition profile, to prove convergence from the critical tridiagonal model to this canonical system, and to understand whether the limits $s \to \pm\infty$ recover the expected sine and Airy regimes.

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Global law for heavy-tailed random features matrices

Vanessa Piccolo
(joint work with Alice Guionnet)

A two-layer neural network can be written in the form

$$f(x; W, a) = \frac{1}{\sqrt{p}} a^\top \sigma(Wx),$$

where $W \in \mathbb{R}^{p\times d}$ and $a \in \mathbb{R}^p$ denote the first- and second-layer weights, respectively, and $\sigma\colon \mathbb{R} \to \mathbb{R}$ is an activation function applied entrywise. Given data samples $x_1, \dots, x_n \in \mathbb{R}^d$, collected as the columns of a matrix $X \in \mathbb{R}^{d\times n}$, the corresponding feature matrix is

$$Y = \sigma(WX) \in \mathbb{R}^{p\times n}.$$

In the classical *random features model*, W is sampled at random and then kept fixed, while only the readout vector a is trained. Although the map $x \mapsto \sigma(Wx)$ is nonlinear, the optimization problem is therefore linear in the trainable parameters. The associated Gram and conjugate-kernel matrices are

$$K = \frac{1}{p} Y^\top Y \quad \text{and} \quad C = \frac{1}{n} Y Y^\top.$$

Up to normalization and possible zero eigenvalues, they contain the same nonzero spectral information. Their spectra control, among other things, the conditioning of the regression problem and enter naturally into formulas for training and generalization errors.

The random features model also gives rise to an interesting nonlinear random matrix problem. Even when W and X have independent entries, the entries of $Y = \sigma(WX)$ are neither independent nor linear functions of the underlying random variables. One might therefore expect the spectrum of Y to depend in a complicated way on the activation function and on the full distributions of W and X. Since all three dimensions are large in applications, we study this problem in the proportional high-dimensional regime, in which the input dimension d, the number of neurons p and the number of samples n tend to infinity at comparable rates. Consider first the classical light-tailed random features regime. Suppose that W and X have i.i.d. centered entries with unit variance and sufficiently light tails, and that σ is smooth and centered with respect to the standard Gaussian measure. Then the global spectral behavior of

$$\frac{1}{\sqrt{n}} \sigma\left(\frac{WX}{\sqrt{d}}\right)$$

agrees asymptotically with that of a much simpler Gaussian-equivalent model [9, 8, 1]:

$$\frac{1}{\sqrt{n}} \left(\sqrt{\theta_2(\sigma)} \frac{\tilde{W}\tilde{X}}{\sqrt{d}} + \sqrt{\theta_1(\sigma) - \theta_2(\sigma)}\, \tilde{Z} \right),$$

where $\tilde{W}$, $\tilde{X}$, and $\tilde{Z}$ are independent random matrices with i.i.d. standard Gaussian entries. Here, the parameters $\theta_1(\sigma)$ and $\theta_2(\sigma)$ correspond to the Gaussian mean of

σ^2 and the square of the Gaussian mean of σ', respectively. Thus, at the level of the global spectrum, the nonlinearity survives only through these two scalar Gaussian parameters. Extensions of this light-tailed picture cover additive biases, variance-profile models, and related nonlinear models built from orthogonally invariant matrices; see respectively [10, 2, 11]. Related equivalence principles also hold at the level of learning: in several high-dimensional random-features models, the asymptotic training and generalization errors coincide with those of a Gaussian model having the same covariance structure; see e.g. [7, 5, 3].

Empirical studies suggest that trained weight matrices can differ substantially from the classical i.i.d. light-tailed model used at initialization [6]. Training may give rise to correlations between the weights and the data, while regularization and pruning may produce sparse or highly heterogeneous weight matrices. Trained networks have also been observed to exhibit heavy-tailed spectral behavior. Motivated by these considerations, in joint work with Alice Guionnet [4], we study the spectral behavior of the model when the first-layer weights are no longer light-tailed. An i.i.d. heavy-tailed model does not capture all the structures produced by training, but it provides a tractable setting in which large non-Gaussian contributions and correlations already arise in the feature matrix.

We assume that X has light-tailed i.i.d. centered entries, while the entries of W are i.i.d. and symmetric and satisfy

$$d \log \mathbb{E}\left[\exp(i\lambda W_{ij})\right] \to \Phi(\lambda), \qquad \lambda \in \mathbb{R},$$

for some limiting characteristic exponent Φ. The dimension-dependent normalization is therefore included in the law of W. This setting includes, for example, symmetric α-stable weights, with $\alpha \in (0,2)$, as well as sparse matrices with an order-one number of nonzero entries per row, on average. A quadratic Φ gives the usual light-tailed scaling. Our main result establishes the convergence of the moments of the empirical eigenvalue distribution of

$$C = \frac{1}{n} Y Y^\top, \quad Y = \sigma(WX).$$

Under an additional assumption ensuring that these moments determine a unique measure, the empirical spectral distribution also converges almost surely. We assume here that σ is smooth, bounded, and odd. The limiting moments admit an explicit combinatorial description. More precisely, expanding

$$\frac{1}{p} \operatorname{Tr} C^k = \frac{1}{pn^k} \operatorname{Tr}(YY^\top)^k$$

leads to sums indexed by closed walks on bipartite graphs. In the light-tailed case, only a small class of graphs contributes in the limit. Heavy-tailed and sparse weights allow many more graphs to contribute. These graphs can be organized as trees of blocks, with an even-degree condition within each block.

Several properties of the limiting measure are not yet apparent from this formula. Although all its moments are finite under our assumptions, we conjecture that its support is typically unbounded. Can one determine its precise tail behavior or obtain a more analytic characterization, for instance through an equation

for its Stieltjes transform? The behavior of the largest eigenvalues, the possible emergence of outliers, local eigenvalue statistics, and fluctuations of linear spectral statistics also remain largely open. Another natural direction is to replace the i.i.d. matrix W by a finite-rank deformation, a matrix with a variance profile, or a model with more general correlations, and to understand the corresponding limiting spectrum.

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Large Deviations for Random Matrices: Recent Results with the Spherical Integral Method

Jana Reker

(joint work with Alice Guionnet, Jonathan Husson)

Over the last few years, the method of tilting by spherical integrals (or 'spherical integral method') has been established as a versatile tool for proving large deviations principles for the largest eigenvalue of various Hermitian random matrix models. Originally introduced in [4] to study the large deviations of the largest eigenvalue for Wigner matrices with sharp sub-Gaussian entries, the results were later generalized to Wigner matrices with less restrictive hypotheses on the Laplace transform [1, 2], as well as matrices beyond the classical Wigner model (e.g., [6]).

This talk reports on recent progress on generalizing the method to structured matrices, namely the work of Ducatez, Guionnet, and Husson [3] on sharp sub-Gaussian $N \times N$ matrices with a variance profile, and the article [5] on Gaussian Kronecker random matrices. The latter concerns $NL \times NL$ matrices of the form

$$\mathbf{X}_N := \sum_{j=1}^{k} A_j \otimes W_j^{(N)} + A_0 \otimes \mathrm{Id}_N,$$

where $A_0, \ldots, A_k$ are deterministic real symmetric (resp. complex Hermitian) $L \times L$ matrices, Id_N denotes the $N \times N$ identity matrix, and $W_1^{(N)}, \ldots, W_k^{(N)}$ are i.i.d. random matrices sampled from the GOE (resp. GUE) of size $N \times N$. Note that this yields a correlated and deformed model. For both types of random matrices, i.e., the model with a variance profile and the Gaussian Kronecker matrices, one obtains a large deviation principle with speed N and a good rate function. This motivates further investigation into structured models, e.g., polynomials in independent GOE or GUE matrices via the linearization trick or the (considerably more ambitious) question of extending the spherical integral method to more general correlated models.

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The Zigzag Strategy for Random Band Matrices

Volodymyr Riabov

(joint work with László Erdős)

One-dimensional Random Band Matrices (RBM) have entries concentrated in a narrow band of width W around the diagonal, modeling systems with spatially localized interactions. By varying the bandwidth W from 1 to N, they interpolate between the celebrated Anderson model (N random degrees of freedom) and the mean-field random matrix ensembles (N^2 degrees of freedom). In 1990, it was conjectured that random band matrices undergo an Anderson phase transition around the critical bandwidth $W \sim N^{1/2}$ in the bulk spectrum [2, 5]. This conjecture has remained a prominent problem in the field, motivating decades of research and yielding a series of insightful partial results.

Heuristically, the critical bandwidth threshold is predicted by the *Thouless criterion*. If the complex eigenvectors (quantum waves) are localized on the scale $\xi \le N$ (with $\xi = N$ corresponding to delocalization), the system can be thought of as a superposition of N/ξ essentially independent localization boxes. Under the unitary time evolution $\mathrm{e}^{\mathrm{i}tH}$ effectively restricted to the localization box, the time required for the system to resolve individual eigenvalues is of order ξ. For a one-dimensional RBM with bandwidth W, the diffusive spread of a wave packet at time $t \ge 0$ is of order

$$\ell(t) := \min\{W\sqrt{t}, N\}.$$

Roughly speaking, the scale ξ is determined by balancing $\xi \sim \ell(\xi)$, which leads to

$$W \gg \sqrt{N} \Longrightarrow \xi = N, \qquad W \ll \sqrt{N} \Longrightarrow \xi \sim W^2,$$

predicting complete delocalization above the critical bandwidth and localization below it.

Over the past two decades, major progress towards understanding the delocalized phase has been achieved by extending modern resolvent techniques from mean-field random matrix theory to band matrices. In particular, *static* self-consistent equations for the resolvent $G(z) := (H - z)^{-1}$, which proved remarkably successful in the mean-field setting, were used to establish complete delocalization for RBM with $W \gg N^{3/4}$ [1], with this intermediate threshold seemingly marking the technical limitation of the static methods. More recently, with the advent of the *dynamical* approach to analyzing the resolvent, this limitation has been overcome: [6] established delocalization for all $W \gg N^{1/2}$ in the setting of block-band matrices with Gaussian entries.

In [4], we prove complete eigenvector delocalization, Wigner–Dyson universality of the bulk eigenvalue statistics, and Quantum Unique Ergodicity for one-dimensional Hermitian RBM with $W \gg N^{1/2}$. Our results apply to general variance profiles and arbitrary entry distributions satisfying standard moment assumptions, in both the real-symmetric and complex-Hermitian symmetry classes.

Using the aforementioned resolvent techniques developed in the mean-field setting, we derive bulk eigenvalue universality and the desired properties of the corresponding eigenvectors from the *local laws* – optimal concentration estimates on $G(z)$, down to the optimal spectral scale $\Im z \gg N^{-1}$. Single resolvent local laws capture the strong self-averaging of $G(z)$ by asserting that it concentrates around a deterministic matrix $M(z)$, given, in the case of RBM, by $m_{\mathrm{sc}}(z)\, I$. However, proving such concentration results for the RBM resolvent poses a serious difficulty, since all relevant objects exhibit diffusion-like decay on an emergent length scale $\ell(z)$, satisfying

$$\ell(z) \sim \min\bigl\{W\eta^{-1/2}, N\bigr\}, \qquad \eta := |\Im z|.$$

This length scale is closely related to the phenomenon of *quantum diffusion*, and corresponds to the spread of the wave packet at times $t \sim |\Im z|^{-1}$. We find that maintaining sufficient control on this decay is the key difficulty in the proof. We establish quantum diffusion for RBM in the following form.

Theorem 1 (Quantum Diffusion, Theorem 3.3 in [4])**.** *Let H be a random band matrix with bandwidth $W \gg N^{1/2}$. Then, for z in the bulk spectrum, with $\eta := |\Im z| \gg N^{-1}$, the resolvent $G(z)$ satisfies the local law*

$$|G_{xy}(z)|^2 \approx \frac{1}{\ell(z)\eta}\Upsilon\left(\frac{|x-y|}{\ell(z)}\right),$$

where Υ is a fast-decaying spatial profile on a length scale $\ell(z)$, see Figure 1.

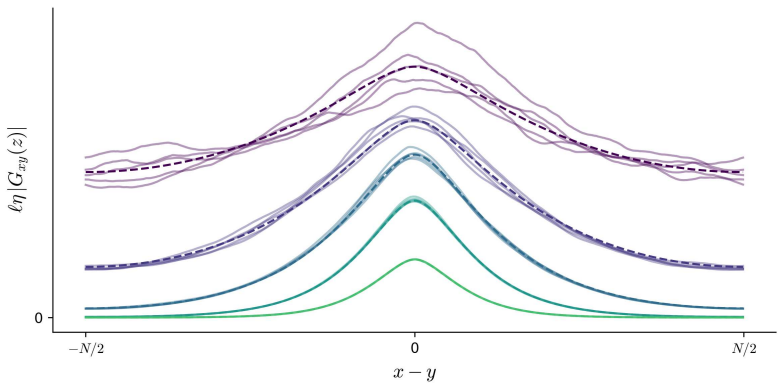

FIGURE 1. Concentration of the resolvent entries $\ell\eta|G_{xy}(z)|^2$ of a random band matrix around a deterministic decay profile (dashed lines) at different levels of η (increasing from top to bottom). Note that the concentration is stronger at larger η and that the fluctuations around the deterministic profile also exhibit off-diagonal spatial decay.

Single-resolvent local laws alone are insufficient for establishing quantum diffusion. The object encoding the off-diagonal decay naturally involves two resolvents, namely, with $(S^x)_{ab} = \delta_{ab}S_{xa}$ and $S_{xa} := \mathrm{Var}[H_{xa}]$,

$$\sum_a S_{xa}|G_{ay}(z)|^2 = \big(G(\overline{z})S^x G(z)\big)_{yy}\ .$$

We call such products of resolvents interlaced with deterministic matrices *resolvent chains*. In fact, along the proof, it is essential to establish local laws for arbitrarily long chains. We find that the profile Υ also governs the spatial decay of both the deterministic approximation and the fluctuations in longer resolvent chains.

To establish the necessary local laws, we extend the recently introduced *Zigzag strategy*. The strategy involves two principal steps. First, a tandem of a matrix-valued Ornstein–Uhlenbeck process and a deterministic *characteristic flow* in the space of spectral parameters is used to propagate local laws dynamically from coarse spectral scales down to smaller $|\Im z|$. This process introduces a Gaussian component to the distribution of the matrix entries, which is removed in the second step using Green function comparison arguments. An alternating application of these two steps allows us to obtain local laws at the optimal scale $|\Im z| \gg N^{-1}$.

The application of the dynamical method to multi-resolvent local laws, together with the analysis of the resulting hierarchy of evolution equations, was introduced in [3].

Along the dynamics, the fluctuations of the resolvent chains are governed by a potentially infinite coupled hierarchy of nonlinear evolution equations. We show that, after appropriate truncation, this system becomes *self-improving*. In fact, we essentially reduce the analysis of the hierarchy to a series of key estimates on the action of the associated propagator semigroup.

Furthermore, we show that the analysis of the deterministic approximations to the long chains can also be conducted using the same dynamical approach, resulting in deterministic evolution equations governed by the same propagator, thus providing a unified toolkit for establishing multi-resolvent local laws.

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Low complexity smoothed analysis

Mark Rudelson

(joint work with Shabarish Chenakkod, Michal Derezinski, and Xiaoyu Dong)

Smoothed analysis introduced by Spielman and Teng [1] is considered the most efficient method to escape the worst case scenarios arising in analysis of many numeric linear algebra algorithms. Consider, for example, solving the linear system

$$Ax = b,$$

where A is an $n \times n$ matrix and x, b are n-dimensional real vectors. If the vector b is known with a certain error, then this error gets magnified by the *condition number* of the matrix A:

$$\kappa(A) = \frac{s_{\max}(A)}{s_{\min}(A)} = \|A\| \cdot \|A^{-1}\|,$$

where $s_{\max}(A), s_{\min}(A)$ are its maximal and minimal singular values. The condition number of a matrix is a priori unbounded which makes solving the system

computationally challenging. Spielman and Teng suggested to replace the given matrix A by its noisy version

$$M = A + \varepsilon R,$$

where R is a random matrix of the norm commensurate with that of A, and $\varepsilon > 0$ is the magnitude of the noise. Using rescaling, we can assume that $\|A\| = 1$ and $\|R\| \leq 1$ with high probability.

A theorem proved by Spielman and Teng asserts that if R is taken to be the standard Gaussian matrix, then

$$\kappa(A + \varepsilon R) \leq \frac{n}{\varepsilon\delta}$$

with probability at least $1 - \delta$. Moreover, the bound $\kappa(A + \varepsilon R) = O(n)$ is the best possible for any random matrix if no a priori information about A besides its norm is available.

Although adding the Gaussian noise yields the best magnitude of the condition number, it is not the most practical from the computational point of view due to the high cost in both randomness and space requirements.

- Constructing a Gaussian matrix requires generating n^2 independent normal random variables which is computationally expensive.
- Modern iteration methods of solving linear systems rely on evaluating Ax_k, or A^*y_k, where x_k and y_k are vectors obtained at the k-th iteration. These evaluations can be performed in $O(n)$ size memory if the matrix A is stored efficiently. At the same time, evaluation of Gx_k for a Gaussian matrix G requires storing the whole matrix which takes memory of the size $O(n^2)$.
- Since the normal random variables are continuous, they have to be discretized, and the precision of the discretization should be sufficient to ensure the required accuracy of the solution. This results in additional increase of the memory requirements.

In the last years, several random matrix models requiring a smaller memory compared to the Gaussian matrix have been introduced. A natural strategy for reducing the complexity of a random matrix is to sparsify it by replacing most of its entries with zeros. In [2], Shah, Srivastava, and Zeng analyzed relaxation with a sparse Gaussian matrix R having $O(n \log^2 n)$ non-zero entries in average. Although it made the necessary storage be relatively close to the optimal size $O(n)$, this came at the cost of a super-polynomial dependence in the condition number $\kappa(A + \varepsilon R) = n^{O(\log n/\log\log n)}$. They also observed that $n \log n$ non-zero entries is the natural limit for this approach since below that limit the resulting perturbation becomes singular due to coupon collector problem.

We construct an $n \times n$ random matrix R with $\|R\| \leq 1$ which requires only $O(n)$ independent random variables ($O(n \log n)$ random bits) and such that the condition number of $A + \varepsilon R$ is of order $O(n)$ with high probability. Our analysis is centered on lower bounding the smallest singular value of the perturbed matrix, $s_n(M)$ for $M = A + \varepsilon R$, which is the infimum of $\|Mx\|_2$ over all x in the unit sphere. To this end, we introduce two independent random matrices R_1 and R_2

and set $R = R_1 + R_2$. These matrices allow to obtain the lower bounds for $\|Mx\|_2$ over x in different parts of the unit sphere. More precisely, while bounding the infimum over the vectors x close to sparse, we condition on R_2 and view M as a perturbation of the signal matrix $A + \varepsilon R_2$ be the random matrix εR_1. To consider the infimum of $\|Mx\|_2$ over the rest of the sphere, we switch the roles of R_1 and R_2 considering the former as a part of the signal, and the latter as the noise.

The matrix R_1 is constructed in the form $R_1 = D_1 V D_2$, where D_1, D_2 are diagonal matrices with independent ± 1 random variables on the diagonal, and V is a dense matrix having certain quasi-randomness properties. The matrices D_1 and D_2 require generating and storing $2n$ independent ± 1 random variables. The matrix V can be deterministic, and thus does not need random variables for its generation. This, however, will require keeping all the entries V in memory. Instead we produce a random construction of V requiring only $O(n)$ independent random variables ($O(n \log n)$ random bits) for generation and storage. The perturbation R_1 is used for handling $\|Mx\|_2$ for compressible , i.e., close to sparse vectors.

In order to handle the incompressible vectors, we use the matrix R_2. To construct it, we return to the sparse perturbations, and show that $O(n)$ non-zero entries are sufficient in this case. A uniformly random choice of $O(1)$ entries per column would require generating and storing $O(n)$ independent random variables, which would be optimal. This approach, however, hits an obstacle: it leads to a few *heavy rows*, i.e., those with disproportionately more non-zeros than the rest, which blows up the norm of the perturbation. To address this, we replace entire heavy rows with rows of zeros. This produces the necessary bound for the norm of the matrix R_2, but introduces a non-trivial dependency between the entries requiring a new argument to handle it.

The random matrix $R = R_1 + R_2$ constructed this way requires generation of $O(n)$ independent random variables ($O(n \log n)$ random bits) and no discretization. The main outcome of our analysis is the following theorem showing that the matrix R matches the performance of the Gaussian matrix in terms of the dependence of the condition number on n.

Theorem. *For any $\delta > 0$, the random matrix R whose construction is described above has the following property. Let $\varepsilon > 0$, and let A be any matrix such that $\|A\| \le 1$. Then*

$$\kappa(A + \varepsilon R) \le O\left(\frac{n}{\varepsilon^3 \delta^2}\right)$$

with probability at least $1 - \delta$.

We also show that neither of the constructions similar to R_1 and R_2 alone can achieve the same performance.

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On the Spielman–Teng conjecture

Julian Sahasrabudhe
(joint work with Ashwin Sah, Mehtaab Sawhney)

In this talk I discussed an asymptotic version of the Spielman–Teng conjecture. In [1] Sah, Sawhney and myself proved that if M is an $n \times n$ matrix with iid subgaussian entries with mean zero and variance one then

$$\mathbb{P}\big(\sigma_n(M) \leqslant \varepsilon n^{-1/2}\big) = \big(1 + o(1)\big)\varepsilon + e^{-\Omega(n)}$$

for all $e^{-\Omega(n)} \leqslant \varepsilon \ll 1$, where the implicit constants depend only on the underlying distribution of the entries. Here $\sigma_n(M)$ denotes the least singular value of the $n \times n$ matrix M.

To situate this result, we recall some of the history of this problem, which goes back to the seminal work of von Neumann (see [2, pg. 14, 477, 555]) on approximate solutions to linear systems of equations. He suggested that if B is an $n \times n$ random matrix with iid uniform entries in $\{-1, 1\}$, one typically has

$$\sigma_n(B) \approx n^{-1/2}. \tag{1}$$

This was later formally conjectured by Smale [3] and then proved by Szarek [4] and Edelman [5] when B is replaced with the $n \times n$ matrix G with iid *standard Gaussian* entries. In this case, Edelman found an *exact* expression for the density of the least singular value, which implies that

$$\mathbb{P}(\sigma_n(G) \leqslant \varepsilon n^{-1/2}) \leqslant \varepsilon, \tag{2}$$

for all $\varepsilon \geqslant 0$ (see e.g. [6]). While this provides us with an essentially complete understanding of the *Gaussian* case, one encounters serious difficulties when trying to extend (2) to other distributions. Indeed, Edelman's proof relies fundamentally on an exact description of the joint distribution of the singular values that is particular to the Gaussian setting.

In the last 20 years, intense study of the least singular value of iid random matrices has been undertaken with the overall goal of proving an appropriate version of (2) for different entry distributions and models of random matrices. This particular project was formalized by Spielman–Teng who put forward their conjecture that,

$$\mathbb{P}(\sigma_n(B) \leqslant \varepsilon n^{-1/2}) \leqslant \varepsilon + e^{-cn}, \tag{3}$$

for all $\varepsilon \geqslant 0$ and for some absolute $c > 0$. Here the "e^{-cn}" term reflects a challenging feature of *discrete* entry distributions, which is that the matrix can be singular with non-zero probability. In fact, the problem of determining the singularity probability (i.e. the case $\varepsilon = 0$) has been, on its own, the subject of intense activity, going back to the pioneering work of Komlós in the 1960s, which was developed intensely in the years since [7].

For general ε, a key breakthrough was made by Rudelson [8] and independently by Tao and Vu [9] which gave bounds on the least singular value beyond the Gaussian case.

The starting point of our work is the pair of beautiful papers of Rudelson and Vershynin [10] and Tao and Vu [11]. Rudelson and Vershynin [10] proved the conjecture of Spielman–Teng is true up to a constant factor $C > 0$ by proving that

$$\mathbb{P}(\sigma_n(B) \leqslant \varepsilon n^{-1/2}) \leqslant C\varepsilon + e^{-cn}. \tag{4}$$

Tao and Vu [11] proved the Spielman–Teng conjecture holds for all $\varepsilon \geqslant n^{-c}$, for some $c > 0$.

The methods of these two papers are quite different and we will draw upon ideas from both. One interesting feature of [11] is that they don't "directly" prove the Spielman-Teng Conjecture but rather *compare* B to the corresponding event for a Gaussian matrix G. Specifically, they prove

$$\left|\mathbb{P}(\sigma_n(B) \leqslant \varepsilon n^{-1/2}) - \mathbb{P}(\sigma_n(G) \leqslant \varepsilon n^{-1/2})\right| = O(n^{-c_0}), \tag{5}$$

for some $c_0 > 0$ and then draw on the work of Edelman, mentioned above, to derive the Spielman–Teng conjecture in this range. This sort of comparison is sometimes called a "universality" phenomenon in random matrix theory.

In [1] we proved the Spielman–Teng conjecture, up to a $1 + o(1)$ factor, by showing that this universality phenomenon holds all the way down to exponentially small scales in probability.

Theorem 1. *Let M be an $n \times n$ random matrix with iid subgaussian entries ξ of mean 0 and variance 1 and let G be an $n \times n$ matrix with iid standard normal entries. Then, for all $\varepsilon \geqslant 0$,*

$$\mathbb{P}\big(\sigma_n(M) \leqslant \varepsilon n^{-1/2}\big) = \big(1 + O(\delta_n)\big)\mathbb{P}\big(\sigma_n(G) \leqslant \varepsilon n^{-1/2}\big) + e^{-\Omega_\xi(n)}, \tag{6}$$

where $\delta_n = (\log n)^{-\Omega(1)}$ and is independent of ε.

This theorem then implies the Spielman–Teng conjecture, up to a $1+o(1)$ factor, for all $\varepsilon \geqslant 0$. In fact, we prove that it holds for the wider class of random matrices M with iid *subgaussian* entries.

Corollary 2. *Let M be an $n \times n$ random matrix with iid subgaussian entries distributed as ξ, with mean 0 and variance 1. Then, for all $\varepsilon \geqslant 0$,*

$$\mathbb{P}\big(\sigma_n(M) \leqslant \varepsilon n^{-1/2}\big) \leqslant \big(1 + \delta_n\big)\varepsilon + e^{-\Omega_\xi(n)},$$

where $\delta_n = (\log n)^{-\Omega(1)}$ is independent of ε.

While many of the above results hold for this wider class ([10, 11] for example), we draw attention to this more general form here, as we believe that our theorem is best possible in the sense that the $1+o(1)$ factor *cannot* be removed for general subgaussian ξ. Indeed, for *complex-valued* ξ, examples were given by Edelman, Guionnet and Péché [12], suggesting that the $1+o(1)$ factor *is* indeed necessary. In other words, our suspicion is that the specific bound (3) conjectured by Spielman–Teng, if true, is *not* a "universal phenomenon" but rather a special property of

matrices with iid entries which are uniform in $\{-1, 1\}$. However, we have not been able to confirm this suspicion.

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Characteristic polynomials of non-Hermitian random band matrices

Tatyana Shcherbina

(joint work with Mariya Shcherbina)

We consider non-Hermitian Gaussian random band matrices (RBM), i.e $N \times N$ matrices H_N whose entries H_{ij} are independent Gaussian random complex variables with mean zero and variance determined by

$$\mathbf{E}\left\{H_{jk}\bar{H}_{jk}\right\} = J_{jk} \tag{1}$$

with J_{jk} taken to be small when $|j-k| \gg W$.

We consider $J = \left(-W^2\Delta + 1\right)^{-1}$ with Δ being the discrete Laplacian on $[1, N] \cap \mathbb{Z}$ with Neumann boundary conditions. Notice that $J_{jk} \approx C_1 W^{-1} \exp\{-C_2|j-k|/W\}$, so it is exponentially small when $|j-k| \gg W$, as $W \to \infty$. Thus, matrices H_N indeed can be considered as a special case of non-Hermitian random band matrices with the bandwidth W.

The Hermitian analog of matrices (1) plays an important role in mathematical physics. Having nonzero entries only in the strip of width W around the main diagonal, Hermitian RBM provide a natural model to study eigenvalue statistics and quantum transport in disordered systems as they interpolate between classical

Wigner matrices, i.e., Hermitian random matrices with iid elements, and random Schrödinger operators, where the randomness only appears in the diagonal potential. In particular, Hermitian RBM can be used as a prototype of the celebrated Anderson metal-insulator phase transition even in dimension one: for $W \gg \sqrt{N}$ the eigenvectors are delocalized and the eigenvalues have universal GUE local statistics, while the localized eigenvectors and Poisson statistics occur for $W \ll \sqrt{N}$. The recent mathematical results justifying this conjecture for the Hermitian RBM in the dimension one can be found in [7], [13], [3], [4] and references therein.

Despite the recent progress in studying universality of the local eigenvalue statistics for non-Hermitian matrices with iid entries, the eigenvalue statistics of non-Hermitian matrices with a non-trivial spatial structure is much less accessible. In particular, for the non-Hermitian RBM (1) even justification of the expected convergence of the empirical spectral distribution to the circular law, i.e., to the uniform distribution on a unit disk appearing as a limiting distribution for the non-Hermitian matrices with iid entries, is a highly non-trivial task. For RBM with the specific block tridiagonal structure the circular law is proved in [6] for $W \gg n^{\varepsilon}$, however, for the general band profile the best recent result [5] shows such convergence for $W \gg N^{1/2+c}$ only.

In this talk we discuss another spectral characteristic of the non-Hermitian RBM (1), namely, the correlation functions of characteristic polynomials defined as

$$\Theta_k(z_1, \dots, z_k) = \mathbf{E}\Big\{ \prod_{s=1}^{k} \det(H_N - z_s) \det(H_N - z_s)^* \Big\}. \tag{2}$$

More precisely, we are interested in the asymptotic behavior of Θ_2 for matrices (1), as $W, N \to \infty$, and

$$z_1 = z + \zeta/N^{1/2},\ z_2 = z - \zeta/N^{1/2}, \quad |z| < 1, \tag{3}$$

with ζ varying in a compact set in $\mathbb{C}$.

The interest in the characteristic polynomials of random matrices is stimulated by its connections to the number theory, quantum chaos, integrable systems, combinatorics, representation theory and others. In addition, although Θ_k is not a local object in terms of eigenvalue statistics, it is also expected to be universal in a certain sense. In particular, it was proved in [1] (see also [2] for the Gaussian (Ginibre) case) that for non-Hermitian random matrices H with iid complex entries with mean zero, variance one, and $2k$ finite moments for any $z_j = z + \zeta_j/\sqrt{N}$, $j = 1, .., k$ and $|z| < 1$ we get

$$\lim_{N\to\infty} N^{-\frac{k^2-k}{2}} \frac{\Theta_k(z_1, \dots, z_k)}{\prod_j \Theta^{1/2}(z_j, z_j)} = C_k \frac{\det(K(\zeta_i, \zeta_j))_{i,j}^k}{|\Delta(\zeta)|^2}. \tag{4}$$

Here

$$K(w_1, w_2) = e^{-|w_1|^2/2 - |w_2|^2/2 + w_1 \bar{w}_2}, \tag{5}$$

$\Delta(\zeta)$ is a Vandermonde determinant of $\zeta_1, \dots, \zeta_k$, and C_k is a constant depending only on the fourth cumulant $\kappa_4 = \mathbf{E}[|H_{11}|^4] - 2$ of the elements distribution, but not

on the higher moments. In particular, this means that the local limiting behavior (4) for non-Hermitian matrices with iid entries coincides with that for the Ginibre ensemble as soon as the first four moments of elements distribution are Gaussian, i.e., the local behavior of the correlation functions of characteristic polynomials also exhibits a certain form of universality. Similar results were obtained for many classical Hermitian random matrix ensembles.

Notice that for the Hermitian analog of RBM the local behaviour of the correlation function of characteristic polynomials exhibits the crossover at $W \sim \sqrt{N}$ similar to the crossover in the local eigenvalue statistics: it coincides with that for GUE/GOE ensemble for $W \gg \sqrt{N}$, and factorized (which means that the limit in the r.h.s. of (4) is equal to 1) as $1 \ll W \ll \sqrt{N}$ (see [11], [8], [12]).

In this talk we present a similar result for non-Hermitian RBM (1):

Theorem 1. *For the second correlation function of characteristic polynomials (2) of 1d non-Hermitian Gaussian RBM (1) we have*

$$\lim_{N\to\infty} \frac{\Theta_2(z_1,z_2)}{\Theta_2^{1/2}(z_1,z_1)\Theta_2^{1/2}(z_2,z_2)} = \begin{cases} \frac{1-e^{-4|\zeta|^2}}{4|\zeta|^2}, & N^{1-\varepsilon} \gg W \gg N^{1/2}\cdot\log N; \\ (e^{\mathbb{A}_0}\mathbf{1},\mathbf{1}), & W = \kappa_* N^{1/2}(1+o(1)); \\ e^{-2|\zeta|^2}, & N^{\varepsilon} \ll W \ll \sqrt{\dfrac{N}{\log N}}, \end{cases}$$

where z_1, z_2 are defined in (3), the differential operator $\mathbb{A}_0$ has the form

$$\mathbb{A}_0\varphi(z) = \frac{1}{8(\kappa_* u_*)^2}\Big((1-z^2)\frac{d^2}{dz^2} - 2z\frac{d}{dz}\Big)\varphi(z) + 2|\zeta|^2 z\varphi(z), \quad z\in[-1,1],$$

with a boundary conditions $|\varphi(1)| < \infty$, $|\varphi(-1)| < \infty$, $\mathbf{1}(z) \equiv 1$, and $u_* = \sqrt{1-|z|^2}$.

The theorem is the first important steps towards the proof of bulk universality and Anderson's type transition for the non-Hermitian RBM.

The result is established in two papers [9], [10]. The method we use is based on the SUSY transfer matrix approach developed in [8] for the Hermitian case.

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Diffusion of a Tagged Toda Fluid Particle

Herbert Spohn
(joint work with S. Chahal, A. Brollo, I. Mukherjee, A. Dhar, A. Kundu)

The Toda fluid is an integrable many-particle system which is governed by the equations of motion

$$\tfrac{d}{dt}q_j = p_j, \quad \tfrac{d}{dt}p_j = e^{-(q_j-q_{j-1})} - e^{-(q_{j+1}-q_j)}, \quad j \in \mathbb{Z}, \tag{1}$$

positions $q_j \in \mathbb{R}$, momenta $p_j \in \mathbb{R}$. Introducing the Flaschka variables $a_j = \exp((q_j - q_{j+1})/2)$, integrability is based on the Lax matrix, L, which is a hermitian Jacobi matrix with $L_{j,j} = p_j$, $L_{j,j+1} = L_{j+1,j} = a_j$, and $L_{i,j} = 0$ otherwise. Under the Toda flow

$$\frac{d}{dt}L(t) = [B(t), L(t)], \tag{2}$$

where $B_{j,j} = 0$, $-B_{j,j+1} = B_{j+1,j} = \frac{1}{2}a_j$. Since $B(t)$ generates a unitary group, the eigenvalues, λ_α, of L are conserved. In particularly, the density of the locally conserved fields are given by $(L^n)_{j,j}$, $n = 1, 2, \ldots$. Furthermore, $\log(a_j)^2$ is the density of the stretch field, which is also conserved. More details are available in [1], covering also other many-body integrable systems with random initial conditions.

In the physics literature, the Toda fluid on a coarsened scale is viewed as a gas of quasiparticles, having a sequence of collisions realized in an one-dimensional gas of hard rods. This means a motion linear in time interrupted by jumps either to the right or left. Hard rods have a constant scattering shift, whereas for the Toda fluid the scattering shift is momentum-dependent and given by $\log|p_1 - p_2|^2$. A mathematical definition of quasiparticles was discovered only very recently by A. Aggarwal [2]. He argued that quasiparticles are labelled by the eigenvalues of L and move according to the associated eigenvectors of the Lax matrix. More precisely, if $L(t)\psi_\alpha(t) = \lambda_\alpha\psi_\alpha(t)$, then the quasiparticle position, $Q_\alpha(t)$, is defined by

$$Q_\alpha(t) = \sum_{j\in\mathbb{Z}} q_j(t)|\psi_\alpha(j,t)|^2. \tag{3}$$

If the initial data are random according to thermal equilibrium, i.e. p_j i.i.d. Gauss and a_j i.i.d. χ distributed, then almost surely L has exponentially localized

eigenvectors. The quasiparticle position is slightly blurred, but moves according to the underlying fluid.

In [3] the effective velocity for a single tagged quasiparticle is proved and in [4] Brownian motion relative to this linear motion. In our contribution [5] these results are confirmed numerically with high precision. We also determine the velocity autocorrelation function of a quasiparticle, which turns out to have exponential decay. Studied is the motion of a tagged particle, which has mean zero, diffuses, and has a velocity autocorrelation function decaying as $|t|^{-3}$. Tagging a particle is numerically trivial, but tagging a quasiparticle can be accomplished only through a spectrally resolving simulation scheme.

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Random covers of hyperbolic surfaces

Ramon van Handel

(joint work with Michael Magee and Doron Puder)

Let A be the adjacency matrix of a d-regular finite graph, so that its largest eigenvalue is $\lambda_1(A) = d$. How small can $\lambda_2(A)$ be? This question is of significant practical interest, since it controls for example the maximal rate at which a random walk on a d-regular graph can mix. It was understood in the 1980s that random walks on d-regular graphs cannot mix arbitrarily quickly: a classical lemma of Alon–Boppana [1] states that there is a fundamental speed limit

$$\lambda_2(A) \geq \lambda_1(\mathbb{T}_d) - o(1)$$

for *any* sequence of d-regular graphs with diameter going to infinity. Here the infinite d-regular tree $\mathbb{T}_d$ arises as the universal cover of the finite graphs, and a calculation of Kesten [2] yields that $\lambda_1(\mathbb{T}_d) = 2\sqrt{d-1}$.

That this speed limit can in fact be attained for a sequence of graphs with diameter going to infinity is not at all obvious. The original examples of graphs with optimal spectral gap, due to Lubotzky–Phillips–Sarnak [3] and Margulis [4], were deterministic constructions whose analysis was based on some of the deepest advances in number theory of the 20th century. In contrast, Friedman [5] has shown that a *random* d-regular graph with N vertices already satisfies

$$\lambda_2(A) \leq \lambda_1(\mathbb{T}_d) + o(1)$$

with probability going to one as $N \to \infty$. Thus not only do d-regular graphs with optimal spectral gaps exist, but in fact all but a vanishing fraction of all d-regular

graphs have this property. Very recently, the work of Huang–McKenzie–Yau [6] has achieved an exact understanding of the error term in Friedman's theorem, providing essentially the last word on this subject.

Progress on analogous questions in the setting of manifolds has been much slower, despite that these questions long predate their graph counterparts. To describe these problems, let X be a closed hyperbolic n-manifold, and denote by $\lambda_i(X)$ the eigenvalues of its Laplacian. The analogue of the Alon–Boppana lemma in this setting, proved by Huber [7] and Cheng [8] in the 1970s, states that

$$\lambda_1(X) \leq \lambda_0(\mathbb{H}^n) + o(1)$$

for *any* sequence of closed hyperbolic n-manifolds with diameter going to infinity. Here the hyperbolic n-space $\mathbb{H}^n$ arises again as the universal cover of the manifolds, and an explicit calculation shows that $\lambda_0(\mathbb{H}^n) = (\frac{n-1}{2})^2$.

Can the speed limit for hyperbolic n-manifolds be attained? Until very recently, there were *no* known results on this subject. To be sure, number theory provides us with concrete candidates of sequences of finite volume hyperbolic n-manifolds (at least in dimensions $n = 2$ and 3) that are conjectured to have optimal spectral gaps, dating back to the seminal work of Selberg in the 1960s [9]. However, the proof that these manifolds do indeed have optimal spectral gaps would require the resolution of some of the most challenging open problems in contemporary number theory, which appear to be far outside the bounds of current technology.

As in the case of graphs, we can also consider random constructions. In this setting, remarkable progress has been made in the past few years *in the case of surfaces* (that is, in dimension $n = 2$). A major breakthrough in this area, due to Hide and Magee in 2023 [10], constructed the first examples of closed hyperbolic surfaces with optimal spectral gaps by working with a random model of *non-compact* hyperbolic surfaces and then applying a compactification procedure. This work left open, however, the question whether *typical* closed hyperbolic surfaces have an optimal spectral gap, in analogy with Friedman's theorem.

In 2025, this question was answered affirmatively in several natural models of random surfaces: for the Weil-Petersson model by Anantharaman–Monk [11] and Hide–Macera–Thomas [12], for the random cover model by Magee–Puder–Van Handel [13], and for the Brooks–Makover model by Shen–Wu [14]. In each of these models (which are roughly analogous to the uniform, permutation/random lift, and configuration models of random regular graphs), the result is that

$$\lambda_1(X) \geq \lambda_0(\mathbb{H}^2) - o(1)$$

with probability going to one as the diameter tends to infinity.

With the exception of the work of Anantharaman–Monk [11] which was based on the original approach used by Friedman for random graphs, all the above works are based on a new advance in random matrix theory: the *polynomial method* of Chen–Garza-Vargas-Tropp–Van Handel [15] and its strengthened form due to Magee–Puder–Van Handel [13]. In the case of random cover models, the question actually reduces (using the method developed by Hide–Magee [10]) precisely to a strong converge problem for a family (S_a, S_b, S_c, S_d) of random permutation

matrices that are conditioned to satisfy the identity

$$S_a S_b S_a^* S_b^* S_c S_d S_c^* S_d^* = \mathbf{1}.$$

The complicated dependence structure between these matrices captures the fact that the limiting behavior of random surfaces is not governed by free probability, as is the case in almost all of classical random matrix theory, but is rather governed by the (non-free) fundamental group of the underlying surface. Consquently, the spectral statistics that arise in the setting of random hyperbolic surfaces do not satisfy any self-consistent equations, which is a major obstacle to their analysis using classical techniques. One of the interesting features of these questions for random matrix theorists is that they require the development of random matrix methods that do not rely, explicitly or implicitly, on freeness.

For closed hyperbolic manifolds of dimensions $n \geq 3$, the analogous questions remain wide open. Another major challenge is to achieve a precise understanding of the fluctuations of the smallest nontrivial eigenvalue of a random hyperbolic surface, as was done in the setting of graphs by Huang–McKenzie–Yau [6].

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Delocalization of non-mean-field random operators with Gaussian entries

Jun Yin

(joint work with Horng-Tzer Yau, Fan Yang, Steven Khang Truong, Kevin Yang, Sofiia Dubova, and Jiaqi Fan)

The goal of this talk was to describe recent progress on the localization and delocalization problem for random operators which are not of mean-field type, with particular emphasis on Gaussian block models. The guiding example is Anderson's random Schrödinger operator

$$H_\lambda = -\Delta + \lambda V$$

on a lattice: the Laplacian produces transport, while the random potential tends to localize eigenvectors. Anderson's conjecture predicts a transition between localized and delocalized phases in dimension at least three. Related questions arise for random band matrices and Wegner orbital models, which interpolate between disordered lattice Hamiltonians and Wigner matrices.

One theme of the talk was that several familiar models can be viewed inside a common block framework. Let the underlying lattice be partitioned into boxes of internal dimension W^d. We consider operators of the form

$$H_{V,A} = \begin{pmatrix} V_1 & A_{12} & A_{13} & \cdots \\ A_{12}^* & V_2 & A_{23} & \cdots \\ A_{13}^* & A_{23}^* & V_3 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

where the diagonal blocks V_i are independent random matrices and the off-diagonal blocks A_{ij} describe interactions between boxes. The law of A_{ij} is assumed to depend on the distance $|i-j|$. When the A_{ij} are deterministic nearest-neighbor identity blocks one obtains a block Anderson-type model; when the A_{ij} are random blocks one recovers random band or Wegner orbital models. Thus the main distinction is not whether the hopping term is random or deterministic, but rather the strength and range of the coupling relative to the randomness inside each block.

This point of view suggests a single effective parameter for the bulk localization length. In a simplified nearest-neighbor situation one may write

$$\Gamma = \frac{\|A\|_{\mathrm{HS}}^2}{\|V\|_{\mathrm{HS}}^2 W^d}.$$

More generally, at an energy E one expects

$$\Gamma_E = \frac{\sum_j \|A_{1j}\|_{\mathrm{HS}}^2 j^{d_*}}{\|V\|_{\mathrm{HS}}^2 W^d} \frac{\rho_H(E)}{\rho_H(\mathrm{bulk})},$$

where d_* reflects the dimension and hopping range. The conjectural picture is

$$\ell_E \sim \begin{cases} W, & \Gamma_E \lesssim 1, \\ W\Gamma_E, & d = 1,\ \Gamma_E \gg 1, \\ \exp(c\Gamma_E), & d = 2,\ \Gamma_E \gg 1, \\ \infty, & d \geq 3,\ \Gamma_E \gg 1. \end{cases}$$

This formulation matches the one-dimensional random band matrix prediction $\ell \asymp W^2$ and is also consistent with long-range power-law band matrices. For instance, if

$$\mathbb{E}|H_{xy}|^2 \propto (|x-y|/W+1)^{-1-\alpha}, \qquad \alpha > -1,$$

then the predicted one-dimensional localization length is infinite for $-1 < \alpha < 1$, of order $W^{\alpha/(\alpha-1)}$ for $1 < \alpha < 2$, and of order W^2 for $\alpha > 2$.

I then discussed recent rigorous results. For the one-dimensional Gaussian random band matrix, Yau and Yin proved the conjectured delocalization threshold: if $W \geq L^{1/2+\mathfrak{c}}$, then all bulk eigenvectors are delocalized, in the sense that for every fixed small $\tau > 0$,

$$\max_{\alpha} \frac{\|u_\alpha\|_\infty}{\|u_\alpha\|_2} \leq L^{-1/2+\tau}$$

with overwhelming probability; the bulk local statistics are also universal. For power-law random band matrices, Fan, Yang and Yin obtained lower bounds matching the delocalized side of the above prediction. In dimensions $d \geq 2$, work with Dubova, Kevin Yang, Fan Yang and Yau proves super-polynomial lower bounds on localization lengths for Gaussian random band matrices and related non-mean-field models.

For random block Schrödinger operators, the same philosophy leads to sharp coupling thresholds. Consider $H = V + \lambda\Psi$, where V is block diagonal with independent $W^d \times W^d$ Gaussian blocks and Ψ describes hopping between neighboring blocks. In dimensions one and two, Truong, Yang and Yin prove lower bounds on the bulk localization length under $\lambda \gg W^{-d/2}$; in dimension two this matches the known localization scale. In higher dimensions, recent work with Dubova, Yang and Yau extends delocalization to $d \geq 3$.

I also explained the new loop hierarchy method. The starting point is a Brownian interpolation of the matrix. Write, schematically,

$$(H_t)_{xy} = S_{xy}^{1/2} B_{xy}(t), \qquad G_t = (H_t - z_t)^{-1},$$

where $S_{xy} = \mathbb{E}|H_{xy}|^2$ and the spectral parameter z_t is chosen so that the flow ends at the desired resolvent. If $x_i \sim a_i$ denotes a local average over a block near a_i, define loop observables by

$$L^{(k)}_{a_1\cdots a_k}(t) := \Big\langle G_t(x_1,x_2)G_t(x_2,x_3)\cdots G_t(x_k,x_1)\Big\rangle_{x_i\sim a_i}.$$

The case $k = 2$ contains the two-point quantity which controls the spatial spread of eigenvectors. Applying Itô's formula and using

$$\frac{\partial G_{xy}}{\partial H_{st}} = -G_{xs}G_{ty}$$

gives a system of differential identities for the loops. This is the loop hierarchy: the time derivative of a loop is expressed in terms of other loop observables, together with martingale and error terms.

The main difficulty is that the hierarchy is not closed in a naive sense. Differentiating a loop may create longer loops, so a direct induction on the length fails. The new idea is to introduce a primitive hierarchy. Its unknowns $K^{(k)}$ solve the same leading transport equations as the true loops $L^{(k)}$, but only the tree-growing part of the right-hand side is retained. In this primitive system no equation depends on a loop longer than the one on the left-hand side, and the solution admits an explicit tree representation. One then writes

$$L^{(k)} = K^{(k)} + \left(L^{(k)} - K^{(k)}\right)$$

and proves dynamically that the purple correction $L^{(k)} - K^{(k)}$ is smaller than the primitive tree term. This reduction turns the apparently unstable loop hierarchy into a usable expansion for the two-point function. It is the key mechanism behind the optimal one-dimensional band matrix result and the subsequent higher dimensional extensions.

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Reporter: Ella Hiesmayr

Participants

Prof. Dr. Gernot Akemann
Faculty of Physics
Bielefeld University
Postfach 100131
33501 Bielefeld
GERMANY

Prof. Dr. Johannes Alt
Institut für Angewandte Mathematik
Universität Bonn
Endenicher Allee 60
53115 Bonn
GERMANY

Prof. Dr. Fanny Augeri
Université de Paris
Institut de Mathématiques
8 place Aurélie Nemours
75013 Paris Cedex 13
FRANCE

Prof. Dr. Afonso S. Bandeira
Departement Mathematik
ETH-Zentrum
Rämistr. 101
8092 Zürich
SWITZERLAND

Prof. Dr. Zhigang Bao
Department of Mathematics
The University of Hong Kong
Pokfulam Road
Hong Kong
CHINA

Prof. Dr. Gerard Ben Arous
Courant Institute of Mathematical
Sciences
New York University
251, Mercer Street
New York, NY 10012-1185
UNITED STATES

Dr. Lucas Benigni
Dept. of Mathematics and Statistics
University of Montreal
CP 6128, succ. Centre Ville
Montréal QC H3C 3J7
CANADA

Prof. Dr. Paul Bourgade
Courant Institute of
Mathematical Sciences
New York University
251, Mercer Street
New York, NY 10012
UNITED STATES

Thomas Buc—d'Alché
Dept. de Mathématiques, IRMA
Université de Strasbourg
7 rue René-Descartes
67084 Strasbourg Cedex
FRANCE

Dr. Mireille Capitaine
Institut de Mathématiques de Toulouse,
CNRS
Université de Toulouse
118 route de Narbonne
31062 Toulouse Cedex 9
FRANCE

Prof. Dr. Djalil Chafaï
Département de Mathématiques
Université Paris-Dauphine
Place du Maréchal de Lattre de Tassigny
75775 Paris Cedex 16
FRANCE

Prof. Dr. Giorgio Cipolloni
Department of Mathematics
University of Roma Tor Vergata
Roma 00133
ITALY

Prof. Dr. Benoît Collins
Department of Mathematics
Graduate School of Science
Kyoto University
Kitashirakawa, Oiwake-cho, Sakyo-ku
Kyoto 606-8502
JAPAN

Prof. Dr. Nicholas A. Cook
Department of Mathematics
Duke University
P.O. Box 90320
Durham 27708-0320
UNITED STATES

Stéphane Dartois
University of Bordeaux
351 cours de la liberation
33400 Talence Cedex
FRANCE

Sofiia Dubova
Department of Mathematics
Lunt Hall
Northwestern University
2033 Sheridan Road
Evanston, IL 60208-2730
UNITED STATES

Dr. Raphaël Ducatez
Université Claude Bernard Lyon 1
21 Av. Claude Bernard,
69100 Villeurbanne
FRANCE

Prof. Dr. László Erdős
Institute of Science and
Technology Austria (IST Austria)
Am Campus 1
3400 Klosterneuburg
AUSTRIA

Prof. Dr. Friedrich Götze
Fakultät für Mathematik
Universität Bielefeld
Postfach 100131
33501 Bielefeld
GERMANY

Tamara Grava
Woodland Rd, Bristol BS8 1UG
School of Mathematics
University of Bristol
Bristol BS8 1TW
UNITED KINGDOM

Prof. Dr. Alice Guionnet
Département de Mathématiques, UMPA
École Normale Supérieure de Lyon
46, allée d'Italie
69364 Lyon Cedex 07
FRANCE

Prof. Dr. Razvan-Gheorghe Gurau
Institut für Theoretische Physik
Universität Heidelberg
Philosophenweg 19
69120 Heidelberg
GERMANY

JProf. Dr. Anna Gusakova
Institut für Mathematische Stochastik
Universität Münster
Orléans-Ring 10
48149 Münster
GERMANY

Joscha Henheik
Section de Mathématiques
Université de Genève
Rue Du Conseil Général 7-9
1205 Genève
SWITZERLAND

Dr. Ella Hiesmayr
Département de Mathématiques, UMPA
École Normale Supérieure de Lyon
46, allée d'Italie
69364 Lyon Cedex 07
FRANCE

Prof. Dr. Jiaoyang Huang
Department of Statistics
The Wharton School
University of Pennsylvania
3730 Walnut Street
Philadelphia, PA 19104-6340
UNITED STATES

Prof. Dr. Jonathan Husson
Laboratoire de Mathématiques Blaise Pascal
UMR 6620 du CNRS
Université Clermont-Auvergne
63177 Aubière Cedex
FRANCE

Jonas Jalowy
Institut für Mathematik
Universität Paderborn
Warburger Str. 100
33098 Paderborn
GERMANY

Prof. Dr. Jonathan Keating
Mathematical Institute
Oxford University
Woodstock Road
Oxford OX2 6GG
UNITED KINGDOM

Prof. Dr. Antti Knowles
Section de Mathématiques
Université de Genève
CP 64
Rue du Conseil-Général 7-9
1205 Genève 4
SWITZERLAND

Oleksii Kolupaiev
Institute of Science and Technology Austria
Am Campus 1
3400 Klosterneuburg
AUSTRIA

Prof. Dr. Holger Kösters
Institut für Mathematik
Universität Rostock
Ulmenstraße 69 (Haus 3)
18057 Rostock
GERMANY

Prof. Dr. Torben Krüger
Lehrstuhl Mathematische Stochastik
Department Mathematik
Universität Erlangen-Nürnberg
Cauerstr. 11
91058 Erlangen
GERMANY

Gaultier Lambert
Department of Mathematics
KTH Royal Institute of Technology
Lindstedtsvägen 25
100 44 Stockholm
SWEDEN

Dr. Theo McKenzie
Department of Mathematics
Stanford University
Stanford, CA 94305-2125
UNITED STATES

Mohammed Osman
School of Mathematical Sciences
Queen Mary University of London
Mile End Road
London E1 4NS
UNITED KINGDOM

Elliot Paquette
Dept. of Mathematics and Statistics
McGill University
Montréal QC H3A 2K6
CANADA

Vanessa Piccolo
Information, Learning and Physics laboratory
Physics Department
EPFL Switzerland
Station 11
1015 Lausanne
SWITZERLAND

Jana Reker
Dept. de Mathématiques, U.M.P.A.
École Normale Superieure de Lyon
46, Allee d'Italie
69364 Lyon Cedex 07
FRANCE

Volodymyr Riabov
Institute of Science and Technology Austria (ISTA)
Am Campus 1
3400 Klosterneuburg
AUSTRIA

Prof. Dr. Mark Rudelson
Department of Mathematics
University of Michigan
530 Church Street
Ann Arbor, MI 48109-1043
UNITED STATES

Prof. Dr. Julian Sahasrabudhe
Dept. of Math. and Math. Stats. (DPMMS), University of Cambridge
Wilberforce Road
Cambridge CB3 0WA
UNITED KINGDOM

Prof. Dr. Mariya Shcherbina
Institute for Low Temperatures
47 Lenin's Ave.
310164 Kharkov
UKRAINE

Dr. Tatyana Shcherbina
Department of Mathematics
University of Wisconsin – Madison
480 Lincoln Drive
Madison WI 53706
UNITED STATES

Prof. Dr. Roland Speicher
Fachrichtung Mathematik
Universität des Saarlandes
Gebäude E2 4
Postfach 151150
66041 Saarbrücken
GERMANY

Prof. Dr. Herbert Spohn
Zentrum Mathematik
Technische Universität München
Boltzmannstraße 3
85748 Garching bei München
GERMANY

Prof. Dr. Ramon van Handel
Princeton University
Fine Hall, Room 207
Princeton, NJ 08544
UNITED STATES

Prof. Dr. Jun Yin
Department of Mathematics
University of California, Los Angeles
Los Angeles CA 90095-1555
UNITED STATES