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Topologie

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Abstract. The workshop covered the areas of algebraic and geometric topology, homotopy theory, geometric group theory, and manifold topology, as well as applications of topological methods to neighbouring areas of mathematics. The goal of the workshop was to survey recent trends in the field of Topology as a whole, to explain breakthrough results in an accessible way, and to strengthen ties between topologists who might otherwise not have an opportunity to interact.

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Introduction by the Organizers

Many exciting recent developments in topology have taken place at the interface with other areas such as algebraic geometry, number theory, differential geometry, mathematical physics, group theory, and representation theory. The workshop covered aspects of geometric topology, homotopy theory, manifold topology, as well as applications of topological methods in neighbouring areas. The major challenge and benefit of such a broad workshop is to strengthen the ties between topologists who might otherwise not have an opportunity to interact. To strike a balance, we therefore focused on topics of current interest that emphasize the interactive sides of topology and on topics that bring together different areas of research within topology.

In addition to individual research talks, the workshop featured a three-lecture series by Tomer Schlank on recent progress in chromatic homotopy theory and

an extensive gong show on Tuesday afternoon with short presentations by junior participants.

The lecture series by Tomer Schlank outlined the fundamental ideas and algebraic, arithmetic, and perfectoid analogies behind recent breakthroughs in the study of asymptotic phenomena in the stable homotopy groups of spheres. These techniques are an intriguing extension of the long-standing interaction between topology and arithmetic geometry.

In addition to Schlank's lecture series there were several further talks on homotopy theory. Gijs Heuts explained the notion of Euler characteristics of groups with respect to generalised cohomology theories, as a generalisation of orbifold Euler characteristic. Robert Burklund described a comically simple construction of $\mathbb{E}_1$-algebra structures on spectra such as $\mathbb{S}/2^4$. Ulrich Bunke described a six-functor formalism in the context of E-theory of C^*-algebras.

In algebraic topology more broadly, Diarmuid Crowley explained an application of stable homotopy theory to resolving the final ambiguities in the structure of the groups of homotopy spheres. Kelly Wang described an infinite family of homology classes on $\mathrm{GL}_n(\mathbb{F}_2)$, obtained using methods from higher algebra. Søren Galatius described a possible space-level definition of Kontsevich's graph complex. Kirsten Wickelgren explained several WDVV-type equations in the enriched enumerative geometry made possible by $\mathbb{A}^1$-homotopy theory. Dan Petersen explained the application of many methods from algebraic topology to resolve (the function field version of) a conjecture of Conrey–Farmer–Keating–Rubinstein–Snaith in arithmetic statistics.

In low-dimensional topology, İnanç Baykur explained recent progress on the question of whether every 4-manifold admits either no or infinitely-many smooth structures. Thomas Kragh explained progress towards defining a TQFT structure for the Bauer–Furita invariant. Kathryn Mann constructed Anosov foliations on certain 4-manifolds which disprove a conjecture of Verjovsky. Arunima Ray explained why the compactly supported mapping class group of the universal exotic $\mathbb{R}^4$ is trivial. JungHwan Park gave an overview of several results on the general questions of extending symmetries from the boundary of a 4-manifold to its interior.

In symplectic topology, Daniel Álvarez-Gavela and Noah Porcelli described complementary perspectives on the problem of controlling the normal invariant of a homotopy equivalence arising from a symplectomorphism between cotangent bundles.

The gong show gave junior participants the opportunity to showcase their research in brief talks. Presentations were held by Márton Beke, Max Blans, András Csépai, Anupam Datta, João Lobo Fernandes, Christian Kremer, and Jin Miayazawa. These talks reflected the breadth of topology, covering geometric topology, homotopy theory, and K-theory.

The workshop was attended by 47 participants (one online), representing a wide range of interests in topology and geographical backgrounds. Beyond the

rich mathematical interaction in smaller groups, the participants also enjoyed a self-organised soccer afternoon and a table-tennis tournament.

Acknowledgement: The organizers are grateful to Márton Beke for handling the video stream and to the reporter Christian Kremer.

Workshop: Topologie

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Abstracts

Three lectures on recent progress in chromatic homotopy theory

Tomer Schlank

Lecture 1. The stable homotopy groups of spheres form a fundamental but still mysterious sequence of finite abelian groups. Their individual computation is notoriously difficult, but chromatic homotopy theory provides a conceptual framework for organizing them into layers of increasing complexity. I will begin by introducing spectra, which may be regarded as higher analogues of abelian groups, and explain how they allow us to formulate stable homotopy theory in algebraic terms. I will then describe the basic ideas of chromatic and telescopic localization and explain how this structure helps us study asymptotic phenomena in the stable homotopy groups of spheres.

Lecture 2. I will continue the study of the telescopic layers of stable homotopy theory, focusing on chromatic ambidexterity and Galois theory. We shall also discuss their interaction with algebraic K-theory. I will explain how these ideas combine to distinguish telescopic localization from Morava K-theoretic localization, leading to the failure of Ravenel's Telescope Conjecture at all heights at least two. I will also describe how the same methods produce large families of elements in the stable homotopy groups of spheres and yield new asymptotic lower bounds for the sizes of the stable stems.

Lecture 3. The $K(n)$-local sphere is one of the basic building blocks of chromatic homotopy theory, but its homotopy groups remain extremely difficult to compute. I will discuss how modern perfectoid techniques from p-adic geometry can be used to study computational problems in the $K(n)$-local category. In particular, I will describe joint work with Barthel, Stapleton, and Weinstein in which we determine the rational homotopy groups of the $K(n)$-local sphere at every chromatic height and every prime.

Does the symplectic topology of a cotangent bundle remember the differential topology of the base?

Daniel Álvarez-Gavela

Let M be a smooth closed manifold. The cotangent bundle T^*M is equipped with a canonical differential 1-form called the *tautological* Liouville form $\lambda \in \Omega^1(T^*M)$. It is uniquely characterized by the tautological property that for any 1-form on M, say $\alpha \in \Omega^1(M)$, viewed as a map $M \xrightarrow{\alpha} T^*M$, there holds $\alpha^*\lambda = \alpha$. If $q_1, \ldots, q_n$ are local coordinates on M and $p_1, \ldots, p_n$ are the dual coordinates then $\lambda = \sum_i p_i dq_i$. Its differential $\omega = d\lambda = \sum_i dp_i \wedge dq_i$ is a symplectic form on T^*M, which equips the cotangent bundle of M with its standard symplectic structure. To what extend does the symplectic geometry of T^*M (with its standard structure) determine the smooth topology of M? More concretely, recall that for (X_i, ω_i)

$i = 1, 2$, symplectic manifolds a diffeomorphism $\varphi : X_1 \to X_2$ is called a *symplectomorphism* when $\varphi^*\omega_2 = \omega_1$.

Question A. Let M and N be closed smooth manifolds. Does the existence of a symplectomorphism between their cotangent bundles $T^*M \simeq T^*N$ imply the existence of a diffeomorphism $M \simeq N$?

The first major progress towards this question was obtained by Abouzaid in [A08] and strengthened by Ekholm, Kragh and Smith in [ES16, EKS16]. The results use J-holomorphic curves and give constraints in the case in which M is an odd dimensional homotopy sphere. Concretely, the conclusion is that if the cotangent bundles T^*M and T^*N of odd dimensional homotopy spheres M and N are symplectomorphic, then $[M] = \pm[N]$ in the group θ_n/bP_{n+1} [KM63].

The first constraint for general M was obtained in joint work with M. Abouzaid, S. Courte and T. Kragh [AACK25] using ideas related to Waldhausen's manifold approach to the algebraic K-theory of spaces.

Theorem A. If M and N are closed, simply connected manifolds and $\varphi : T^*M \to T^*N$ is a symplectomorphism, then the normal invariant of the associated homotopy equivalence $M \to N$ factors as a composition $M \to B(G/O) \to G/O$ where the map $B(G/O) \to G/O$ is 2-torsion.

Example 1. So if M and N are homotopy spheres, not necessarily odd dimensional, this theorem states that in particular $[M] - [N]$ is a 2-torsion element of θ_n/bP_{n+1}.

Example 2. The factoring through $B(G/O)$ has other applications beyond the fact that the map out of it is 2-torsion, for example it follows that if Σ is the unique exotic 8-dimensional sphere then $T^*\Sigma$ is *not* symplectomorphic to T^*S^8.

Remark 1. The same result was proved independently using Floer homotopy theory by Porcelli and Smith [PS24, PS25]. In fact they identify the map $B(G/O) \to G/O$ as the η-map with respect to the infinite loop structure of G/O coming from Whitney sum. However in [AACK25] a further factoring is established through a map $B(\mathcal{T}, \mathcal{Q}) \to B(G/O)$ where $\mathcal{T}$ is the Waldhausen space (monoid) of tubes and $\mathcal{Q}$ is the subspace (submonoid) of non-degenerate quadratic forms.

One may also consider variants of the motivating question in which one remembers additional structure. Indeed, the standard symplectic structure ω on T^*M has the distinguished λ, which enjoys the following special property: the vector field Z which is ω-dual to λ is gradient-like for an exhausting potential $h : T^*M \to \mathbf{R}$. For example we could take $h = \|p\|^2$ for some Riemannian metric on M. Exact symplectic manifolds with a chosen primitive for the symplectic structure which satisfy a precise formulation of the above property are called *Weinstein manifolds*. This is an important class of open symplectic manifolds which includes all smooth affine complex manifolds, i.e. Stein manifolds, and hence in particular includes all smooth affine complex varieties. The natural notion of equivalence in the Weinstein category is that of *Weinstein deformation equivalence* (roughly speaking this

allows for isomorphism as well as homotopy of λ). Therefore a natural counterpart to the motivating Question A is the following.

Question B. Let M and N be closed smooth manifolds. Does the existence of a Weinstein deformation equivalence between their cotangent bundles $T^*M \simeq T^*N$ imply the existence of a diffeomorphism $M \simeq N$?

A $2n$-dimensional Weinstein manifold X is the union of standard symplectic handles. In particular it has the homotopy type of a CW complex of dimension $\leq n$. In fact, X may be viewed as a symplectic thickening of a certain stratified subset $K \subset X$ called the *skeleton*, whose strata are isotropic (hence of dimension $\leq n$) submanifolds of X. For example, when $n = 1$ any open Riemann surface is a Weinstein manifold and its skeleton is a graph. If the graph is equipped with a cyclic orientation of the half-edges incident at each vertex, then the Weinstein structure may be recovered from the skeleton alone. The skeleton of a cotangent bundle T^*M with its standard Weinstein structure is the zero section M, and the smooth topology of M alone determines the Weinstein structure of T^*M.

A long-standing dream of the field is to understand an arbitrary Weinstein manifold as *the cotangent bundle of its skeleton*. One serious difficulty in implementing this vision is the highly singular nature of a generic skeleton of a Weinstein manifold of dimension > 4. Instead of trying to work with a generic skeleton, the arborealization program aims to deform the Weinstein structure so that the singularities of the skeleton become as simple as possible. Concretely, the goal is to obtain a skeleton whose singularities are *arboreal singularities*, a particularly simple class of Lagrangian singularities introduced by Nadler. Further, the arborealization program aims to not only establish existence of arboreal skeleta, but also to establish their uniqueness up to a suitable list of *Reidemeister moves*.

The existence theorem was proved in joint work with Eliashberg and Nadler [AGEN20].

Theorem B. If a Weinstein manifold X admits a polarization, then it admits an arboreal skeleton.

Further, the article [AGEN21] joint with Eliashberg and Nadler established that arboreal singularities have no moduli. In particular, there are finitely many in each dimension up to symplectomorphism. More precisely, we proved that a geometric definition of arboreal singularities (iterated transverse Liouville cones) admits the following remarkable rigidity:

Theorem C. For each combinatorial type, arboreal singularities are characterized up to contractible choice of symplectomorphism by a unique local model. The combinatorial type is classified by a finite rooted tree (with edges not adjacent to the root equipped with a ± 1 decoration).

From Theorem C it follows that the skeleta produced by Theorem A have singularities symplectomorphic to the normal forms in [AGEN21], i.e. that they are *arboreal spaces*. In fact, they are *positive arboreal spaces* as they are modeled on the normal forms for positive arboreal singularities.

Work in progress with Eliashberg and Nadler aims to complete the arborealization program to establish the following uniqueness result.

Theorem D. (in progress) The positive arboreal skeleton of a polarized Weinstein manifold is unique up to positive Reidemeister moves.

At this stage the study of the symplectic topology of polarized Weinstein manifolds up to polarized Weinstein homotopy will have been reduced to the study of the differential topology of positive arboreal spaces up to positive Reidemeister moves. There may therefore be a purely differential topological approach to Question B, which would go through the development of surgery theory in the category of positive arboreal spaces and would need to understand the effect of positive Reidemeister moves on the conjectural surgery invariants.

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Generalized Euler characteristics and duality for infinite groups

Gijs Heuts

(joint work with Irakli Patchkoria)

The aim of this talk was to discuss a family (indexed on the natural numbers) of generalizations of the notion of *orbifold Euler characteristic* of a group G [9]. The classical orbifold Euler characteristic, studied by Wall and Serre [1, 2], is defined as follows. If G admits a torsion-free subgroup $\Gamma \leq G$ of finite index, then one can define

$$\chi_{\mathrm{orb}}(G) := \frac{\chi^{\mathbb{Q}}(\underline{E}G/\Gamma)}{[G:\Gamma]},$$

where $\chi^{\mathbb{Q}}(\underline{E}G/\Gamma)$ denotes the usual (rational homology) Euler characteristic of the finite CW complex $\underline{E}G/\Gamma \simeq B\Gamma$. Alternatively, it can be computed by Quillen's

formula

$$\chi_{\mathrm{orb}}(G) = \sum_{G\sigma} (-1)^{d_\sigma} \frac{1}{|G_\sigma|},$$

where the sum is over the G-orbits of cells, d_σ is the dimension of such a cell σ, and G_σ denotes the stabilizer of σ in $X = \underline{E}G$. This formula is the origin of the term orbifold Euler characteristic: if X is a manifold, then $X//G$ is an orbifold and the formula above computes its Euler characteristic. Note that this is a rational number and is usually different from the classical Euler characteristic of X/G, which is always an integer. In fact, the number $\chi_{\mathrm{orb}}(G)$ can be defined through the above formula for an group G admitting a finite *classifying space for proper actions* $\underline{E}G$. In more detail, this $\underline{E}G$ should be a finite G-CW complex where each cell has finite stabilizer and for each finite subgroup $H \leq G$, the fixed points $\underline{E}G^H$ are contractible. Important examples include arithmetic groups and mapping class groups.

The generalizations we introduce are easily described by an analog of Quillen's formula. For $n \geq 0$, the *Morava E-theory orbifold Euler characteristic* of G satisfies the following:

$$\chi^{E_n}_{\mathrm{orb}}(G) := \sum_{G\sigma} (-1)^{d_\sigma} |BG_\sigma|_{E_n}.$$

Here $|BG_\sigma|_{E_n}$ denotes the *E_n-cardinality* of the space BG_σ, as defined by Ben-Moshe–Carmeli–Schlank–Yanovski [3, 4, 5, 6]. It admits the following elementary formula:

$$|BH|_{E_n} = \frac{|H_{n,p}|}{|H|}.$$

Here $H_{n,p}$ denotes the set of commuting n-tuples of elements in H of p-power order; in other words, it is the set of continuous homomorphisms $\mathbb{Z}_p^n \to H$.

The two main points of the talk can be summarized as follows:

(1) The numbers $\chi^{E_n}_{\mathrm{orb}}(G)$ arise naturally when studying the cohomology of the group G with coefficients not in rational cohomology, but the theories E_n (the *Morava E-theories*) that play a central role in chromatic homotopy theory.

(2) The numbers $\chi^{E_n}_{\mathrm{orb}}(G)$ can be explicitly computed in many interesting examples, such as the mapping class group Γ^1_g of a genus g surface with a marked point, and many arithmetic groups. Previous work of Patchkoria [7] shows how to derive several interesting number-theoretic congruences from the values of these numbers.

Still, our work leaves open a general calculation for arithmetic groups in the style of Harder's result [8] and, for example, the calculation of $\chi^{E_n}_{\mathrm{orb}}(G)$ for $G = \Gamma_g$, the unpointed mapping class group, at heights $n \geq 2$. It would be very interesting to determine these numbers.

There are two main technical ingredients in our work, both having to do with duality between homology and cohomology of (infinite) groups. The first is a generalization of 'Tate vanishing for finite groups', which was proved by Greenlees–Sadofsky for the $K(n)$-local category and Kuhn for the $T(n)$-local category. We

prove that if G is a group with finite $\underline{E}G$, then for any $T(n)$-local spectrum X with G-action (in the naive sense $X \in \mathrm{Fun}(BG, \mathrm{Sp}_{T(n)})$), the norm map

$$\mathrm{Nm}_{BG} \colon (X \otimes \mathbf{D}_{BG})_{hG} \to X^{hG},$$

is a $T(n)$-local equivalence. Our second innovation is to introduce a new duality functor on the ∞-category of proper G-spectra, for which we write $D \colon \mathrm{Sp}_G^{\mathrm{prop}} \to (\mathrm{Sp}_G^{\mathrm{prop}})^{\mathrm{op}}$. This duality is distinct from Spanier–Whitehead duality (except for the case where G is finite); it provides a notion of duality adapted to proper G-spectra, for which the orbit cells $\Sigma_+^\infty G/H$, with $H \leq G$ a finite subgroup, exhibit the appropriate duality behavior even when G is infinite. The role of this functor D in our work is that if we consider the unit $\mathbb{S}_G \in \mathrm{Sp}_G^{\mathrm{prop}}$, then $D(\mathbb{S}_G)$ is a genuine (proper) equivariant refinement of Klein's dualizing complex $\mathbf{D}_{BG}$. These two pieces of work allow us to understand $\chi_{\mathrm{orb}}^{E_n}$ through the category $\mathrm{Fun}(BG, \mathrm{Sp}_{K(n)})$, but should also be of independent interest in the study of the generalized (co)homology of infinite groups.

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Splitting the Kervaire-Milnor Sequence

Diarmuid Crowley

(joint work with R. Burklund, I. Levy)

Background: Let Θ_m be the abelian group of (oriented) h-cobordism classes of homotopy m-spheres. For simplicity we assume that $m \geq 5$. Kervaire and Milnor [11] placed Θ_m it into a short exact sequence

$$0 \to bP_{m+1} \to \Theta_m \to \mathrm{Ker}(K_m) \to 0, \tag{1}$$

where bP_{m+1} is the subgroup of those homotopy spheres bounding parallelisable manifolds, and K_m is a subquotient of the stable m-stem, π_m^s, defined as follows.

For $J_m\colon \pi_m(SO) \to \pi_m^s$ the J-homomorphism, $K_m\colon \mathrm{Coker}(J_m) \to \mathbb{Z}/2$ is the homomorphism

$$K_m := \begin{cases} \text{The Kervaire Invariant} & m = 4k{+}2, \\ 0 & \text{else.} \end{cases}$$

Kervaire and Milnor proved that bP_{m+1} is a cyclic group, and adding work of Mahowald [13], or more generally Quillen's solution of the Adam's Conjecture [16], the results of [11] give

$$bP_m \cong \begin{cases} \mathbb{Z}/a_k(2^{2k-2})(2^{2k-1}{-}1)\mathrm{Num}(B_k/4k) & m = 4k-1, \\ \mathrm{Coker}(K_{m+1}) & m = 4k+1, \\ 0 & m \text{ is even.} \end{cases}$$

Here $a_k = \frac{1-(-1)^k}{2}$ and B_k is the kth Bernoulli number (topologist's indexing) and $\mathrm{Num}(B_k/4k)$ is the numerator of $B_k/4k$, when expressed in reduced form.

The Kervaire Invariant $K_n\colon \pi_n^s \to \mathbb{Z}/2$ was shown to vanish by Browder [2] when $n \neq 2^{j+1}{-}2$, and by Hill, Hopkins and Ravenel [9] when $2^{j+1}{-}2 \geq 254$. In the other direction, Kervaire and Milnor already noted in their initial paper that K_2, K_6 and K_{14} are all non-zero, and then K_{30}, K_{62} and K_{126} were shown to be non-zero by, respectively, Mahowald and Tangora [14], Barratt, Jones and Mahowald [1], and recently Lin, Wang and Xu [12].

Away from the "magic dimensions" $m = 2^{j+1}{-}3$, the Kervaire-Milnor Sequence (1) was proven to be split by Brumfiel in a sequence of papers [4, 5, 6]. The case where $m = 4k{-}1$ was also independently solved by Frank [8]. The above results left the splitting of the Kervaire-Milnor sequence open only in the magic dimensions $m(j) = 2^{j+1} - 3 \geq 253$; i.e. $j \geq 7$.

Results: Our main result is the following

Theorem A. The Kervaire-Milnor sequence splits for $m = 253$ and $m = 509$.

Remark. We expect that the Kervaire-Milnor sequence splits in all dimensions, which relies on delicate and involved computations which are work in progress. We discuss the overall strategy briefly below.

The proof of Theorem A involves two main steps, summarised as Theorems B and C below. We first translate splitting the Kervaire-Milnor sequence into a problem in stable homotopy theory, analogous to Browder's translation of the Kervaire Invariant Problem [2]. Then we must then make progress on the homotopy theoretic problem.

We first state the homotopy theoretic translation, and then explain the concepts and notation used.

Theorem B. Let $m = 2^{j+1} - 3$ for $j \geq 7$ and supposed that $K_m = 0$, equivalently by the results above, that $j \geq 7$. The following are equivalent:

(1) The Kervaire-Milnor sequence for Θ_m splits;
(2) The twisted Kervaire invariant $K^{\mathrm{t}}_{m+1}\colon \Omega^{\mathrm{fr}}_{m+1}(S^1;\eta) \to \mathbb{Z}/2$ is zero;
(3) $h_j^2 e_0$ is not a permanent cycle in the Adam's Spectral Sequence for $\pi_*(\mathbb{S}/2)$.

Here $\Omega^{\mathrm{fr}}_{m+1}(S^1;\eta)$ is the $(m+1)$-dimensional "η-twisted framed bordism" group, which is the bordism group of closed smooth manifold M with normal maps to (S^1,η), where η denotes the non-trivial stable vector bundle over S^1; K^{t} is Brown's Generalisation of the Kervaire invariant [3], which is defined since the relevant Wu class, $v_{((m+1)/2)+1}$, is zero for obvious dimensional reasons; $\mathbb{S}/2 := \mathbb{S}^0 \cup_2 \mathbb{D}^1$, which is naturally homotopy equivalent to $\mathrm{Th}(\eta)$, the Thom spectrum of η; and for $e_0 \in H_0(\mathbb{S}/2;\mathbb{Z}/2)$ the generator, $h_j^2 e_0$ is the corresponding cycle on the E^2-page of the Adams spectral sequence for $\pi_*(\mathbb{S}/2)$.

Theorem A follows from Theorem B and the following

Theorem C. For $j = 7, 8$, $h_j^2 e_0$ is not a permanent cycle in the Adams Spectral Sequence for $\pi_*(\mathbb{S}/2)$.

Recalling that Hill, Hopkins and Ravenel proved that h_j^2 is not a permanent cycle in the Adam's Spectral Sequence for $\pi_*(\mathbb{S})$ [9], our strategy to prove Theorem C is to modify the proof outline of Hill, Hopkins and Ravenel from the sphere, to the sphere mod 2. Hill, Hopkins and Ravenel constructed a spectrum Ω and map $\iota\colon \mathbb{S} \to \Omega$ with three key properties:

(1) periodicity - $\Omega_* \cong \Omega_{*+256}$;
(2) the gap - $\pi_{-3}\Omega = \pi_{-2}\Omega = \pi_{-1}\Omega = 0$;
(3) detection - if $x \in \pi_{2^{j+1}-2}(\mathbb{S})$ represents h_j^2 then $\iota_*(x) \neq 0 \in \pi_{2^{j+1}-2}(\Omega)$.

Hill, Hopkins and Ravenel worked with equivariant stable homotopy for the group $\mathbb{Z}/8$, starting from the spectrum $MU \wedge MU \wedge MU \wedge MU$ using the conjugation action on MU and the cyclic permutation of the factors. Heuristically speaking, the Kervaire invariant corresponds to the class of order two in $H^2(\mathbb{Z}/8;\mathbb{Z}) \cong \mathbb{Z}/8$, which vanishes mod 2. Hence we first move to the semi-direct product $G := \mathbb{Z}/8 \tilde{\times} \mathbb{Z}$ where $\mathbb{Z}$ acts on $\mathbb{Z}/8$ by -1. We have $H^2(G;\mathbb{Z}) \cong \mathbb{Z}/2$ with generator mapping to $[4] \in H^2(\mathbb{Z}/8;\mathbb{Z})$. We then observe that G is a subgroup of the automorphism group of $\mathbb{E}_4$, Morava E-theory at chormatic height 4. Starting from the homotopy orbits $\mathbb{E}_4^{hG}$, we construct a spectrum $\overline{\Omega}$ and a map $\bar{\iota}\colon \mathbb{S}/2 \to \overline{\Omega}$ such that $\overline{\Omega}$ is 256-periodic and has gaps at π_{-2} and π_{-1}, which suffice. The final property of detection for $\overline{\Omega}$ requires non-trivial calculations, which we have made for $j = 7, 8$.

Additional Motivation: By smoothing theory [10], for $m \geq 5$ there is a canonical isomorphism

$$\Theta_m \to \pi_m(PL/O)$$

where PL/O is the homotopy fibre of the natural map $BO \to BPL$, which classifies the process forgetting the linear structure on a stable vector bundle, and considering it only as a stable piecewise linear bundle. The space PL/O plays an important role in geometric topology, and the odd primary part of Brumfiel's splitting results was recovered in the work of May et. al, in [15], which investigated the homotopy type classifying spaces for manifold theory, including the space PL/O. Theorem A above can be regarded as a basis datum for the investigation of the 2-local homotopy type of PL/O.

The space PL/O also arises in the study of spaces of diffeomorphisms, since $PL/O = \lim_{d\to\infty} PL_d/O_d$ and by Morlet's Theorem [7], there is a homotopy equivalence $\mathrm{Diff}_\partial(D^d) \to \Omega^{d+1} PL_d/O_d$, from the space of diffeomorphisms of the d-disc, which are the identity near the boundary, to the $(d+1)$-fold loop space of PL_d/O_d.

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A dichotomy in dimension four

R. İnanç Baykur

(joint work with András I. Stipsicz, Zoltán Szabó)

A closed topological manifold in any dimension other than four admits at most finitely many smooth structures. In dimension four the situation is dramatically different: many topological four-manifolds are known to carry infinitely many pairwise non-diffeomorphic smooth structures, while many others admit no smooth structure at all. No smoothable closed four-manifold is currently known to carry exactly finitely many smooth structures. This motivates the following question:

Question: *Does every closed oriented topological four-manifold admit either no smooth structure or infinitely many smooth structures?*

Our Oberwolfach talk reported on joint work with A. Stipsicz and Z. Szabó [5], giving further evidence for this dichotomy in the case of finite cyclic fundamental groups.

We use the following conventions. A smooth four-manifold X is an *exotic copy* of a smooth four-manifold Z if X and Z are homeomorphic but not (orientation-preserving) diffeomorphic. We say that a topological four-manifold satisfies the above dichotomy if it admits either no smooth structure or infinitely many smooth structures, up to diffeomorphism.

Recall that the intersection form Q_X of a closed oriented four-manifold X is called odd if $Q_X(\alpha, \alpha)$ is odd for some $\alpha \in H_2(X; \mathbb{Z})$; otherwise it is even. For the odd forms appearing below, the relevant smoothable homeomorphism types have intersection form

$$a\langle 1\rangle \oplus b\langle -1\rangle, \qquad a = b_2^+(X), \quad b = b_2^-(X).$$

Finally, a smooth four-manifold is called irreducible if in any smooth connected sum splitting $X = X_1 \# X_2$, one of the summands is a homotopy four-sphere.

Our main result is the following. Let X be a closed, oriented topological four-manifold with

$$\pi_1(X) \cong \mathbb{Z}_{4m+2}$$

for some $m \geq 0$, and suppose that X has odd intersection form. Unless

$$b_2^+(X) = b_2^-(X) \leq 7,$$

the manifold X admits either no smooth structure or infinitely many distinct smooth structures [5].

By the classification results of Freedman and Hambleton–Kreck [8, 10], this leaves only seven smoothable homeomorphism types for each fixed group $\mathbb{Z}_{4m+2}$ for which we currently do not know how to produce more than one smooth structure.

The novel part of the construction is largely in the cases where both $b_2^+(X)$ and $b_2^-(X)$ are even. In many earlier constructions with finite cyclic fundamental group, the examples were symplectic or almost complex, and the usual Seiberg–Witten invariants were effective on the manifolds in question. When b_2^+ is even (and $b_1 = 0$, as in our case), there is no almost complex structure on the four-manifold that facilitates the calculation of nontrivial Seiberg–Witten invariants. In these cases, we instead distinguish the smooth structures by looking at finite covers. That is, the examples are constructed with virtually distinct Seiberg–Witten invariants: the invariants of suitable covers distinguish the smooth structures on the quotients.

Our basic input is a collection of simply connected four-manifolds with non-trivial Seiberg–Witten invariants and free orientation-preserving involutions, including examples arising from [4, 15]. These are combined using equivariant versions of normal connected sum and circle sum operations, together with Luttinger and torus surgeries. The construction is arranged so that the free action survives the surgery and the quotient has the desired finite cyclic fundamental group and odd intersection form. Blow-ups and orientation reversal then allow us to fill out the required odd geography. The homeomorphism type is identified by the

Freedman–Hambleton–Kreck classification [8, 10], while non-diffeomorphism, as promised, follows from the Seiberg–Witten invariants of the covers, using product formulas along tori [12]. This is the point at which the even b_2^+ cases become accessible.

A second result we have concerns irreducible fake projective planes. Here a fake projective plane means a smooth four-manifold X with

$$H_*(X;\mathbb{Q}) \cong H_*(\mathbb{CP}^2;\mathbb{Q}), \qquad X \ncong \mathbb{CP}^2.$$

Complex fake projective planes, beginning with Mumford's example [13], are ball quotients by Yau's theorem [16], and the work of Prasad–Yeung and Cartwright–Steger gives exactly 50 such complex examples, up to isometry [14, 6]. This naturally yields the question, raised by Fintushel and Stern, whether there are irreducible fake projective planes beyond the complex ones [7].

Our equivariant constructions also produce infinite families of pairwise non-diffeomorphic irreducible fake projective planes with various fixed fundamental group [5]. These groups include

$$R \times \mathbb{Z}_2,$$

where R is the fundamental group of any rational homology three-sphere, and also

$$\mathbb{Z}_m \times D_m$$

for every odd m, where D_m denotes the dihedral group of order $2m$. Our fake projective planes admit neither complex nor symplectic structures.

A few related developments and open directions are worth mentioning. Recall that the above dichotomy question remains widely open for simply connected four-manifolds. In the finite cyclic case however, there is further progress: the recent work of Ladu and Tagliente extends our results to groups $\mathbb{Z}_{4k}$, for each $k > 0$ [11]. Related work of Arabadji and Morgan studies the geography of irreducible four-manifolds with fundamental group of order two, including both odd and even intersection forms [1, 2]. Further, Bais and Torres have produced irreducible fake real projective four-spaces [3].

A related symplectic question is also worth isolating. In the complex category, Gieseker's moduli theorem implies finiteness of diffeomorphism types for surfaces of general type with fixed Chern numbers [9]. In the symplectic category one can ask for an analogous dichotomy:

Question: For a prescribed pair of Chern numbers, are there either no minimal symplectic four-manifolds or infinitely many pairwise non-diffeomorphic ones?

Many geography regions support positive answers, but the borderline case $c_1^2 = 3c_2$, which includes fake projective planes, remains especially mysterious. The examples described above give non-complex, non-symplectic fake projective planes, but the existence of symplectic non-complex fake projective planes remains an intriguing question, which is the subject of our ongoing work.

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Stable homotopy refinement of Seiberg–Witten invariants

Thomas Kragh

(joint work with Stefan Behrens and Alice Hedenlund)

The Bauer–Furuta invariant, defined in [2], is often used to distinguish smooth 4-manifolds. That is, two smooth 4-manifolds can be homeomorphic, but can be detected as non-diffeomorphic if they do not have the same Bauer–Furuta invariant. This invariant is a stable homotopy refinement of the Seiberg–Witten invariant, also known as the monopole invariant. The Bauer–Furuta invariant has even been used by Jianfeng Lin to detect exotic diffeomorphisms. Indeed, he detected the first example of a diffeomorphism $X \to X$ which is topologically isotopic to the identity, but even after taking connect sum with the identity on $S^2 \times S^2$ it is still not smoothly isotopic to the identity. However, this crucially uses the stable homotopy refinement as the obstruction is not detected by the usual monopole counts.

There is a TQFT lift of the Seiberg–Witten invariants given by Seiberg–Witten Floer homology. Indeed, this associates to each closed, oriented spin^c 3-manifold

Y three chain complexes whose homology groups fit into a long exact sequence. A spin^c cobordism then provides an induced map of these chain complexes, which provides natural gluing formulas for these invariants. An example of the usefulness of this type of thinking was given by Morgan–Mrowka–Szabo in [1], where they derive gluing formulas for gluing two 4-manifolds with boundary T^3 together along their boundary.

A similar TQFT structure for the Bauer–Furuta invariants has only been partially defined. It should heuristically be some sort of stable S^1-equivariant object X, and the three types of Seiberg–Witten Floer theory should be recovered by the long exact sequence of homology groups associated with the norm cofiber sequence of spectra

$$X_{hS^1} \to X^{hS^1} \to X^{tS^1}$$

where these, respectively, are: the homotopy orbits, the homotopy fixed points, and the Tate spectrum.

In [3] Manolescu constructed a spectrum associated to any 3-dimensional rational homology sphere. In the general case, there is an obstruction to actually defining S^1-equivariant spectra. Indeed, this is essentially given by a map $P := H^1(Y;\mathbb{R})/H^1(Y;\mathbb{Z}) \to U = \operatorname{colim}_n U(n)$. In [5] Sasahira–Stoffregen gave a construction, only assuming that this map is null-homotopic. In [4] Khandhawit–Lin–Sasahira defined the so-called fully unfolded version, which involves passing to the universal cover of P, so that this obstruction vanishes. This, however, only defines part of the expected spectrum.

In this talk we sketch how to use the map $P \to U$ as a twisting map, to generally define a twisted S^1-equivariant parametrized spectrum over P.

I will also discuss the following conjectural target category for a TQFT lift of the Bauer–Furuta invariant. Its objects are triples (P, τ, X) where P is a torus, $\tau : P \to U$ is a map, and X is a parametrized S^1-equivariant spectrum twisted by the map τ. A morphism from (P_1, τ_1, X_1) to (P_2, τ_2, X_2) should be a span of tori

$$\begin{tikzcd} & P \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ P_1 & & P_2 \end{tikzcd}$$

together with the data (h, f), where $h : P \times I \to U$ is a homotopy from $\tau_1 \circ p_1$ to $\tau_2 \circ p_2$, and $f : p_1^* X_1 \to p_2^! X_2$ is a map of twisted parametrized S^1-equivariant spectra, which explicitly uses the given identification h of the two twisting maps.

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Anosov flows and codimension 1 foliations

Kathryn Mann
(joint work with Sergio Fenley, Rafael Potrie)

This talk describes a family of Anosov flows on 4-manifolds constructed recently in [FMP], which provide counterexamples to the following well-known conjecture in dynamical systems:

Conjecture (Verjovsky, [Ver])**.** *If a manifold M of dimension at least 4 supports a* codimension one Anosov flow*, then this flow is necessarily orbit equivalent to a suspension of a linear toral automorphism.*

The technical terms *codimension one Anosov flow* and *suspension* are explained below. We produce not one but many counterexamples, all in dimension 4:

Theorem. *There exist codimension one Anosov flows on compact 4-manifolds which are not orbit equivalent to suspension flows. These occur on infinitely many distinct manifolds, even up to taking finite covers. Additionally, there exist some compact 4-manifolds which support infinitely many codimension one Anosov flows, all distinct up to orbit equivalence.*

Below, I describe some context and motivation surrounding the study of Anosov flows and why these are of interest in topology, followed by a number of questions raised by these new examples.

Context. Anosov flows are fundamental examples in smooth dynamics and geometric topology. The motivation for the study of Anosov flows comes from the remarkable behavior of geodesic flow on the unit tangent bundle of a manifold of negative curvature. Negative curvature implies that geodesics themselves – orbits of this flow – have "chaotic" behavior in the sense that small changes in initial direction lead to exponentially divergent trajectories. However, the flow as a whole is not chaotic but stable. Specifically, any C^1-small perturbation gives a new flow which differs only by time-change and conjugation by a homeomorphism of the ambient space. This result is due to Anosov; the essential property used in the proof is that geodesic flows in negative curvature have invariant transverse directions of expansion and contraction, abstracted as follows:

Definition. *A flow ϕ^t on a closed Riemannian manifold M is* Anosov *if generated by a nonvanishing vector field X such that the tangent bundle has a $D\phi^t$-invariant splitting $TN = E^s \oplus \mathbb{R}X \oplus E^u$ with the property that there are constants $c, \lambda > 0$ such that*

$$\|D\phi^t v\| \leq ce^{-\lambda t}\|v\| \quad \forall t \geq 0, v \in E^s, \text{ and}$$
$$\|D\phi^t v\| \leq ce^{\lambda t}\|v\| \quad \forall t \leq 0, v \in E^u.$$

In [Sma] Smale posed the question to classify Anosov flows up to *time change and conjugacy* this is formally known as *orbit equivalence*. Since then there have been many constructions in dimension 3, obtained by surgery techniques These include Dehn surgery along a transverse annulus (Goodman), cut-and-paste along quasi-transverse tori (initiated by Handel and Thurston and since pushed to many generalizations), or modifications to produce transverse tori (the first examples due to Franks-Williams) followed by gluing construcitons. There has also been some recent successes in the classification program in dimension 3 using topological invariants, see for instance [BFM]. Anosov flows also provide interesting families of foliations. It is a general fact that the distributions $E^s \oplus \mathbb{R}X$ and $E^u \oplus \mathbb{R}X$ are always integrable, tangent to a pair of transverse foliations saturated by orbits. In dimension 3, and more generally when one of these foliations has codimension 1, Haefliger's foliation theory can be used to give some structural results. For instance, Plante and Thurston build on this to show that the fundamental group of a manifold with a codimension 1 Anosov flow must have exponential growth.

Relatively few examples are known on higher dimensional manifolds. One construction applicable in all dimensions is that called *suspension flows*. Given a hyperbolic linear toral automorphism A (that is, $A \in \mathrm{SL}(n, \mathbb{Z})$ and has all eigenvalues off the unit circle), the *suspension flow* of A is the flow ϕ^t defined on the mapping torus

$$(\mathbb{R}^n/\mathbb{Z}^n) \times [0,1]/(x,1) \sim (Ax, 0)$$

by $\phi^t(x,s) = (x, s+t)$ modulo $\sim$. Prior to our work, the only examples of Anosov flows known on manifolds of dimension at least 4 (up to orbit equivalence) were suspensions, geodesic flows, and one family on odd-dimensional manifolds constructed by Barthelmé, Bonatti, Gogolev and Rodriguez-Hertz [BBGH]. Verjovsky's conjecture is the statement that, provided the dimension of M is at least 4, and the cxdimension of $E^s \oplus X$ or of $E^u \oplus X$ is equal to 1, supensions are the only examples. This is motivated, perhaps, by the corresponding statement for Anosov diffeomorphisms, which was proved by Franks and Newhouse [Fra, New], as well as a general philosophy about rigidity in low codimension. Ghys [Ghy] proved Verjovsky's conjecture under the additional assumption that the flow is volume preserving and the codimension one subbundle is of class C^2. This result was improved by Simic [Sim] who replaced the C^2 hypothesis with $C^{1+\alpha}$ for every $\alpha \in (0,1)$.

Idea of construction. The idea of our construction is as follows. For any hyperbolic 3-manifold N supporting a quasigeodesic pseudo-Anosov flow, Cannon-Thurston and Fenley [CT, Fen] constructed $\pi_1(N)$-equivariant maps f from the boundary of the *orbit space* of the flow, which is a topological circle, to the boundary of the universal cover $\mathbb{H}^3 = \widetilde{N}$ of N, which is a topological sphere.

We use this data to build a flow on a S^1 bundle over N by defining a flow on $\widetilde{N} \times S^1$ via the vector field which on each "horizontal" slice $\widetilde{N} \times \{p\}$ is tangent to geodesics with forward endpoint $f(p) \in \partial_\infty \widetilde{N}$, then passing to the quotient of $\widetilde{N} \times S^1$ by the diagonal action of $\pi_1(N)$ by deck transformations on the first factor,

and the induced action on the boundary of the orbit space (of the pseudo-Anosov flow) on the second. We show that, for an appropriate choice of smooth structure on this bundle (really, an appropriate choice of coordinates on S^1), this flow has an arbitrarily small perturbation which is Anosov.

Questions. Below are some concrete questions for further study.

(1) What is the optimal regularity of the weak-stable foliation of an example constructed by our methods?
(2) All of our examples lie on circle bundles over hyperbolic 3-manifolds. Is every non-suspension codimension 1 example on a 4-manifold supported on a circle bundle over a hyperbolic 3-manifold? What if one assumes that the codimension 1 foliation is *uniform* and also *R-covered*? Uniform means that for any pair of leaves L_1, L_2 in the lifted foliation to $\widetilde{M}$, in the lifted metric L_1 and L_2 are bounded distance apart. *R-covered* means that the leaf space of this lifted foliation is homeomorphic to $\mathbb{R}$, equivalently (in this context) that the leaf space is Hausdorff.
(3) Do there exist any non R-covered codimension 1 examples on 4 or higher dimensional manifolds?
(4) Invariants for classification: In [BM], [BFM] it is shown that for transitive Anosov flows on 3-manifolds, the set of free homotopy classes of periodic orbits essentially determines the flow up to orbit equivalence. This is not true for our 4-dimensional examples, however they are all determined by the set of free homotopy classes of periodic orbits counted with multiplicity. Is this always a complete invariant in codimension one? (The proofs referenced above are specific to dimension 3 and offer no evidence.)
(5) Higher dimensional Cannon-Thurston maps: Do there exist actions of $\pi_1(N)$, for N a negatively curved compact n-manifold, $n > 3$, on the circle, and equivariant S^{n-1}-filling curves $S^1 \to \partial\widetilde{N}$? If so, one could produce higher dimensional codimension 1 examples.

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An Unexpected Rational Blowdown

Márton Beke
(joint work with Olga Plamenevskaya and Laura Starkston)

I describe recent work regarding the fillability of certain normal surface singularity links with symplectic rational homology balls ($\mathbb{Q}HD$'s). Algebraically there is the question of deforming a normal surface singularity into a rational homology ball, i.e. obtaining a $\mu = 0$ smoothing of the singularity where mu is the Milnor number of the deformation. A general formula of Durfee immediately implies that the singularity has to be *rational*. Early work in the area was done by J. Wahl in [Wah81], describing a linear family of graphs with boundary $L(p^2, p-1)$ admitting a $\mu = 0$ smoothing.

New interest in these constructions arose from work of Fintushel-Stern [FS97]. They define the notion of a *rational blowdown*; meaning the removal of the resolution graph of one of the above mentioned Wahl singularities, and gluing back its $\mathbb{Q}HD$ filling. They compute the Seiberg-Witten invariants of the resulting manifold under this operation, enabling the creation of "smaller" exotic 4-manifolds in terms of χ or σ.

Motivated by this application Stipsicz–Szabó–Wahl [SSW08] found combinatorial restrictions on the possible plumbing graphs, which may be used for the rational blowdown. They are classified into 7 families denoted $\mathcal{G}, \mathcal{W}, \mathcal{N}, \mathcal{M}, \mathcal{A}, \mathcal{B}, \mathcal{C}$. They also find $\mathbb{Q}HD$ smoothings for the first four families, and a few subfamilies of the latter three. Later Bhupal–Stipsicz [BS11] showed, that for the *"star shaped"* graphs (i.e. when there is one vertex of valency ≥ 3, and all other vertices have valency ≤ 2) this list is complete. See also [Fow13] for the complex analytic classification in the star-shaped case.

Conjecture 1 (Wahl, '80s)**.** *If a normal surface singularity link admits a $\mathbb{Q}HD$ smoothing, then its resolution graph is star-shaped.*

We probed this conjecture using the deformation theory of de Jong–van Straten [DJVS98].

Definition 2 ([Spi90])**.** A negative definite resolution graph Γ is *sandwiched* if it can be augmented by new -1 leaves, such that the graph blows down to the empty graph.

To such a graph de Jong–van Straten associate a plane curve singularity germ via blowdown. Conversely any plane curve germ, with each component decorated by a positive integer encodes one such graph uniquely. Moreover this is true for any analytic singularity type with a given topological type.

They prove a functorial identification between certain deformations of decorated plane curve germs, and the deformations of the corresponding surface singularity. Since the deformation theory of plane curves is much better understood, this makes tackling the problem more tractable.

Homological information of the smoothing is encoded by intersection data of the decorated curves, which is in turn encoded in the graph. Thus the first attempt was to understand the combinatorics of such curve configurations. This allowed a new proof of the Bhupal-Stipsicz classification theorem for star-shaped graphs using the intersection combinatorics of certain configurations of smooth and cuspidal curves ([Bek25]). Moreover I obtained a partial generalization of the well-known theorem, that a star-shaped graph with 3 legs and central node with framing ≤ -5 cannot admit a $\mathbb{Q}HD$ smoothing (see e.g. [BS13, Theorem 2.5]).

Theorem 3 ([Bek25])**.** *A negative definite resolution graph with at most 4 nodes of valency* 3 *and framing* ≤ -5*, or one node of valency 4, framing* ≤ -6 *and at most 3 of valency 4 and framing* ≤ -5 *cannot admit a* $\mathbb{Q}HD$ *smoothing.*

The proof rests on a reduction theorem of "combinatorial" $\mathbb{Q}HD$ smoothings. For a fixed number of nodes one has to check that no combinatorial solutions exist for the combinatorial $\mathbb{Q}HD$ fillability question of only finitely many graphs, which is then done via computer search. I also show, that some of the restrictions of a singularity link admitting a $\mathbb{Q}HD$ smoothing is already forced by the satisfiability of its combinatorial $\mathbb{Q}HD$ fillability, i.e.

Theorem 4 ([Bek25])**.** *If a sandwiched graph* Γ *has a combinatorial* $\mathbb{Q}HD$ *smoothing, the canonical cycle satisfies* $K^2+|\Gamma| = 0$*, and the intersection form* Q_Γ *embeds into a unimodular lattice of rank* $|\Gamma|$*.*

During these computations I also found combinatorial solutions to some graphs with multiple nodes, where some nodes had "small" framing, meaning that valency + framing ≥ 0. This means, that the intersection combinatorics are not enough to completely settle Wahl's conjecture.

Using recent results of Min-Roy-Wang [MRW24] on *"spinal" open books*, Plamenevskaya–Starkston [PS25] obtained a generalisation of the dJvS theory for symplectic fillings of sandwiched singularity links. The plane curve germ associated to a sandwiched singularity is interpreted symplectically, and Stein fillings are encoded by certain factorisations of the boundary braid of the curve monodromy. They develop the language of *braided wiring diagrams with tangency*, a more general framework began by Arvola [Arv92] used to study plane line arrangements.

Using this theory we developed diagrammatic moves, which change the topology of the filling, but leave the total monodromy unchanged. We were able to construct a two parameter family of surface singularity links contained in the $\mathcal{C}$ family, which admit a Stein $\mathbb{Q}HD$ filling.

Theorem 5 ([BPS25]). *Let $G_{k,n}$ be a dual resolution graph as in Figure 1 and $(Y_{k,n}, \xi_{k,n})$ the link of the corresponding singularity with its canonical contact structure. Then for all $k \geq 0, n \geq 1$, $(Y_{k,n}, \xi_{k,n})$ admits a Stein filling which is a rational homology disk, despite the fact that the surface singularities of the given topological type admit no rational homology disk smoothing.*

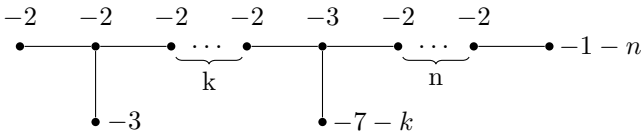

FIGURE 1. The family $G_{k,n}$ for $k \geq -1, n \geq 0$, where $G_{k,n}$ has $k + n + 6$ vertices, with k vertices between the two nodes. The graph $G_{-1,1}$ has a single node of valency 4, with 6 vertices total.

Our obstruction is coming from Wahl [Wah11, Theorem 8.6], where it is shown, that no surface singularty with two valency 3 vertices can admit a $\mathbb{Q}HD$ smoothing.

In [GGP21] it is shown, that the normal surface singularities, which admit a (classical) planar open book is precisely the set of so called *minimal* rational singularities, meaning each vertex has framing at most the negative of its valency. In the sandwiched terminology this means, that the graph admits a description using a plane curve germ with only *smooth* components. Motivated by this we asked the question: is it true that the set of normal surface singularity links admitting a planar spinal open book is the set of sandwiched singularities? The answer no in general, but during the construction in [PS25] only a part of the general spinal package was used. Restating the question: is the set of sandwiched singularities the same as the set of singularity links admitting a fibered *multilink* (in the sense of Eisenbud-Neumann [EN85]) whose fiber has genus 0? The answer turns out to be no, but we find a complete classification of such singularities [BP26], and describe the complex analytic functions on the germ producing such genus 0 fibrations. We also compute

Theorem 6 ([BP26]). *If the surface singularity link (L, ξ_{can}) with its canonical contact structure admits a planar multilink open book, its Heegaard-Floer contact invariant vanishes.*

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The Goodwillie transform and spectral Lie algebras

Max Blans

(joint work with Thomas Blom, Gijs Heuts)

The spectral Lie operad $\mathbb{L}$ is an operad in spectra that lifts the ordinary Lie operad in abelian groups to topology. The homotopy theory $\mathrm{Alg}_{\mathbb{L}}(\mathrm{Sp})$ of spectral Lie algebras bears a close resemblance to the homotopy theory $\mathcal{S}_*$ of pointed spaces. An explanation for this is given by the theorem of Ching [5], which states that the Goodwillie derivatives of the identity functor $\partial_*\mathrm{id}_{\mathcal{S}_*}$ are equivalent to $\mathbb{L}$.

In recent work with Blom [2], we generalize this theorem by showing that for all differentiable ∞-categories $\mathcal{C}$, the Goodwillie derivatives $\partial_*\mathrm{id}_{\mathcal{C}}$ admit the structure of an operad, and that for every reduced filtered colimit preserving functor $F\colon \mathcal{C} \to \mathcal{D}$, the derivatives $\partial_* F$ admit the structure of an $(\partial_*\mathrm{id}_{\mathcal{D}}, \partial_*\mathrm{id}_{\mathcal{C}})$-bimodule. Moreover, we show that the derivatives satisfy a chain rule, generalizing work of Arone and Ching [1] and proving a conjecture of Lurie [6]. One way to formulate our results is as follows.

Theorem 1 (B.–Blom)**.** Let Diff denote the $(\infty, 2)$-category of differentiable ∞-categories and reduced filtered colimit preserving functors. There exists a functor of $(\infty, 2)$-categories

$$\mathcal{G}_{\mathrm{Alg}} \colon \mathrm{Diff} \to \mathrm{Diff}$$

that sends $\mathcal{C}$ to $\mathrm{Alg}_{\partial_* \mathrm{id}_{\mathcal{C}}}(\mathrm{Sp}(\mathcal{C}))$, and $F \colon \mathcal{C} \to \mathcal{D}$ to the functor that takes a $\partial_* \mathrm{id}_{\mathcal{C}}$-algebra X to the $\partial_* \mathrm{id}_{\mathcal{D}}$-algebra $\partial_* F \circ_{\partial_* \mathrm{id}_{\mathcal{C}}} X$.

We call $\mathcal{G}_{\mathrm{Alg}}$ the *Goodwillie transform*. In the upcoming paper [3], we show that it preserves a lot of structure.

Theorem 2 (B.–Blom)**.** The Goodwillie transform preserves products and cotensors by small ∞-categories. It preserves colimits and finite limits on the level of mapping categories.

One consequence of this result is that a symmetric monoidal structure on $\mathcal{C}$ induces one on $\mathrm{Alg}_{\partial_* \mathrm{id}_{\mathcal{C}}}(\mathrm{Sp}(\mathcal{C}))$. For example, applying the Goodwillie transform to the smash product on $\mathcal{S}_*$ gives the smash product on $\mathrm{Alg}_{\mathbb{L}}(\mathrm{Sp})$; a priori, it is not clear this defines a symmetric monoidal structure on spectral Lie algebras. Together with Heuts, we show that $\mathbb{L}$ admits a characterization in terms of this symmetric monoidal structure [4].

Theorem 3 (B.–Heuts)**.** Let $\mathcal{O}$ be an operad in spectra such that $\mathcal{O}_0 = 0$ and $\mathcal{O}_1 = \mathbb{S}$. Then there is an equivalence of operads $\mathcal{O} \simeq \mathbb{L}$ if and only if the free $\mathcal{O}$-algebra functor

$$\mathrm{free}_{\mathcal{O}} \colon \mathrm{Sp} \to \mathrm{Alg}_{\mathcal{O}}(\mathrm{Sp})$$

admits a symmetric monoidal structure with respect to the tensor product of spectra and the smash product of $\mathcal{O}$-algebras.

We also prove that versions of the James splitting, the Hilton–Milnor theorem, the EHP sequence, and Mather's second cube lemma hold in spectral Lie algebras by applying the Goodwillie transform to these results in $\mathcal{S}_*$.

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Quasi-holomorphic homotopies of immersions of 3-manifolds into 5-manifolds

András Csépai

The notion of (stable) quasi-holomorphic maps and their cobordisms was introduced in [2]. These are real smooth maps between real smooth manifolds equipped with a structure that imitates the singularities of (stable) holomorphic maps. The smallest positive codimensional case where singularities appear is that of cobordisms between immersions $M^3 \looparrowright N^5$. Such a cobordism naturally appears between the *link immersion* $S^3 \looparrowright S^5$ of a holomorphic map germ $(\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$ and the standard embedding $S^3 \subset S^5$ (see [4], [1]).

A stable quasi-holomorphic cobordism in these dimensions is a map $F\colon W^4 \to N^5 \times [0,1]$, which has simple regular points almost everywhere, transverse double regular points along a surface $\widetilde{D}(F)$ (in the target; double covered by its preimage), and transverse triple points and cross-cap (complex Whitney umbrella) points in discrete sets $\widetilde{T}(F)$ and $\widetilde{C}(F)$ respectively (again in the target). The *quasi-holomorphic homotopy* of two immersions $M^3 \looparrowright N^5$ is defined as a homotopy $f_s\colon M^3 \to N^5$ ($s \in [0,1]$) such that the map $F\colon M^3 \times [0,1] \to N^5 \times [0,1]$ given by $F(p,s) := (f_s(p), s)$ is a stable quasi-holomorphic cobordism. We say that f_s has a *triple point move* or a *cross-cap move* at a point $q \in N^5 \times \{s\}$, if q is in $\widetilde{T}(F)$ or $\widetilde{C}(F)$ respectively; and f_s has an *elliptic* or *hyperbolic self tangency move* at q, if the surface $\widetilde{D}(F)$ is not transverse to the level $N \times \{s\}$ at q and its change when passing this level is a 0- or 2-handle attachment or a 1-handle attachment respectively. (Note that regular homotopies are precisely the quasi-holomorphic homotopies without cross-cap moves; our naming is consistent with [3].)

One important invariant of an immersion $f\colon M^3 \looparrowright N^5$ is its *knot group* defined as $\pi_1(N^5 \setminus f(M^3))$. Concerning the change of the knot group under quasi-holomorphic homotopies, in [1] we prove:

Theorem. Let $f_s\colon M^3 \looparrowright N^5$ ($s \in [0,1]$) be a quasi-holomorphic homotopy.

(1) If f_s has no elliptic self-tangency moves, then the knot groups of f_0 and f_1 are isomorphic.
(2) If f_s has one elliptic self-tangency move, then there are presentations of the knot groups of f_0 and f_1 whose generator sets coincide and their relation sets only differ in the commutator $[a,b]$ for two generators a, b.

A noteworthy corollary of this is that if two 3-knots (i.e. embeddings $S^3 \hookrightarrow S^5$) are quasi-holomorphically homotopic, then they have the same knot group.

In an ongoing joint work with G. Pintér we study quasi-holomorphic cobordisms of links of holomorphic map germs, and found that all classical germ invariants are determined by the above described four moves. Our final goal is to topologically classify these germs; the above theorem is an important step in this project.

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Atiyah-Bott-Shapiro for $\mathbb{Z}/2\mathbb{Z}$-graded, Real C^*-algebras

Anupam Datta

Among the several variants of Kasparov's KK-theory, the one that has had the most applications in analysis and index theory is the *graded* theory. However, the graded Real KK-category was not known to have a universal property in the literature. In fact, the following result shows that the analogue of the universal properties from the ungraded theories *do not* work in the graded setting.

Theorem 1. ([3]) The canonical functor

$$\widehat{C^*}\text{-alg} \to \widehat{KK}$$

is not a Dwyer-Kan localization at *any* class of morphisms. Here, $\widehat{C^*}$-alg denotes the category of graded Real C^*-algebras, and $\widehat{KK}$ is Kasparov's graded, Real category. The same conclusion holds for Connes-Higson's graded, Real category $\widehat{E}$.

Bunke-Duenzinger suggest the following definition of the complex, graded bivariant K-theory categories. Define

$$\widehat{KK} := \mathrm{coMod}_{KK^{C_2}}(\mathcal{S}), \widehat{E} := \mathrm{coMod}_{E^{C_2}}(\mathcal{S}),$$

where KK^{C_2} resp. E^{C_2} is the stable ∞-category representing C_2-equivariant KK-theory resp. E-theory ([2]), $\mathcal{S} := C_0(\mathbb{R})$ with the even-odd grading, and $\mathrm{coMod}_c(\mathcal{C})$ denotes the ∞-category of comodules over a cocommutative coalgebra c in a symmetric monoidal ∞-category $\mathcal{C}$.

The proof that the above definition recovers classical (Kasparov's) graded KK-category crucially hinges on works of Ulrich Haag ([4], [5]). The papers do not generalize to the Real case, and some details therein are a bit sketchy. So some new techniques are needed for proving the following.

Theorem 2. ([3]) The analogues of the above definitions also work in the Real case, and coincide with Kasparov's (Connes-Higson's) Real KK- (Real E-) theory.

An important corollary of the categorical perspective on Real, graded KK-theory is that one can formulate an Atiyah-Bott-Shapiro ([1]) type result for a general C^*-algebra, as follows.

Corollary 3. ([3]) For A a graded, Real C^*-algebra, we have natural fiber sequences of spectra

$$\mathbf{K}(A_u) \to \mathbf{K}((A \hat{\otimes} \mathrm{Cl}^{0,1})_u) \to \mathbf{K}^{\mathrm{gr}}(A)$$

and

$$\mathbf{K}^{\mathrm{gr}}(A^{\mathrm{op}}) \to \mathbf{K}((A\hat{\otimes}\,\mathrm{Cl}^{0,1})_u) \to \mathbf{K}(A_u),$$

where $(-)_u$ is the underlying (ungraded) C^*-algebra functor, $\mathbf{K}(-)$ is the ungraded K-theory functor, and $\mathbf{K}^{\mathrm{gr}}(-)$ is the graded K-theory functor, A^{op} denotes the opposite algebra of A, and $\mathrm{Cl}^{p,q}$ denotes the Clifford algebra on $p+q$ generators where p generators square to -1, and q generators square to 1.

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On the space of long knots in codimension 2

João Lobo Fernandes

(joint work with Samuel Muñoz-Echániz)

The homotopy types of the spaces of embeddings $\mathrm{Emb}_\partial(M,N)$ between compact smooth manifolds M^m and N^n extending a fixed embedding $\iota : \partial M \hookrightarrow \partial N$ have become central objects of study in high-dimensional geometric topology. When the codimension $n-m \geq 3$, *Goodwillie–Weiss embedding calculus* [11, 3, 4] provides a systematic approach to studying this space by producing a converging tower of homotopy-theoretic approximations, in terms of *configuration space* data of the manifolds M and N. This approach has lead to many computations of homotopy groups of embedding spaces. For example, Fresse–Turchin–Willwacher [2] computed the rational homotopy groups $\pi_*^{\mathbf{Q}}(\mathrm{Emb}_\partial(D^{d-k}, D^d))$, where D^n denotes the n-dimensional disc and $\iota : \partial D^{d-k} \hookrightarrow \partial D^d$ is the standard embedding, in terms of the homology of certain *graph complexes* when $k \geq 3$.

In this talk, I presented a recent computation of the rational homotopy groups of an embedding space in codimension 2 where the embedding calculus tower is known to *not* converge, coming from joint work with Samuel Muñoz-Echániz.

Theorem 1 ([8])**.** *For $d \geq 6$ and $* \geq 1$, we have an isomorphism*

$$\pi_*^{\mathbf{Q}}(\mathrm{Emb}_\partial(D^{d-2}, D^d), u) \cong \begin{cases} 0 & \text{for } d \text{ even and } * < d-3, \\ \mathrm{K}_*^{\mathbf{Q}}(\mathbf{Z}) & \text{for } d \text{ odd and } * < d-4, \end{cases}$$

where u denotes the unknot, and $\mathrm{K}_*^{\mathbf{Q}}(\mathbf{Z})$ denotes the rationalized algebraic K-theory groups of $\mathbf{Z}$. Furthermore, in degree $* = d-3$ for d even and degree $* = d-4$ for d odd, the left-hand side is an infinite-dimensional $\mathbf{Q}$-vector space.

In [8], we exploit the *parameterized isotopy extension* fiber sequence induced by the unknot $u : D^{d-2} \hookrightarrow D^d$, which takes the form

$$\mathrm{Emb}_\partial(D^{d-2}, D^d)_u \to \mathrm{BDiff}_\partial(S^1 \times D^{d-1}) \to \mathrm{BDiff}_\partial(D^d)_\iota$$

provided $d \geq 4$, where we have identified the complement of a neighborhood of u with the solid torus $S^1 \times D^{d-1}$. Here, the subscript $(-)_u$ denotes the path component of the unknot and $(-)_\iota$ denotes the subgroup of those path components of $\mathrm{Diff}_\partial(D^d)$ which are hit by the map from $\mathrm{Diff}_\partial(S^1 \times D^{d-1})$.

Theorem 1 is obtained by combining the works of Krannich [5], Kupers–Randal-Williams [7], and Krannich–Randal-Williams [6] on the rational homotopy groups of $\mathrm{BDiff}_\partial(D^d)$, with our analysis of the rational homotopy groups of $\mathrm{BDiff}_\partial(S^1 \times D^{d-1})$ resulting in the following theorem.

Theorem 2 ([8])**.** *For $d \geq 6$, there is an isomorphism*

$$\pi_*^{\mathbf{Q}}(\mathrm{BDiff}_\partial(S^1 \times D^{d-1})) \cong \begin{cases} 0 & \text{if } * < d-3, \\ \mathrm{K}^{\mathbf{Q}}_{*+1}(\mathbf{Z}[\mathbf{Z}]) & \text{if } * < d-4, \end{cases}$$

Furthermore, in degree $* = d-3$ for d even and degree $* = d-4$ for d odd, the left-hand side is an infinite-dimensional $\mathbf{Q}$-vector space.

Our method to prove Theorem 2 is inspired by the aforementioned works on moduli spaces of discs and by previous work of Bustamante–Randal-Williams [1] on moduli spaces of even-dimensional solid tori. The main new ingredients in our approach are an analog of the Weiss–Williams theorem for embedding spaces, recently obtained by Muñoz-Echániz [10], and an odd-dimensional analog of the work of Galatius–Randal-Williams, recently obtained by the author [9].

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Approximate Fibrations

Christian Kremer
(joint work with Marco Volpe)

This short talk gave an introduction to the notion of *approximate fibrations*, with the goal of sketching their use to construct actions of finite groups on topological manifolds, motivating joint work with Marco Volpe on approximate fibrations between ∞-topoi.

A map $f\colon X \to Y$ between ANRs is called an approximate fibration if it is proper and for every open $U \subset Y$ the square

$$\begin{tikzcd} f^{-1}(U) \arrow[r] \arrow[d] & X \arrow[d] \\ U \arrow[r] & X \end{tikzcd}$$

is a homotopy pullback. The original definition, given by Coram and Duvall, was in terms of approximate lifting properties, hence the name. In [CV], a characterisation in terms of higher topos theory is given.

Theorem ([CV])**.** *A map $f\colon X \to Y$ between ANRs is an approximate fibration if and only if the induced geometric morphism $f_*\colon \mathrm{Sh}(X) \to \mathrm{Sh}(Y)$ is internally corepresentable, compatible with base change.*

A main motivation to study approximate fibration is that if E and L are topological manifolds, and $p\colon E \to L \times \mathbb{R}$ is an approximate fibration, there is a topological space $E \cup_p L$ with desirable properties. The underlying set of $E \cup_p L$ is $E \cup L$ and the topology is minimal so that $E \subset E \cup_p L$ is an open embedding and the map $p \cup (\mathrm{id_L} \times \{+\infty\})\colon E \cup_p L \to L \times (0, \infty]$ is continous. This topoloigal space is the *teardrop construction* on p. See [HTWW] for more information and applications to stratified spaces.

If the homotopy fibres of p are spheres and the dimension of E is at least six, then $E \cup_p L$ is a topological manifold. This is useful to construct manifolds with interesting group actions. Suppose, E comes with a free action of a finite group G. Then if $p\colon E/G \to L \times \mathbb{R}$ is an approximate fibration, so is $E \to E/G \to L \times \mathbb{R}$, and we see that the teardrop construction $E \cup_p L$ can be equipped with a G-action which is free on E and trivial on $L \subset E \cup_p L$. This construction is used by the author in upcoming work to construct solutions for the Nielsen realisation problem for high-dimensional aspherical manifolds in special cases.

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Real Seiberg–Witten theory and exotic P^2-knots

Jin Miyazawa

In 4-dimensional topology, the following conjecture is known to be difficult and important:

Conjecture 1 (Smooth unknotting conjecture)**.** *Let $f\colon S^2 \to S^4$ be a smooth embedding. If $\pi_1(S^4 \setminus f(S^2)) \cong \mathbb{Z}$, then f is smoothly isotopic to the unknot.*

An embedding of S^2 into S^4 is called a 2-knot. The unknot is a 2-knot which bounds an embedded D^3 in S^4.

Theorem 1. [4] *There is an infinite family of embeddings $f_n\colon \mathbb{R}P^2 \to S^4$ for all $n \in \mathbb{N}$ such that*

- *f_0 is the standard one,*
- *$\{f_n\}_n$ are topologically ambient isotopic to each other,*
- *if $n \neq m$, then f_n and f_m are not smoothly isotopic.*

The standard embedding of $\mathbb{R}P^2$ is given by the composition of the mapsto

$$\mathbb{R}P^2 \hookrightarrow \mathbb{C}P^2 \to S^4$$

where the second map is the quotient of the complex conjugation and the first map is an embedding to the fixed point set of the complex conjugation.

The construction of f_n is as follows:

- Let f_0 be the standard embedding of $\mathbb{R}P^2$ into S^4.
- Let $g\colon S^2 \to S^4$ be an embedding called 1-roll spun knot of $P(-2,3,7)$, where $P(-2,3,7) \subset S^3$ is the $(-2,3,7)$-pretzel knot.
- We set $f_n := f_0 \# \#^n g$ where $\#$ denotes the inner connected sum.

Note that $\#^n g\colon S^2 \to S^4$ is not topologically isotopic to the unknot. However, once we take an inner connected sum of the standard $\mathbb{R}P^2$ embedding with g, the fundamental group of the complement of the embedding becomes $\mathbb{Z}/2$. From the result of Lawson [3], they are topologically ambient isotopic.

To prove theorem 1, we use Real Seiberg–Witten theory, which is a variant of the ordinary Seiberg–Witten theory. To explain this, let us recall ordinary Seiberg–Witten theory. For simplicity, we only consider the invariants on closed 4-manifolds.

Let X be a closed 4-manifold. The input datum for this invariant is a spin^c structure on X. This is a tangential structure and always exist for 4-manifolds. The set of the isomorphism classes of spin^c structures on X can be identified with $H^2(X,\mathbb{Z})$. Note that this identification is not canonical. If the dimension $b^+(X)$ of the space of self-dual harmonic 2-form $\mathcal{H}^2_+(X)$ is bigger than 1, one obtains an integer $SW(X,\mathfrak{s})$ for each spin^c structure $\mathfrak{s}$ on X.

In the case when $b^+(X) = 1$, we have to fix additional data c called a chamber and we have an invariant $SW(X,\mathfrak{s},c)$ which depends on the chamber.

These invariants are obtained by counting the solutions up to gauge transformations because the Seiberg–Witten equations have a symmetry of an action of

the gauge group $\mathcal{G} = \mathrm{Map}(X, S^1)$. It is necessary to take the quotient of the set of the solutions by this action otherwise the set of the solutions is too large.

The reason we need the condition $b^+(X) > 0$ is to avoid *reducible solutions* of the Seiberg–Witten equations. Reducible solutions are solutions whose isotoropy groups are non-trivial. In other words, the action of the group of gauge transformations is not free. In the negative definite case, we cannot avoid reducible solutions so it is difficult to define a counting invariant of the solutions up to $\mathcal{G}$-action.

We now explain the Real Seiberg–Witten theory. The input data are the following: A closed 4-manifold X with an orientation-preserving involution $\iota\colon X \to X$, a spin^c structure on X, and an anti-linear lift I of ι on the spinor bundle $S = S^+ \oplus S^-$ of $\mathfrak{s}$. We assume $I^2 = 1$.

We call a pair $(\mathfrak{s}, I)$ a Real spin^c structure. Unlike ordinary spin^c structures, the existence of a Real spin^c structure is a non-trivial problem. At least, if the fixed point set of ι has codimension 2, the action of ι^* on $H^1(X, \mathbb{Z})$ is trivial, and there is a spin^c structure $\mathfrak{s}$ satisfying $\iota^*\mathfrak{s} \cong \overline{\mathfrak{s}}$, then we have an anti-linear lift I of ι such that $I^2 = 1$ ([2, Lemma 2.9]).

Let $(\mathfrak{s}, I)$ be a Real spin^c structure on (X^4, ι). Using I, we can define an involution $\mathcal{I}$ on the solution set of the Seiberg–Witten equations. Let us consider the fixed points of $\mathcal{I}$. As we mentioned before, there is an action of the gauge transformation $\mathcal{G} = \mathrm{Map}(X, U(1))$ on the solutions. We consider the subgroup of $\mathcal{G}$ which preserves the fixed point set of $\mathcal{I}$ setwise. We denote the subgroup by $\mathcal{G}^I$.

Here, let us point out the difference between the ordinary Seiberg–Witten equations and Real one. Assume the formal dimension of the moduli space is 0 and $H^1(X, \mathbb{Z}) = 0$.

In the ordinary case, let us consider the framed moduli space. This is a quotient of the set of the solutions by a subgroup $\mathcal{G}_0$ of $\mathcal{G}$ such that $\mathcal{G}/\mathcal{G}_0 \cong U(1)$ and acts freely on $\mathcal{C}$. If we take a good perturbation of the equations, this is diffeomorphic to some copies of $U(1)$ and a reducible point $*$ if $b^+(X) = 0$. Moreover, the number of the copies of $U(1)$ depends on the metric and perturbations. Therefore this is not an invariant of $(X, \mathfrak{s})$.

In the Real case, the framed moduli space is finitely many copies of $\mathbb{Z}/2 \cong \{\pm 1\}$ and a reducible point. This is because $\mathcal{G}^I/\mathcal{G}_0^I \cong \{\pm 1\}$ where $\mathcal{G}_0^I$ is a subgroup acting freely on $\mathcal{C}^I$. In this case, the number of irreducible solutions is not an invariant as in the ordinary case. However, if we forget ± 1 action on the framed moduli space and simply signed count of them, this is independent of the metric and is well defined up to overall sign. Let us denote the absolute value of this signed count by

$$|\deg(X, \iota, \mathfrak{s}, I)|. \tag{1}$$

This invariant implies that $(S^4, f_n(\mathbb{R}P^2))$s in theorem 1 are not pairwise diffeomorphic. Let X_n be the double branched cover of S^4 along f_n with opposite orientation to the natural orientation from the construction and ι_n be the covering transformation. From the computation in [4], we have a unique Real spin^c structure $(\mathfrak{s}_n, I_n)$ such that the formal dimension of the Real Seiberg–Witten equations

is 0 and

$$|\deg(X_n, \iota_n, \mathfrak{s}_n, I_n)| = 3^n.$$

Note that X_n is a negative definite 4-manifold and we cannot obtain such non-trivial invariant for negative definite manifolds. (We can see that the Real Seiberg-Witten invariant and $|\deg(\cdot)|$ are trivial for $-X_n$.) Moreover, Hughes–Kim–Miller showed that X_n is diffeomorphic to $-\mathbb{C}P^2$ [1]. Therefore, to prove exoticness of f_n, we cannot ignore the information of the involutions.

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Quotient Algebras via "Stability"

Robert Burklund

This talk described a technique for constructing $\mathbb{E}_1$-algebra structures on nil-thickenings of $\mathbb{A}_2$-algebras. The main theorem presented was the universal case of the main theorem of [B] for $\mathbb{E}_1$-algebras. By naturality it can be used to draw conclusions like, as $\mathbb{S}/4$ admits an $\mathbb{A}_2$-algebra structure, $\mathbb{S}/16$ admits an $\mathbb{E}_1$-algebra structure. The argument has been greatly simplified by the removal of unnecessary technology.

Theorem. *Let R be an $\mathbb{E}_0$-algebra in a stably monoidal category $\mathcal{C}$. Let $\mathcal{I} \xrightarrow{\alpha} 1 \to R$ be the fiber-of-unit sequence of R. There are for $k \geq 2$ iteratively defined obstructions*

$$\theta_k \colon \Sigma^{2k-3}\mathcal{I}^{\otimes k} \to R$$

whose joint vanishing allows us to equip R with an $\mathbb{E}_1$-structure.

The proof given was essentially as in [B].

Theorem. *Let $\mathcal{C}$ denote the free monoidal category with an $\mathbb{A}_2$-algebra. Let R denote this universal $\mathbb{A}_2$-algebra. Let $\mathrm{St}(\mathcal{C})$ denote the free stably monoidal category with an $\mathbb{A}_2$-algebra. Let $\mathcal{I} \xrightarrow{\alpha} 1 \to R$ be the fiber-of-unit sequence of R in $\mathrm{St}(\mathcal{C})$. We show that the $\mathbb{E}_0$-algebra $R^{(2)} = 1/\alpha^{\otimes 2}$ admits a refinement to an $\mathbb{E}_1$-algebra.*

Proof. We apply the obstruction theory from the first theorem. In order to verify that the obstructions vanish we will show that the mapping spectra

$$\mathrm{Map}^{\mathrm{Sp}}_{\mathrm{St}(\mathcal{C})}(\Sigma^{2k-3}\mathcal{I}^{\otimes 2k}, R^{(2)}).$$

are 1-connective.

Extend the top left square below into the indicated grid of cofiber sequences.

$$\begin{tikzcd}
\mathcal{I}^{\otimes 2} \arrow[r, "\alpha\otimes\mathcal{I}"] \arrow[d, "\alpha^{\otimes 2}"] & \mathcal{I} \arrow[r] \arrow[d, "\alpha"] & R\otimes\mathcal{I} \arrow[d] \\
1 \arrow[r, equal] \arrow[d] & 1 \arrow[r] \arrow[d] & 0 \arrow[d] \\
R^{(2)} \arrow[r] & R \arrow[r] & R\otimes\Sigma\mathcal{I}
\end{tikzcd}$$

The bottom row lets us reduce to showing that

$$\mathrm{Map}^{\mathrm{Sp}}_{\mathrm{St}(\mathcal{C})}(\Sigma^{2k-3}\mathcal{I}^{\otimes 2k}, R\otimes\mathcal{I}^{\otimes\epsilon})$$

with $\epsilon \in \{0,1\}$ is 1-connective.

Using the $\mathbb{A}_2$-algebra structure on R these mapping spectra are retracts of

$$\mathrm{Map}^{\mathrm{Sp}}_{\mathrm{St}(\mathcal{C})}(R\otimes\Sigma^{2k-3}\mathcal{I}^{\otimes 2k}, R\otimes\mathcal{I}^{\otimes\epsilon}).$$

The $\mathbb{A}_2$-algebra structure on R also provides a splitting of the cofiber sequence

$$R\otimes 1 \to R\otimes R \to R\otimes\Sigma\mathcal{I}.$$

From this we learn that the mapping spectra above are themselves retracts of

$$\mathrm{Map}^{\mathrm{Sp}}_{\mathrm{St}(\mathcal{C})}(\Sigma^{-3}R^{\otimes 2k+1}, \Sigma^{-\epsilon}R^{\otimes 1+\epsilon}).$$

Knowing that tensor powers of R are in the image of $\mathcal{C}\to\mathrm{St}(\mathcal{C})$ we have

$$\mathrm{Map}^{\mathrm{Sp}}_{\mathrm{St}(\mathcal{C})}(\Sigma^{-3}R^{\otimes 2k+1}, \Sigma^{-\epsilon}R^{\otimes 1+\epsilon}) \cong \Sigma^{3-\epsilon}\Sigma^\infty_+\mathrm{Map}_{\mathcal{C}}(R^{\otimes 2k+1}, R^{\otimes 1+\epsilon}).$$

Connectivity of suspension spectra completes the proof. □

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On the mapping class groups of the universal smoothing of $\mathbb{R}^4$

Arunima Ray

(joint work with Robert E. Gompf and Patrick Orson)

The Euclidean space $\mathbb{R}^n$ has a unique smooth structure, up to diffeomorphism, for all $n \neq 4$ by results of Radó, Moise, Munkres, and Stallings [1, 2, 3, 4]. In contrast, the 4-dimensional Euclidean space $\mathbb{R}^4$ has uncountably infinitely many smooth structures. This is a result of Taubes [5], building on work of Casson, Freedman, and Donaldson [6, 7, 8]. The smooth 4-manifolds which are homeomorphic but not diffeomorphic to $\mathbb{R}^4$ are called *exotic* $\mathbb{R}^4$*s*. We denote the topological space $\mathbb{R}^4$ endowed with its standard smooth structure by $\mathbb{R}^4_{\mathrm{std}}$. Freedman and Taylor proved in [9] that there exists a *universal exotic* $\mathbb{R}^4$, which we denote by $\mathcal{U}$, which contains every other exotic $\mathbb{R}^4$ as a smooth submanifold.

Given an oriented, smooth, noncompact manifold M, one can study a few different versions of its mapping class group. The first is the typical *mapping class*

group, denoted by $\mathrm{MCG}(M)$, consisting of the orientation preserving diffeomorphisms of M up to isotopy. One also has the *compactly supported mapping class group*, denoted by $\mathrm{MCG}_c(M)$, consisting of orientation preserving diffeomorphisms of M which are *compactly supported*, meaning that they equal the identity outside of some compact subset of M, up to compactly supported isotopy. Finally, there is the *mapping class group at infinity*, denoted by $\mathrm{MCG}^\infty(M)$. This group consists of germs of orientation preserving diffeomorphisms of M, up to isotopy as embeddings into M. Here a *germ* of a diffeomorphism is the equivalence class of diffeomorphisms which agree outside some compact subset of M. We show that these three mapping class groups are related by the following exact sequence:

$$\mathrm{MCG}_c(M) \xrightarrow{i} \mathrm{MCG}(M) \xrightarrow{r} \mathrm{MCG}^\infty(M).$$

One sees by a standard argument that $\mathrm{MCG}(\mathbb{R}^4_{\mathrm{std}}) = 1$. In contrast, it is an open question whether $\mathrm{MCG}_c(\mathbb{R}^4_{\mathrm{std}}) = 1$. Indeed, $\mathrm{MCG}_c(\mathbb{R}^4_{\mathrm{std}}) \cong \mathrm{MCG}(S^4_{\mathrm{std}})$, and understanding the latter group is an important open question in 4-manifold topology. Similarly, it is open whether $\mathrm{MCG}^\infty(\mathbb{R}^4_{\mathrm{std}}) = 1$. Gompf showed in [10] that $\mathrm{MCG}^\infty(\mathbb{R}^4_{\mathrm{std}}) = 1$ if and only if the 4-dimensional Schoenflies conjecture is true. This motivates the study of the various mapping class groups of exotic $\mathbb{R}^4$s. This was begun by Gompf in [10] who showed that there exists an exotic $\mathbb{R}^4$, which we denote by $\mathcal{R}$, such that $\mathrm{MCG}(\mathcal{R})$ and $\mathrm{MCG}^\infty(\mathcal{R})$ are uncountably infinite. Gompf's proof constructed diffeomorphisms which are not compactly supported. It was shown by Konno–Lin–Mukherjee–Muñoz-Echániz and Shivkumar that for each finite $n \geq 1$, there exists an exotic $\mathbb{R}^4$, which we denote by $\mathcal{R}_n$, such that $\mathrm{MCG}_c(\mathcal{R}_n)$ contains a subgroup of rank n [11, 12].

We study the mapping class groups of $\mathcal{U}$. In contrast to other exotic $\mathbb{R}^4$'s we show the two following results.

Theorem 1. $\mathrm{MCG}_c(\mathcal{U}) = 1$.

Theorem 2. *The restriction map* $r\colon \mathrm{MCG}(\mathcal{U}) \to \mathrm{MCG}^\infty(\mathcal{U})$ *is an isomorphism.*

The surjectivity of r was proven by Gompf in [10]. The proof of Theorem 1 and the injectivity of r proceeds as follows. First we prove that an arbitrary compactly supported diffeomorphism of an arbitrary exotic $\mathbb{R}^4$ is pseudo-isotopic to the identity, via a pseudo-isotopy which is fixed outside a compact set K. Here recall that a *pseudo-isotopy* between diffeomorphisms $f_0, f_1\colon M \to M$ of a closed, smooth manifold M consists of a diffeomorphism $F\colon M \times [0,1] \to M \times [0,1]$ such that $F|_{M\times\{i\}} = f_i$ for $i = 0, 1$. A level-preserving pseudo-isotopy is an isotopy. Next we would like to apply Quinn's theorem that pseudo-isotopy implies stable isotopy for diffeomorphisms for compact, simply connected 4-manifolds [13]. In other words, if M is a compact, simply connected 4-manifold, a pseudo-isotopy between diffeomorphisms $f_0, f_1\colon M \to M$ can be upgraded to an isotopy between $f_0\#\mathrm{id}$ and $f_1\#\mathrm{id}$ defined on $M\#n(S^2\times S^2)$ for some finite $n \geq 0$. The proof of this theorem was recently corrected by Gabai–Gay–Hartman–Krushkal–Powell in [14]. Of course, our manifold $\mathcal{U}$ is not compact – this can be rectified by considering the compact support K instead, but K is not simply connected. We are also not

willing to modify $\mathcal{U}$ by connected sum with copies of $S^2 \times S^2$. Thus each step of Quinn's proof and Gabai–Gay–Hartman–Krushkal–Powell's correction must be scrutinised and appropriately modified, leveraging the construction and properties of $\mathcal{U}$.

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Periodic families in the homology of $\mathrm{GL}_n(\mathbb{F}_2)$

Kelly Wang

In [3] Quillen completely determined the cohomology ring $H^*(\mathrm{GL}_n(\mathbb{F}_q);\mathbb{F}_l)$ of general linear group over a finite field $\mathbb{F}_q$ with coefficient $\mathbb{F}_l$ where $l \neq p$, p being the $\mathbb{F}_q$. At the characteristic Quillen showed the cohomology is stably trivial: $\widetilde{H}^*(\mathrm{GL}_\infty(\mathbb{F}_q);\mathbb{F}_p) = 0$; and by the work of Quillen, Maazen, Galatius-Kupers-Randal-Williams we know $H_*(\mathrm{GL}_n(\mathbb{F}_q);\mathbb{F}_p)$ satisfy homological stability: $H_d(\mathrm{GL}_n(\mathbb{F}_q), \mathrm{GL}_{n-1}(\mathbb{F}_q);\mathbb{F}_p) = 0$ for $d < n+$(some explicit constant) if $q \neq 2$ and for $d < \frac{2}{3}(n-1)$ if $q = 2$ [1]. But very little is known outside of this stable range (see [2]), and so is the sharpness of homological stability results in general. Our results contributes towards those questions in the case $q = 2$.

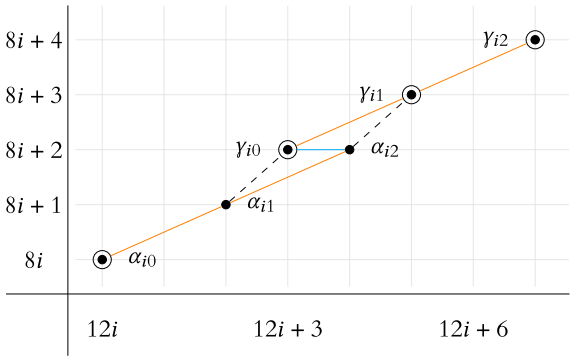
Theorem 1. *We describe an endomorphism*

$$\varphi : H_{d-8}(\mathrm{GL}_{n-12}(\mathbb{F}_2), \mathrm{GL}_{n-13}(\mathbb{F}_2); \mathbb{F}_2) \to H_d(\mathrm{GL}_n(\mathbb{F}_2), \mathrm{GL}_{n-1}(\mathbb{F}_2); \mathbb{F}_2)$$

such that

$$\alpha_{ij} := \varphi^i(Q^1(\sigma)^j.\omega_{0,0}) \quad \textit{and} \quad \gamma_{ij} := \varphi^i(Q^1(\sigma)^j.\omega_{3,2})$$

where $i \geqslant 0$, $j = 0, 1, 2$, are all zero.



Here $\sigma \in H_0(\mathrm{GL}_1(\mathbb{F}_2); \mathbb{F}_2)$ is the stabilisation class, see [5] *for explicit description of the relative homology class $\omega_{0,0}$ and $\omega_{3,2}$ (and the meaning of the solid and dotted lines). Here we are using the module structure of the relative homology over the absolute ones.*

Theorem 2. *The circled classes in the figure above are all non-zero under the boundary map $H_d(\mathrm{GL}_n(\mathbb{F}_2), \mathrm{GL}_{n-1}(\mathbb{F}_2); \mathbb{F}_2) \to H_{d-1}(\mathrm{GL}_{n-1}(\mathbb{F}_2); \mathbb{F}_2)$.*

In particular Theorem 1 tells us the $\frac{2}{3}$ stability slope is sharp. Our work also have such consequences for general linear groups over the integer and automorphism group of free groups.

Theorem 3. (1) *$H_d(\mathrm{GL}_n(\mathbb{Z}); \mathbb{F}_2)$ does not have stability of slope $> \frac{2}{3}$.*
(2) *$H_d(\mathrm{Aut}(F_n); \mathbb{F}_2) = 0$ does not have stability of slope $> \frac{2}{3}$.*

Very vaguely speaking, our approach recognises chains $\bigoplus_{n \geqslant 0} C_*(\mathrm{GL}_n(\mathbb{F}_2); \mathbb{F}_2)$ on the classifying spaces as an $\mathbb{E}_\infty$-algebra and implement the new homotopical viewpoint of homological stability.

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Extending Symmetries from 3-Manifolds to 4-Manifolds

JungHwan Park
(joint work with Sungkyung Kang, Masaki Taniguchi)

The theme of the talk was the following extension problem. Let Y be a closed oriented 3-manifold equipped with a symmetry, for instance a diffeomorphism, a finite group action, or a homotopy coherent finite group action. If Y bounds a smooth compact 4-manifold X, when can the symmetry on Y be extended over X? This question provides a useful common language for two seemingly different topics: sliceness problems for knots and exotic Dehn twists on 4-manifolds.

The first example comes from knot concordance. If a knot $K \subset S^3$ bounds a smoothly embedded disk $D \subset B^4$, then the branched cover of B^4 along D is a homology-ball filling of the branched cover of S^3 along K. Moreover, the deck transformation on the boundary extends over this filling. Thus an obstruction to the existence of such an equivariant filling gives an obstruction to sliceness.

We applied this point of view to cables of the figure-eight knot E. The family $E_{2n,1}$ is especially subtle: these knots are rationally slice [8] and were shown by Miyazaki not to be ribbon [9], so they are natural test cases for the slice-ribbon problem. The main result discussed in the first part of the talk is that every nontrivial cable of the figure-eight knot has infinite order in the smooth concordance group [5]. In particular, every nontrivial cable is not smoothly slice. This extends the earlier nonsliceness result for $E_{2,1}$ due to Dai, Kang, Mallick, Park and Stoffregen [1], and the odd-parameter result obtained using real Seiberg–Witten Frøyshov invariants [4].

The proof uses higher branched covers. For the even cables, write $2n = 2^k m$ with m odd and consider the 2^k-fold branched cover. The natural involution obtained from the deck group leads to a family of concordance invariants

$$\kappa_R^{(k)} \colon \mathcal{C} \longrightarrow \frac{1}{16}\mathbb{Z},$$

defined using real Seiberg–Witten Floer K-theory. These invariants generalize the real K-theoretic Frøyshov invariant of Konno, Miyazawa and Taniguchi [3]. A key topological input is a cobordism in a twice-punctured $2\mathbb{CP}^2$, which relates the branched covers of $E_{2n,1}$ to Seifert fibered spaces. The relevant Floer-theoretic computations are then made possible by an $O(2)$-equivariant version of the lattice homotopy type, building on the lattice homotopy model of Dai, Sasahira and Stoffregen [2]. The same strategy gives stronger obstructions to equivariant homology-ball fillings of the branched covers.

The second part of the talk considered 4-dimensional Dehn twists. Let Y be a closed 3-manifold and let ϕ be a loop in $\mathrm{Diff}^+(Y)$. The loop gives a diffeomorphism of $Y \times [0,1]$, and whenever $Y \times [0,1]$ is embedded in a 4-manifold X, this diffeomorphism extends by the identity to a self-diffeomorphism of X, called a Dehn twist along Y. A boundary Dehn twist is obtained when $Y = \partial X$.

For $Y = S^3$, such twists have order at most two. For Seifert fibered homology spheres, however, rotations along the Seifert fibers give loops in the diffeomorphism group with much richer behavior. The extension problem above reappears as follows: if a boundary Dehn twist has finite order, then, after choosing a sufficiently large prime p, one obtains a homotopy coherent $\mathbb{Z}/p$-action on the filling whose restriction to the boundary is the action induced by fiber rotation. Families Seiberg–Witten theory and equivariant Frøyshov-type inequalities can obstruct the existence of such an extension.

This leads to infinite-order exotic Dehn twists. In joint work with Kang and Taniguchi, we proved that boundary Dehn twists along Seifert homology spheres, except for the 3-sphere, give infinite-order exotic diffeomorphisms on simply connected positive-definite fillings [6]. In particular, these diffeomorphisms are topologically isotopic to the identity rel. boundary but are not smoothly isotopic to the identity rel. boundary, and no nonzero power becomes smoothly trivial.

Finally, we discussed the stabilization problem. Exotic phenomena in dimension four often disappear after taking connected sums with copies of $S^2 \times S^2$. Using a $\mathrm{Pin}(2) \times \mathbb{Z}/2$- equivariant refinement of the lattice homotopy type, we construct a compact contractible 4-manifold X with boundary $\Sigma(3,5,19)$ such that odd iterates of the boundary Dehn twist remain exotic even after two stabilizations:

$$X\#(S^2 \times S^2)\#(S^2 \times S^2).$$

This gives, in particular, a relative exotic diffeomorphism on a contractible 4-manifold that survives two stabilizations [7].

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A moduli space of metric graphs, and a Koszul pre-dual

Søren Galatius

Moduli of metric graphs

In earlier work [1], joint with Brown, Chan, and Payne, we introduced a moduli space M_{Gr}, a topological space whose points are in bijection with isometry classes of pairs (G, ℓ), where G is a finite graph and $\ell \in (0, \infty)^{EG}$ is a "metric". The space M_{Gr} is topologized as the quotient of the disjoint union of copies of $[0, \infty]^{EG}$, where the quotient map $[0, \infty]^{EG} \to M_{\mathrm{Gr}}$ collapses any edge $e \in EG$ with $\ell(e) = 0$ and deletes any edge with $\ell(e) = \infty$. The space M_{Gr} is filtered by closed subspaces $F_g M_{\mathrm{Gr}}$, defined by the condition $b_1(G) \leq g$.

The filtration induces a spectral sequence with

$$E^1_{s,t} = H_{s+t}(F_s M_{\mathrm{Gr}}, F_{s-1} M_{\mathrm{Gr}}; \mathbb{Q}) \quad \Rightarrow \quad H_{s+t}(M_{\mathrm{Gr}}; \mathbb{Q}),$$

and in [1] it was explained how to define a commutative product and a coassociative coproduct on this spectral sequence, and how to relate its E^1 page to the Grothendieck–Teichmüller group using [2]. The motivation in op.cit. was the study of $W_0 H^*_c(\mathcal{A}_g)$, but the filtered space M_{Gr} and associated spectral sequence is worth studying in its own right.

In [1], the official definition of M_{Gr} is as a topological space, a "coarse" moduli space of metric graphs. It admits a refinement to an orbifold "moduli stack" of metric graphs, which keeps track of isometry groups, and we will use the same notation M_{Gr} for this refinement.

A Koszul predual. The coproduct on $E^1_{*,*}$ is not cocommutative. One can try to understand it better by looking for a "Koszul pre-dual": an E_∞ space X equipped with an E_∞ map $g : X \to \mathbb{N}$, or equivalently a decomposition $X = \coprod_{g \in \mathbb{N}} X_g$ with $X_a X_b \subset X_{a+b}$, and a filtered equivalence

$$BX \xrightarrow{\simeq} M_{\mathrm{Gr}}.$$

By abstract nonsense, an equivalence of this form promotes the filtered space M_{Gr} to an E_1 coalgebra and an E_∞ algebra, explaining the algebraic structure on the spectral sequence.

An analogous "Koszul pre-dual" for the setting of Quillen's rank spectral sequence in algebraic K-theory was introduced recently in [4] and [5], see in particular [5, Theorem 10.11] for the connection to Quillen's spectral sequence. See also [1, Remark 1.14].

My talk discussed how to implement a multi-object version of this strategy for M_{Gr}: I sketched the definition of an E_∞ category $\mathcal{C}$ enriched in $\mathbb{N}$-graded spaces and outlined a proof of a filtered equivalence

$$B\mathcal{C} \xrightarrow{\simeq} M_{\mathrm{Gr}}.$$

The objects of $\mathcal{C}$ are finite sets. Morphisms from S to T are generated by pairs $(G, [\ell])$ where G is a graph with $EG \neq \emptyset$ and specified bijections $VG = S$ and $\pi_0(G) = T$, and $[\ell] \in \mathbb{R}_+^{EG}/\mathbb{R}_+$ is the projective class of a metric. A general

element of the morphism space $\mathcal{C}(S,T)$ is a formal word $(G_1,[\ell_1])\circ\cdots\circ(G_k,[\ell_k])$ of length $k\geq 0$, with specified identifications $S=V(G_1)$, $T=\pi_0(G_k)$, and $\pi_0(G_j)=V(G_{j+1})$ for all $j\in\{1,\ldots,k-1\}$. Informally, the identification $\pi_0(G_j)=V(G_{j+1})$ model that the connected components of G_j sit as "infinitesimally small" graphs inside the vertices of G_{j+1}, and the topology imposed on $\mathcal{C}(S,T)$ mirrors this intuition (as a family of projective metrics $[\ell]\in\mathbb{R}_+^{EG}/\mathbb{R}_+$ on a graph G degenerates with $\ell(e)/\ell(e')\to 0$, the corresponding generating morphism $(G,[\ell])$ degenerates to a composition where e is infinitesimally small compared to e'). More formally, if $EG=\{e_1,\ldots,e_n\}$, then the space $\mathbb{R}_+^n/\mathbb{R}_+=\Delta^{n-1}\setminus\partial\Delta^{n-1}$ is compactified as the closure of the image of the embedding $r:\Delta^{n-1}\setminus\partial\Delta^{n-1}\to[0,1]^{n^2}$ with coordinates $r_{ij}=\min(1,\ell(e_i)/\ell(e_j))$. Denoting this compactification as $P^{n-1}\subset[0,1]^{n^2}$, the boundary $\partial P^{n-1}=P^{n-1}\setminus(0,1]^{n^2}$ is then glued to decomposable morphisms.

The morphism spaces $\mathcal{C}(S,T)$ are not compact, but decompose as

$$\mathcal{C}(S,T)=\bigsqcup_{g\in\mathbb{N}}\mathcal{C}_g(S,T)$$

according to loop order $g=b_1(G)$, and each $\mathcal{C}_g(S,T)$ is a compact orbifold. Indeed, the total number of edges $n=|EG_1|+\cdots+|EG_k|$ is determined as $n=g+|S|-|T|$, so $\mathcal{C}_g(S,T)$ is the quotient of a finite disjoint union of products of the form $P^{n_1-1}\times\cdots\times P^{n_k-1}$ with $n_1+\cdots+n_k=g+|S|-|T|$.

Comparison. One method for establishing the filtered equivalence

$$B\mathcal{C}\xrightarrow{\simeq}M_{\mathrm{Gr}}$$

is inspired by the "flow category" of a Morse function $f:M\to\mathbb{R}$ on a smooth compact manifold, introduced in [3] and denoted $\mathcal{C}_f$ there. In that situation, the flow of a gradient-like vector field gives an action of the group $G=(\mathbb{R},+)$ on M, whose fixed points are the critical points of f, and they are the objects of the flow category. The function f is strictly decreasing along non-constant orbits, each non-constant orbit is a generating morphism of the flow category, and a general morphism is a formal composition of generating morphisms. If $a,b\in M$ are critical points, then the morphism space $\mathcal{C}_f(a,b)$ embeds into the space of lifts in the diagram

$$\begin{tikzcd} & M \arrow[d, "f"] \\ {[f(a),f(b)]} \arrow[ur, dashed] \arrow[r, hook] & \mathbb{R}, \end{tikzcd}$$

topologized as a subspace of the mapping space. The point is that any orbit closure or concatenation of such has a canonical parametrization given as the inverse of f. With respect to this embedding of $\mathcal{C}_f(a,b)$, composition of morphisms becomes concatenation of paths.

In our setting, we define an action of $G=(\mathbb{R},+)$ on M_{Gr} as

$$t.(G,\ell)=(G,e^{-t}\ell).$$

The fixed points of this action are precisely the graphs with $EG = \emptyset$, i.e., the entire graph G consists of just the finite set VG, which we regard as an object of $\mathcal{C}$. The non-constant orbits for the action of G on M_{Gr} correspond to the generating morphisms of $\mathcal{C}$. As an analogue of the Morse function, we may take $f : M_{\mathrm{Gr}} \to \mathbb{R}/\mathbb{Z}$ defined by

$$f(G, \ell) = \sum_{e \in EG} \rho \circ \ell(e) \mod 1$$

for some increasing homeomorphism $\rho : (0, \infty) \to (0, 1)$, for instance $\rho(t) = t/(1 + t)$. This function is continuous, and locally decreasing along non-constant orbits of the scaling action. Similarly to the flow category of a Morse function, the morphism space $\mathcal{C}_g(S, T)$ embeds into the space of lifts in the diagram

$$\begin{tikzcd} & M_{\mathrm{Gr}} \arrow[d, "f"] \\ {[0,n]} \arrow[ur, dashed] \arrow[r, hook] & \mathbb{R} \arrow[r, two heads] & \mathbb{R}/\mathbb{Z}, \end{tikzcd}$$

topologized as a subspace of the mapping space, where again $n = g + |S| - |T|$.

This association of paths in M_{Gr} to morphisms in $\mathcal{C}$ sends composition of morphisms to concatenation of paths. This can be used to define a filtered equivalence $B\mathcal{C} \to M_{\mathrm{Gr}}$.

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Normal invariants of nearby Lagrangians

Noah Porcelli

(joint work with Ivan Smith)

Background. One of the primary questions in symplectic geometry is to classify Lagrangian submanifolds of a symplectic manifold (X, ω), up to diffeomorphism or isotopy. We often restrict to the case $\omega = d\lambda$ is exact, and study *exact* Lagrangians $L \subseteq X$, meaning $\lambda|_L \in \Omega^1(L)$ is also exact.

The cases $X = \mathbb{C}^n$ and $X = T^*Q$ (for Q any closed manifold) play special roles in the Lagrangian classification problem: Darboux' theorem states $\mathbb{C}^n$ provides a local model near any point in any symplectic manifold, and Weinstein's theorem that T^*Q provides a local model near any Lagrangian diffeomorphic to Q.

In 1985, Gromov [8] introduced holomorphic curve techniques into symplectic geometry, and proved that there are no closed exact Lagrangians in $\mathbb{C}^n$.

In cotangent bundles, we have a conjectural classification:

Nearby Lagrangian Conjecture. *Let $L \subseteq T^*Q$ be a closed exact Lagrangian submanifold. Then L is isotopic to the zero-section, through exact Lagrangians.*

In particular, L is diffeomorphic to Q.

Abouzaid and Kragh [1, 10] proved that any such Lagrangian L must be homotopy equivalent to Q. This was later refined to *simple* homotopy equivalence [4], with a parametrised version being proved using different methods in [7].

Given a simple homotopy equivalence $f : M \to N$ of closed manifolds, the next invariant one should compute if one wants to prove M is diffeomorphic to N is the *normal invariant* $\mathcal{NI}(f) \in [N, G/O]$; if N is simply-connected of even dimension at least 8, vanishing of $\mathcal{NI}(f)$ is sufficient to guarantee diffeomorphism. The normal invariant is bordism-theoretic in nature: for example, $\mathcal{NI}(f)$ vanishes if there is a framed bordism from M to N living over N.

Theorem ([3],[15])**.** *Let $L \subseteq T^*Q$ be a closed exact Lagrangian submanifold. Then the normal invariant of the homotopy equivalence $L \to Q$ is 2-torsion.*

In fact, both [3, 15] proved related but slightly different refinements of this, using totally different methods. [3] uses methods from parametrised Morse theory, whilst [15] uses Floer homotopy theory and spectral Fukaya categories, developed in various forms in [11, 12, 2, 13, 6, 14, 15, 5].

Fukaya categories. Let X be a Liouville domain (e.g. $X = T^*Q$). The *integral Fukaya category* of X has as objects closed exact Lagrangians L[1] equipped with a rank-1 $\mathbb{Z}$-local system ξ. Morphism groups consist of Floer homology groups, and are constructed by considering certain smooth compact moduli spaces of holomorphic bigons, and counting their 0-dimensional components. These effectively detect cohomological data, and one of the main results of [1, 10] is that any nearby Lagrangian is isomorphic to the zero section. In general, using a trace-type map one can show that isomorphic Lagrangians determine the same fundamental class in homology of X.

New techniques. A key technical input is the *spectral Fukaya category.* Given a Thom ring spectrum $M\Theta$ and a polarisation on X, there is a *spectral Fukaya category.* An object is a Lagrangian (satisfying some conditions on TL, depending on Θ), equipped with a rank-1 $M\Theta$-local system ξ, and morphisms are $\pi_* M\Theta$-modules.

In [15], it is proved (very roughly) that if $L \cong K$ are isomorphic objects in the integral Fukaya category and L can be lifted to the spectral Fukaya category, then the isomorphism can be lifted to an isomorphism $L \cong (K, \xi)$ in the spectral Fukaya category, for some rank-1 $M\Theta$-local system ξ.

[1]Satisfying some technical assumptions.

Combined with a computation in [14], this implies L and K have the same fundamental classes in $M\Theta_*(X)$, up to an extra term depending on ξ, which is always 2-torsion. Here the 2-torsion arises in spirit because to the special role of the Hopf element $\eta \in \pi_1(\mathbb{S}) \cong \mathbb{Z}/2$ in the trace map in K-theory, applied to the space of rank-1 $M\Theta$-local systems [16].

Further applications in [12, 15] of these methods include constraints on symplectic mapping class groups, as well as constraints on the homotopy types and smooth structures of Lagrangians in some other symplectic manifolds such as plumbings of cotangent bundles.

Outlook. The main bottleneck in applying methods of [12, 13, 14, 15] to larger classes of examples in the exact setting is that we effectively rely on bootstrapping from computations in integral Fukaya categories, a hard but classical problem.

One of the key steps in passing between the geometry of holomorphic curves and stable homotopy theory is the manifold-theoretic construction of a category equivalent to $M\Theta$-modules [9] (cf. also [13] for a weak prototype and [2] for the framed case). There are many natural further directions not yet in the literature, including manifold-theoretic constructions of $\mathbb{E}_\infty$-structures, equivariant or global-equivariant analogues, or incorporating Baas-Sullivan singularities.

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WDVV equation over a field

Kirsten Wickelgren
(joint work with Erwan Brugallé, Johannes Rau)

A degree d, genus 0, plane curve is an algebraic map $f : \mathbb{P}^1 \to \mathbb{P}^2$ where f is given by $f[s,t] = [f_0(s,t), f_1(s,t), f_2(s,t)]$ for f_i homogenous polynomials of degree d. So in degree 1, such curves are lines, and in degree 2, they are conics. There is a unique line through 2 points in the plane $\mathbb{P}^2(\mathbb{C})$, and a unique conic through 5. These counts generalize to a count N_d of degree d, genus 0, plane curves passing through $3d-1$ points. Surprisingly, the problem of determining these numbers is connected to mathematical physics, and it was not until the 1990's that it was completely solved. For example, Kontsevich determined them with a celebrated recursive WDVV equation [10]

$$N_d = \sum_{d_1+d_2=d, d_i>0} N_{d_1} N_{d_2} (d_1^2 d_2^2 \binom{3d-4}{3d_1-2} - d_1^3 d_2 \binom{3d-4}{3d_1-1}). \tag{1}$$

This equation results from the following geometric picture. Choose distinct lines L_1, L_2 in $\mathbb{P}^2_{\mathbb{C}}$ and points $a_3, \ldots, a_{3d}$ in general position. Let $\mathcal{M}_{0,3d}^{L_i,a_i}(\mathbb{P}^2, d)$ denote the moduli stack parametrizing $(f : C \to \mathbb{P}^2, p_1, \ldots, p_{3d})$ where C is a smooth, genus 0 curve; f is a degree d plane curve; and the p_i are points on C such that $f(p_i)$ lies on L_i for $i = 1, 2$, and $f(p_i) = a_i$ for $i = 3, 4, \ldots, 3d$. Let $r : \mathcal{M}_{0,3d}^{L_i,a_i}(\mathbb{P}^2, d) \to \mathbb{P}^1$ denote the cross-ratio map, defined to be the image of p_3 under the unique Möbius transformation taking p_1 to 0, p_2 to ∞, and p_4 to 1. The moduli space $\mathcal{M}_{0,3d}^{L_i,a_i}(\mathbb{P}^2, d)$ admits a compactification $\overline{\mathcal{M}}_{0,3d}^{L_i,a_i}(\mathbb{P}^2, d)$ parametrizing stable maps $(f : C \to \mathbb{P}^2, p_1 \ldots, p_{3d})$, meaning C is a tree of $(\mathbb{P}^1)$'s, the p_i are in the smooth locus, and any component sent to a point by f has at least three nodes or marked points. The map r extends to

$$\overline{r} : \overline{\mathcal{M}}_{0,3d}^{L_i,a_i}(\mathbb{P}^2, d) \to \mathbb{P}^1$$

and, thanks to the compactification, the cardinality of $\overline{r}^{-1}(0)$ equals the cardinality of $\overline{r}^{-1}(1)$. One can see that the cardinality of $\overline{r}^{-1}(0)$ is

$$|\overline{r}^{-1}(0)| = \sum_{d_1+d_2=d, d_i>0} N_{d_1} N_{d_2} d_1^2 d_2^2 \binom{3d-4}{3d_1-2}$$

in the following manner. Consider the points of $\overline{r}^{-1}(0)$. A stable map in $\overline{r}^{-1}(0)$ has cross ratio 0, which (generally) implies that $f : C_1 \cup C_2 \to \mathbb{P}^2$ where p_1 and p_3 lie on C_1 and p_2 and p_4 lie on C_2. See Figure 1.

Such a map is formed from two maps $f_i : C_i \to \mathbb{P}^2$ with the degree of f_i equal to d_i for $i = 1, 2$ such that $d_1 + d_2 = d$. There are $\binom{3d-4}{3d_1-2}$ ways to choose an additional $3d_1 - 2$ points out of the a_i for f_1 to interpolate. Then there are N_{d_1} choices of such curves f_1, times N_{d_2} choices for a degree d_2 curve interpolating the

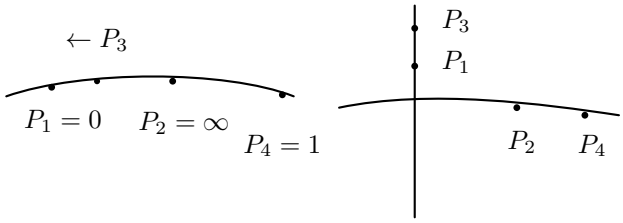

FIGURE 1. Cross ratio 0.

remaining a_i. There are $d_i = |f_i(C_i) \cap L_i|$ choices for p_i with $i = 1, 2$. Finally, there are $d_1 d_2 = |f_1(C_1) \cap f_2(C_2)|$ choices for the node $C_1 \cap C_2$. See Figure 2. One can see similarly that

$$|\overline{r}^{-1}(1)| = N_d + \sum_{d_1+d_2=d, d_i>0} N_{d_1} N_{d_2} d_1^3 d_2 \binom{3d-4}{3d_1-1}$$

showing (1).

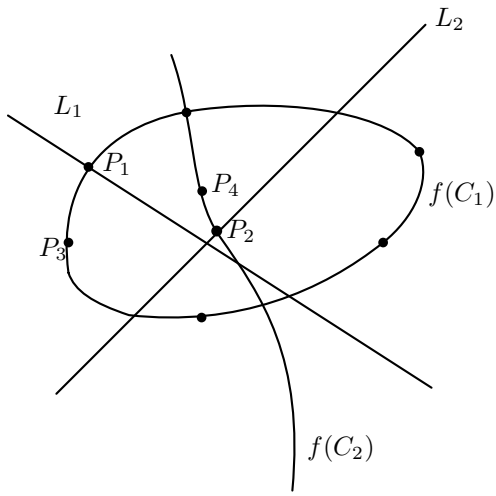

FIGURE 2. Stable maps in $\overline{r}^{-1}(0)$.

Such formulas are valid when the rational curves are defined with complex coefficients. Some of the solutions may be real, or integral, or defined over Q[i], but the fixed count only sees the number of curves over the algebraic closure. $\mathbb{A}^1$-homotopy theory of schemes provides a framework to record arithmetic information about the field of definition of the curves or solutions. Morel defined an $\mathbb{A}^1$-degree homomorphism

$$\deg^{\mathbb{A}^1} : [\mathbb{P}^n/\mathbb{P}^{n-1}, \mathbb{P}^n/\mathbb{P}^{n-1}] \to GW(k)$$

from the homotopy classes of endormophisms of $\mathbb{P}^n/\mathbb{P}^{n-1}$ valued in the Grothendieck–Witt group $GW(k)$ of a field k. The Grothendieck–Witt group is the group completion of the semiring of isomorphism classes of non-degenerate, symmetric bilinear forms $\beta : V \times V \to k$ over k under perpendicular direct sum. Over a field k, $GW(k)$ is generated by the class of rank 1 forms $\langle a\rangle : k \times k \to k$ defined

$$\langle a\rangle(x, y) = axy$$

for a in k^*. There is a rank homomorphism rank : $GW(k) \to \mathbb{Z}$ taking β to the dimension of V. For $k \subset \mathbb{R}$, there is a signature homomorphism sign : $GW(k) \to \mathbb{Z}$ taking β to $p-n$ where $\beta\otimes\mathbb{R} = p\langle 1\rangle - n\langle -1\rangle$. Morel's degree of a map f is related to the topological degrees of both the induced maps on the complex and real points of schemes by

$$\text{rank deg}^{\mathbb{A}^1} f = \text{deg } f(\mathbb{C}) \qquad \text{sign deg}^{\mathbb{A}^1} f = \text{deg } f(\mathbb{R}).$$

For $k \subseteq L$ a separable field extension, there is a transfer map $\mathrm{Tr}_{L/k} : GW(L) \to GW(k)$ taking $\beta : V \times V \to L$ to $V \times V \xrightarrow{\beta} L \xrightarrow{\mathrm{Tr}} k$, where $L \xrightarrow{\mathrm{Tr}} k$ is the trace map (defined by setting $\mathrm{Tr}(\ell)$ to be the trace of the k linear transformation $L \to L$ given by multiplication by ℓ).

In joint work with Kass, Levine, and Solomon [8] [9], we define a count $N_d^{\mathbb{A}^1}$ of degree d plane curves valued in $GW(k)$ for k a perfect base field of characteristic not 2 or 3. Because of the relationship with quadratic forms, enumerative results enriched in $GW(k)$ can be called *quadratically enriched*. More specifically, fix an étale k algebra A of degree $3d-1$. Then A is a product of fields $A \cong \prod_{i=1}^{j} K_i$. Consider the degree d plane curves passing through any generally chosen points $p_1, \ldots, p_j$ with $k(p_i) \cong K_i$. Assign a weight in $GW(k)$ to each of these curves. For $f : \mathbb{P}^1 \to \mathbb{P}^2$, the weight is defined to be $\mathrm{Tr}_{k(f)/k} \prod_{p \text{ node}} \langle m(p)\rangle$. In words, this weight is the transfer of the product of over the nodes p of $f(\mathbb{P}^1)$ of $\langle m(p)\rangle$, where $m(p)$ is defined by the field of definition of the two tangent directions of the nodes being $k(p) \subset k(p)[\sqrt{m(p)}]$. The sum over the genus 0, degree d plane curves through $p_1, \ldots, p_j$ of the weights equals $N_d^{\mathbb{A}^1}(A)$ in $GW(k)$

$$N_d^{\mathbb{A}^1}(A) = \sum_{\substack{f \text{ degree } d \\ \text{rational plane curve} \\ \text{through the points} \\ p_1,\ldots,p_j}} \mathrm{Tr}_{k(f)/k} \prod_{p \text{ node of } f(\mathbb{P}^1)} m(p). \tag{2}$$

In particular, this sum does not depend on the choice of $p_1, \ldots, p_j$. For any A, the rank of $N_d^{\mathbb{A}^1}$ recovers N_d,

$$\text{rank } N_d^{\mathbb{A}^1} = N_d,$$

and N_d is an example of a Gromov–Witten invariant. For $k \subset \mathbb{R}$, the signature of $N_d^{\mathbb{A}^1}$ recovers Welschinger's invariant signed count of real rational plane curves [12] [1]. Welschinger's invariants $\mathrm{Wel}_s(\mathbb{P}^2, d)$ depend on an integer $s \leq (3d-1)/2$. The invariant $\mathrm{Wel}_s(\mathbb{P}^2, d)$ is a signed count of the degree d, genus 0 plane curves passing through generally chosen points consisting of s pairs of complex conjugate

points and $3d-1-2s$ real points. For $A \otimes_k \mathbb{R} \cong \mathbb{C}^s \times \mathbb{R}^{3d-1-2s}$, we have

$$\text{sign } N_d^{\mathbb{A}^1} = \text{Wel}_s(\mathbb{P}^2, d).$$

Welschinger invariants are also open Gromov–Witten invariants. Solomon found WDVV equations for open Gromov–Witten invariants [11].

Example: For $d = 4$, we have $N_4 = 620$, and $\text{Wel}_s(\mathbb{P}^2, 4) = 240, 144, 80, 40, 16, 0$ for $s = 0, 1, 2, 3, 4, 5$ respectively. The quadratically enriched count for degree 4 and $A = k^7 \times k[\sqrt[4]{d}]$ is given

$$N_4^{\mathbb{A}^1}(k^7 \times k[\sqrt[4]{d}]) = 32 + 238h + 22\langle d\rangle + 80\langle 2\rangle + 10\langle 2d\rangle,$$

where $h = \langle 1\rangle + \langle -1\rangle$ denotes the hyperbolic form. Note that $\text{rank} N_4^{\mathbb{A}^1}(k^7 \times k[\sqrt[4]{d}]) = 32 + 2 * 238 + 22 + 80 + 10 = 620 = N_4$. For $k = \mathbb{Q}$ and $d > 0$, we can also compute the signature, $\text{sign} N_4^{\mathbb{A}^1}(k^7 \times k[\sqrt[4]{d}]) = 144 = \text{Wel}_1(\mathbb{P}^2, 4)$. Note that for d positive, we have $(\mathbb{Q}^7 \times \mathbb{Q}[\sqrt[4]{d}]) \otimes_{\mathbb{Q}} \mathbb{R} \cong \mathbb{C} \times \mathbb{R}^9$, so we take $s = 1$.

In fact, analogous quadratically enriched Gromov–Witten invariants $N_{X,d}^{\mathbb{A}^1}(A)$ of degree d, genus 0 curves on X are defined for any $\mathbb{A}^1$-connected del Pezzo surface X over k of degree $K_X \cdot K_X \geq 4$ and any $d \in \text{Pic } X$. With some further restrictions, the definition extends for del Pezzo surfaces X with $K_X \cdot K_X \geq 2$. See [8] [9].

In joint work with Brugallé and Rau, we completely determine $N_{X,d}^{\mathbb{A}^1}(A)$ when $K_X \cdot K_X \geq 6$ and X is k-rational and we provide WDVV equations [6] [5] [4] [3]. To explain the quadratically enriched WDVV equations, consider again the WDVV equation (1). In the derivation of this equation, a degree d curve through a set of points degenerates into two curves of degrees d_1 and d_2. It is important to choose some subset of the points to interpolate with the degree d_1 curve. The rest of the points are interpolated by the genus 0 curve of degree d_2. In the quadratically enriched setting, the curve of degree d runs through a set of points Spec $A \hookrightarrow \mathbb{P}^2$. This leads to the question:

Question: From an étale k-algebra of rank n, in how many ways can we choose an étale k-algebra of rank j?

Here is an answer. Galois theory can be rephrased as an equivalence of categories

$$F : \text{Et}_k \xrightarrow{\simeq} G_k - \text{FSet}$$

between the category Et_k of finite étale k-algebras and the category $G_k - \text{FSet}$ of finite sets equipped with a continuous action of the absolute Galois group. For a set S of size n, the number of ways to choose a subset of size j is the cardinality of

$$\binom{S}{j} := \{S' \subset S : |S'| = j\}.$$

When S is equipped with a continuous G_k-action, so is $\binom{S}{j}$. Thus for a finite étale k-algebra A, we define another finite étale k-algebra

$$\binom{A}{j} := F^{-1}\binom{F(A)}{j}.$$

In Grothendieck–Witt, the corresponding binomial coefficient is

$$\binom{A}{j}^{GW} := \mathrm{Tr}_{\binom{A}{j}/k}\langle 1\rangle.$$

The binomial coefficients $\binom{A}{j}^{GW}$ respect pullback, so it is useful to consider them as Witt invariants in the sense of Serre, which are defined as follows. Let Et_n be the functor from the category of fields over k to the category sets which takes a field extension L to the set of finite étale L-algebras A of degree n. Let W also be a functor from fields over k to sets. W takes K to $W(K)$, defined to be the Witt group of K, given by $W(K) = \mathrm{GW(K)}/(\mathbb{Z}h)$. (The Witt group is a ring because $\mathbb{Z}h$ is an ideal.) Serre defined a *Witt invariant* to be a natural transformation

$$\mathrm{Et}_n \to \mathrm{W}.$$

For example $A \mapsto \binom{A}{j}^{GW}$ is a Witt-invariant, which we denote $\binom{\mathrm{Et}_n}{j}$. Let Inv_n denote the ring of all Witt invariants, where the addition and multiplication are pointwise. We promote the Witt invariant $\binom{\mathrm{Et}_n}{j}^{GW}$ to a product,

$$(\times) : \mathrm{Inv}_{n_1} \times \mathrm{Inv}_{n_2} \to \mathrm{Inv}_{n_1+n_2}$$

as follows. Note that the fiber of $\binom{A}{j} \otimes A$ consists of the set of pairs $\{(S', x) : S' \subseteq F(X), |S'| = j, x \in F(X)\}$. This fiber can be expressed as the disjoint union

$$F(\binom{A}{j} \otimes A) = \{(S', x) : x \in S'\} \coprod \{(S', x) : x \notin S'\}$$

of those (S', x) where x is in S' and those pairs for which x is not in S'. It follows that

$$\binom{A}{j} \otimes A \cong A_1 \times A_2$$

where A_1, A_2 are finite étale algebras over $\binom{A}{j}$ of degrees j and $n-j$ respectively. We may thus make the following definition.

Definition: For α_i in Inv_{n_i} for $i = 1, 2$, and $A \in \mathrm{Et}_{n_1+n_2}(K)$, let $\alpha_1(\times)\alpha_2$ in $\mathrm{Inv}_{n_1+n_2}$ be the Witt invariant defined by

$$\alpha_1(\times)\alpha_2(A) = \mathrm{Tr}_{\binom{A}{n_1}/K}(\alpha_1(A_1)\alpha_2(A_2)).$$

Example: $\binom{\mathrm{Et}_{n_1}}{j_1}(\times)\binom{\mathrm{Et}_{n_2}}{j_2} = \binom{\mathrm{Et}_{n_1+n_2}}{j_1, n_1-j_1, j_2, n_2-j_2}$ where the right hand side is the Witt invariant corresponding to the multinomial coefficient $\binom{n_1+n_2}{j_1, n_1-j_1, j_2, n_2-j_2}$ in a manner analogous to the above. In particular $1(\times)1 = \binom{\mathrm{Et}_{n_1+n_2}}{n_1}$.

In Solomon's work on WDVV equations for open Gromov–Witten invariants, it is important to count curves interpolating pairs of complex conjugate points. This leads to the following adaptation of the previous question:

Question: For an étale k-algebra of rank $2j$, in how many ways can we choose a decomposition into a pair of étale k-algebras of rank j which are conjugate under $\mathrm{Gal}(k[\sqrt{d}]/k)$?

Here is an analogous answer. Let A be an étale k-algebra of degree $2j$. The G_k-set $\binom{F(A)}{j}$ has an additional action of $\mathbb{Z}/2$ given by taking a subset $S' \subset F(A)$ of size j to its complement $F(A) \setminus S'$. This action commutes with the action of G_k, giving rise to an action of $\mathbb{Z}/2 \times G_k$ on $\binom{F(A)}{j}$. We define a twisted G_k-action on $\binom{F(A)}{j}$ by having G_k act through the action of $\mathbb{Z}/2 \times G_k$ by the homorphism

$$G_k \to \mathrm{Gal}(k[\sqrt{d}]/k) \times G_k \to \mathbb{Z}/2 \times G_k$$

where the first map is the product of the canonical quotient and the identity, and the second is the product of the canonical map $\mathrm{Gal}(k[\sqrt{d}]/k)$ and the identity. Let $\binom{F(A)}{j}^{\sqrt{d}}$ denote the set $\binom{F(A)}{j}$ together with its twisted action of G_k. We define

$$\binom{A[\sqrt{d}]}{j} := F^{-1}\Big(\binom{F(A)}{j}^{\sqrt{d}}\Big)$$

and a corresponding twisted binomial coefficient

$$\binom{\mathrm{Et}_{2j}\sqrt{\mathrm{Et}_2}}{j}^{GW} : \mathrm{Et}_{2j} \times \mathrm{Et}_2 \to GW$$

$$A \times K[\sqrt{d}] \mapsto \mathrm{Tr}_{\binom{A[\sqrt{d}]}{j}/K}\langle 1\rangle.$$

We have a canonical decomposition $\binom{A[\sqrt{d}]}{j} \otimes_K k[\sqrt{d}] \otimes_K A \cong A_1 \times A_2$ of finite étale algebras over $\binom{A[\sqrt{d}]}{j} \otimes_K K[\sqrt{d}]$. Let $N^{\sqrt{d}} : \mathrm{GW}(\binom{A[\sqrt{d}]}{j} \otimes_K K[\sqrt{d}]) \to \mathrm{GW}(\binom{A[\sqrt{d}]}{j})$ be Rost's multiplicative norm on the Grothendieck–Witt group. We define an associated operation:

Definition: For α in Inv_j and $d \in k^*$, let $\alpha^{\sqrt{d}}$ in Inv_{2j} be the Witt invariant defined

$$\alpha^{\sqrt{d}}(A) = \mathrm{Tr}_{\binom{A[\sqrt{d}]}{j}/K} N^{\sqrt{d}}\alpha(A_1).$$

We introduce some auxiliary operations on Witt-invariants to state our theorem: For α in Inv_n, define

- $\widetilde{\alpha}$ in Inv_{n-1} by $\widetilde{\alpha}(A) = \alpha(A \times K)$ for $A \in \mathrm{Et}_{n-1}(K)$
- $\alpha^{[1]}$ in Inv_{n-2} by $\alpha^{[1]}(A) = \alpha(A \times K[\sqrt{-1}])$ for $A \in \mathrm{Et}_{n-2}(K)$
- $\alpha^{(1)}$ in Inv_{n-2} by $\alpha^{(1)} = \widetilde{\widetilde{\alpha}}$.

Theorem 1 (Brugallé-Rau-W. [6]) Let k be a perfect field of characteristic not 2 or 3. Let X be a k-rational del Pezzo surface of degree ≥ 6. Then there is a complete calculation of $N^{\mathbb{A}^1}_{X,d}(A)$ and a WDVV equation. For example, let $X = \mathbb{P}^2$, and let N_d abbreviate the Witt invariant $N^{\mathbb{A}^1}_{X,d}$. Then the following WDVV equations hold and completely determine N_d.

$$\widetilde{N}^{[1]}_d = \widetilde{N}^{\sqrt{-1}}_{d/2}(d/2)^2(-1)^{d/2+1} + \sum_{\substack{d=d_0+2d'\\ d'\geq 1}} (-1)^{d'+1}(d')^3 d_0 N^{\sqrt{-1}}_{d'}(\times)\widetilde{N}_{d_0} + \sum_{\substack{d=d_1+d_2\\ d_1,d_2\geq 1}} \left(d_1 d_2 \widetilde{N}_{d_1}(\times)\widetilde{N}_{d_2} - d_1^2 N_{d_1}(\times) N^{(1)}_{d_2}\right)$$

$$N^{[2]}_d = \sum_{\substack{d=2d_1+d_2\\ d_1,d_2\geq 1}} d_1^2 d_2^2 (-1)^{d_1} (\widetilde{N})^{\sqrt{-1}}_{d_1}(\times) N_{d_2} + \sum_{\substack{d=2d_1+d_2\\ d_1,d_2\geq 1}} (-1)^{d_1-1} d_1^3 d_2 N^{\sqrt{-1}}_{d_1}(\times) N^{[1]}_{d_2} + \sum_{\substack{d=d_1+d_2\\ d_1,d_2\geq 1}} d_1 d_2 \widetilde{N}_{d_1}(\times) N^{[1]}_{d_2} - \sum_{\substack{d=d_1+d_2\\ d_1,d_2\geq 1}} d_1^2 N_{d_1}(\times)\widetilde{N}^{[1]}_{d_2}.$$

The proof of this theorem uses work of Jaramillo Puentes, Markwig, Pauli, and Röhrle along with Jake Solomon's WDVV equations for open Gromov–Witten invariants in an essential manner.

Example: For $\delta \in k$, let $Q(\delta) \hookrightarrow \mathbb{P}^3$ denote the scheme determined by the quadric equation $x^2 - y^2 + z^2 - \delta w^2 = 0$. See Figure 3.

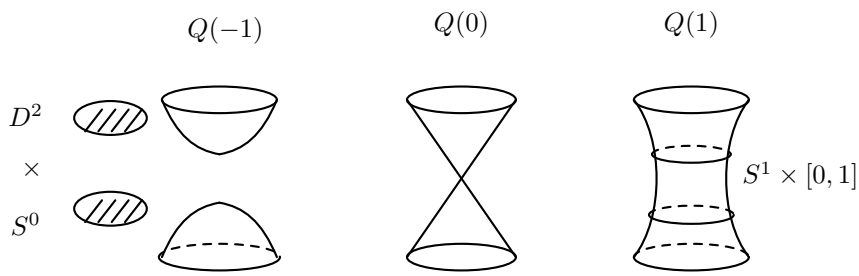

FIGURE 3. Real points of quadrics.

Then

$$N^{\mathbb{A}^1}_{Q(\delta),4\mathcal{O}(1)}(k^7) = 1212h + 255\langle 2\delta\rangle + 831\langle 2\rangle.$$

Note that the discriminant of $N^{\mathbb{A}^1}_{Q(\delta),4\mathcal{O}(1)}(k^7)$ is a multiple of an odd power of δ. It follows that $N^{\mathbb{A}^1}_{Q(\delta),4\mathcal{O}(1)}(k^7)$ must be ramified at primes dividing δ an odd number of times.

Given a ramified Witt-invariant, if the geometric reason for the ramification can be identified, then this Witt invariant can be replaced by a multi-variable one with additional variables controlling the ramification. For example, the proof of Theorem 1 replaces the Witt invariant $N^{\mathbb{A}^1}_{Q(\delta),4\mathcal{O}(1)}$ by an unramified multivariable Witt invariant

$$\mathrm{Et}_7 \times \mathrm{Et}_2 \to \mathrm{GW}$$

which can be evaluated at $(k^7, k[\sqrt{\delta}])$ to recover $N^{\mathbb{A}^1}_{Q(\delta),4\mathcal{O}(1)}$. These unramified Witt invariants come from considering the moduli space of all k-rational del-Pezzo surfaces of a given degree. The functor Et_7 encodes the point configeration (with the étale algebra A above) and the functor Et_2 encodes the del Pezzo surface X.

We describe one additional part of the proof. At the "boundary" of the moduli space of del Pezzo surfaces are surfaces with a single curve E of self-intersection -2. Working over the real numbers $\mathbb{R}$, there are "two directions" to leave such a surface $\widetilde{X}$ and travel into the space of del Pezzo surfaces. These directions can be distinguished by the topology of the real points of the surfaces in these two directions, which can be expressed

$$X_{-1} \simeq \widetilde{X}(\mathbb{R}) \#_E Q_{-1}(\mathbb{R}) \qquad\qquad X_1 \simeq \widetilde{X}(\mathbb{R}) \#_E Q_1(\mathbb{R})$$

by removing a small neighborhood of $E(\mathbb{R})$ from $\widetilde{X}(\mathbb{R})$, removing a small neighborhood of a $(1,1)$-curve from $Q(\pm 1)(\mathbb{R})$, and gluing together along the boundaries. Over a field k, there are directions associated to every δ in $k^*/(k^*)^2$ in a certain sense, and del Pezzo surfaces X_δ, which one might label as $X_\delta = \widetilde{X} \#_E Q_\delta$. See [3] [1] [2] for a rigorous discussion. The weighted count of degree d, genus 0 curves on X_δ analogous to (2) can be shown to equal a weighted count of reducible curves on $\widetilde{X}$ and $Q(\delta)$. In these counts, it is useful to introduce notation $N^{\sigma,\mathbb{A}^1}_{\tilde{X},d}$ for a weighted count of genus 0 curves $f : C \to \widetilde{X}$ interpolating the chosen points which meet the line E in some given étale algebra σ. In contrast to the other weighted counts discussed here, which can be expressed as a degree or a cohomology class, these counts $N^{\sigma,\mathbb{A}^1}_{\tilde{X},d}$ depend on the chosen points, rendering their computation effectively impossible (from this point of view). However, $N^{\mathbb{A}^1}_{X_\delta,d}$ can be expressed in terms of the uncomputable $N^{\sigma,\mathbb{A}^1}_{\tilde{X},d}$. The identity

$$\binom{A[\sqrt{\delta}]/k}{j}^{GW} = \binom{A/k}{j}^{GW} \langle \delta^j \rangle + (\langle 2 \rangle - \langle 2\delta \rangle) \sum_{l=0}^{j-1} (-1)^l \binom{A/k}{l}^{GW}$$

in quadratically enriched binomial coeffficients allows one to express $N^{\mathbb{A}^1}_{X_\delta,d}$ in terms of $N^{\mathbb{A}^1}_{X_1,d}$ by making all the uncomputable terms cancel!

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Moments in families of L-functions over function fields via homotopy theory

Dan Petersen

(joint work with Bergström–Diaconu–Westerland and Miller–Patzt–Randal-Williams)

The following conjecture is due to Conrey–Farmer–Keating–Rubinstein–Snaith [1]:

Conjecture 1. *Let $\zeta(s)$ be Riemann's zeta-function. For each integer $k \geq 1$ there exists an explicit polynomial P_k of degree k^2 such that*

$$\frac{1}{T}\int_0^T |\zeta(\tfrac{1}{2}+it)|^{2k}\mathrm{d}t = P_k(\log T) + o(1).$$

In fact, this conjecture is a special case of a general "recipe" that is supposed to apply also to moments in *families of L-functions.* One well-studied family is the quadratic family, associated to the family of number fields $\mathbf{Q}(\sqrt{d})$ as d varies.

Conjecture 2. *For each integer $k \geq 1$ there exists an explicit polynomial Q_k of degree $k(k+1)/2$ such that*

$$\frac{1}{N}\sum_{\substack{|d|\leq N\\ d \text{ squarefree}}} L(\tfrac{1}{2},\chi_d)^k = Q_k(\log N) + o(1).$$

Conjecture 2 admits an exact function-field analogue.

Conjecture 3. *Let q be an odd prime power. For each integer $k \geq 1$ there exists an explicit polynomial $R_{k,q}$ of degree $k(k+1)/2$ such that*

$$\frac{1}{q^{2g+1}} \sum_{\substack{d \in \mathbf{F}_q[t] \\ d \text{ squarefree, monic} \\ \deg d = 2g+1}} L(\tfrac{1}{2}, \chi_d)^k = R_{k,q}(g) + o(1).$$

The goal of the two papers [2, 3] is to prove Conjecture 3 for q sufficiently large with respect to k, using methods of homological stability and homotopy theory. More explicitly, we prove the equality in Conjecture 3 with an error term of the form $O(4^{g(k+1)} q^{-Ag+B})$, with A and B positive constants. Once q is large enough, we get a power-saving error term. In particular, the error term is good enough that we "see" all coefficients of all the polynomials R_k. This is perhaps the strongest vindication yet that the Conrey–Farmer–Keating–Rubinstein–Snaith "recipe" is correct.

Here are some ideas of the proof. The left-hand side of Conjecture 3 can be interpreted homologically, by the Grothendieck–Lefschetz trace formula, as

$$\mathrm{Tr}\big(\mathrm{Frob}_q \mid H_*^{\acute{e}t}(\mathrm{Conf}_{2g+1}(\mathbb{A}^1)_{\overline{\mathbf{F}}_q}; \mathcal{L}_k)\big),$$

where $\mathrm{Conf}_n(\mathbb{A}^1)$ is the configuration space of n distinct unordered points on the affine line,[1] and $\mathcal{L}_k$ is a certain (étale) local system, depending on k. The local system $\mathcal{L}_k$ lifts to characteristic zero, and one has a comparison isomorphism

$$H_*^{\acute{e}t}(\mathrm{Conf}_{2g+1}(\mathbb{A}^1)_{\overline{\mathbf{F}}_q}; \mathcal{L}_k) \cong H_*^{sing}(\mathrm{Conf}_{2g+1}(\mathbf{C}); \mathcal{L}_k) = H_*(\beta_{2g+1}; \mathcal{L}_k),$$

where β_n denotes the braid group on n strands. Thus the left-hand side of Conjecture 3 is expressed as a question about homology of braid groups as the number of strands goes to infinity.

This is a problem that screams out be studied via homological stability.

The local system $\mathcal{L}_k$ decomposes into summands V_λ, indexed by irreducible representations of the symplectic group. The multiplicity of V_λ in $\mathcal{L}_k$ depends on k and n, but the family of summands $\{V_\lambda\}$ are the same regardless.[2] It is thus natural to study each summand $H_*(\beta_n; V_\lambda)$ individually.

What is done in [2]**.** The very general stability machine of Randal-Williams–Wahl [4] implies that the sequence $H_*(\beta_n; V_\lambda)$ satisfies homological stability:

$$H_k(\beta_n; V_\lambda) \to H_k(\beta_{n+1}; V_\lambda) \text{ is an isomorphism for } k < \tfrac{1}{2}(n - |\lambda|). \tag{1}$$

It is thus meaningful to ask what the stable homology of $H_*(\beta_n; V_\lambda)$ is. We compute the stable homology, by mostly well-established methods. We imitate a method used by Randal-Williams to calculate the stable homology of the mapping class groups with symplectic coefficients $H_*(\mathrm{Mod}_g^1; V_\lambda)$: the trick is that these

[1] Equivalently, the moduli space of squarefree monic polynomials of degree n.

[2] The polynomial dependence on n in Conjecture 3 arises only from this: the multiplicity of V_λ in $\mathcal{L}_k$ is given by a degree $k(k+1)/2$ polynomial in n.

groups can be read off if one computes the stable homology of the space of surfaces with a map to an Eilenberg–MacLane space $K(V,n)$, where V is a $\mathbf{Q}$-vector space, considered as a functor of V. We are thus led to consider similarly the space parametrizing a configuration of points in the plane, and a map from the branched double cover of the plane branched at these points, to $K(V,n)$. These spaces are very natural to analyse using *scanning*, and one can compute the stable homology; in fact, even together with its action of $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$. The answer we obtain is compatible with Conjecture 3, but the range of stability (1) is not good enough to imply Conjecture 3. We show that Conjecture 3 would follow from our topological calculations if we knew a stable range of the form

$$H_k(\beta_n; V_\lambda) \to H_k(\beta_{n+1}; V_\lambda) \text{ is an isomorphism for } k < An - B; \tag{2}$$

what we call "uniform stability bound".

What is done in [3]**.** We prove (2). Most prior work on homological stability with twisted coefficients has focused on stability for *polynomial coefficient systems*. Easy examples show that for a typical polynomial coefficient system, the stable range depends on the degree of polynomiality. The V_λ are polynomial of degree $|\lambda|$, which gives rise to the dependence on $|\lambda|$ in (1). We thus need a new approach, not relying on polynomiality. The idea is that the coefficients V_λ are "pulled back" from the family of groups $\{\mathrm{Sp}_{2g}(\mathbf{Z})\}$, and on this family of groups they all have a uniform stable range by Borel's work on stable real cohomology of arithmetic groups [5]. We prove a general theorem that allows you to "propagate" a stable range when pulling back a coefficient system from one family of groups to another, under certain conditions. More precisely, we consider systems of groups $\mathcal{G} = \{\Gamma_n\}$ and $\mathcal{Q} = \{Q_n\}$, and representations $V(n)$ of Q_n, and ask for the following:

(1) Both $\mathcal{G}$ and $\mathcal{Q}$ form braided monoidal groupoids, satisfying the conditions of [4], both having highly connected destabilisation complexes.
(2) There is a monoidal functor $\mathcal{G} \to \mathcal{Q}$, given by a system of surjections $\Gamma_n \to Q_n$.
(3) The representations $V(n)$ assemble to a coefficient system V for the system of groups $\mathcal{Q}$, and there is a stability axiom for the *shifts* of V; that is,

$$H_*(Q_n, Q_{n-1}; \Sigma^k V(n), \Sigma^k V(n-1)) = H_*(Q_n, Q_{n-1}; V(n+k), V(n+k-1)) = 0$$

for $*$ and k in a range depending on n.

Under these hypotheses we deduce a stable range for $H_*(\Gamma_n; V(n))$ depending on the connectivities in (i) and the stable range assumed in (iii), but no polynomiality is assumed or used. In fact, in the usual approach to twisted stability, polynomiality is used precisely to control the behaviour under shifts, and (iii) in a sense replaces polynomiality by bluntly assuming good behaviour.

Condition (iii) is verified using Borel's work, combined with *branching rules* for classical groups. In fact, understanding the shifts $\Sigma^k V_\lambda$ essentially entails understanding how the irreducible representation V_λ of the symplectic group decompose when restricted to a smaller symplectic group; the decomposition is (very nearly)

independent of n, which leads to shifts $\Sigma^k V_\lambda$ having (very nearly) the same range of stability as V_λ itself.

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The six functor formalism for E-theory of X-C^*-algebras

Ulrich Bunke

This talk is based on [Bun26] and the joint work [BV] with Marco Volpe.

1. E-theory

We consider the symmetric monoidal category $C^*\mathrm{Alg}$ of C^*-algebras with the maximal tensor product. By $\mathrm{Mod}_{C^*\mathrm{Alg}}(\mathrm{Cat})_*$ we denote the category of pointed $C^*\mathrm{Alg}$-modules in Cat. Let $f : \mathcal{A} \to \mathcal{B}$ be a morphism in $\mathrm{Mod}_{C^*\mathrm{Alg}}(\mathrm{Cat})_*$.

Definition 1.1. *The morphism f is called E-admissible if it preserves finite products, bifibre sequences and filtered colimits.*

Definition 1.2. *An E-theory context is an object $\mathcal{A}$ of $\mathrm{Mod}_{C^*\mathrm{Alg}}(\mathrm{Cat})_*$ such that $A \otimes - : C^*\mathrm{Alg} \to \mathcal{A}$ for every A in $\mathcal{A}$, and $- \otimes N : \mathcal{A} \to \mathcal{A}$ for every N in $C^*\mathrm{Alg}$, are E-admissible.*

We let EC denote the category of E-theory contexts.

Let L be a symmetric monoidal category.

Definition 1.3. *A lax symmetric monoidal functor to E-theory contexts is a functor $\mathcal{A} : L \to \mathrm{EC}$ together with a lax symmetric monoidal refinement of the composition $L \xrightarrow{\mathcal{A}} \mathrm{EC} \to \mathrm{Cat}$ such that, for every l, h in L and A in $\mathcal{A}(l)$, the functor $A \otimes - : \mathcal{A}(h) \to \mathcal{A}(l \otimes h)$ is E-admissible.*

Let $\mathcal{A}$ be an E-theory context. Using the functor

$$\mathrm{CH}^{\mathrm{op}} \to C^*\mathrm{Alg}\ , \quad W \mapsto C(W)$$

and the $C^*\mathrm{Alg}$-module structure of $\mathcal{A}$, the category $\mathcal{A}$ acquires a topological enrichment, and therefore a notion of homotopy equivalence. For a one-dimensional projection p in $K(\ell^2)$ and A in $\mathcal{A}$, the map

$$A \cong A \otimes \mathbb{C} \overset{-\otimes(\lambda \mapsto \lambda p)}{\to} A \otimes K(\ell^2)$$

in $\mathcal{A}$ is called a corner embedding.

Let $\mathcal{C}$ be a cocomplete stable ∞-category and $\mathcal{A}$ be an E-theory context.

Definition 1.4. *A functor $F : \mathcal{A} \to \mathcal{C}$ is called homological if it is reduced, sends homotopy equivalences and corner embeddings to equivalences, preserves finite products and filtered colimits, and sends bifibre sequences to fibre sequences.*

Theorem 1.5. *There is a universal homological functor $\mathrm{e}_{\mathcal{A}} : \mathcal{A} \to \mathrm{E}(\mathcal{A})$.*

It is uniquely characterized by

$$\mathrm{e}_{\mathcal{A}}^* : \mathrm{Fun}^{\mathrm{colim}}(\mathrm{E}(\mathcal{A}), \mathcal{C}) \xrightarrow{\simeq} \mathrm{Fun}^{\mathrm{homol}}(\mathcal{A}, \mathcal{C})$$

for any cocomplete stable ∞-category $\mathcal{C}$. It turns out that $\mathrm{E}(\mathcal{A}) \in \mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$. One can check that the homotopy category of $\mathrm{E}(C^*\mathrm{Alg})^{\aleph_1}$ (the $\aleph_1$-compact objects in $\mathrm{E}(C^*\mathrm{Alg})$) is equivalent to the classical E-theory category introduced by Higson [Hig90a], see [Bun24].

Theorem 1.6. *If $\mathcal{A} : L \to \mathrm{EC}$ has the structure of a lax symmetric monoidal functor to E-theory contexts, then the universal homological functors, applied to the values of $\mathcal{A}$, combine to a lax symmetric monoidal functor*

$$E(\mathcal{A}(-)) : L \to \mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}} \ .$$

2. X-C^*-algebras and functor formalisms

Let X be a locally compact Hausdorff space. Following [Kas88]:

Definition 2.1. *An X-C^*-algebra is a pair (A, ϕ) consisting of a C^*-algebra A and a surjective and A-bilinear homomorphism $\phi : C_0(X) \otimes A \to A$.*

We get the category $XC^*\mathrm{Alg}$ of C^*-algebras and structure-preserving homomorphisms.

For two X-C^*-algebras (A, ϕ) and (A', ϕ') we can define a balanced tensor product whose underlying C^*-algebra is $A \otimes_{C_0(X)} A'$. This equips $XC^*\mathrm{Alg}$ with a symmetric monoidal structure. Furthermore, for a map $f : X \to Y$ of locally compact Hausdorff spaces, we obtain functors $f^* : YC^*\mathrm{Alg} \to XC^*\mathrm{Alg}$ and $f_! : XC^*\mathrm{Alg} \to YC^*\mathrm{Alg}$. The symmetric monoidal functor f^* sends (B, ψ) in $YC^*\mathrm{Alg}$ to an object of $XC^*\mathrm{Alg}$ with underlying C^*-algebra $C_0(X) \otimes_{C_0(Y)} B$. The functor $f_!$ sends (A, ϕ) in $XC^*\mathrm{Alg}$ to the object (A, ψ), where

$$\psi : C_0(Y) \otimes A \xrightarrow{f^*} C_b(X) \otimes A \xrightarrow{\hat{\phi}} A$$

and $\hat{\phi}$ is the canonical extension of ϕ.

The triple (LCH, I, P) of the category of locally compact Hausdorff spaces with the subcategories of open inclusions and proper maps is called a Nagata context.

Theorem 2.2. *The functor*

$$(-)C^*\mathrm{Alg} : \mathrm{LCH}^{\mathrm{op}} \to \mathrm{Cat}$$

is a three functor formalism on the Nagata context (LCH, I, P).

This says that the functor is lax symmetric monoidal, that f^* is a left-adjoint if $f \in P$, a right-adjoint if $f \in I$, and that the canonical projection and base-change formulas are satisfied. The respective adjoints are given by the functor $f_!$ in both cases. By the theory of [HM24], [Sch25], we know that three functor formalisms on the Nagata context (LCH, I, P) are the same as three functor formalisms on the geometric context $(\mathrm{LCH}, \mathrm{All})$.

Corollary 2.3. *The functor*

$$E(-) := E((-)C^*\mathrm{Alg}) : \mathrm{LCH}^{\mathrm{op}} \to \mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$$

is a six functor formalism on the Nagata context (LCH, I, P).

3. Comparison with sheaves

Following [Zhu25]:

Definition 3.1. *A lax symmetric monoidal functor*

$$D : \mathrm{LCH}^{\mathrm{op}} \to \mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$$

is called a coefficient system if it is a sheaf and such that closed-open decompositions of objects in LCH *induce recollements.*

Theorem 3.2. *The functor* $E : \mathrm{LCH}^{\mathrm{op}} \to \mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$ *is a coefficient system.*

By [Vol25], for every $\mathcal{C}$ in $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}})$, we have a six functor formalism

$$\mathrm{Shv}(-, \mathcal{C}) : \mathrm{LCH}^{\mathrm{op}} \to \mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}} \ .$$

It is in addition a coefficient system. Generalizing a result of [Zhu25], we have:

Theorem 3.3 ([BV])**.** *For every coefficient system D, there is a unique comparison map of coefficient systems*

$$\mathcal{B} : \mathrm{Shv}(-, D(*)) \to D(-) \ .$$

To a six functor formalism D we associate the homology theory

$$\Gamma^D : \mathrm{LCH}^{\mathrm{op}} \to D(*) \ , \quad X \mapsto p_{X,*}1_X,$$

where $p_X : X \to *$ is the projection and 1_X in $D(X)$ is the tensor unit. We say that Γ^D is finitary if, for every cofiltered system $(X_i)_{i\in I}$ in CH, we have

$$\mathrm{colim}_{i\in I}\Gamma^D(X_i) \simeq \Gamma^D(\lim_i X_i) \ .$$

We say that D is section-determined if, for every X in LCH, the family

$$(p_{U,*}j^*_{:U\to X} : D(X) \to D(*))_{U\in \mathrm{Open}(X)}$$

is jointly conservative.

Theorem 3.4 ([BV]). *If $D : \mathrm{LCH}^{\mathrm{op}} \to \mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$ is a six functor formalism which is a coefficient system, section-determined and such that Γ^D is finitary and $D(*)$ is dualizable, then the comparison map $\mathrm{Shv}(-, D(*)) \to D(-)$ is an equivalence of six functor formalisms.*

Theorem 3.5. *The six functor formalism E is section-determined, and its associated cohomology theory Γ^E is finitary.*

By [BD24] we know that $E(*)$ is dualizable.

Corollary 3.6. *We have an equivalence of six functor formalisms*

$$\mathrm{Shv}(-, E(*)) \simeq E(-) \ .$$

This result enables us to transfer results and constructions between sheaf theory and homotopy theory of C^*-algebras.

Combining with a result of [Efi24] we obtain:

Corollary 3.7. *For every X in* LCH*, the category $E(X)$ in $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$ is dualizable.*

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