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Geometrie

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Abstract. The workshop Geometrie was well attended with 47 in-person and two virtual participants whose interests covered a considerable part of current research in Differential Geometry. We had talks in such diverse subjects as geometric evolution equations, Kähler geometry, minimal surfaces, RCD spaces and positive scalar curvature metrics.

Mathematics Subject Classification (2020): 53-XX, 32Qxx, 49Qxx, 58Jxx.

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Introduction by the Organizers

There were 19 talks in total: 16 one-hour talks, and three 45-minute talks by Ph.D. students. The participants reflected a broad geographic and gender representation and contributed to a very convivial atmosphere.

The conference focused on several central themes in modern differential geometry, geometric analysis and partial differential equations (PDEs). A recurring topic throughout the talks was the deep interaction between curvature, geometric flows and nonlinear PDEs, especially in both smooth and singular settings.

Part of the program was devoted to Kähler geometry and complex Monge-Ampère equations. Jian Song's talk focused on uniform geometric estimates, such as bounds on diameter and Sobolev constants, for families of Kähler metrics under uniform L^p control of volume measures. These results extend to singular Kähler varieties with mild singularities, such as normal or klt singularities, leading to compactness theorems for families of Kähler metrics. Moreover, these techniques

connect with the theory of metric measure spaces satisfying RCD conditions, bridging complex geometry with synthetic notions of Ricci curvature.

In Tristan Collins' talk, optimal transport theory played a key role, particularly in the study of boundary regularity for solutions to the real Monge–Ampère equation. New monotonicity formulas arising from the geometric formulation of optimal transport were presented, with applications to the study of degenerations of Calabi–Yau metrics. This highlights a deep interplay between geometric PDEs, optimal transport of measures, and Kähler geometry. The geometry of the space of measures, equipped with the Hellinger–Kantorovich metric, was another interesting direction. In this setting, Theo Sturm presented explicit formulas for sectional curvature, revealing a decomposition into a classical geometric component and an additional nonnegative contribution arising from transport effects.

The program also covered aspects of Lipschitz and Alexandrov geometry, where recent work by Mohammad Alattar extends classical topological and geometric results to non-smooth settings, as well as applications to geometric topology.

Another important theme concerned Einstein manifolds with symmetry. In his talk, Christoph Böhm explained how to exploit symmetry assumptions in order to reduce the Einstein equation to simpler systems, allowing for partial classification results. Different behaviors arise depending on the sign of the Einstein constant: negative curvature leads to rigidity phenomena, and various existence results are known in the positive case. The Ricci-flat case was discussed in Mingyang Li's talk. In particular, reductions of the Einstein equations to harmonic map systems were used to construct examples of Ricci-flat manifolds and to study gravitational instantons, often producing spaces with rich topology.

Several talks focused on Ricci flow and Ricci solitons. One line of work, presented by Man-Chun Lee, studies the smoothing of singular spaces, such as polyhedral geometries, using Ricci flow, producing smooth metrics that are stable under perturbations of the singular structure. Other results of Lee and of Adam Thompson concern the construction of new examples of expanding Ricci solitons, shedding light on collapsing phenomena and the formation of singularities.

Free boundary problems were investigated from both variational and PDE perspectives through the talks of Giada Franz and Zihui Zhao. These problems, in which the domain is determined by the solution itself, exhibit intricate interactions between regularity and singularity formation. The analysis of model singularities and rigidity properties provides insight into the structure of solutions and their boundaries.

Further topics included generalizations of the positive mass theorem in geometric analysis, the use of homogenization techniques in spectral geometry to study PDEs on heterogeneous media, and the application of Morse theory to minimal hypersurfaces. In the latter case, minimal surfaces are viewed as critical points of the area functional, and their associated flow lines and instability directions are analyzed via geometric flows such as mean curvature flow.

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Workshop: Geometrie

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Abstracts

Einstein metrics with symmetry

Christoph Böhm

A Riemannian manifold (M^n, g) is called Einstein, if its Ricci tensor $\mathrm{ric}(g)$ is a multiple of Riemannian metric g, that is if

$$\mathrm{ric}(g) = \lambda \cdot g\,, \tag{1}$$

where $\lambda \in \mathbb{R}$. This is a (weakly) elliptic PDE of second order.

In dimensions $n = 2, 3$ Einstein implies constant sectional curvature and in dimension $n = 4$ there exist topological obstructions to the existence of Einstein metrics on certain compact M^4. In higher dimensions no such obstructions are known for $\lambda < 0$. Showing existence of Einstein metrics is highly non-trivial and always based on some Ansatz, like assuming Kähler Einstein, Sasaki Einstein, special holonomy, glueing or symmetry.

In this talk we will focus on symmetry assumptions. If (M^n, g) is a G-homogeneous space, that is if G acts isometrically on (M^n, g) with cohomogeneity zero (that is transitively) then (1) turns into a system of algebraic equations. By [2] we know in that then M^n must be diffeomorphic to $\mathbb{R}^n$ if $\lambda < 0$. By [1], a homogeneous Ricci flat space is flat. The case $\lambda > 0$ is wide open, even if there exists a classification up to dimension $n = 12$: see [3] where also several open questions are stated. Note that variational methods can be applied in this case, leading to general existence results for G-invariant Einstein metrics: see again [3].

We call (M^n, g) a G-cohomogeneity one manifold, if G acts isometrically and with cohomogeneity one, that is $\dim M^n/G = 1$. Then, (1) turns into a dynamical system, the cohomogeneity one Einstein ODE, which we will focus on in this talk. Note that essentially no general structure theorems exists concerning existence or non-existence.

We call (M^n, g) a G-cohomogeneity two manifold, if G acts isometrically and with cohomogeneity two, i.e. $\dim M^n/G = 2$. Then, (1) turns into a true PDE on $B = M^n/G$ a space which a smooth interior but with possible boundary. Let us mention the spectacular construction of T^2-invariant gravitational instantons $(M_i^4, g_i)_{i\in\mathbf{N}}$ (satisfying $\mathrm{ric}(g_i) = 0$) by Li and Sun [5]. Concerning compact Einstein metrics in dimension $n = 4$ it is known by [6] that a simply-connected closed 4-manifold admitting a T^2-invariant Einstein metric with nonnegative sectional curvature is locally symmetric. By [4] we obtain a classification of compact 4-manifolds whose isometry group has dimension at least 3.

We turn now back to G-cohomogeneity one Einstein manifolds (M^{n+1}, g). Then, on the regular part $I \times P^n$ of M^{n+1} (the set of principal orbits $G/H = P^n$) the metric g can be written as $dt^2 + g(t)$ where $g(t)$ is a G-invariant metric on G/H for each $t \in I$. We are mostly interested in the case that G is compact and that $I = (0, T)$ for finite T (implying that M^{n+1} is compact as well) or that $I = (0, \infty)$ (implying that M^{n+1} is non-compact). We denote by $L(t)$ the shape operator of

$P_t = t \times P^n$, by $H(t) = \mathrm{tr}L(t)$ the mean curvature of P_t and by $\mathrm{Ric}(g(t))$ the Ricci endomorphism of $(P_t, g(t))$. Then (1) implies the following dynamical system

$$g'(t) = 2g(t) \cdot L(t) \tag{2}$$

$$L'(t) = -H(t) \cdot L(t) + \mathrm{Ric}(g(t)) - \lambda \cdot I_n \tag{3}$$

$$(n-1) \cdot \lambda = \mathrm{scal}(g(t)) + \mathrm{tr}(L^2(t)) - H^2(t) \tag{4}$$

The system (2),(3) is a first order ODE for $(g(t), L(t))$ and that (4) is an invariant submanifold. Thus the phase space has dimension $d = 2N - 1$, N denoting the dimension of the space of G-invariant metrics on $P^n = G/H$.

One is interested in maximal solutions of (2),(3), in particular in those solutions which can be extended smoothly to the boundary of I, meaning that they define smooth complete cohomogeneity one Einstein manifolds (M^{n+1}, g).

Question 1: Does there exist at least one complete G-invariant Einstein metric on a G-cohomogeneity one manifold M^{n+1} with $M^{n+1}/G = [0, \infty)$ and $\lambda < 0$.

It is know that there exist examples, where not all smooth G-invariant Einstein metrics on such M^{n+1} with $\lambda < 0$ are complete: see [7]. In the Ricci flat case there are examples, such that no smooth G-invariant Einstein metric is complete. Still, one can ask the same question for Ricci-flat metrics of course and hope to obtain at least classes of examples admitting at least one complete solution.

The case that $I = (0, T)$ for finite T and $\lambda > 0$ is of course much more difficult. Here, one must construct a maximal solution which can be extended smoothly at both boundary points $t = 0$ and $t = T$. There exist many very intriguing examples, but mostly only for 3-dimensional phase spaces, except one assume Kähler (or conformally Kähler).

Problem 2: Develope an existence theory for compact cohomomogeneity one Einstein manfolds, which works for arbitrary high-dimensional phase spaces.

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Ricci solitons and Harmonic maps

Adam Thompson
(joint work with Ramiro A. Lafuente)

An expanding Ricci soliton is a triple (M, g, X), where (M, g) is a Riemannian manifold and $X \in \mathfrak{X}(M)$ is a complete vector field, satisfying

$$\mathrm{Ric}(g) + \mathcal{L}_X g = -\tfrac{1}{2} g. \tag{1}$$

Expanding Ricci solitons give rise to self-similar solutions of the Ricci flow: if $\{\eta_t\}_{t\in\mathbb{R}}$ is the one-parameter family integrating X, the curve of metrics $g_t := \eta_t^* g$ satisfies the normalised Ricci flow equation

$$\partial_t\, g_t = -2\,\mathrm{Ric}(g_t) - g_t.$$

Besides generalising Einstein metrics and modelling de-singularisation of singularities that form along the Ricci flow, expanding solitons are interesting since they arise as limits of solutions that exist for all $t \in (0, \infty)$ (i.e. *immortal* solutions). These are well-studied in the Kähler case [6] and in dimension 3 [3, 4].

In general, the limit of an immortal Ricci flow (in the Gromov–Hausdorff sense) can have strictly lower dimension; that is, they may exhibit collapsing. In this case, the Gromov–Hausdorff limit loses geometric information and more geometric information is obtained by considering "unwrapped limits" [3, 5].

To our knowledge, there were no known non-trivial examples beyond the homogeneous and Kähler cases. Our main result is the existence of infinitely many such examples in the case the base $B = \Sigma^2$ is a closed orientable Riemann surface of genus ≥ 3.

Theorem 1 (Lafuente–T.)**.** *For all closed Σ^2 with* $\mathrm{genus}(\Sigma) \geq 3$*, there exists on $\mathbb{R}^n \to M^{n+2} \to \Sigma^2$ an infinite family of affine expanding Ricci solitons with a local nilpotent symmetry that acts transitively on the fibres.*

Roughly speaking, a Ricci soliton on a twisted principal bundle is called affine if the one-parameter family of diffeomorphisms given by integrating the soliton vector field are affine maps. Our main tool for proving Theorem 1 is a dimensional reduction theorem that works for general closed B. Namely, we prove that affine expanders are in one-to-one correspondence with solutions to the harmonic-Einstein equations:

$$\begin{cases} \mathrm{tr}_{g_B}(\nabla dh) & = 0\,, \\ \mathrm{Ric}(g_B) & = -\tfrac{1}{2} g_B + h^* g_{\mathsf{sym}}. \end{cases} \tag{2}$$

When $B = \Sigma^2$ is a closed oriented surface, this reduces the problem of constructing affine expanders to finding branched minimal surfaces. In this case, we can appeal to an existence result for branched minimal surfaces by Labourie [2] to prove the existence of a large family of solitons:

Corollary 1. *Assume that $\rho \in \mathrm{Hom}(\pi_1(\Sigma), \mathsf{SL}_n(\mathbb{R}))$, $n \geq 3$, is in the Hitchin component. Then, there exists a ρ-equivariant branched minimal immersion $\Sigma \to$*

$\mathsf{SL}_n(\mathbb{R})/\mathsf{SO}(n)$, *and hence* $M^{n+2} = \tilde{\Sigma} \times_\rho \mathbb{R}^n$ *admits an expanding Ricci soliton metric.*

Hitchin components are homeomorphic to an open balls of dimension $(2g-2)(n^2-1)$, where g is the genus of Σ. Since the underlying manifold $M^{n+2} = \tilde{\Sigma} \times_\rho \mathbb{R}^n$ remains constant when ρ varies in a fixed component, Corollary 1 yields the existence of a $(2g-2)(n^2-1)$-parameter family of complete expanding Ricci solitons on M^{n+2} admitting a free, local, isometric $\mathbb{R}^n$-action.

We are also able to use Hitchin's complete description [1] of the moduli space of $\mathsf{SL}_2(\mathbb{R})$-Higgs bundles to classify 4-dimensional twisted principal bundles:

Corollary 2. *Let* Σ^2 *be a closed surface of genus* $g \geq 3$. *Then, there exists a family, depending on* $10g - 14 = 4(g-2) + 6(g-1)$ *real parameters, of affine Ricci solitons on twisted principal bundles* $M^4 = \tilde{\Sigma} \times_\rho \mathbb{R}^2$ *over* Σ, *none of which is gradient or locally homogeneous.*

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X–ADM mass and X–positive mass theorem

Francesca Oronzio

(joint work with Carlo Mantegazza)

The positive mass theorem is one of the central results in scalar curvature geometry and mathematical relativity. Its classical Riemannian form asserts that a complete asymptotically flat manifold with nonnegative scalar curvature has nonnegative ADM mass, with equality only for Euclidean space. The theorem was proved by Schoen–Yau [13] using minimal hypersurfaces and by Witten [16] via spinors. Several variants have subsequently appeared, among them positive mass theorems for weighted manifolds [2, 6, 11] and mass–charge inequalities in the presence of an electric field [5, 7, 8, 9, 10, 14].

We introduce the X–ADM mass and the associated positive mass theorem (briefly, X–PMT). The X–ADM mass is a geometric extension of the ADM mass that unifies several positive mass results within a single framework. It is obtained by replacing the scalar curvature with a natural expression involving an arbitrary

vector field X, so that the ADM flux acquires the corresponding boundary contribution. Different choices of X in the X–PMT recover the previously known theorems, thereby providing the same proof for all of them. The argument uses neither minimal hypersurfaces nor spinors; instead, it is a direct potential–theoretic method based on the level sets of a Green's function for a drift Laplacian. In this sense the theorem is both a generalization and a unification of existing positive mass results through an elementary approach.

Throughout this paper, (M, g) is a smooth, connected Riemannian 3–manifold without boundary. We denote by R its scalar curvature, by ∇ its Levi–Civita connection, and by $\Delta = \mathrm{tr}\nabla^2$ the Laplace–Beltrami operator. Repeated indices are summed. We say that (M, g) is *asymptotically flat* (AF) if there exists a closed and bounded subset $K \subset M$, an open ball $B \subset \mathbb{R}^3$ centered at the origin, and a diffeomorphism $\Phi : M \setminus K \to \mathbb{R}^3 \setminus \overline{B}$ such that $(\Phi_* g)_{ij} = \delta_{ij} + O_2(|x|^{-\tau})$ for every $i, j \in \{1, 2, 3\}$ and some $\tau > 1/2$. In this case, we call $(M \setminus K, \Phi)$ an asymptotically flat chart (of order τ). We say that a smooth vector field X on an asymptotically flat manifold (M, g) is *admissible* if, in some asymptotically flat chart, $X^i = O_1(|x|^{-1-\tau_0})$ for any $i \in \{1, 2, 3\}$ and some $\tau_0 > 1/2$. If, moreover, $\mathrm{R} + 2\mathrm{div}X \in L^1(M, g)$, its X–ADM mass is

$$m_X(g) = \frac{1}{16\pi} \lim_{r\to\infty} \int_{\{|x|=r\}} \left(\partial_j g_{ij} - \partial_i g_{jj} + 2X^i\right) \frac{x^i}{|x|}\, d\sigma_{\mathrm{eucl}}. \tag{1}$$

The X–ADM mass is well defined: by an argument analogous to those used by Bartnik [3] and Chruściel [4] for the ADM mass, the limit in (1) exists, is finite, and is independent of the chosen asymptotically flat chart of order greater than 1/2.

For $k \in \mathbb{R} \setminus \{0\}$, we set

$$\mathrm{R}_X^{(k)} = \mathrm{R} + 2\mathrm{div}X - \left(1 + \frac{1}{k}\right) |X|^2.$$

This is the scalar expression naturally associated with the generalized k–Bakry–Émery tensor obtained by replacing df with $X^\flat$.

We can now state the positive mass theorem for the X–mass.

X–positive mass theorem [12]**:** Let (M, g) be an orientable, connected, complete, asymptotically flat Riemannian 3–manifold, and let X be an admissible smooth vector field. Assume that

(1) $\mathrm{R} + 2\mathrm{div}X \in L^1(M, g)$;
(2) $\mathrm{R}_X^{(k)} \geq 0$ for some $k \in \mathbb{R} \setminus (-2, 0]$;
(3) $H_2(M; \mathbb{Z})$ contains no spherical classes.

Then

$$m_X(g) \geq 0. \tag{2}$$

Moreover, the following rigidity statements hold.

(a) If $\mathrm{R}_X^{(k)} \geq 0$ for some $k \in \mathbb{R} \setminus [-2, 0]$, then $m_X(g) = 0$ if and only if $X \equiv 0$ and (M, g) is isometric to $(\mathbb{R}^3, g_{\mathrm{eucl}})$.

(*b*) If $\mathrm{R}_X^{(-2)} \geq 0$, then $m_X(g) = 0$ if and only if there exists a smooth function f such that $X = \nabla f$ and $(M, e^{-f}g)$ is isometric to $(\mathbb{R}^3, g_{\mathrm{eucl}})$.

The theorem contains, as special cases, several known results in their three dimensional, time–symmetric version: the classical positive mass theorem, the weighted positive mass theorem and the positive mass theorem with charge.

The classical positive mass theorem. If the stronger decay

$$X^i = O_1(|x|^{-1-2\tau_0}), \qquad \tau_0 > 1/2,$$

holds, the term in (1) involving the components of X vanishes, and $m_X(g) = m_{\mathrm{ADM}}(g)$. In particular, when $X \equiv 0$, the X–PMT recovers the classical 3–dimensional positive mass theorem, under the additional assumption on $H_2(M;\mathbb{Z})$. When $X \not\equiv 0$, it still yields the classical positive mass inequality under the same topological assumption, while allowing for weaker conditions on the scalar curvature. Furthermore, the vanishing of the ADM mass does not imply flatness, under the threshold condition $\mathrm{R}_X^{(-2)} \geq 0$. Indeed, there exist nontrivial examples exhibiting $X = O_1\big(|x|^{-1-2\tau_0}\big)$, $\mathrm{R}_X^{(-2)} \geq 0$, and $m_X(g) = m_{\mathrm{ADM}}(g) = 0$; by rigidity, such manifolds must be diffeomorphic to $\mathbb{R}^3$ and conformally isometric to the Euclidean space.

The weighted positive mass theorem. A weighted Riemannian manifold (M^3, g, f) is a Riemannian manifold endowed with the measure $e^{-f}d\mu$. It is said to be asymptotically flat if (M, g) is asymptotically flat and the weight f tends to zero at infinity with a suitable decay rate—more precisely, $f \in C^{2,\alpha}_{-\tau}(M)$ in the notation of Baldauf–Ozuch [2]—so that ∇f is an admissible vector field and the weighted mass is well defined under the additional assumption $\mathrm{R} + 2\Delta f \in L^1(M, g)$. Following Baldauf and Ozuch [2], the (unnormalized) weighted mass is

$$m_f(g) = \lim_{r\to\infty} \int_{\{|x|=r\}} \big(\partial_j g_{ij} - \partial_i g_{jj} + 2e^{-f}\partial_i f\big) \frac{x^i}{|x|}\, d\sigma_{\mathrm{eucl}}.$$

Notice that, under the prescribed decay for f, the factor e^{-f} in the limit may be replaced by 1. Consequently, due to the normalization (1), we have

$$m_{\nabla f}(g) = m_f(g)/(16\pi).$$

With $X = \nabla f$, the X–PMT includes, in dimension three and under its topological assumption, the weighted positive mass theorem of Law–Lopez–Santiago [11].

The positive mass theorem with charge. Let $\mathcal{E}$ be an admissible electric field. We suppose that

$$\mathrm{R} + \mathrm{div}\mathcal{E} \in L^1(M, g), \qquad \mathrm{R} - 4|\mathrm{div}\mathcal{E}| - 2|\mathcal{E}|^2 \geq 0,$$

and define the total charge by

$$\mathcal{Q} = \frac{1}{4\pi} \lim_{r\to\infty} \int_{\{|x|=r\}} \mathcal{E}^i \frac{x^i}{|x|}\, d\sigma_{\mathrm{eucl}}.$$

Under these assumptions, the positive mass theorem with charge asserts that

$$m_{\mathrm{ADM}}(g) \geq |\mathcal{Q}|. \tag{3}$$

Notice that replacing $\mathcal{E}$ with $-\mathcal{E}$ changes the sign of $\mathcal{Q}$ while leaving the assumptions unchanged; hence, we may assume without loss of generality that $\mathcal{Q} \geq 0$. Since the curvature condition yields $\mathrm{R}^{(-2)}_{-2\mathcal{E}} \geq 0$ and the identity

$$m_{-2\mathcal{E}}(g) = m_{\mathrm{ADM}}(g) - \mathcal{Q}$$

holds, the conclusion (3) follows directly from the X–PMT by setting $X = -2\mathcal{E}$. We also note that a rigidity statement is obtained.

We next outline the elementary proof of inequality (2). It is based on a monotonicity argument, which extends the Green's function approach developed in [1] to the drift Laplacian operator associated with X. Full details are provided in [12].

Fix $o \in M$. Let $\mathcal{G}_o$ be the minimal positive Green's function with pole at o, vanishing at infinity, for the operator $\Delta - (1/2)\nabla_X$. We consider

$$u = 1 - 4\pi\mathcal{G}_o.$$

Thus, $u \to -\infty$ at the pole o and $u \to 1$ at infinity.

For every $t > 0$ such that $1 - 1/t$ is a regular value of u, we denote by Σ_t the regular level set $\{u = 1 - 1/t\}$ of u, and let $\nu = \nabla u/|\nabla u|$ be the ∞–pointing unit normal vector field along Σ_t. For any such $t > 0$, we define

$$F(t) = 4\pi t + t^3 \int_{\Sigma_t} |\nabla u|^2 \, d\sigma - t^2 \int_{\Sigma_t} |\nabla u| \mathrm{H} \, d\sigma + t^2 \int_{\Sigma_t} g(X, \nabla u) \, d\sigma$$

where H is the mean curvature of Σ_t with respect to ν.

Then, the first–variation formulas, the Bochner formula, and the traced Gauss equation yield

$$\begin{aligned} F'(t) = & \int_{\Sigma_t} \left[\frac{\mathrm{R}^{(-2)}_X}{2} + \left| \frac{\nabla^\top |\nabla u|}{|\nabla u|} + \frac{X^\top}{2} \right|^2 + \frac{|\mathring{\mathrm{h}}|^2}{2} \right] d\sigma \\ & + \frac{3}{4} \int_{\Sigma_t} \left(\frac{2|\nabla u|}{1-u} - \mathrm{H} + g(X, \nu) \right)^2 d\sigma + 4\pi - \int_{\Sigma_t} \frac{\mathrm{R}_{\Sigma_t}}{2} \, d\sigma. \end{aligned}$$

Here, $\mathring{\mathrm{h}}$ denotes the trace–free second fundamental form, while the superscript $\top$ denotes tangential projection onto Σ_t.

The curvature term is nonnegative because

$$\mathrm{R}^{(-2)}_X = \mathrm{R}^{(k)}_X + \frac{k+2}{2k}|X|^2 \geq 0 \qquad \text{for } k \in \mathbb{R} \setminus (-2, 0].$$

Moreover, the homological assumption, together with the maximum principle, controls the topology of the regular level sets. Indeed, if Σ_t is disconnected, none of its connected components can be a sphere. The Gauss–Bonnet theorem therefore gives

$$4\pi - \int_{\Sigma_t} \frac{\mathrm{R}_{\Sigma_t}}{2} \, d\sigma \geq 0.$$

Consequently,

$$F'(t) \geq 0$$

for every $t > 0$ such that $1 - 1/t$ is a regular value of u.
Now, the set of critical values of u has zero Lebesgue measure by Sard's theorem, and F admits a locally absolutely continuous representative on $(0, +\infty)$. Hence, the monotonicity extends across all critical levels.

Writing $r = d_g(o, \cdot)$, we have

$$4\pi\mathcal{G}_o = \frac{1}{r} + o_2(r^{-1}),$$

where the notation $f = o_2(r^{-1})$ means that $|\nabla^j f| = o(r^{-1-j})$ for every $j = 0, 1, 2$. Consequently, the equation $u = 1 - 1/t$ implies $r = t + o(t)$, and hence

$$|\Sigma_t| = 4\pi t^2 + o(t^2), \qquad |\nabla u| = \frac{1}{t^2} + o(t^{-2}), \qquad \mathrm{H} = \frac{2}{t} + o(t^{-1})$$

on Σ_t. At the same time, the term involving X is $O(t^2)$. Joining all, we then obtain

$$\lim_{t \to 0^+} F(t) = 0. \tag{4}$$

In an asymptotically flat coordinate chart, we have

$$u = 1 - \frac{B}{|x|} + O_2\big(|x|^{-1-\tau}\big) \quad \text{with} \quad B > 0, \tag{5}$$

which yields

$$|\nabla u| = \frac{B}{|x|^2} + O\big(|x|^{-2-\tau}\big), \qquad \mathrm{H} = \frac{2}{|x|} + O\big(|x|^{-1-\tau}\big). \tag{6}$$

The asymptotic expansion (5) also implies that

$$|x| = Bt + O\big(t^{1-\tau}\big) \tag{7}$$

on the level set $\Sigma_t = \{u = 1 - 1/t\}$. Since the surfaces Σ_t and the coordinate spheres $\{|x| = R\}$ are area outer–minimizing for sufficiently large t and R, it follows that

$$\mathrm{Area}\big(\{u = 1 - 1/t\}\big) = 4\pi B^2 t^2 + O(t^{2-\tau}). \tag{8}$$

We consider the following equivalent expression for $F(t)$, given by

$$F(t) = M_X(t) + E(t),$$

where

$$M_X(t) = \frac{t}{4}\left(16\pi - \int_{\Sigma_t} \mathrm{H}^2 \, d\sigma + 4\int_{\Sigma_t} \frac{g(X, \nabla u)}{(1-u)} \, d\sigma\right),$$

$$E(t) = \frac{t}{4}\int_{\Sigma_t} \left(\frac{2|\nabla u|}{1-u} - \mathrm{H}\right)^2 d\sigma.$$

The asymptotic expansions (5) and (6), along with the estimates (7) and (8), imply that $E(t)$ tends to zero as $t \to +\infty$, since $\tau > 1/2$.

By expanding on Σ_t the mean curvature H, the area form, and the X–term with respect to the Euclidean metric $g_{\mathrm{eucl}} = \delta_{ij} dx^i \otimes dx^j$ in the chosen asymptotically flat coordinate chart, we obtain

$$M_X(t) = \frac{t}{4}\left(16\pi - \int_{\Sigma_t} \overline{\mathrm{H}}^2 \, d\overline{\sigma}\right) + \frac{1}{2B}\int_{\Sigma_t} \overline{\mathrm{div}}_{\Sigma_t} \omega^\top \, d\overline{\sigma} + \frac{1}{2B}\int_{\Sigma_t} \left(\partial_j g_{ij} - \partial_i g_{jj} + 2X^i\right) \overline{\nu}^i \, d\overline{\sigma} + O(t^{1-2\tau}),$$

where $\gamma = (g_{ij} - \delta_{ij})\, dx^i \otimes dx^j$, ω is the 1–form defined by $\omega = \gamma_{jk}\overline{\nu}^k dx^j$, and the overbarred quantities are computed with respect to the Euclidean metric. Thus, the expression for $M_X(t)$ contains three terms: an Euclidean Willmore defect, which is nonpositive by the Willmore inequality [15]; a tangential divergence, whose integral vanishes; and the X–ADM flux. Hence

$$\lim_{t\to\infty} F(t) \le \frac{8\pi}{B} m_X. \tag{9}$$

Combining (4), (9), and the monotonicity of F along the values $t > 0$ for which $1 - 1/t$ is a regular value of u yields

$$0 = \lim_{t\to 0^+} F(t) \le \lim_{t\to\infty} F(t) \le \frac{8\pi}{B} m_X,$$

and therefore $m_X \ge 0$.

If $m_X = 0$, then F is constant and every nonnegative term in the right–hand side of the formula for $F'(t)$ vanishes on each regular level set. After a careful analysis, this leads to the rigidity results.

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The Geroch conjecture for initial data sets

Sven Hirsch

In general relativity, an initial data set for the Einstein equations consists of a Riemannian manifold (M, g) together with a symmetric 2-tensor k. Such initial data sets are typically assumed to satisfy the dominant energy condition

$$\mu \geq |J|.$$

Here, the energy and momentum densities are given by

$$\mu = \frac{1}{2}\big(\mathrm{scal} + (\mathrm{tr}_g\, k)^2 - |k|^2\big), \qquad J = \mathrm{div}_{\mathrm{g}}\big(k - (\mathrm{tr}_g\, k)g\big).$$

This raises the fundamental question of which manifolds M admit initial data (g, k) satisfying $\mu \geq |J|$, or even $\mu > |J|$, and what geometric rigidity can be expected in the limiting case.

In the important Riemannian special case $k = 0$, the dominant energy condition reduces to

$$\mathrm{scal} \geq 0.$$

Manifolds satisfying scalar curvature bounds have been studied extensively, and this naturally leads to analogous questions and conjectures for general initial data sets.

Some results in scalar curvature geometry, such as the positive mass theorem, have direct analogues for initial data sets, whereas others, such as the Penrose conjecture, remain notoriously open. Results concerning closed manifolds, however, do not carry over directly to the case $k \neq 0$. Indeed, for any closed Riemannian manifold (M, g), the initial data set

$$(M, g, \Lambda g)$$

satisfies $\mu > |J|$ for all sufficiently large Λ. In particular, the torus T^n admits initial data sets satisfying the strict dominant energy condition, even though the classical Geroch conjecture rules out metrics of positive scalar curvature on T^n.

In joint work with Simone Cecchini, Bernhard Hanke, Lukas Schönlinner, and Rudolf Zeidler, we showed the following.

Theorem 1. *Consider the initial data set* (T^n, g, k) *satisfying the dominant energy condition. Suppose that* $\mathrm{supp}(k)$ *is simply connected. Then* (T^n, g, k) *isometrically embeds into the Minkowski torus with second fundamental form* k.

The proof uses methods from spin geometry. A crucial part is to rule out compact pp-waves satisfying the dominant energy condition.

A geometric theory of complex Monge-Ampère equations

Jian Song

In this talk, we will discuss recent developments in uniform geometric estimates for solutions of complex Monge-Ampère equations. In the classical work of Kolodziej, the L^p bound of the Monge-Ampère volume measure implies L^∞ bound for the Kähler potential. We establish various geometric estimates such as uniform bounds for diameter, Green's function and Sobolev constants under the same L^p assumption, without any curvature assumptions. We also establish connections between the PSH condition for the Monge-Ampère volume measure and RCD structures, which leads to topological applications to singular Calabi–Yau spaces. This is a joint work with Guo–Phong–Sturm and Fu–Guo–Wang.

Sectional Curvature for Kantorovich-Wasserstein and Hellinger-Kantorovich Geometries

Karl-Theodor Sturm

We will analyze in detail curvature properties of the space $\mathcal{M}_0 = \mathcal{M}_0(\mathsf{M})$ of finite measures on a (Euclidean or Riemannian) space M. Equipped with the Hellinger-Kantorovich metric HK, the space of measures is a geodesic metric space of fundamental importance which has attracted a lot of attention in recent years, both from theoretical perspectives in unbalanced optimal transport problems and from applications, including computational optimal transport, statistical optimal transport, applications to data science and machine learning, modeling cellular dynamics, birth-death dynamics for sampling, and interaction-force transport gradient flows, [LMS16, LMS18, LMS23], [KMV16], [CPSV18a, CPSV18b], [DPST25], [GV18], [KV19], [DS25], [LM26], [GGV25].

Most importantly, the Hellinger-Kantorovich metric admits an intuitive interpretation as a relaxation of the Kantorovich-Wasserstein metric W_2 with optional creation and annihilation of mass, the cost of the latter being measured in terms of the Kakutani-Hellinger metric. For a measure $\mu \in \mathcal{M}_0$, the 'analytic tangent space' is given by the Sobolev space $\mathbb{T}_\mu \mathcal{M}_0 = H^1(\mu)$ and the 'extended tangent space' is $\mathbb{T}_\mu^{\mathsf{ext}} \mathcal{M}_0 = H^1(\mu) \otimes \mathcal{M}_0\big(\mathsf{M} \setminus B_{\pi/2}(\mathsf{supp}[\mu])\big)$.

a. We will show that for every $\mu \in \mathcal{M}_0(\mathsf{M})$ and every pair of tangent vectors $\varphi, \psi \in \mathcal{C}_c^\infty(\mathsf{M}) \subset H^1(\mu)$, the sectional curvature is given as

$$\mathsf{Sec}_\mu(\varphi,\psi) := \lim_{t\to 0} \frac{3}{\Lambda t^4}\left[t^2|\varphi-\psi|^2_{H^1(\mu)} - \mathsf{HK}^2\Big(\exp_\mu^{\mathcal{M}_0}(t\varphi), \exp_\mu^{\mathcal{M}_0}(t\psi)\Big)\right]$$

with $\Lambda := |\varphi|^2_{H^1(\mu)} \cdot |\psi|^2_{H^1(\mu)} - \langle\varphi,\psi\rangle^2_{H^1(\mu)}$, and decomposes into a lifted and a twisted part

$$\mathsf{Sec}_\mu(\varphi,\psi) = \mathsf{Sec}_\mu^\uparrow(\varphi,\psi) + \mathsf{Sec}_\mu^\nabla(\varphi,\psi).$$

b. The lifted part vanishes if $\dim \mathsf{M} = 1$ and otherwise it is given in terms of the sectional curvature of the underlying Riemannian manifold M:

$$\mathsf{Sec}_\mu^\uparrow(\varphi,\psi) = \frac{1}{\Lambda}\int_{\mathsf{M}} \Big(\mathsf{Sec}_x^{\mathsf{M}}(\nabla\varphi,\nabla\psi) - 1\Big)\cdot\Big[|\nabla\varphi|_x^2\,|\nabla\psi|_x^2 - \langle\nabla\varphi,\nabla\psi\rangle_x^2\Big]\,d\mu(x).$$

c. The twisted part is always nonnegative:

$$\mathsf{Sec}_\mu^\nabla(\varphi,\psi) \geq 0.$$

In the Euclidean case with absolutely continuous μ, it is explicitly given as

$$\begin{aligned}\mathsf{Sec}_\mu^\nabla(\varphi,\psi) &= \frac{3}{\Lambda}\inf\left\{\int_{\mathsf{M}}\Big[|F-\nabla\eta|^2 + 4\eta^2\Big]\,d\mu :\ \eta\in H^1(\mathbb{R}^n,\mu)\right\}\\ &= \frac{12}{\Lambda}\int\Big|\Big(-\Delta_\mu(4-\Delta_\rho)\Big)^{-1/2}\mathrm{div}_\mu F\Big|^2\,d\mu\end{aligned}$$

with $F := \frac{1}{2}\big(\nabla^2\varphi\nabla\psi - \nabla^2\psi\nabla\varphi\big)$. Even for 1-dimensional spaces, Sec_μ^∇ does not vanish.

d. In particular for $\mathsf{M} = \mathbb{R}^n$, there will be a competition between the two parts: $\mathsf{Sec}_\mu^\nabla(\varphi,\psi) \geq 0$ and $\mathsf{Sec}_\mu^\uparrow(\varphi,\psi) \leq 0$. So far, it was only known that $(\mathcal{M}_0(\mathbb{R}^n), \mathsf{HK})$ for $n \geq 2$ does not have nonnegative curvature in the sense of Alexandrov. We show that it also does not have nonpositive curvature in the sense of Alexandrov.

e. More detailed insights are obtained for the multi-dimensional torus $\mathsf{M} = \mathbb{T}^n$ and for its normalized volume measure μ. Let $(e_k)_{k\in\mathbb{Z}^n}$ denote the Fourier basis of $L^2(\mathbb{T}^n,\mu)$. Then $\mathsf{Sec}_\mu(e_k,e_\ell) = \frac{1}{2^n}\sum_{\sigma\in\{-1,1\}} S(k^\sigma,\ell)$ with $k^\sigma = (\sigma_p k_p)_{p=1,\dots,n} \in \mathbb{Z}^n$ and

$$S(k,\ell) = \frac{-|k|^2\,|\ell|^2 + \langle k^2,\ell^2\rangle + \frac{3}{4}\left[|k-\ell|^2 - \frac{\left(|k|^2-|\ell|^2\right)^2}{4+|k+\ell|^2}\right]\langle k,\ell\rangle^2}{(4+|k|^2)\,(4+|\ell|^2)}$$

for any pair $k,\ell\in\mathbb{Z}^n$ which is 'generic' in the sense that $k_p \neq \pm\ell_p, k_p\ell_p \neq 0$ for all $p = 1,\dots,n$. Moreover, $-\left(\frac{3}{2}\right)^n \leq \mathsf{Sec}_\mu\big(e_k,e_\ell\big) \leq \frac{3}{2}\,4^n\left(|k|^2+|\ell|^2\right)$ for arbitrary k,ℓ with $k\neq\ell$. In particular, we prove that

- there are plenty of cases with negative sectional curvature: if $n \geq 2$, $k,\ell \neq 0$, and $k_p\ell_p = 0$ $(\forall p = 1,\dots,n)$, then

$$\mathsf{Sec}_\mu(e_k,e_\ell) \in \big(-1,-1/25\big];$$

- in average, the sectional curvature will be positive: if $|k_1|, |k_2| \geq C$, then
$$\sum_{\ell \in \mathbb{Z}^2, |\ell| \leq L} \mathsf{Sec}_\mu(e_k, e_\ell) \geq \frac{1}{1000}\, C^2\, L^2;$$
- for all $k \in \mathbb{Z}^2$ with sufficiently large $|k_1|, |k_2|$, the Ricci curvature diverges
$$\mathsf{Ric}_\mu(e_k, e_k) := \|e_k\|^2_{H^1(\mu)} \cdot \lim_{L \to \infty} \sum_{\ell \in \in \mathbb{Z}^2, |\ell| \leq L} \mathsf{Sec}_\mu(e_k, e_\ell) = +\infty;$$
- for every $k \in \mathbb{Z}^n$ and every $s > 2 + \frac{n}{2}$, the re-normalized Ricci curvature $\mathsf{Ric}^{(s)}_\mu(e_k, e_k)$ exists and is finite.

f. Our approach to sectional curvature on the space of finite measures is based on the observation that in the case of a smooth Riemannian manifold (M, g), the sectional curvature $\mathsf{Sec}^{\mathsf{M}}_z(v, w)$ is given in purely metric terms as
$$\mathsf{Sec}^{\mathsf{M}}_z(v, w) = 3 \lim_{t \to 0} \frac{t^2 |v - w|^2 - \mathsf{d}^2\big(\exp_z(tv), \exp_z(tw)\big)}{t^4\big(|v|^2 |w|^2 - \langle v, w\rangle^2\big)}$$
for every $z \in \mathsf{M}$ and every linearly independent pair of tangent vectors $v, w \in \mathbb{T}_z\mathsf{M}$. Inspired by this observation, therefore for each geodesic metric space (X, d), each $z \in \mathsf{X}$ and each 'independent' pair of geodesics α, β emanating from z we define
$$\mathsf{Sec}^{\mathsf{X}}_z(\alpha, \beta) := 3 \lim_{t \to 0} \frac{t^2 \mathsf{d}^2_z(\alpha, \beta) - \mathsf{d}^2\big(\alpha_t, \beta_t\big)}{t^4\big(|\dot\alpha|^2 |\dot\beta|^2 - \langle \alpha, \beta\rangle^2_z\big)}$$
where $\mathsf{d}_z(\alpha, \beta) := \limsup_{t \to 0} \frac{1}{t} \mathsf{d}(\alpha_t, \beta_t)$ denotes the directional distance of the geodesics, $|\dot\alpha|, |\dot\beta|$ their metric speed, and $\langle \alpha, \beta\rangle_z := \frac{1}{2}\big(|\dot\alpha|^2 + |\dot\beta|^2 - \mathsf{d}^2_z(\alpha, \beta)\big)$. One easily verifies that for every complete geodesic space (X, d),

- $\mathsf{Sec}(\mathsf{X}, \mathsf{d}) \geq \kappa$ Alexandrov sense $\Rightarrow \forall z \in \mathsf{X}, \alpha, \beta \in \mathring{\mathbb{T}}^{\mathsf{dir}}_z\mathsf{X}:\ \mathsf{Sec}_z(\alpha, \beta) \geq \kappa$,
- $\mathsf{Sec}(\mathsf{X}, \mathsf{d}) \leq \lambda$ Alexandrov sense $\Rightarrow \forall z \in \mathsf{X}, \alpha, \beta \in \mathring{\mathbb{T}}^{\mathsf{dir}}_z\mathsf{X}:\ \mathsf{Sec}_z(\alpha, \beta) \leq \lambda$.

g. This concept of sectional curvature for geodesic metric spaces will be analyzed in detail for two spaces: the Hellinger-Kantorovich space $(\mathcal{M}_0(\mathsf{M}), \mathsf{HK})$ of finite measures on M and the Kantorovich-Wasserstein space $(\mathcal{P}_2(\mathsf{M}), \mathsf{W}_2)$ of probability measures on M. In both cases, for convenience M is assumed to be a smooth Riemannian manifold.

In the latter case, which is significantly easier than the former, we can indeed confirm and extend fundamental insights of Otto [Ott01] and Lott [Lot08]. Based on a formal differential-geometric calculus on the Kantorovich-Wasserstein space $(\mathcal{P}_2(\mathbb{R}^n), \mathsf{W}_2)$ — which nowadays is called Otto calculus — Otto [Ott01] deduced the formula
$$\mathsf{Sec}^{\nabla}_\mu(\nabla\varphi, \nabla\psi) = \frac{3}{\Lambda} \cdot \inf\left\{ \int_{\mathsf{M}} \big|\nabla^2\psi\nabla\varphi - \nabla\eta\big|^2 d\mu :\ \eta \in \mathcal{C}^\infty_c(\mathsf{M}) \right\}$$
with $\Lambda := |\nabla\varphi|^2_{L^2(\mu)} |\nabla\psi|^2_{L^2(\mu)} - \langle \nabla\varphi, \nabla\psi\rangle^2_{L^2(\mu)}$ for the sectional curvature of $\mathcal{P}_2$ (which in the Euclidean case coincides with its twisted part) and provided an interpretation of it in terms of O'Neill's formula for Riemannian submersions.

Lott [Lot08] extended these formal calculations to analyze the sectional curvature of $\mathcal{P}_2(\mathsf{M})$ for general Riemannian manifolfds M. He deduced the formulas for the twisted part as before for $\mathsf{M} = \mathbb{R}^n$ and in addition now for the lifted part

$$\mathsf{Sec}^{\uparrow}_{\mu}(\nabla\varphi, \nabla\psi) = \frac{1}{\Lambda} \int_{\mathsf{M}} \mathsf{Sec}^{\mathsf{M}}_{x}(\nabla\varphi, \nabla\psi) \cdot \left[|\nabla\varphi|^2_x \, |\nabla\psi|^2_x - \langle \nabla\varphi, \nabla\psi \rangle^2_x \right] d\mu(x).$$

Extensions of these concepts and results beyond the setting of absolutely continuous measures μ where e.g. discussed in [Gig12] and [Lot17].

h. The PDE approach of the Otto calculus, however, is limited to sufficiently regular measures μ and to directions $\nabla\varphi, \nabla\psi$ in the analytic tangent space (which correspond to Monge transport problems) whereas our OT approach also applies to general measures μ and to general directions P, Q in the geometric tangent space (which correspond to Kantorovich transport problems). This, for instance, allows us to prove that

$$\mathsf{Sec}^{\mathcal{P}_2}_{\delta_z}(P, Q) \leq 0$$

for all $z \in \mathsf{M}$ and all linearly independent $P, Q \in \mathring{\mathbb{T}}^{\mathsf{geo}}_{\delta_z}\mathcal{P}_2$ provided that $\mathsf{Sec}^{\mathsf{M}}_{x} \leq 0$ for all $x \in \mathsf{M}$. In particular, $\mathsf{Sec}^{\mathcal{P}_2}_{\delta_z} = 0$ for all $z \in \mathsf{M}$ in the Euclidean case $\mathsf{M} = \mathbb{R}^n$.

i. In the spirit of the Otto calculus, formal calculations on the space $(\mathcal{M}_0(\mathsf{M}), \mathsf{HK})$ have been proposed in [GV18] towards identifying the sectional curvature and establishing a Riemannian submersion property as expressed in O'Neill's formula. Our OT based approach provides rigorous proofs and plenty of novel deep insights.

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Partial Regularity of Stable Stationary Harmonic Maps into Certain Lie Groups

Jacob Krantz

Let $u : M \to (G, g)$ be a stable stationary harmonic map between a compact manifold and a compact simple Lie group with bi-invariant metric. In general, u will not be smooth everywhere. For example, the projection map $u : B_1^4(0) \to (S^3, g_{\text{round}}) : x \mapsto x/|x|$ has a singularity at the origin and is in fact minimizing with respect to its boundary data. Therefore, it is natural to ask for an upper bound on the Hausdorff dimension of the singular set of u.

Suppose now that G is not of type C_n for $n \geq 8$, E_8, F_4, or G_2 . We discussed a proof that every stable stationary harmonic map $u : M \to G$ is smooth away from a Hausdorff codimension four set. To prove this, we used a compactness theorem of Hsu [2] and a factorization theorem of Burstall and Rawnsley [1].

Hsu's compactness theorem allows us to use dimension reduction in the case of stable stationary harmonic maps into compact Lie groups with bi-invariant metrics. In particular, let $\phi : S^2 \to G$ be nonconstant and harmonic. Call its homogeneous extension to $\mathbb{R}^3$ the cone map associated to ϕ. We say that ϕ is cone stable if its cone map is stable. Hsu's theorem then says that as long as every such ϕ is neither stable nor cone stable, any stable stationary harmonic map $u : M \to G$ must be smooth away from a Hausdorff codimension four set. It is well-known that ϕ can never be stable, so we focused on the proof that ϕ can never be cone stable.

The proof then proceeds by contradiction. Assuming ϕ is harmonic and cone stable, we are able to achieve the following cone stability inequality:

$$\int_{S^2} \frac{1}{4}|\sigma|^2 + |\nabla^\phi \sigma|^2 - g(R^G(\sigma, \nabla_i \phi)\nabla_i \phi, \sigma) \geq 0 \tag{1}$$

for all $\sigma \in \Gamma(S^2, \phi^* TG)$. Thus, it suffices to find a variation σ of ϕ that does not satisfy the above inequality. To find σ, we used a factorization theorem of Burstall and Rawnsley. Interpreting their factorization theorem correctly provides a variation that perturbs our given ϕ towards a harmonic map of lower energy. Moreover, the energy along this variation can be computed explicitly. From this we were able to estimate $\nabla^2 E[\phi](\sigma, \sigma) \leq -8\pi n_G < 0$, where n_G only depends on the Lie algebra and the normalization of the metric. We then discussed the key properties of σ that allowed us to estimate its L^2 norm, and we estimated this norm explicitly for $G = \mathrm{SU}(n)$ with the Killing metric.

With these computations in hand, we evaluated our cone stability inequality. Our choice of σ did not satisfy the cone stability inequality, so we finally observed that when $G = \mathrm{SU}(n)$ such a map ϕ can never be cone stable. This completed the proof of codimension four regularity for $G = \mathrm{SU}(n)$. Similar computations can be done for any compact simple Lie group G that is not of type C_n for $n \geq 8$, E_8, F_4, or G_2 . We concluded with the fact that this result is sharp: there are stable stationary harmonic maps into these groups with codimension four singular sets.

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Continuous smoothing of some spherical polyhedra and its application to solitons existence

Man-Chun Lee

(joint work with Pak-Yeung Chan, Yi Lai)

Let (X, d) be a complete metric space. We are particularly interested in spaces with an appropriate curvature lower bound in a weak sense. Examples include Ricci limit spaces, $\mathrm{RCD}(k, N)$ spaces, or Alexandrov spaces with curv $\geq -k$. These spaces arise naturally when studying the space of smooth metrics with a curvature lower bound via the Gromov pre-compactness theorem, or manifolds with corners. Thanks to the efforts of Cheeger, Colding, Naber, Jiang [3, 4, 5, 6, 7, 8, 9, 10] and others, the structure of Ricci limit spaces is relatively well understood. This naturally raises the question of whether a singular space is naturally a Ricci limit space. In general, we ask the following:

Question 1. *Given a (non-collapsed) singular space (X^n, d, p) with curvature bounded from below by k in a weak sense, can we find a sequence of smooth manifolds (M_i, g_i, p_i) with curvature bounded from below by k in the classical sense such that (M_i, g_i, p_i) converges to (X, d, p) in the pointed Gromov–Hausdorff sense as $i \to +\infty$?*

In the context of Alexandrov geometry, a variant of this question has been around since the eighties (see [14, 15]). It remains widely open except in two dimensions (see [16] for example). In the RCD setting, it was explicitly raised in [1, Question 1.16]. The question turns out to be more involved in general. When $n = 3$, it was shown by Simon–Topping [18] (see also [17]) that a non-collapsed Ricci limit space must be a topological manifold, using a Ricci flow approach. In comparison, the metric cone over $\mathbb{RP}^2$ is a non-collapsed $\mathrm{RCD}(0, 3)$ space, but it cannot be a Ricci limit space. From this, we see that X^n being a topological manifold is a necessary condition for Question 1 to hold. Without the assumption of being a manifold, the topological regularity of $\mathrm{RCD}(k, n)$ spaces with $n \leq 3$ has also been studied in [13, 11].

From the tangent cone viewpoint, if X is a topological manifold, the tangent cones are modeled by $C(\Sigma^{n-1})$, where the link Σ^{n-1} is a metric space that is a topological sphere with a curvature lower bound in a weak sense. When $n = 3$, the weak Ricci and sectional curvature lower bounds coincide by [19]. In order to make full use of this, we need to know whether such a metric cone is smoothable, and moreover, how this infinitesimal smoothability leads to fixed-scale smoothability.

We are particularly interested in the special case where X^n is a polyhedral manifold of spherical or Euclidean type:

Definition 1. *$(\mathcal{P}, d_{\mathcal{P}})$ is said to be a spherical (resp. Euclidean) polyhedral manifold if $(\mathcal{P}, d_{\mathcal{P}})$ is a complete length space homeomorphic to a topological n-manifold that admits a locally finite triangulation in which each n-simplex is isometric to an n-simplex in the fixed model space $\mathbb{M}^n(\kappa)$ with constant curvature $\kappa > 0$ (resp. 0).*

This corresponds to the case where the asymptotic tangent cone is almost the manifold itself at a fixed small scale. One of the interesting features of polyhedral manifolds is an approximation theorem by Petrunin [22], which states that any smooth manifold with a curvature bounded from below (in a strong sense) can be Lipschitz approximated by polyhedral spaces. In [20], the authors raised the question of whether the converse is true, which constitutes a special case of Question 1:

Conjecture 1. *Assume $\mathcal{P}$ is a compact polyhedral space with nonnegative curvature in the sense of Alexandrov. Then it can be presented as a limit of Riemannian orbifolds with nonnegative sectional curvature.*

In [20], this question was answered in three dimensions using gluing and the Ricci flow techniques from [17]. To get a sense of the subtlety of the smoothability problem, consider the very special case of the three-dimensional American football.

In multiply warped product coordinates, its metric is expressed as

$$g_{0,\beta} := g_{0,(\beta_1,\beta_2,\beta_3)} := \beta_1^2\Big(dx_1^2 + \beta_2^2 \sin^2 x_1\big(dx_2^2 + \beta_3^2 \sin^2 x_2\, dx_3^2\big)\Big) \tag{1}$$

where $\beta_i \in (0,1]$. The higher-dimensional analog is completely analogous. At $x_1 = 0$, the singularity is conical, while along $x_2 = 0$, the singularity is of edge type. Singularities are, in general, merged together. Moreover, the complexity of these singularities stacks up whenever a spherical suspension is taken to the next dimension, making general smoothing while preserving the curvature lower bound highly challenging.

To the best of the authors' knowledge, there are two primary approaches toward the smoothing problem: either by obtaining estimates on the harmonic radius or through Ricci flow smoothing. Ricci flow smoothing—or more precisely, smoothing by expanders—is particularly powerful in handling conical singularities. Intuitively, this is because the Ricci flow induced by an expander takes the form $2t\,\mathrm{Ric}(g(t)) + g(t) = \nabla^2 u$, which models a cone as $t \to 0$. When the link Σ of the cone $C(\Sigma^{n-1})$ is a smooth metric with $\mathrm{Rm} > 1$, smoothability was confirmed by Deruelle [12] using PDE methods.

Motivated by the problem of smoothing polyhedral manifolds, we initiate in [2] the smoothing problem for metrics of the form (1). We show that homogeneous spherical polyhedrons are smoothable, and more importantly, that the smoothing is stable:

Theorem 1 (Chan–Lai–L.)**.** *For all $\delta > 0$ and $n \geq 3$, there exist constants $C(n,\delta), S(n,\delta) > 0$ such that the following holds: For all $\beta = (\beta_1, \dots, \beta_{n-1}) \in [\delta, 1]^{n-1}$, there is a one-parameter family of metrics $g_\beta(t)$ on $\mathbb{S}^{n-1} \times (0, S]$ coming out of $g_{0,\beta}$ such that:*

(i) $1 \leq \mathrm{Rm}(g_\beta(t)) \leq Ct^{-1}$ *and* $\mathrm{inj}(g_\beta(t)) \geq \sqrt{C^{-1}t}$*;*
(ii) $g_\beta(t) \to g_{0,\beta}$ *in the GH sense as* $t \to 0$*;*
(iii) *If* $|\alpha - \beta| = o(1)$*, then there exists a diffeomorphism* $\Psi_{\alpha,\beta}$ *such that*

$$(1 - o(1))g_\beta(t) \leq \Psi_{\alpha,\beta}^* g_\alpha(t) \leq (1 + o(1))g_\beta(t);$$

(iv) $g_\beta(t)$ *shares the same symmetry group as* $g_{0,\beta}$*.*

This is based on a new weak stability result for the Ricci flow, where the central Ricci flow is allowed to be a Ricci flow smoothing originating from a cone with a non-smooth link. It also builds upon the cone smoothing framework of Gianniotis–Schulze [23]. Such stability is crucial to the smoothability question since the link Σ_x of the tangent cone $C(\Sigma_x)$ at different points x varies at most in the Gromov–Hausdorff sense, and any valid smoothing must be compatible with the variations of the initial data, i.e., the link Σ_x.

As an important application, we use this framework to produce new examples of steady Ricci solitons, extending the flying wing existence results of Lai [21] to higher dimensions with broader symmetry groups.

Theorem 2. *Let $n \geq 4$ and $1 = i_1 < i_2 < \cdots < i_k < i_{k+1} = n$ where $k \leq n-1$. There exists a family of $O(i_2 - i_1) \times \cdots \times O(i_k - i_{k-1}) \times O(i_{k+1} - i_k + 1)$-symmetric*

n-dimensional gradient steady solitons (M, g, f, p) with $\mathrm{Rm} \geq 0$*, parametrized by the eigenvalues of* Ric *at p.*

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Singularity analysis in the one-phase free boundary problem

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(joint work with Max Engelstein, Daniel Restrepo)

Free boundary problems are natural problems arising from physical phenomena and have close ties with both PDEs and geometric variational problems. For this talk, let Ω_0 be a fixed domain in $\mathbb{R}^d$, and let φ be a non-negative function on $\partial\Omega_0$. We consider the following variational problem

$$\min\left\{\int_{\Omega_0} |\nabla u|^2 + \chi_{\{u>0\}}\, dx \mid u = \varphi \text{ on } \partial\Omega_0\right\},$$

first studied by Alt and Caffarelli in [1]. A critical point of the functional satisfies the Euler-Lagrange equation

$$\text{(EL)} \qquad \begin{cases} -\Delta u = 0, & \text{in } \Omega := \{u > 0\}, \\ |\nabla u| = 1, & \text{on } \partial\Omega \cap \Omega_0, \end{cases}$$

an over-determined elliptic PDE. A special feature of free boundary problems in general is that the regularity of the solution u is intertwined with the regularity of the freely-evolving boundary $\Gamma := \partial\{u > 0\}$; and singularity, rather than something to avoid, is where interesting behaviors occur.

For any $x_0 \in \Gamma$, blow-ups of the form

$$\frac{u(x_0 + r_k x)}{r_k}, \qquad r_k \to 0,$$

converge along subsequences to one-homogeneous global solutions of (EL) satisfying $b(r\omega) = rb(\omega)$; furthermore, these homogeneous *tangent functions* b retain the variational property (minimizing or stable) of the original function u. Thus free boundary singularities are modeled by homogeneous solutions.

A trivial example of homogeneous solution is the flat solution $b_f(x) = (x_1)^+$, which corresponds to *regular* free boundary points. The axisymmetric solutions b_{as} are the simplest non-flat homogeneous solutions, and they are proven to be unstable in low dimensions $d \leq 6$ in [2] and minimizing when $d \geq 7$ in [3]. This points to the conjecture that 7 is the critical dimension for the free boundary regularity of stable or minimizing solutions, analogous to how Simons cone points to the critical dimension in the regularity theory of minimal hypersurfaces. Recently, more examples of such solutions are constructed in [8] and their variational properties are also studied. In an upcoming work [9] with Firester and Tsiamis, we construct a rich family of singular solutions using the foliation of the sphere by isoparametric hypersurfaces. These new examples will become natural testing grounds for open problems about the one-phase free boundary problem.

Since singularity is in general unavoidable in high dimensions, it is natural to ask the following questions: is the tangent function at a singular point unique, and knowing the singularity model, what information can we deduce about the free boundary? We answer these questions in [7].

Uniqueness of the tangent function. Let b be a one-homogeneous solution whose free boundary is smooth away from the origin. We say b is *integrable through rotations* if the degree-one kernel elements of the linearized equation at b are generated by rotations. Our first result proves the uniqueness of the tangent function under this hypothesis, which arises naturally from minimal surfaces. An analogous result has been proven previously in [5] for *minimizing* solutions.

Theorem 1. *Let b be a one-homogeneous solution in $\mathbb{R}^d$ with isolated singularity at the origin, which is integrable through rotations. Suppose that u is a non-degenerate weak solution asymptotic to b at infinity, namely*

$$\text{there exists } R_k \to +\infty \text{ such that } \frac{u(R_k x)}{R_k} \to b(x) \quad \text{in } L^2_{\mathrm{loc}}.$$

Then there exists $\alpha \in (0,1)$ such that

$$\frac{u(x)}{|x|} = b\left(\frac{x}{|x|}\right) + O(|x|^{-\alpha}) \qquad \text{for } |x| \gg 1.$$

In particular, the blow-down is unique. An analogous statement holds for blow-ups at a singular free boundary point, with a power rate of convergence.

Our proof takes advantage of the Weiss monotonicity formula

$$W(u,r) := \frac{1}{r^d}\int_{B_r} \left(|\nabla u|^2 + \chi_{\{u>0\}}\right) dx - \frac{1}{r^{d+1}}\int_{\partial B_r} u^2 \, d\mathcal{H}^{d-1},$$

which satisfies

$$\frac{d}{dr}W(u,r) = \frac{2}{r^{d+2}}\int_{\partial B_r} (x\cdot\nabla u - u)^2 \, d\mathcal{H}^{d-1} \geq 0.$$

Quantitative estimates on annuli relate the oscillation of $W(u,r)$ to the distance of u from a rotation of b. The summability of the monotone quantity then allows us to stitch together the rotations at different scales, thus proving the uniqueness with a power decay.

Classification of solutions near the axisymmetric cone. In [4] as well as [6], the authors construct two smooth global minimizers u_- and u_+ associated with b_{as}, which are natural solutions to smooth out the singularity at the origin. They satisfy $u_- \leq b_{as} \leq u_+$, and the free boundaries of their homothetic rescalings

$$u^\lambda_\pm(x) := \frac{u_\pm(\lambda x)}{\lambda}, \qquad \lambda > 0,$$

form foliations on the two sides of b. We prove the following rigidity result.

Theorem 2. *Let b_{as} be the axisymmetric minimizing solution in $\mathbb{R}^d$, $d \geq 7$. Suppose that u is a non-degenerate global weak solution asymptotic to b_{as} at infinity. Then, modulo translations in $\mathbb{R}^d$, u either agrees with b_{as} or is a homothetic rescaling of u_- or u_+.*

We emphasize that no minimizing property and no one-sidedness[1] with respect to b_{as} is assumed for u. Moreover, we give a more general necessary condition for the rigidity statement, and verify the condition for b_{as}; in future work, we plan to apply the theory to the many new examples of singularity models.

The proof is based on a linear analysis at an homogeneous solution b with isolated singularity. If $\Omega = \{b > 0\}$, then the linearization of (EL) is

$$\text{(L)} \qquad \begin{cases} -\Delta v = 0, & \text{in } \Omega, \\ \partial_\nu v + Hv = 0, & \text{on } \partial\Omega, \end{cases}$$

where ν is the unit normal pointing into Ω, and H is the mean curvature of $\partial\Omega$ oriented towards its complement. Since Ω is a cone, solutions separate as $v(x) = |x|^\gamma v(x/|x|)$; writing $\Sigma = \Omega \cap \mathbb{S}^{d-1}$, the spherical part and the radial exponent satisfies

$$\begin{cases} -\Delta_{\mathbb{S}^{d-1}} v = \lambda v, & \text{in } \Sigma, \\ \partial_\nu v + Hv = 0, & \text{on } \partial\Sigma, \end{cases} \qquad \gamma^2 + (d-2)\gamma - \lambda = 0.$$

For the axisymmetric cone, the linear modes relevant to solutions approaching b_{as} are completely determined: they arise from the foliation and translations. Our perturbation analysis allows one to pass this statement to quantitative decay estimates to the nonlinear equation. Then, the foliation by the leaves of $u_\pm$ and a maximum principle argument enables us to bring the information near ∞ to compact regions and conclude.

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[1] In fact, assuming that u lies on one side of b is a strong assumption, under which the above classification holds for a much more general class of singularity models.

Morse theory for minimal surfaces: emanating flow lines

Giada Franz
(joint work with Theodora Bourni)

Let (M^{m+1}, g) be a compact Riemannian manifold with (possibly empty) convex boundary. Morse theory provides a framework for studying the space $\mathcal{H}$ of hypersurfaces embedded in M through the analysis of the area functional (see e.g. [17, 14, 12, 4]). Within this framework, *free boundary minimal hypersurfaces* play a distinguished role, as they are precisely the critical points of the area functional.

By analogy with classical Morse theory, one expects that a free boundary minimal hypersurface Σ of Morse index[1] I should admit an I-dimensional unstable manifold emanating from it. Namely, one expects the existence of an I-parameter family of solutions of the gradient flow of the area functional, converging to Σ backwards in time, as $t \to -\infty$. As these flow lines exist for all negative times, they are called *ancient solutions*.

The flow in this setting is the *free boundary mean curvature flow*, introduced by Huisken [9], with first properties studied by Stahl [15, 16]. More precisely, an ancient solution to the free boundary mean curvature flow is a family of hypersurfaces $\{\Sigma_t\}_{t\in(-\infty,0]}$ such that Σ_t intersects ∂M orthogonally for all $t \in (-\infty, 0]$ and the normal velocity $\partial X/\partial t$ for $X \in \Sigma_t$ is equal to $\vec{H}(X,t)$, the mean curvature vector of Σ_t at X.

However, the free boundary mean curvature flow is only the formal gradient flow of the area functional, and the area functional is not a Morse function on $\mathcal{H}$, which is a very complicated infinite-dimensional space. These are major obstacles to making the above heuristic precise. In our work, we are able to overcome these difficulties, by constructing and characterizing ancient solutions to free boundary mean curvature flow.

Theorem 1 (Theorem A in [2])**.** *Let Σ^m be a compact, properly[2] embedded, free boundary minimal hypersurface in a compact Riemannian manifold (M^{m+1}, g) with convex boundary.*

There exists an I-parameter family of ancient solutions converging exponentially fast to Σ as $t \to -\infty$. The convergence is smooth in the graphical sense. Moreover, these are all ancient solutions backward converging graphically smoothly to Σ such that the convergence is faster than $1/|t|$ in C^0-norm.

The uniqueness part of the statement is sharp, in the sense that there exist ancient solutions of free boundary mean curvature flow C^0-converging to Σ with rate $1/|t|$ as $t \to -\infty$. These solutions reflect the fact that the area functional is generally not Morse. Indeed, they are not present in the case when Σ is non-degenerate (i.e., Σ does not admit null-directions for the second derivative of the area).

[1]We recall that the Morse index of a free boundary minimal hypersurface is the maximal dimension of a subspace of normal variation on which the second derivative of the area functional is negative definite.

[2]Here, properly embedded means that $\partial\Sigma = \Sigma \cap \partial M$.

Our result is inspired by the work of Choi–Mantoulidis [6] in the closed case and of Han [8], Choi–Huang–Lee [5] in the noncompact case without boundary. Similar results are known in several other nonlinear parabolic settings [11]. Moreover, there is a rich literature on the classification of ancient solutions to mean curvature flow, especially in the Euclidean setting, motivated by their central role in the analysis of singularity formation (see e.g. [7, 10, 1, 3]).

Let us briefly present the main ideas of the proof. Denote by

$$\lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots \leq \lambda_I < 0 \leq \lambda_{I+1} \leq \cdots \nearrow \infty$$

the discrete spectrum[3] of the Jacobi operator on Σ and $\{\varphi_k\}_{k\geq 1}$ an associated $L^2(\Sigma)$-orthonormal basis of eigenfunctions. Let $\tilde{\Sigma}_t$ be the graph of the function[4] $\varphi(x,t) = \sum_{k=1}^{I} a_k \varphi_k(x) e^{-\lambda_k t}$ for every $t \in (-\infty, 0]$. Then, $\{\tilde{\Sigma}_t\}_{t\in(-\infty,0]}$ is a solution to the free boundary mean curvature flow up to lower-order error terms. However, these hypersurfaces generally fail to satisfy the free boundary condition. Consequently, they cannot even be used as initial data for the flow. We overcome this difficulty by a careful application of the implicit function theorem, proving that we can slightly perturb $\tilde{\Sigma}_t$, thereby recovering the orthogonality condition.

We build an ancient solution $\{\Sigma_t\}_{t\in(-\infty,0]}$ "approximating" (in a suitable sense) $\{\tilde{\Sigma}_t\}_{t\in(-\infty,0]}$ by a careful application of the fixed point theorem. To do so, we need several elliptic and parabolic estimates for graphs over a free boundary minimal hypersurface. The uniqueness part of the statement is then based on an ODE lemma due to Merle–Zaag (Lemma A.1 in [13]), as adapted by Choi–Mantoulidis.

Since the first eigenfunction φ_1 has a sign, we can assume $\varphi_1 > 0$. Therefore, the graph $\tilde{\Sigma}_t$ of the function $\varphi(x,t) = \varphi_1(x)e^{-\lambda_1 t}$ is mean convex for every $t \in (-\infty, 0]$ and so is the corresponding ancient solution Σ_t. We give an alternative proof of the existence of this mean convex ancient solution, based on the implicit function theorem argument mentioned above together with a barrier argument. This provides a more detailed geometric description of this solution.

Many interesting questions remain open, even in the mean convex setting. For instance, we conjecture that every free boundary minimal hypersurface admits at most one mean convex ancient solution that converges to it in $C^{1,\alpha}$. More generally, one would like to classify mean convex ancient solutions in a fixed ambient manifold without assuming convergence to a smooth minimal hypersurface. What are the possible backward limits of such solutions?

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Einstein cone manifolds

Ursula Hamenstädt

A *cone manifold* is a smooth manifold equipped with a metric which is Riemannian outside of a subset of positive codimension and whose singular locus is locally modeled on the quotient of a Riemannian manifold by a finite group of isometries.

Particular cone manifolds are such that the singular locus is a smooth submanifold of codimension two that is totally geodesic for the cone metric and that in suitable charts, the singular locus is modeled by the quotient of a smooth metric under a finite cyclic group of isometries with codimension two fixed point set.

We begin with discussing closed hyperbolic cone manifolds of dimension three. Particular such hyperbolic cone manifolds arise as cyclic covers of a smooth hyperbolic manifolds, branched along a closed geodesic which is homologous to zero. By hyperbolization, the branched covering has a *smooth* hyperbolic metric in its own right which by Mostow rigidity admits a cyclic group of isometries. We explain how this can be used to show that any closed hyperbolic 3-manifold admits infinitely many mutually distinct structures as a hyperbolic orbifold [3].

We then pass to the Kähler setting and quotients of the ball (the complex hyperbolic space). Cyclic coverings of a closed complex hyperbolic manifolds are hard to construct as evaluation of the Kähler form shows that a complex codimension one totally geodesic submanifold is never nullhomologous. A result of Stover and Toledo [4] shows that such branched coverings exist nevertheless. This was used by Guenanica and Hamenstädt to construct on suitable such branched coverings a Kähler Einstein metric of negative sectional curvature, starting from Kähler Einstein cone manifolds [1].

We explain a new approach to the result of Stover and Toledo using the Albanese map of complex hyperbolic manifolds which gives a more flexible method for the construction of branched covering. A particular good control can be obtained in complex dimension two using the fact that in this case, closed totally geodesic submanifolds are just closed real hyperbolic surfaces We use this to show that the universal coverings of some of these examples in complex dimension two are domains in $\mathbb{C}^2$ with negatively curved Kähler Einstein metrics whose boundary are weakly convex (in the sense of CR-geometry) but not rectifiable.

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Gravitational instantons and harmonic maps

Mingyang Li
(joint work with Song Sun)

Gravitational instantons were first introduced by S. Hawking [1] in the study of Euclidean quantum gravity. In [1] they were roughly described as "solutions of the classical Einstein equations which are non-singular on some section of complexified spacetime and in which the curvature dies away at large distances". Mathematically, a gravitational instanton is a complete, noncompact, 4-dimensional oriented Riemannian manifold (M, g) that satisfies the following conditions:

- Ricci-flat: $\mathrm{Ric}_g \equiv 0$;
- quadratic curvature decay: $|\mathrm{Rm}_g| = O(r^{-2})$.

Most examples of gravitational instantons have extra special structures. Recall that the Riemannian curvature tensor can be viewed as a self-adjoint endomorphism of the bundle of 2-forms

$$\mathrm{Riem} : \Lambda^2 \to \Lambda^2.$$

In dimension four with a fixed orientation, the bundle of 2-forms further decomposes into the self-dual part Λ^+ and the anti-self-dual part Λ^-. Under this decomposition, one can write Riem as a matrix

$$\mathrm{Riem} = \begin{pmatrix} \mathrm{W}^+ + \frac{\mathrm{Sc}}{12} & \mathring{\mathrm{Ric}} \\ \mathring{\mathrm{Ric}} & \mathrm{W}^- + \frac{\mathrm{Sc}}{12} \end{pmatrix}.$$

Here $\mathrm{W}^\pm$ is the self-dual/anti-self-dual Weyl curvature, $\mathring{\mathrm{Ric}}$ is the trace-free Ricci curvature and Sc is the scalar curvature. For a Ricci-flat 4-manifold, only the Weyl curvature terms remain. According to the number of eigenvalues of W^+, we divide 4-dimensional Ricci-flat metrics into the following three Types.

(1) Type I: $\mathrm{W}^+ \equiv 0$. That is, g is anti-self-dual.
(2) Type II: W^+ has exactly one eigenvalue of multiplicity two everywhere.
(3) Type III: W^+ has three distinct eigenvalues generically.

It was proved by [3] that if W^+ has exactly two distinct eigenvalues on an open domain, then this must hold everywhere. Thus every oriented Ricci-flat 4-manifold falls into one of the above three types.

In the first two situations, the Ricci-flat metric g has extra structures. In the Type I case the Ricci-flat metric is locally hyperkähler and in the Type II case the Ricci-flat metric is conformally Kähler. We therefore say a gravitational instanton has special geometry if it is Type I or II. Gravitational instantons with special geometry are intimately related to complex geometry. Most previously known examples of gravitational instantons have special geometry. Moreover, thanks to work of many authors over the past decade, simply connected Type I and Type II gravitational instantons are now essentially classified.

In joint work with Song Sun, we provide a systematic construction of Type III gravitational instantons. In particular we get infinitely many new asymptotically flat gravitational instantons without special geometry. Let X_n be the oriented 4-manifold diffeomorphic to

$$X_n := \begin{cases} (S^2 \times \mathbb{R}^2)\#k\mathbb{CP}^2\#k\overline{\mathbb{CP}^2}, & \text{if } n = 2k+1; \\ (S^2 \times \mathbb{R}^2)\#k\mathbb{CP}^2\#(k+1)\overline{\mathbb{CP}^2}, & \text{if } n = 2k+2. \end{cases} \tag{1}$$

We proved

Theorem 1 ([2]). *For any $\beta \in (0,1)$ and any $n \in \mathbb{Z}_{\geq 1}$, there exists an AF_β gravitational instanton on X_n.*

Here, AF_β (asymptotically flat with angle β) refers to the flat model space $\mathbb{R} \times (\mathbb{R}^2 \times \mathbb{R})/\mathbb{Z}$, where the generator of the isometric $\mathbb{Z}$-action rotates $\mathbb{R}^2$ by angle $2\pi\beta$ and translates $\mathbb{R}$ by one. Saying that a gravitational instanton is AF_β means that it is asymptotic to this model space at a polynomial rate. When $n \geq 3$, the gravitational instanton produced in Theorem 1 is Type III.

The gravitational instantons in Theorem 1 are constructed via axisymmetric harmonic maps. It was first observed by physicists that vacuum Einstein equation for stationary axisymmetric black holes reduces to axisymmetric harmonic maps

from Euclidean $\mathbb{R}^3$ into the hyperbolic space $\mathcal{H}^2$. There is a Riemannian analogue of this reduction, which reduces gravitational instantons with T^2-symmetry to axisymmetric harmonic maps. The topology of the conical gravitational instanton is encoded in a so-called *rod structure* $\mathfrak{R}$. Given a rod structure $\mathfrak{R}$, there is a unique harmonic map that is compatible with $\mathfrak{R}$. By the regularity of Ricci-flat metrics, one can show that such harmonic maps produce gravitational instantons with conical singularities along certain totally geodesic 2-spheres. The main technical issue is that one does not have explicit information about the cone angles, so it is hard to know if the conical gravitational instanton is actually smooth.

We overcome this difficulty by degenerating the rod structures. In the construction of conical gravitational instantons, one can vary the rod structure to adjust the cone angles. By studying bubble limits of axisymmetric harmonic maps, we can understand the limiting behavior of cone angles when the rod structure degenerates. With this information, a topological fixed point argument then shows there exists a rod structure $\mathfrak{R}$ such that the corresponding cone angles are all 2π. This produces a smooth gravitational instanton.

A topological invariant of rod structures called *degree* is also defined in [2]. It is proved that for toric conical gravitational instantons, the rod structure has degree zero if and only if it is Type I. The rod structure has degree one if and only if it is Type II.

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Enumerative problems for minimal surfaces of prescribed genus

Adrian Chun-Pong Chu
(joint work with Yangyang Li, Zhihan Wang)

Enumerative problems in geometry have a long history. In 1849, A. Cayley and G. Salmon showed that every smooth cubic surface has exactly 27 copies of $\mathbb{CP}^1$. For over a century, enumerative geometry has been an active branch of algebraic geometry, with even relations to string theory in physics. In geometric analysis, we have some well-known conjectures of a similar flavor:

- Poincaré (1905): Every S^2 has at least 3 embedded closed geodesics.
- S.-T. Yau (1982): Every S^3 has at least 4 embedded minimal spheres.
- B. White (1989): Every S^3 has at least 5 embedded minimal tori.

Note, "every" here means the sphere is equipped with an arbitrary Riemannian metric.

Among these three conjectures, only the first one was fully solved: Building on the work of Lyusternik-Schnirelmann [1], M. Grayson gave a proof [2] using curve

shortening flow. Yau's conjecture was solved for bumpy metrics and metrics with positive Ricci curvature by Wang-Zhou [3]. Joint with Y. Li, we recently resolved White's conjecture for metrics with positive Ricci curvature:

Theorem. [4] Every Riemannian 3-sphere with positive Ricci curvature contains at least 5 minimal tori.

We employ the Simon-Smith min-max theory, and one key difficulty is to overcome genus degeneration in order to obtain minimal surfaces of non-zero genus. Due to the limitation of space, we refer readers to [4] for references on the numerous foundational works on min-max theory, and the previous works on constructing minimal submanifolds of controlled topological types under general metrics.

The above conjectures naturally leads to the following question:

Question. Given any Riemannian 3-sphere, what is the minimum number of embedded minimal surfaces of genus g that must exist independently of the choice of ambient metric?

The author provided a lower bound for every genus, under positive Ricci curvature:

Theorem. [5] Every Riemannian 3-sphere with positive Ricci curvature contains a minimal surface of any genus.

Moreover, joint with Y. Li and Z. Wang, we made progress in the genus 2 case.

Theorem. [7] Every Riemannian 3-sphere with positive Ricci curvature contains at least 4 minimal surfaces of genus 2.

To product multiple minimal surfaces of prescribed genus, we develop an *enumerative min-max theorem.* Let M be a closed orientable 3-manifold. Let $\mathcal{S}(M)$ be the set of all separating surfaces in M that are smoothly embedded except at possibly finitely many points, and let $\mathcal{S}_g(M)$ (resp. $\mathcal{S}_{\leq g}(M)$) be the subset of elements with genus g (resp. $\leq g$). The following theorem *turns the problem of constructing minimal surfaces into understanding topological properties of the relative pair of spaces* $(\mathcal{S}_{\leq g}(M), \mathcal{S}_{\leq g-1}(M))$. For the actual formulation with precise terminology, see [7].

Theorem. (Enumerative min-max theorem [6] [7]). Let M be a Riemannian 3-manifold with positive Ricci curvature, and g a positive integer. Let

$$\Phi : (X, \partial X) \to (\mathcal{S}_{\leq g}(M), \mathcal{S}_{\leq g-1}(M))$$

be a "continuous" map, where X is a manifold with boundary, and suppose we have p cohomology classes $\lambda_1, ..., \lambda_p \in H^*(X; \mathbb{Z}_2)$. Assume the following:

(1) For any relative cycle $(Y, \partial Y) \subset (X, \partial X)$ that represents the Poincaré dual of
$$\lambda_1 \smile \cdots \smile \lambda_p \in H^*(X; \mathbb{Z}_2),$$
the subfamily $\Phi|_Y$ cannot be "deformed" into $\mathcal{S}_{\leq g-1}(M)$.

(2) For any subcomplex $C \subset X$ such that the family Φ maps into $\mathcal{S}_g(M)$, if $\lambda_i|_C \neq 0$ for some i, then the family $\Phi|_C$ must be "non-trivial".

Then M has at least $p+1$ embedded minimal surfaces of genus g.

Lastly, let us mention an open problem from Yau's list:

Yau's problem #96 (1982). In the standard 3-sphere, what is the structure of the space of all minimal surfaces with a fixed genus?

We propose to use the relative topological structure of $(\mathcal{S}_{\leq g}(S^3), \mathcal{S}_{\leq g-1}(S^3))$ to attack the above problem. Due to the large symmetry of the standard 3-sphere, one should first adapt the above enumerative min-max theorem to the equivariant setting. Then we may ask, how to describe "all the relative topological structure" of $(\mathcal{S}_{\leq g}(S^3), \mathcal{S}_{\leq g-1}(S^3))$? Must "every non-trivial topology" produce some minimal surface? Conversely, must every minimal surface exist due to some topological reason involving the set $S_{\leq g}(S^3)$?

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Deforming Lipschitz Homeomorphisms

Mohammad Alattar

In the 1960's, Cernavskiĭ proved several foundational results on the local contractibility of homeomorphism groups of manifolds [3]. Cernavskiĭ's techniques were rather involved and elaborate. Shortly after, Kirby introduced a general and elegant technique of deforming homeomorphisms using the torus [6]. This technique allowed Kirby [6] and Edwards and Kirby [5] to reprove Cernavskiĭ's results.

In addition to having several important applications to topology (see for example [7]), these results on deforming homeomorphisms, and their proofs, turn out to be deeply and intimately related to establishing finiteness and counting results in geometry and topology. For example, using the results of Edwards and Kirby, Cheeger and Kister proved that up to homeomorphism, there are countably many compact topological manifolds [4].

Shortly after Edwards and Kirby announced their results, Siebenmann [8] observed that the "essence" of several of the results on deforming homeomorphisms

of a given topological space X is a property $D(X)$. He introduced a general class of stratified spaces, which he showed, using the torus, to satisfy the property $D(X)$.

One of Perelman's insights is that if X is a finite-dimensional Alexandrov space, then X is a stratified set in the sense of Siebenmann (more precisely, a WCS set). As a direct corollary, one obtains that $D(X)$ holds if X is a finite-dimensional Alexandrov space. Using the property $D(X)$, Perelman was able to establish his stability theorem, which asserts that given a sequence $\{X_i^n\}_{i=1}^{\infty}$ of compact n-dimensional Alexandrov spaces with a uniform lower curvature bound, if the X_i converge in the Gromov–Hausdorff sense without collapse to some compact Alexandrov space X, then for all large i, X_i is homeomorphic to X.

In this talk, we shall explain how to extend the results of Perelman and Siebenmann to the Lipschitz category, using the work of Sullivan [9]. The content of the talk is based on, and is an extension of, our work on deforming Lipschitz homeomorphisms [1, 2].

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Homogenization theory in spectral geometry

Alexandre Girouard

(joint work with Jade Brisson, Bruno Colbois, Katie Gittins, Antoine Henrot, Mikhail Karpukhin, Jean Lagacé)

For a compact Riemannian manifold Ω with boundary, the Steklov problem is

$$\Delta u = 0 \quad \text{in } \Omega, \qquad \partial_\nu u = \sigma u \quad \text{on } \partial\Omega.$$

Its eigenvalues σ_k are those of the Dirichlet-to-Neumann map $D : C^\infty(\partial\Omega) \to C^\infty(\partial\Omega)$. The question to find sharp upper bound for perimeter-normalized Steklov

eigenvalues is rich and fecund. Homogenization enters because maximizing sequences frequently develop fine boundary microstructure instead of converging to a smooth maximizer.

Perforations of manifolds. In [3], we considered periodically perforated Euclidean domains $\Omega^\varepsilon \subset \Omega \subset \mathbb{R}^d$. If the new boundary $\partial\Omega^\varepsilon \setminus \partial\Omega$ has a non-trivial limiting constant density $\beta \in (0,\infty)$, the Steklov eigenvalues $\sigma_k(\Omega^\varepsilon)$ converge to the eigenvalues $\Sigma_k(\beta)$ of the dynamical problem

$$-\Delta U = A_d \beta \Sigma U \quad \text{in } \Omega, \qquad \partial_\nu U = \Sigma U \quad \text{on } \partial\Omega, \tag{1}$$

where $A_d = |\partial B^d(0,1)|$. Remarkably, the spectral parameter Σ appears both in the interior and on the boundary. This family interpolates between the perimeter-normalized Steklov problem (at $\beta = 0$) and the volume-normalized Neumann problem (as $\beta \to \infty$). Taking the unit disk $\Omega = \mathbb{D}$ leads to $\Omega^\varepsilon \subset \mathbb{D}$ with $\sigma_1(\Omega^\varepsilon)L(\partial\Omega^\varepsilon) \xrightarrow{\varepsilon\to 0, \beta\to+\infty} \mathrm{Area}(\mathbb{D})\mu_1(\mathbb{D}) \approx \pi \times 3.39$, where μ_1 is the first non-zero Neumann eigenvalue. The problem of determining $\sup_{\Omega\subset\mathbb{R}^2} \sigma_1(\Omega)L(\partial\Omega)$ becomes tantalizing.

On a closed Riemannian manifold M of dimension d, a similar construction was developped in [5], where lack of periodicity is addressed by using a maximal ε-separated set $S^\varepsilon \subset M$. The associated Voronoĭ cells V_p^ε (with $p \in S^\varepsilon$) tessellate M and they satisfy $\mathrm{Vol}(V_p^\varepsilon) \asymp \varepsilon^d$. Given $0 < \beta \in C^\infty(M)$, remove from each cell the geodesic ball $B_{r_\varepsilon(p)}(p)$ whose radius is determined by $\mathcal{H}^{d-1}\big(\partial B_{r_\varepsilon(p)}(p)\big) = \beta(p)\,\mathrm{Vol}(V_p^\varepsilon)$. If $\Omega^\varepsilon = M \setminus \bigcup_{p\in S^\varepsilon} B_{r_\varepsilon(p)}(p)$, then the total volume removed tends to zero, while the boundary measures satisfy $\mathrm{d}A_{\partial\Omega^\varepsilon} \overset{*}{\rightharpoonup} \beta\,\mathrm{d}V$, and we proved the convergence of $\sigma_k(\Omega^\varepsilon)$ to the eigenvalues $\lambda_k(M,\beta)$ of the β-weighted Laplace eigenvalue problem on M: $-\Delta\phi = \lambda\beta\phi$.

A later measure-theoretic continuity theory was developed in [4]. It places Laplace, Neumann and Steklov problems in a unified framework in which variational eigenvalues $\lambda_k(\mu)$ are associated with a Radon measure μ, and convergence of measures $\mu_\varepsilon \to \mu$ in an appropriate Sobolev-dual topology implies $\lambda_k(\mu_\varepsilon) \to \lambda_k(\mu)$. On a closed Riemannian surface (M, g_M) this leads to

$$\sup_{\Omega\subset M} \sigma_k(\Omega)L(\partial\Omega) = \sup_{g\in[g_M]} \lambda_k(M,g)\,\mathrm{Area}_g(M),$$

thereby linking the shape optimisation problem for σ_k to the conformal geometry of M. For bounded planar domains one obtains the sharp isoperimetric upper bounds $\sigma_k(\Omega)L(\partial\Omega) \le 8\pi k$, saturated by suitably perforated domains with an increasing number of small holes. For $k = 1$, the sharp upper bound is $8\pi = \mathrm{Area}(\mathbb{S}^2)\lambda_1(\mathbb{S}^2)$, the area-normalized first Laplace eigenvalue of the unit round sphere, thereby completely answering the above question from [3].

Recent papers building on this framework include Robin-to-Schrödinger homogenization results [9], operator estimates [8], and the expansion in [6], whose next-order term is a Coulomb-type interaction energy.

Oscillations of Riemannian metrics. In ongoing work with Brisson, Colbois, Gittins and Lagacé [1], we consider warped products $\Omega \times_h \Sigma$. That is, the product

$\Omega \times \Sigma$ equipped with the Riemannian metric $g_h = g_\Omega + h^2 g_\Sigma$, where $0 < h \in C^\infty(\Omega)$. Here, Ω is a d-dimensional compact manifold with boundary and Σ is a n-dimensional closed manifold. Denote the spectrum of the Laplace operator on Σ by $(\lambda_j)_{j\geq 0}$. Assuming $d + n \geq 3$, we prove

$$\sup_{\substack{h>0 \\ h=1 \text{ on } \partial\Omega}} \sigma_k(\Omega \times_h \Sigma) = \begin{cases} \lambda_k(\Sigma)\dfrac{\mathrm{Vol}(\Omega)}{\mathrm{Vol}(\partial\Omega)}, & n = 2, \\ +\infty, & n = 1 \text{ or } n \geq 3. \end{cases} \tag{2}$$

Surprisingly, it is the situation when the fibres $\Sigma = S^1$ have dimension $n = 1$ that is the most complicated. Indeed, separation of variables reduces the Steklov problem on the warped product to auxiliary Steklov problems on Ω. For every eigenvalue λ_j of the Laplacian on Σ, the auxiliary eigenvalues $(\sigma_{\lambda_j,\ell}(h))_{\ell\geq 0}$ are the min–max values of the Rayleigh quotient

$$\frac{\displaystyle\int_\Omega \left(h^n|\nabla f|^2 + \lambda_j h^{n-2} f^2\right) \mathrm{d}V}{\displaystyle\int_{\partial\Omega} h^n f^2 \,\mathrm{d}A}. \tag{3}$$

The Steklov spectrum of $\Omega \times_h \Sigma$ is the union of these spectra over all Laplace eigenvalues λ_j. For fibres Σ of dimension $n = 1$, h and h^{-1} compete with one another in (3), and this is precisely where homogenization is used: partition Ω into Voronoĭ cells at scale $\varepsilon > 0$, choose in each cell a ball of radius ε^α with $1 < \alpha < d/(d-1)$, and make $h_\varepsilon \in C^\infty(\Omega)$ very small on these minuscule balls. The values are calibrated so that, for every $1 \leq p < \infty$,

$$h_\varepsilon \longrightarrow h \quad \text{in } L^p(\Omega), \qquad h_\varepsilon^{-1}\,\mathrm{d}V \overset{*}{\rightharpoonup} (h^{-1} + B)\,\mathrm{d}V, \tag{4}$$

where $0 \leq B \in C^\infty(\Omega)$ is a given function. We prove that the auxiliary eigenvalues $\sigma_{\lambda_j,\ell}(h_\varepsilon)$ converge to min-max eigenvalues $\sigma_{\lambda_j,\ell}(h, B)$ of

$$\frac{\displaystyle\int_\Omega \left(h|\nabla f|^2 + \lambda_j (h^{-1} + B) f^2\right) \mathrm{d}V}{\displaystyle\int_{\partial\Omega} f^2 \,\mathrm{d}A}.$$

Thus the microstructure is invisible to the coefficient of the gradient, but its reciprocal creates an effective potential B. The freedom afforded by this homogenization procedure is essential. Indeed, the zero fibre mode $\lambda_0 = 0$ contains no potential term, whereas every nonzero mode contains the reciprocal coefficient h^{-1}. Consequently, choosing B large pushes all nonzero fibre modes above a prescribed finite portion of the zero-mode spectrum, while leaving the zero mode unaffected, leading after some more work to

$$\sup\{\sigma_k(\Omega \times_h S^1) : h \leq R,\ h \equiv 1 \text{ on } \partial\Omega\} = R\sigma_k(\Omega).$$

Taking $R \to +\infty$ gives the $n = 1$ case of the dichotomy (2).

Boundary oscillations and concluding perspective. A complementary mechanism, based on rapidly oscillating boundaries rather than perforations, underlies

the stability and flexibility results in [2, 7]. More generally, both parts of this report exhibit the same variational principle: microscopic geometry changes one component of the Rayleigh quotient while another remains stable. In the perforation problem, boundary mass converges to an effective interior measure; in the one-dimensional warped-product problem, concentration of the reciprocal coefficient creates an effective potential. This separation between energy and mass explains both the sharp constants and the infinite supremum phenomena described above.

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Volume-minimizing submanifolds in Lagrangian geometry: Special Lagrangian and Hamiltonian stationary equations

Arunima Bhattacharya

(joint work with W. J. Ogden)

In $\mathbb{C}^n = \mathbb{R}^n \times \mathbb{R}^n$, with the standard Kähler structure, the Lagrangian phase or angle, Θ, of the Lagrangian submanifold

$$L_u = \{(x, Du) \mid x \in \Omega \subseteq \mathbb{R}^n\},$$

can be expressed as

$$\Theta = \sum_{i=1}^{n} \arctan \lambda_i$$

where the λ_i's are the eigenvalues of the Hessian of u and Ω is a fixed bounded domain in $\mathbb{R}^n$. The mean curvature vector along L_u can be written as

$$\vec{H} = J\nabla_g \Theta$$

where ∇_g is the gradient operator of L_u (see the work of Harvey–Lawson [5, Eq. (2.19)]) and g is the induced metric from the Euclidean metric on $\mathbb{C}^n$, which can be written as

$$g = I_n + (D^2u)^2.$$

If Θ is constant, then L_u is calibrated and hence volume minimizing; the potential satisfies the special Lagrangian equation $F(D^2u) = c$ and L_u is called a special Lagrangian submanifold [5]. The threshold $|c| = (n-2)\pi/2$ is known as the critical value: at or above this value, the level sets are convex and interior Hessian estimates hold [11, 12], whereas below it, i.e. when $|c| < (n-2)\pi/2$, singular solutions can occur [7, 10, 6].

Allowing the phase to vary gives the prescribed Lagrangian mean curvature equation

$$F(D^2u) = \Theta(x).$$

In the critical and supercritical range, one obtains interior gradient and Hessian estimates; for example, by [1], when $\Theta \in C^{1,1}(B_R)$ and $|\Theta| \geq \frac{(n-2)\pi}{2}$, we get

$$|D^2u(0)| \leq C(n,\Theta)\exp\left[C(n,\Theta)\,\frac{\max_{B_R(0)}|Du|^{2n-2}}{R^{2n-2}}\right].$$

See [2] for gradient estimates.

Hamiltonian stationary Lagrangian graphs

Hamiltonian stationary Lagrangian submanifolds can be viewed as fourth order generalizations of special Lagrangian submanifolds and are locally governed by the fourth order PDE (1). A primary problem is to identify submanifolds that are minimal in a given Hamiltonian isotopy class. Unlike unrestricted minimal surfaces, the constraints of this class permit compact submanifolds to be minimal, for example, the Clifford torus in $\mathbb{C}^n$. The maximum principle, governing minimal surface equations, no longer applies to these fourth order PDEs, making them challenging to study. Moreover, these equations lack uniform ellipticity unless the Hessian of the potential is bounded. While second order special Lagrangian equations have been extensively studied, important questions for the fourth order Hamiltonian stationary equation remain largely open due to the lack of comparable analytical and geometric tools.

We say that u solves the Hamiltonian stationary equation when the Lagrangian phase Θ is g-harmonic (cf. Oh [8], Schoen–Wolfson [9, Proposition 2.2]):

$$\Delta_g\Theta = 0. \tag{1}$$

The gradient graph of u is a Hamiltonian stationary Lagrangian submanifold, which is a critical point of the volume functional under Hamiltonian deformations, that is, variations generated by $J\nabla_g\eta$ for compactly supported smooth functions η.

One of the results presented is based on joint work with Ogden [3], where we established optimal regularity for Hölder continuous Hamiltonian stationary Lagrangian graphs in $\mathbb{C}^n$.

Theorem 1. *Let $u \in C^{1,\beta}$ be a weak solution of (1) on the unit ball $B_1 \subset \mathbb{R}^n$ with $\beta \in \left(\frac{1}{3}, 1\right)$ and $|\Theta| \geq \frac{(n-2)\pi}{2} + \delta$. Then u is smooth in $B_{1/2}$ and satisfies the estimate*

$$\|u\|_{C^{k,\alpha}(B_{1/2})} \leq C\big(\alpha, k, n, \delta, \|u\|_{C^{1,\beta}(B_1)}\big)$$

where $k \geq 2$.

The next result we presented proved the sharpness of the above Hölder exponent β by constructing the following counterexamples, which hold even under the strongest convexity condition on the Lagrangian phase, namely,

$$|\Theta| > \frac{(n-1)\pi}{2}.$$

Theorem 2. *On $B_1 \subset \mathbb{R}^n$ $(n \geq 3)$, for each positive integer k there exists a weak solution $u^{(n,k)}$ to (1) with $|\Theta| > \frac{(n-1)\pi}{2}$ such that*

$$u^{(n,k)} \in C^{1,\frac{1}{2k+1}} \setminus C^{1,\frac{1}{2k+1}+\varepsilon} \quad \textit{for all } \varepsilon > 0.$$

The above result shows that a convex solution $u \in C^{1,\frac{1}{3}}$ to the Hamiltonian stationary equation need not be any more regular. This is a notable departure from the theory of special Lagrangians, for which Chen–Shankar–Yuan [4] established that convex viscosity solutions of the special Lagrangian equation must always be smooth.

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Boundary regularity for optimal transport and applications to complete Calabi–Yau metrics

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(joint work with Freid Tong)

This talk concerned joint work with Freid Tong on the boundary regularity of optimal transport maps between convex domains. Let $\Omega, \Omega' \subset \mathbb{R}^n$ be bounded convex domains, equipped with measures

$$d\mu = \rho(x)\,dx, \qquad d\nu = \rho'(y)\,dy,$$

of equal mass. For the quadratic cost, Brenier's theorem [1] gives a unique optimal transport map, defined μ-almost everywhere, of the form $T = \nabla u$ for a convex function u. If the densities are continuous, bounded from above and below by positive constants and the domains are convex, then Caffarelli [4] showed that u solves the second boundary value problem for the Monge–Ampère equation in the Alexandroff sense

$$\det D^2 u(x) = \frac{\rho(x)}{\rho'(\nabla u(x))}, \qquad \nabla u(\Omega) = \Omega'. \tag{1}$$

Caffarelli proved that convexity of the domains is the natural hypothesis for global regularity, obtaining global $C^{1,\delta}$ estimates for densities bounded above and below [2] (see also [10]). Under stronger assumptions on the boundaries, he also established global $C^{2,\alpha}$ regularity [3]. More recently, Chen–Liu–Wang [5] proved global $C^{2,\alpha}$ regularity when the domains have $C^{1,1}$ boundary, while Savin–Yu [10] proved global $C^{1,1-\varepsilon}$ and $W^{2,p}$ estimates for transport between arbitrary planar convex domains with constant densities.

Our first result extends the theorem of Savin–Yu to arbitrary dimension and Dini continuous densities.

Theorem 1 (C.-Tong, [7]). *Suppose that Ω, Ω' are bounded convex domains and ρ, ρ' are Dini-continuous with*

$$C^{-1} \leq \rho, \rho' \leq C, \qquad [\rho]_{\mathrm{Dini},\overline{\Omega}} + [\rho']_{\mathrm{Dini},\overline{\Omega'}} \leq C.$$

for $C > 1$. Then, for every $\varepsilon > 0$ and every $p > 1$,

$$u \in C^{1,1-\varepsilon}(\overline{\Omega}) \cap W^{2,p}(\Omega)$$

with a uniform estimate depending only on C, ϵ, p and the inner and outer radii of the domains.

The exponent in the Hölder estimate is sharp: on general convex domains one cannot expect global $C^{1,1}$ regularity. If some regularity of the boundaries is imposed we can obtain sharp global higher order regularity.

Theorem 2 (C.-Tong, [7]). *Suppose that Ω, Ω' are bounded convex domains with $\partial\Omega$ and $\partial\Omega'$ of class $C^{1,\beta}$ for some $\beta \in (0,1)$. Suppose that the densities ρ, ρ' are Hölder continuous for some $\alpha \in (0,1)$,*

$$C^{-1} \leq \rho, \rho' \leq C, \qquad [\rho]_{C^\alpha(\Omega)} + [\rho']_{C^\alpha(\Omega')} \leq C.$$

for some $C > 1$. Then

$$u \in C^{2,\gamma}(\overline{\Omega}), \qquad \gamma = \min\{\alpha, \beta\}.$$

The main new ingredient is a monotonicity formula based on the pseudo-Riemannian interpretation of optimal transport by Kim–McCann–Warren [9]. In this framework one may regard the graph

$$\Sigma = (x, \nabla u(x))$$

as a space-like volume-maximizing submanifold on $\mathbb{R}^{n,n}$ for the weighted volume function

$$\int_\Sigma e^{\frac{1}{2}\log(\rho\rho')} d\mathrm{Vol}_\Sigma.$$

From this interpretation of optimal transport it is natural to consider the volume density of the area maximizing graph.

Theorem 3 (C.-Tong, [7]). *Assume that $0 \in \partial\Omega \cap \partial\Omega'$, and normalize the transport potential by $u(0) = 0$ and $\nabla u(0) = 0$. Suppose that ρ and ρ' are homogeneous of degrees l and k, respectively. Define the extrinsic balls*

$$D_r(0) = \{(x,y) \in \mathbb{R}^{n,n} : x \cdot y \leq r^2\}$$

Then the quantity

$$\chi(r) = r^{-\kappa} \int_{D_r \cap \Sigma} e^{\frac{1}{2}\log(\rho\rho')} d\mathrm{Vol}_\Sigma, \qquad \kappa = \frac{2(n+l)(n+k)}{2n+l+k}, \tag{2}$$

is monotone non-increasing in r. Furthermore, $\chi(r)$ is constant for $r \in [0, r_0]$ if and only if both domains are conical about the origin and

$$u(tx) = t^{1+\frac{n+l}{n+k}} u(x).$$

Note in particular, when $k = l = 0$, the homogeneous model has degree two.

The study of homogeneous optimal transport maps between convex cones is evidently a key problem for the sharp boundary regularity of optimal transport in general. It turns out that homogeneous optimal transport maps are also closely connected to problems arising in geometry, particularly the existence of certain complete Calabi-Yau manifolds. The Tian–Yau problem is the following: let Y be a Fano manifold and let $D = D_1 + \cdots + D_k \in |-K_Y|$ be a reduced divisor with simple normal crossings singularities. Assume that each D_i is ample. The complement $X = Y \setminus D$ carries a holomorphic volume form with a simple pole along D. The problem of Tian-Yau asks whether X admits a complete Ricci-flat Kähler metric. Tian and Yau answered this when D is smooth and irreducible [11]. Their metric is asymptotic to a Calabi ansatz: if s_D is a defining section of the divisor D, then the leading Kähler potential is given by $(-\log|s_D|_h)^{\frac{n+1}{n}}$ where

$n = \dim_{\mathbb{C}} Y$ and the hermitian metric h on the normal bundle is determined by Yau's solution of the Calabi conjecture [12]. We comment that, up to rescaling, the function $u(x) = x^{\frac{n+1}{n}}$ is precisely an optimal transport map from $(\mathbb{R}_{>0}, dx)$ to $(\mathbb{R}_{>0}, y^{n-1}dy)$.

When D has $k > 1$ components, the author and Li [6] proposed a "generalized Calabi ansatz". Under this ansatz, to leading-order, the Ricci-flat equation reduces to a real Monge–Ampère equation. Assuming for simplicity that $D_i \sim_{\mathbb{Q}} D_j$, the resulting real Monge-Ampère equation is

(3)
$$\det D^2 u \left(\sum_{i=1}^{k} \frac{\partial u}{\partial x_i} \right)^{n-k} = 1 \quad \text{on } (\mathbb{R}_{>0})^k, \qquad \nabla u(\mathbb{R}^k_{>0}) = \{ y \in \mathbb{R}^k : \sum_{i=1}^{k} y_i > 0 \}$$

One seeks convex, homogeneous solutions of this equation. For $k = 1$ this recovers precisely the Tian–Yau model. The case $k = 2$ was used by Collins–Li to construct complete Calabi–Yau metrics on the complement of two divisors [6].

Equation (3) is an optimal transport problem between cones with a degenerate homogeneous density. In joint work with Tong and Yau [8], we constructed homogeneous solutions of this problem by a free-boundary Monge–Ampère equation on a simplex. For the geometric problem one needs sharp control of the boundary regularity for these optimal transport maps, and hence we expect the monotonicity formula to play a role in future developments.

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Optimal decay constant for complete manifolds of positive scalar curvature with quadratic decay

Shuli Chen

In the study of three-dimensional manifolds, a fundamental question is about understanding the topological structure of 3-manifolds that admit a Riemannian metric of positive scalar curvature $R > 0$. In the closed case, a closed orientable 3-manifold admits a Riemannian metric with positive scalar curvature if and only if it decomposes as a connected sum of only spherical manifolds and $\mathbb{S}^2 \times \mathbb{S}^1$ summands. This was addressed by the pioneering work of Schoen and Yau [7, 8] and Gromov and Lawson [4, 5, 6], and finally completed by Perelman's work.

In the non-compact case, a similar decomposition theorem has recently been established for open manifolds admitting complete Riemannian metrics of uniformly positive scalar curvature $R \geq 1$ independently by Gromov [3] and Wang [9], using the μ-bubble technique. Namely, such manifolds decompose as a possibly infinite connected sum of spherical manifolds and $\mathbb{S}^2 \times \mathbb{S}^1$ summands.

One may wish to consider a weaker scalar curvature assumption and ask for a decomposition theorem. One natural choice is at most a quadratic decay at infinity.

Definition 1 ([1, Definition 1.2])**.** *Let M be a complete Riemannian n-dimensional manifold. Fix a basepoint $x \in M$, and denote by $r_x(y) = d(x, y)$ the distance function to x. The scalar curvature R of M has* at most C-quadratic decay at infinity *if there exists a constant $D_0 > 0$ such that for every $y \in M$ with $r_x(y) \geq D_0$,*

$$R(y) > \frac{C}{r_x(y)^2}.$$

In [1], Balacheff, Gil Moreno de Mora Sardà, and Sabourau extended the decomposition theorem as follows. Their proof is deduced from results on filling radius instead of via μ-bubbles.

Theorem 1 ([1, Theorem 1.3])**.** *Let M be a complete connected orientable Riemannian 3-manifold. Suppose that M has positive scalar curvature with at most C-quadratic decay at infinity for some $C > 64\pi^2$. Then M decomposes as a possibly infinite connected sum of spherical manifolds and $\mathbb{S}^2 \times \mathbb{S}^1$ summands.*

This theorem is closely related to the following conjecture of Gromov [3, Section 3.6.1].

Conjecture 1 (Critical Rate of Decay Conjecture [3])**.** *There exists a universal critical constant $C_n > 0$ such that the following holds. Let M be an orientable n-manifold that admits a complete Riemannian metric of positive scalar curvature.*

(1) *For every $0 < C < C_n$, there exists a complete Riemannian metric on M of positive scalar curvature with at most C-quadratic decay at infinity.*
(2) *If M admits a complete Riemannian metric with positive scalar curvature with C-quadratic decay at infinity for $C > C_n$, then M admits a complete Riemannian metric with uniformly positive scalar curvature.*

In [2], we improve the decay constant in Theorem 1 and find the optimal one to be $\frac{2}{3}$.

Theorem 2. *Let M be a complete connected orientable Riemannian 3-manifold. Suppose that M has positive scalar curvature with at most C-quadratic decay at infinity for some $C > \frac{2}{3}$. Then M decomposes as a possibly infinite connected sum of spherical manifolds and $\mathbb{S}^2 \times \mathbb{S}^1$ summands.*

The constant $\frac{2}{3}$ is optimal; the manifold $\mathbb{R}^2 \times \mathbb{S}^1$ admits a complete metric of positive scalar curvature with at most C-quadratic decay for every $0 < C < \frac{2}{3}$.

Note that $\mathbb{R}^2 \times \mathbb{S}^1$ does not decompose as a connected sum of spherical manifolds and $\mathbb{S}^2 \times \mathbb{S}^1$ (see e.g. [1, Proposition 8.2]).

The following rigidity result, which shows case (2) of Conjecture 1 holds when $n = 3$, is a direct consequence of Theorem 2 and an adaptation of Gromov–Lawson's Surgery Theorem [5, Theorem A]. In particular, we find $C_3 = \frac{2}{3}$.

Corollary 3. *Let M be an orientable 3-manifold. If M admits a complete Riemannian metric of positive scalar curvature with at most C-quadratic decay at infinity for some $C > \frac{2}{3}$ with respect to some basepoint, it also admits a complete Riemannian metric with uniformly positive scalar curvature. The constant $C_3 = \frac{2}{3}$ is optimal here.*

Unlike [1], our main tool is the method of μ-bubbles. Our main theorem is a consequence of the following construction result and the new exhaustion result.

Proposition 1. *For every $n \geq 3$, the n-manifold $\mathbb{R}^2 \times \mathbb{T}^{n-2}$ admits a complete metric of positive scalar curvature with at most C-quadratic decay for every $0 < C < \frac{n-1}{n}$.*

Proposition 2. *Let $3 \leq n \leq 7$. Suppose M is a complete orientable n-manifold. Suppose that M has positive scalar curvature with at most C-quadratic decay at infinity for some $C > \frac{n-1}{n}$. Then M admits an exhaustion $K_1 \subset K_2 \subset \cdots \subset M$ such that $M = \cup_{i=1}^{\infty} K_i$, each K_i is a compact codimension-0 smooth submanifold with smooth boundary ∂K_i, and ∂K_i is a compact orientable $(n-1)$-manifold that admits a metric of positive scalar curvature.*

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Stable 2-systoles, scalar curvature, and Spin^c comass bounds

Simone Cecchini
(joint work with Sven Hirsch, Rudolf Zeidler)

This talk was based on joint work with Sven Hirsch and Rudolf Zeidler. I discussed a sharp stable 2-systolic inequality for complex projective space under the scalar curvature lower bound of the normalized Fubini–Study metric.

Let M be a smooth manifold diffeomorphic to $\mathbb{C}P^n$, and let g be a Riemannian metric on M. We normalize the Fubini–Study metric g_{FS} on $\mathbb{C}P^n$ by requiring that every projective line have area π. With this normalization,

$$\mathrm{scal}_{g_{\mathrm{FS}}} = 4n(n+1).$$

The main theorem states that if

$$\mathrm{scal}_g \geq 4n(n+1),$$

then

$$\mathrm{sys}_2^{\mathrm{st}}(M,g) \leq \pi.$$

Moreover, equality is rigid: if $\mathrm{sys}_2^{\mathrm{st}}(M,g) = \pi$, then there exists a complex structure J on M and a biholomorphism

$$B : (M,J) \longrightarrow (\mathbb{C}P^n, J_0)$$

such that

$$g = B^* g_{\mathrm{FS}}.$$

Thus the normalized Fubini–Study metric is characterized, among all Riemannian metrics on manifolds diffeomorphic to $\mathbb{C}P^n$ with scalar curvature bounded below by $4n(n+1)$, as the unique maximizer of the stable 2-systole.

The proof is based on a general scalar-curvature–comass comparison theorem for Spin^c manifolds. Let M^{2m} be a closed connected oriented manifold, and let $c \in H^2(M;\mathbb{Z})$ satisfy

$$c \equiv w_2(TM) \pmod 2$$

and

$$\left\langle [M], e^{c/2}\widehat{A}(TM) \right\rangle \neq 0.$$

If $c_{\mathbb{R}} \in H^2(M;\mathbb{R})$ denotes the image of c, then for every Riemannian metric g on M one obtains

$$\min_M \mathrm{scal}_g \leq 4m\pi \, \|c_{\mathbb{R}}\|_{\mathrm{comass}}.$$

The nonvanishing of the Spin^c index gives a harmonic spinor. Applying the Spin^c Lichnerowicz formula, the scalar curvature term is compared with the curvature of the determinant line bundle. The key analytic input is a sharp pointwise Clifford-algebra estimate which bounds the curvature contribution in terms of the comass of the corresponding real 2-form.

Stable norm–comass duality then converts this scalar-curvature–comass estimate into the stable systolic inequality. In the case of $\mathbb{C}P^n$, applying the estimate to the hyperplane class gives the sharp constant π.

The equality case is governed by equality in the Clifford-algebra estimate. This forces the relevant 2-form to be parallel and Kähler, and the metric becomes Kähler–Einstein. The final identification with the Fubini–Study metric follows from the characterization of complex projective space and the uniqueness of Kähler–Einstein metrics on $\mathbb{C}P^n$.

I also briefly discussed the odd-dimensional analogue for manifolds diffeomorphic to $\mathbb{C}P^n \times S^1$. Under the same scalar curvature lower bound one again obtains

$$\mathrm{sys}_2^{\mathrm{st}}(M, g) \leq \pi,$$

and equality implies that the universal cover is isometric to

$$(\mathbb{C}P^n, g_{\mathrm{FS}}) \times \mathbb{R}.$$

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