

On the Structure of Quantum Affine Superalgebra $U_v(\mathbf{A}(0, 2)^{(4)})$

by

Fengchang LI

Abstract

We study the quantum affine superalgebra $U_v(\mathbf{A}(0, 2)^{(4)})$ defined abstractly by Yamane, when v is not a root of unity. We prove that the defining relations satisfied by the Chevalley generators of $U_v^+(\mathbf{A}(0, 2)^{(4)})$, or equivalently the corresponding Nichols algebra $B(\mathbb{V})$, are the quantum Serre relations, which means the Drinfeld–Jimbo-type realization of $U_v(\mathbf{A}(0, 2)^{(4)})$ is obtained. Based on Lyndon word theory, we construct distinguished imaginary root vectors and give an estimation of their comultiplication in the form of generating functions. With these results, we determine all the root multiplicities and give a PBW basis of $U_v^+(\mathbf{A}(0, 2)^{(4)})$. We also describe concretely all the commutation relations among real root vectors and distinguished imaginary root vectors using generating functions, i.e., the Drinfeld second realization is obtained for $U_v^+(\mathbf{A}(0, 2)^{(4)})$.

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§1. Introduction

Quantum groups were introduced formally in [Dri85] and [Jim85] as deformations of the universal enveloping algebra $U(\mathfrak{g})$ of a Kac–Moody algebra $\mathfrak{g}$. Affine quantum groups are also called quantum affine algebras, and the theories of their representations are very rich, with many important applications in many branches of mathematics and mathematical physics. In addition, quantum affine algebras also have the famous Drinfeld second realizations, first appearing in [Dri87] and then [Bec94, Jin98, Dam12, Dam15, JZ16]. Drinfeld second realizations are very important for the theory of quantum affine algebras because they are crucial for the study of representations; see, e.g., [CP91, CP98].

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F. Li: Department of Mathematics, Nanjing University, Hanko Road 22, 210093 Nanjing, P. R. China; Fachbereich Mathematik und Informatik, Philipps-Universität Marburg, Hans-Meerwein-Strasse 6, 35032 Marburg, Germany;

e-mail: dg1921005@mail.nju.edu.cn

Author IDs: ORCID [0009-0009-0402-0205](https://orcid.org/0009-0009-0402-0205)

Affine Kac–Moody superalgebras are generalizations of affine Kac–Moody algebras, defined and classified by van de Leur [vdL89]. In this classification, there is a special family named $A(m, n)^{(4)}$. The main feature of $A(m, n)^{(4)}$ is that there exist odd imaginary roots and that the imaginary root vectors do not necessarily commute with each other. In addition, when $m = n$, the cardinality of the Serre-type defining relations is infinite (also in the quantum case) due to the 1-dimensional center spanned by the identity matrix.

Quantum superalgebras and quantum affine superalgebras appeared first in [Yam94] and [Yam99], respectively, which are foundational for these theories. In them, Yamane defined quantum (affine) superalgebras abstractly, i.e., by the radical of a bilinear form, which is similar to the non-super case introduced by [Lus93]. Then Yamane got the defining relations satisfied by the Chevalley generators of affine Lie superalgebras and the corresponding quantum affine superalgebras except the cases of $A(m, m)^{(2)}$ and $A(m, m)^{(4)}$, which means the Drinfeld–Jimbo-type realization is also obtained for these quantum affine superalgebras.

For quantum affine superalgebras, the Drinfeld second realizations are also important tools for studying representations. Related work appeared in [Yam99, HSTY08, Zha14, BFK26] etc.

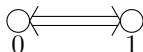
Next, we discuss $U_v(A(m, n)^{(4)})$ when $m \neq n$. When v is transcendental, the defining relations satisfied by the Chevalley generators were known in [Yam99], and the root multiplicities were known in [EG09], which are the same as the root multiplicities of the corresponding affine Lie superalgebra. However, no explicit form of a PBW basis was given. When v is not a root of unity, the defining relations satisfied by the Chevalley generators, the root multiplicities, PBW basis and the Drinfeld second realization were all unknown.

In the last years, Nichols algebras have attracted much interest from many branches of mathematics such as Hopf algebras, quantum groups, algebraic geometry and number theory, and conformal field theory. Concretely, Nichols algebras play an important role in the classification of finite-dimensional pointed Hopf algebras; see [AS10]. It is also well known that the theory of Nichols algebras is closely related to the theory of quantum groups. In the finite case, the positive part of a Drinfeld–Jimbo-type quantum group is a Nichols algebra when v is not a root of unity; see [HS20, Chapter 16]. In the affine case, this is also true by [HK08]. If we define a quantum group abstractly following Lusztig and Yamane, i.e., using the radical of the bilinear form, then the positive part is just a Nichols algebra by definition.

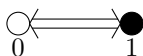
In the present paper, we discuss mainly $U_v(\widehat{\mathfrak{sl}}_2)$ and $U_v(A(0, 2)^{(4)})$ defined abstractly. The structure theory of $U_v(A(0, 2)^{(4)})$ is unavoidable for further

research on $U_v(A(m, n)^{(4)})$. The quantum affine (super)algebras $U_v(\widehat{\mathfrak{sl}}_2)$ and $U_v(A(0, 2)^{(4)})$ look similar and share some common properties, but are also very different.

The Dynkin diagram of $U_v(\widehat{\mathfrak{sl}}_2)$ is



whereas the Dynkin diagram of $U_v(A(0, 2)^{(4)})$ is



As algebras, $U_v^+(\widehat{\mathfrak{sl}}_2)$ is isomorphic to the Nichols algebra $B(\mathbb{U})$, where $\mathbb{U}$ is a braided vector space of diagonal type with braiding matrix

$$\begin{pmatrix} q & q^{-1} \\ q^{-1} & q \end{pmatrix}$$

with respect to a basis $\{x_1, x_2\}$ and $q = v^2$. The isomorphism maps the Chevalley generators E_0, E_1 to x_1 and x_2 . The defining relations satisfied by the Chevalley generators, or equivalently x_1, x_2 , are the quantum Serre relations when q is not a root of unity; for a proof see [HK08].

As algebras, $U_v^+(A(0, 2)^{(4)})$ is isomorphic to the Nichols algebra $B(\mathbb{V})$, where $\mathbb{V}$ is a braided vector space of diagonal type with braiding matrix

$$\begin{pmatrix} q & q^{-1} \\ q^{-1} & -q \end{pmatrix}$$

with respect to a basis $\{x_1, x_2\}$ and $q = v^2$. Again, the isomorphism maps the Chevalley generators E_0, E_1 to x_1 and x_2 . Conjecturally, the defining relations of $U_v^+(A(0, 2)^{(4)})$, or equivalently $B(\mathbb{V})$, are also the quantum Serre relations when q is not a root of unity. In the present paper, this will be proved in Theorem 6.5.

The Nichols algebra $B(\mathbb{V})$ first appeared in Cuntz's classification of rank two affine Nichols algebras [Cun18]. An affine Nichols algebra is a Nichols algebra of diagonal type with an affine Weyl groupoid or equivalently affine Tits arrangement; see [CMW17]. All Nichols algebras in the classification in [Cun18] are quite unexplored except $U_v^+(\widehat{\mathfrak{sl}}_2)$, i.e., $B(\mathbb{U})$, which is very well known. We believe that these Nichols algebras have a Lie-theoretic background.

By the reflection theory of Nichols algebras of diagonal type (see [Hec06]), we conclude that $B(\mathbb{U})$ and $B(\mathbb{V})$ share the same real roots, i.e., $n\delta + \alpha_1$ and $n\delta + \alpha_2$ for $n \geq 0$, where $\delta = \alpha_1 + \alpha_2$. The imaginary roots of $B(\mathbb{U})$ are well known as $n\delta$,

$n > 0$, and the multiplicities are all 1. The only possible imaginary roots of $B(\mathbb{V})$ are also $n\delta$, but the multiplicities have been, in general, unknown. In this paper, all the root multiplicities of $B(\mathbb{V})$ are determined and a PBW basis is given; see Theorem 5.4.

For strategic reasons, we use the notation X_1 and Y_1 for the two Chevalley generators E_0 and E_1 of $U_v^+(\widehat{\mathfrak{sl}}_2)$, or equivalently, the two basis vectors x_1 and x_2 of the braided vector space $\mathbb{U}$. Let $[\ , \]$ be the braided bracket. Let $L_1 = [X_1, Y_1]$. Define $X_{n+1} = [X_n, L_1]$, $Y_{n+1} = [L_1, Y_n]$, for $n \geq 1$. Define $L_n = [X_1, Y_n]$ for all $n \geq 1$. Then, rewriting the relations in the Drinfeld second realization, we obtain the following relations in $B(\mathbb{U})$. Let $\theta = q - q^{-1}$, $[m]_q = \frac{q^m - q^{-m}}{q - q^{-1}}$ and define $L_0 = \frac{1}{\theta}$; define $\bar{L}_n$ for all $n \geq 1$ using the equation

$$\sum_{n=0}^{\infty} \theta L_n u^n = \exp \left(\sum_{n=1}^{\infty} \theta \bar{L}_n u^n \right).$$

Then for all $m, n, k \geq 1$,

- (a) $[L_m, L_n] = 0 = [\bar{L}_m, \bar{L}_n]$,
- (b) $[X_k, \bar{L}_m] = \frac{[m]_q}{m} X_{k+m}$, $[\bar{L}_m, Y_k] = \frac{[m]_q}{m} Y_{k+m}$,
- (c) $[X_n, X_{n+k}] = -[X_{n+k-1}, X_{n+1}]$, $[Y_{n+k}, Y_n] = -[Y_{n+1}, Y_{n+k-1}]$,
- (d) $[X_m, Y_n] = L_{m+n-1}$.

We can define X_n, Y_n, L_n similarly for $B(\mathbb{V})$. It is not difficult to prove that X_n and Y_n are still real root vectors. However, when one looks at the relations in imaginary root spaces, great difficulty appears. At degree 3δ , we have the following relation:

$$(1.1) \quad L_3 = \frac{\theta - 1}{2} [L_2, L_1] + \theta L_1 L_2 - \theta L_1^3,$$

and L_2, L_1 do not commute. Due to this non-commutativity, $U_v(A(0, 2)^{(4)})$ is much more complicated than $U_v(\mathfrak{sl}_2)$. The algebra $U_v(A(0, 2)^{(4)})$ together with this property (difficulty) and with equation (1.1) was first announced by Yamane [Yam18]. Equation (1.1) is equation (3.2a) in Proposition 3.2 of the present paper. How to describe similar relations at higher degrees was unknown. In the present paper, we give a solution; see (3.24a) in Proposition 3.17. Moreover, we obtain the Drinfeld second realization for $B(\mathbb{V})$ or equivalently $U_v^+(A(0, 2)^{(4)})$; see Theorem 7.12.

Our main tool is Lyndon word theory; see [Kha99, Ros99]. There are many excellent applications of the Lyndon word theory in Hopf algebra theory and Lie theory; see [Hec06, HRZ08, Ang15, CHW16, AT24, NT24] etc. In terms of Lyndon word theory, if we define the order by $x_1 < x_2$ for the basis of the braided vector

space $\mathbb{U}$, then X_n, Y_n, L_n are the super-letters corresponding to the Lyndon words $x_1(x_1x_2)^{n-1}$, $(x_1x_2)^{n-1}x_2$, $x_1(x_1x_2)^{n-1}x_2$, respectively.

We also use an important subquotient of Nichols algebras of diagonal type that appeared in [GH07] and [AAH19]. The subquotient is a braided Hopf algebra induced by the braided Hopf algebra structure of the initial Nichols algebra. In [AAH19, AG25, AG23], the subquotient played a central role in the proof of the conjecture in [AAH21] about the equivalence between the finiteness of the root system and the finiteness of the GK dimension for Nichols algebras of diagonal type. The subquotient is very beneficial in the study of $B(\mathbb{V})$ because it has a very manageable comultiplication, which can be used to detect possible relations in $B(\mathbb{V})$, determine the root multiplicities completely and describe the commutation relations in $U_v^+(A(0, 2)^{(4)})$ between real root vectors and imaginary root vectors.

Moreover, in the Drinfeld second realization of $U_v(\widehat{\mathfrak{sl}}_2)$, the loop-like generators of the imaginary roots are described by the exponential series with coefficients in $U_v(\widehat{\mathfrak{sl}}_2)$. In the present paper, this method is also used and is very beneficial. In addition, tangent and some other techniques of generating functions are crucial.

The paper is organized as follows. In Section 2 we introduce some notation and recall some basic theories. In Section 3 we derive some basic relations that we need in $B(\mathbb{V})$. We also discuss the subquotient $K_{\geq 1}/K_{> 1}$ of $B(\mathbb{V})$ and identify the central elements. In Section 4, which is the most important part of this paper, we construct distinguished imaginary root vectors and determine their comultiplication in the subquotient $K_{\geq 1}/K_{> 1}$ using various generating functions. In Section 5 we completely determine all the root multiplicities of $B(\mathbb{V})$ based on the results of Section 4 and a PBW basis of $B(\mathbb{V})$ is obtained. In Section 6 we prove that the defining relations of $B(\mathbb{V})$ are the quantum Serre relations, which means the Drinfeld–Jimbo-type realization of $U_v(A(0, 2)^{(4)})$ is obtained. In Section 7 we obtain the Drinfeld second realization of $B(\mathbb{V})$ or equivalently $U_v^+(A(0, 2)^{(4)})$, also based on the results of Section 4.

§2. Preliminaries

In the whole paper, $q \in \mathbb{C}^\times$ is not a root of unity. Let $\theta = q - q^{-1}$, $[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}$. All brackets $[,]$ in this paper are braided brackets introduced in Section 2.1.

§2.1. Nichols algebras of diagonal type

We refer to [Ang11] for the definition of a braided vector space of diagonal type and the definition of a Nichols algebra.

Let $\mathcal{K}$ be a field. Let $s \in \mathbb{N}$, $X = \{x_1, \dots, x_s\}$. Let $\mathbb{X}$ be the set of words with letters in X . Let V be an s -dimensional vector space with basis $\{x_1, \dots, x_s\}$.

We can identify $\mathcal{K}\mathbb{X}$ with $T(V)$. Then $T(V)$ is a $\mathbb{Z}^s$ -graded algebra determined by $\deg x_i = \alpha_i$, $1 \leq i \leq s$, where $\{\alpha_1, \dots, \alpha_s\}$ is the canonical basis of $\mathbb{Z}^s$.

Let c be a braiding of V . The braided bracket of $x, y \in T(V)$ is defined by

$$[x, y]_c = m \circ (\text{id} - c)(x \otimes y).$$

Assume (V, c) is of diagonal type with braiding matrix $(q_{ij})_{1 \leq i, j \leq s}$ with respect to the basis $\{x_1, \dots, x_s\}$, and let $\chi: \mathbb{Z}^s \times \mathbb{Z}^s \rightarrow \mathcal{K}^\times$ be the bicharacter determined by the condition $\chi(\alpha_i, \alpha_j) = q_{ij}$ for each pair $1 \leq i, j \leq s$. Then for each pair of $\mathbb{Z}^s$ -homogeneous elements $u, v \in \mathbb{X}$, we have $c(u \otimes v) = q_{u,v} v \otimes u$, $q_{u,v} = \chi(\deg u, \deg v) \in \mathcal{K}^\times$. In such a case, the “braided Jacobi identity” and skew derivations hold:

$$\begin{aligned} [[u, v], w] &= [u, [v, w]] - \chi(\alpha, \beta) v[u, w] + \chi(\beta, \gamma) [u, w] v, \\ [u, vw] &= [u, v]w + \chi(\alpha, \beta) v[u, w], \\ [uv, w] &= \chi(\beta, \gamma) [u, w]v + u[v, w], \end{aligned}$$

where u, v, w are homogeneous elements in $T(V)$, and α, β, γ are the degrees of u, v, w . If $\chi(\beta, \alpha)\chi(\alpha, \beta) = 1$, then we have

$$(2.1) \quad [[u, v], w] = [u, [v, w]] - \chi(\alpha, \beta) [v, [u, w]].$$

§2.2. Braided vector space $\mathbb{V}$, the Nichols algebra $B(\mathbb{V})$ and the root system of $B(\mathbb{V})$

Let $\mathbb{V}$ be a 2-dimensional braided vector space of diagonal type with braiding matrix

$$\begin{pmatrix} q & q^{-1} \\ q^{-1} & -q \end{pmatrix},$$

where q is not a root of unity.

It is easy to see $B(\mathbb{V})$ is of affine Cartan type with Cartan matrix $\begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$. We refer to [Hec06] for details. Then the real roots of $B(\mathbb{V})$ are just the positive real roots of $\widehat{\mathfrak{sl}}_2$, namely $(n+1)\alpha_1 + n\alpha_2$ and $n\alpha_1 + (n+1)\alpha_2$ for $n \geq 0$. The Nichols algebra $B(\mathbb{V})$ also has imaginary roots. The only possible imaginary roots are $n\delta$, $\delta = \alpha_1 + \alpha_2$ and $n \geq 1$, but the multiplicities are in general unknown. Note that if β is an imaginary root, then we have $\chi(\beta, \alpha)\chi(\alpha, \beta) = 1$ for all $\alpha \in \mathbb{N}_0^2$, and hence (2.1) holds.

§2.3. Differential operators

We refer to [HS20, Chapter 7].

Definition 2.1. Define the right and left differential operators $\partial_i^R, \partial_i^L: B(\mathbb{V}) \rightarrow B(\mathbb{V})$, $1 \leq i \leq 2$ by

- (1) $\partial_i^R(1) = \partial_i^L(1) = 0$, $\partial_i^R(x_j) = \partial_i^L(x_j) = \delta_{ij}$ for all $1 \leq i, j \leq 2$,
- (2) $\partial_i^R(xy) = x\partial_i^R(y) + \partial_i^R(x)\chi(\alpha_i, \beta)y$ for all $1 \leq i, j \leq 2$ and $x, y \in B(\mathbb{V})$, y is $\mathbb{N}_0^2$ -homogeneous with degree β ,
- (3) $\partial_i^L(xy) = \partial_i^L(x)y + \chi(\alpha, \alpha_i)x\partial_i^L(y)$ for all $1 \leq i, j \leq 2$ and $x, y \in B(\mathbb{V})$, x is $\mathbb{N}_0^2$ -homogeneous with degree α .

Proposition 2.2. For $x \in B(\mathbb{V})$ and x not a constant, in $B(\mathbb{V})$ we have

$$\begin{aligned} x = 0 &\Leftrightarrow \partial_i^R(x) = 0 \quad \forall 1 \leq i \leq 2 \\ &\Leftrightarrow \partial_i^L(x) = 0 \quad \forall 1 \leq i \leq 2. \end{aligned}$$

§2.4. An important subquotient

We follow the notation in [AAH19]. For the braided vector space $\mathbb{V}$, $B(\mathbb{V})$ has a unique $\mathbb{N}_0^2$ -grading as a braided Hopf algebra

$$B(\mathbb{V}) = \bigoplus_{\alpha \in \mathbb{N}_0^2} B^\alpha(\mathbb{V})$$

such that $\deg x_1 = \alpha_1$, $\deg x_2 = \alpha_2$. Now, set

$$B_{\geq 1} = \bigoplus_{\substack{\alpha = a_1\alpha_1 + a_2\alpha_2 \in \mathbb{N}_0^2: \\ a_1 \geq a_2}} B^\alpha(\mathbb{V}), \quad B_{> 1} = \bigoplus_{\substack{\alpha = a_1\alpha_1 + a_2\alpha_2 \in \mathbb{N}_0^2: \\ a_1 > a_2}} B^\alpha(\mathbb{V}),$$

$$K_{\geq 1} = \{x \in B(\mathbb{V}) \mid \Delta(x) \in B_{\geq 1} \otimes B(\mathbb{V})\}, \quad K_{> 1} = K_{\geq 1} \cap B_{> 1}.$$

The subquotient we use is $K_{\geq 1}/K_{> 1}$.

Proposition 2.3. The subquotient $K_{\geq 1}/K_{> 1}$ is a braided Hopf algebra induced by the braided Hopf algebra structure of $B(\mathbb{V})$.

Proof. See [AAH19, Proposition 3.9]. □

§2.5. Lyndon word theory

We follow the notation and terminologies in [HZ18]; for more details and a more general case, see [GH07].

We follow the notation in Section 2.1. Define an order on X by $x_1 < x_2 < \cdots < x_s$, consider the lexicographic ordering on $\mathbb{X}$. Let $\mathbb{X}^\times$ be the set of non-empty words in $\mathbb{X}$. A word $u \in \mathbb{X}^\times$ is a Lyndon word if for any decomposition $u = vw$, $u, v \in \mathbb{X}^\times$, we have $u < w$.

A word $u \in \mathbb{X}^\times$ is a Lyndon word if and only if either $u \in X$, or there exist Lyndon words v, w such that $v < w$ and $u = vw$.

Any word u of length at least two has a unique decomposition into the product of two Lyndon words $u = vw$, where the length of v is minimal. This decomposition is called the Shirshov decomposition.

Recall that we identify $\mathcal{K}\mathbb{X}$ with $T(V)$. For a Lyndon word u , we define $[u] \in T(V)$ inductively as follows:

- (1) $[u] = u$, if $u \in X$,
- (2) $[u] = [[v], [w]]$, where $u = vw$ is the Shirshov decomposition, if $|u| \geq 2$. Here, $[\cdot, \cdot]$ is the braided bracket defined in Section 2.1.

We call $[u]$ super-letters. Then the total order on $\mathbb{X}$ induces a total order on the set of super-letters.

Let $\mathbb{L}$ be the set of Lyndon words. In Lyndon word theory, it is well known that for any $w \in \mathbb{X}^\times$, w has a unique decomposition into monotonic products of Lyndon words: $w = u_1 u_2 \cdots u_k$ for $u_1 \geq u_2 \geq \cdots \geq u_k$ and $u_1, \dots, u_k \in \mathbb{L}$ for some $k \in \mathbb{N}$. Then we define $[w]$ by $[u_1][u_2] \cdots [u_k]$. Denote $M_{>u}$ by the vector space spanned by all $[w] \in T(V)$ such that $w = v_1 v_2 \cdots v_k$, $v_1 \geq v_2 \geq \cdots \geq v_k > u$, $v_1, \dots, v_k \in \mathbb{L}$ for some $k \in \mathbb{N}$.

Proposition 2.4. *Let $u \in \mathbb{L}$. Then $\Delta([u]) - 1 \otimes [u] - [u] \otimes 1 \in T(V) \otimes M_{>u}$.*

Proof. We refer to [GH07, Proposition 3.6]. □

For $\alpha \in \mathbb{Z}^s$, let $o_\alpha \in \mathbb{N} \cup \infty$ be the order of $\chi(\alpha, \alpha)$. Let $O_\alpha = \{1, o_\alpha, \infty\}$.

Definition 2.5. Let $w \in \mathbb{X}^\times$. We say that $[w]$ is a root vector candidate if $w = v^k$ for some Lyndon word v and $k \in O_{\deg v} \setminus \{\infty\}$.

Definition 2.6. A root vector candidate $[w]$ is called a root vector of $B(V)$ if $[w] \in B(V)$ is not a linear combination of elements of the form $[v_k]^{m_k} \cdots [v_1]^{m_1}$, where $k, m_k, \dots, m_1 \in \mathbb{N}$ and $[v_1], \dots, [v_k]$ are root vector candidates or equivalently root vectors with $v_k > \cdots > v_1 > w$.

Theorem 2.7. *Let $\mathbf{R} \subseteq \mathbb{X}^\times$ such that $w \in \mathbf{R}$ if and only if $[w]$ is a root vector. Then the following elements form a vector space basis of $B(V)$:*

$$[v_k]^{m_k} \cdots [v_1]^{m_1}, \quad k \in \mathbb{N}, \quad v_1, \dots, v_k \in \mathbf{R}, \quad v_1 < v_2 < \cdots < v_k, \\ \forall 1 \leq i \leq k, \quad 0 \leq m_i < \min(O_{\deg v_i} \setminus \{1\}).$$

Let X_n, Y_n be the real root vector candidates corresponding to the Lyndon word $x_1(x_1 x_2)^{n-1}$ and $(x_1 x_2)^{n-1} x_2$.

Define L_n by $[X_1, Y_n]$, L'_n by $[X_2, Y_{n-1}]$ (it is convenient to define $L_1 = L'_1 = [X_1, Y_1]$ and $L_0 = L'_0 = \frac{1}{\theta}$). Then L_n is the imaginary root vector candidate at $n\delta$ corresponding to the Lyndon word $x_1(x_1x_2)^{n-1}x_2$. Define $\hat{L}_{n+1} = [L_n, L_1]$, $\tilde{L}_n = \frac{1}{2}(L_n + L'_n)$ (it is convenient to define $\tilde{L}_n = 0$ for $n < 0$).

Define $M_{2n+1} = \text{ad}_{L_2}^n(L_1)$, where $\text{ad}_X(Y)$ means $[X, Y]$ for $X, Y \in B(\mathbb{V})$. Then M_{2n+1} is the imaginary root vector candidate at $(2n+1)\delta$ corresponding to the Lyndon word

$$(x_1x_1x_2x_2)^nx_1x_2.$$

We have the following order in terms of super-letters:

$$\begin{aligned} X_1 &< X_2 < \cdots < X_n < \cdots \\ &< \cdots < L_n < L_{n-1} < L_{n-2} < \cdots < L_2 \\ &< \cdots < M_{2n-1} < M_{2n-3} < \cdots < M_5 < M_3 < M_1 \\ &< \cdots < Y_n < \cdots < Y_2 < Y_1. \end{aligned}$$

Lemma 2.8. *If X_n and Y_n are not 0 in $B(\mathbb{V})$, then they are root vectors at corresponding degrees.*

Proof. Note that X_n and Y_n are the biggest super-letters at corresponding degrees. Then it follows from the definition of root vectors and reasons of degree. $\square$

Lemma 2.9. *In $B(\mathbb{V})$ we have*

$$\Delta(Y_n) \equiv Y_n \otimes 1 + 1 \otimes Y_n \bmod B_{\geq 1} \otimes B(\mathbb{V})$$

and we have $X_n \in K_{\geq 1}$, $L_n \in K_{\geq 1}$, $Y_n \notin K_{\geq 1}$, $X_n \in K_{> 1}$, $L_n \notin K_{> 1}$.

Proof. The expression for $\Delta(Y_n)$ follows from Proposition 2.4. Since $L_n = [X_1, Y_n]$, by the expression for $\Delta(Y_n)$, we have $L_n \in K_{\geq 1}$. In particular, $L_1 \in K_{\geq 1}$, then immediately $X_n \in K_{\geq 1}$. Other conclusions are obvious. $\square$

§3. Relations in $B(\mathbb{V})$

Lemma 3.1. *The following basic relations hold in $B(\mathbb{V})$:*

- (a) $[X_n, L_1^2] = X_{n+2}$ and $[L_1^2, Y_n] = Y_{n+2}$.
- (b) $L'_{2n+1} = L_{2n+1} + [L_{2n}, L_1]$, $L'_{2n+2} = L_{2n+2} - [L_{2n+1}, L_1]$.
- (c) Fix n ; if for all $i \leq n$, $[L_{i-2}, L_1^2] = 0$, then $[X_{2k+1}, Y_{n-2k}] = L_n$, for all $0 \leq k \leq \frac{n-1}{2}$, $k \in \mathbb{N}_0$ and $[X_{2k+2}, Y_{n-(2k+1)}] = L'_n$, for all $0 \leq k \leq \frac{n-2}{2}$, $k \in \mathbb{N}_0$.
- (d) $[M_{2n+1}, M_{2n-1}] = [L_2, M_{2n-1}^2]$ for all $n \geq 1$. If $[L_2, M_{2n-1}^2] = 0$, then $[M_{2n+3}, M_{2n-1}] = -2M_{2n+1}^2$.

Proof. By the braided Jacobi identity, skew derivations directly. $\square$

§3.1. Relations in root space with degree $\leq 4\delta$

We know quantum Serre relations hold in $B(\mathbb{V})$:

$$[X_1, X_2] = [Y_2, Y_1] = 0.$$

In degree $\leq 4\delta$, we have the following relations. These relations are prepared to obtain the central elements in the next subsection and as a starting point for the induction proof of Proposition 3.18.

Proposition 3.2. *In $B(\mathbb{V})$, the following relations hold:*

$$(3.1a) \quad [X_1, L_2] = (q+2)X_3 - \theta L_1 X_2,$$

$$(3.1b) \quad [X_1, L'_2] = qX_3 - \theta L_1 X_2,$$

$$(3.1c) \quad [L_2, Y_1] = qY_3 + \theta Y_2 L_1,$$

$$(3.1d) \quad [L'_2, Y_1] = (q-2)Y_3 + \theta Y_2 L_1,$$

$$(3.2a) \quad 2L_3 - \theta L_1 L'_2 = \theta L_2 L_1 - [L_2, L_1],$$

$$(3.2b) \quad 2L'_3 - \theta L'_2 L_1 = \theta L_1 L_2 + [L_2, L_1],$$

$$(3.2c) \quad \tilde{L}_3 = \frac{1}{2}\theta L_1 \tilde{L}_2 + \frac{1}{2}\theta \tilde{L}_2 L_1,$$

$$(3.3a) \quad [X_2, L_2] = qX_4 + \theta L_1 X_3,$$

$$(3.3b) \quad [X_2, L'_2] = (q-2)X_4 + \theta L_1 X_3,$$

$$(3.3c) \quad [L_2, Y_2] = (q+2)Y_4 - \theta Y_3 L_1,$$

$$(3.3d) \quad [L'_2, Y_2] = qY_4 - \theta Y_3 L_1,$$

$$(3.4a) \quad [L_2, L_1^2] = 0 \quad ([M_3, M_1] = 0),$$

$$(3.4b) \quad [L_3, L_1^2] = 0.$$

Proof. First, $[X_1, L'_2] = [X_1, [X_2, Y_1]] = qX_2 L_1 + q^{-1}L_1 X_2 = qX_3 - \theta L_1 X_2$. Then $[X_1, L_2] = (q+2)X_3 - \theta L_1 X_2$. Now we have proved (3.1a), (3.1b). Similarly, (3.1c), (3.1d) also hold.

On one hand,

$$(3.5) \quad \begin{aligned} [[X_1, L_2], Y_1] &= [(q+2)X_3 - \theta L_1 X_2, Y_1] \\ &= (q+2)L_3 - \theta(L_1 L_2' - q^{-1}Y_2 X_2). \end{aligned}$$

On the other hand,

$$(3.6) \quad \begin{aligned} [[X_1, L_2], Y_1] &= [X_1, [L_2, Y_1]] - [L_2, L_1] \\ &= [X_1, qY_3 + \theta Y_2 L_1] - [L_2, L_1] \\ &= qL_3 + \theta(L_2 L_1 + q^{-1}Y_2 X_2) - [L_2, L_1]. \end{aligned}$$

Comparing (3.5) and (3.6) we get (3.2a) and equivalently (3.2b). Equation (3.2c) is the sum of (3.2a) and (3.2b).

Using $[X_1, \cdot]$ to act on (3.2b) we can get

$$(q + q^{-1} + 1)[X_2, L_2] = q(q + q^{-1} + 1)X_4 + \theta(q + q^{-1} + 1)L_1 X_3.$$

Since q is not a root of unity, we get (3.3a) and then immediately (3.3b). Similarly, we get (3.3c) and (3.3d).

Using $[\cdot, Y_2]$ to act on (3.1a), by similar calculation we can get

$$(3.7) \quad (q+2)[L_2, L_1^2] = \theta L_1 L_3' - \theta L_3 L_1.$$

Using $[\cdot, Y_1]$ to act on (3.3a) we can get

$$q[L_2, L_1^2] = \theta L_3' L_1 - \theta L_1 L_3 - [L_2, L_2'],$$

i.e.,

$$(3.8) \quad (q-2)[L_2, L_1^2] = \theta L_3' L_1 - \theta L_1 L_3.$$

Taking the sum of (3.7) and (3.8) we can get

$$2q[L_2, L_1^2] = \theta[L_2, L_1^2].$$

Since q is not a root of unity, we have $[L_2, L_1^2] = 0$ or equivalently $[M_3, M_1] = 0$, which is (3.4a). Then (3.4b) holds by (3.2a). $\square$

Remark 3.3. We will use the technique in this proof frequently later, to produce new relations in imaginary root space from known relations.

Remark 3.4. Compared with the case of $U_v(\widehat{\mathfrak{sl}_2})$, (3.2a) tells us that L_3 is no longer a root vector in terms of Lyndon word theory. We will see later that for $n > 1$, L_{2n+1} are also no longer root vectors.

§3.2. Central elements in $K_{\geq 1}/K_{>1}$

We will first prove that M_{2m+1}^2 , for all $m \geq 0$ commute with L_n , for all $n > 0$ in $B(\mathbb{V})$.

Lemma 3.5. *Suppose there is a number sequence $\{a_n\}_{n \geq 1}$ in $\mathbb{C}$ and $a_{n+2} = Aa_{n+1} + Ba_n$, for $A, B \in \mathbb{C}$. Suppose the roots x_1, x_2 of $x^2 - Ax - B = 0$ are different; then*

$$a_n = \frac{x_2^{n-1}(a_2 - x_1 a_1) - x_1^{n-1}(a_2 - x_2 a_1)}{x_2 - x_1}.$$

The following proposition plays an important role in this paper. There are several very useful corollaries after that. Moreover, in the next subsection, we will write this proposition and its corollaries in the form of generating functions, which will be crucial for the determination of the root multiplicities and the construction of the Drinfeld second realization.

Proposition 3.6. *For $k \geq 0$, we have the following relations in $B(\mathbb{V})$:*

$$(3.9a) \quad \tilde{L}_{4k+1} = \frac{1}{2}\theta L_1 \tilde{L}_{4k} + \frac{1}{2}\theta \tilde{L}_{4k} L_1 + \frac{1}{4}[\tilde{L}_{4k-2}, M_3],$$

$$(3.9b) \quad [\tilde{L}_{4k}, L_1] = \frac{1}{2}\theta M_3 \tilde{L}_{4k-2} + \frac{1}{2}\theta \tilde{L}_{4k-2} M_3 + \frac{1}{4}[\tilde{L}_{4k-4}, M_5],$$

$$(3.9c) \quad [\tilde{L}_{4k+2}, L_1] = \frac{1}{2}\theta M_3 \tilde{L}_{4k} + \frac{1}{2}\theta \tilde{L}_{4k} M_3 + \frac{1}{4}[\tilde{L}_{4k-2}, M_5],$$

$$(3.9d) \quad \tilde{L}_{4k+3} = \frac{1}{2}\theta L_1 \tilde{L}_{4k+2} + \frac{1}{2}\theta \tilde{L}_{4k+2} L_1 + \frac{1}{4}[\tilde{L}_{4k}, M_3],$$

$$(3.9e) \quad [\tilde{L}_n, L_1^2] = 0 = [L_n, L_1^2] \quad \text{for } n = 4k, 4k+1, 4k+2, 4k+3,$$

$$(3.9f) \quad [\tilde{L}_2, \tilde{L}_{4k}] = 0 \quad \text{and} \quad [\tilde{L}_2, \tilde{L}_{4k+2}] = 0.$$

Proof. We know it holds for $k = 0$ from Proposition 3.2. Suppose it holds for k ; we consider the case of $k + 1$. We denote the anti-bracket by $[\ ,]'$, i.e., $[x, y]' = m \circ (\text{id} + c)(x \otimes y)$.

We know $[L_i, L_1^2] = 0$ for $i \leq 4k + 3$ from the induction hypothesis. Then by Lemma 3.1, $[X_3, Y_{4k+3}] = L_{4k+5}$ and $[X_2, Y_{4k+3}] = L'_{4k+4}$. Using $[\ , Y_{4k+3}]$ to act on (3.1a), on one hand we have

$$(3.10) \quad \begin{aligned} [[X_1, L_2], Y_{4k+3}] &= [(q+2)X_3 - \theta L_1 X_2, Y_{4k+3}] \\ &= (q+2)L_{4k+5} - \theta(L_1 L'_{4k+4} - q^{-1}Y_{4k+4}X_2). \end{aligned}$$

On the other hand, from Lemma 3.1(a), (3.1c) and the induction hypothesis we have $[L_2, Y_{4k+3}] = qY_{4k+5} + \theta Y_{4k+4}L_1$; then

$$(3.11) \quad \begin{aligned} [[X_1, L_2], Y_{4k+3}] &= [X_1, qY_{4k+5} + \theta Y_{4k+4}L_1] - [L_2, L_{4k+3}] \\ &= qL_{4k+5} + \theta(L_{4k+4}L_1 + q^{-1}Y_{4k+4}X_2) - [L_2, L_{4k+3}]. \end{aligned}$$

Comparing (3.10) and (3.11) we get

$$(3.12) \quad 2L_{4k+5} = \theta L_1 L'_{4k+4} + \theta L_{4k+4}L_1 - [L_2, L_{4k+3}].$$

Similarly, using $[, Y_{4k+2}]$ to act on (3.3a), on one hand we have

$$(3.13) \quad \begin{aligned} [[X_2, L_2], Y_{4k+2}] &= [qX_4 + \theta L_1 X_3, Y_{4k+2}] \\ &= qL'_{4k+5} + \theta(L_1 L_{4k+4} + q^{-1}Y_{4k+3}X_3). \end{aligned}$$

On the other hand we have

$$(3.14) \quad \begin{aligned} [[X_2, L_2], Y_{4k+2}] &= [X_2, (q+2)Y_{4k+4} - \theta Y_{4k+3}L_1] - [L_2, L'_{4k+3}] \\ &= (q+2)L'_{4k+5} - \theta(L'_{4k+4}L_1 - q^{-1}Y_{4k+3}X_3) - [L_2, L'_{4k+3}]. \end{aligned}$$

Comparing (3.13) and (3.14) we get

$$(3.15) \quad 2L'_{4k+5} = \theta L'_{4k+4}L_1 + \theta L_1 L_{4k+4} + [L_2, L'_{4k+3}].$$

Taking the sum of (3.12) and (3.15) we get

$$\begin{aligned} \tilde{L}_{4k+5} &= \frac{1}{2}\theta L_1 \tilde{L}_{4k+4} + \frac{1}{2}\theta \tilde{L}_{4k+4}L_1 + [L_2, [L_{4k+2}, L_1]] \\ &= \frac{1}{2}\theta L_1 \tilde{L}_{4k+4} + \frac{1}{2}\theta \tilde{L}_{4k+4}L_1 + [L_2, [\tilde{L}_{4k+2}, L_1]] \\ &= \frac{1}{2}\theta L_1 \tilde{L}_{4k+4} + \frac{1}{2}\theta \tilde{L}_{4k+4}L_1 + [\tilde{L}_{4k+2}, M_3]. \end{aligned}$$

Then (3.9a) holds. Taking the difference of (3.12) and (3.15) we get

$$(3.16) \quad \begin{aligned} [L_{4k+4}, L_1] &= [\tilde{L}_{4k+4}, L_1] = [L_2, \tilde{L}_{4k+3}] \\ &= \left[L_2, \frac{1}{2}\theta L_1 \tilde{L}_{4k+2} + \frac{1}{2}\theta \tilde{L}_{4k+2}L_1 + \frac{1}{4}[\tilde{L}_{4k}, M_3] \right] \\ &= \frac{1}{2}\theta M_3 \tilde{L}_{4k+2} + \frac{1}{2}\theta \tilde{L}_{4k+2}M_3 + \frac{1}{4}[\tilde{L}_{4k}, M_5]. \end{aligned}$$

That is (3.9b) holds. Then we have

$$\begin{aligned} [\tilde{L}_{4k+4}, L_1^2] &= [[\tilde{L}_{4k+4}, L_1], L_1] \\ &= \frac{1}{2}\theta[M_3, [\tilde{L}_{4k+2}, L_1]]' + \frac{1}{4}[[\tilde{L}_{4k}, M_5], L_1] \end{aligned}$$

$$\begin{aligned}
&= \left(\frac{1}{2}\theta\right)^2 [M_3^2, \tilde{L}_{4k}] + \frac{1}{2}\theta\frac{1}{4}[M_3, [\tilde{L}_{4k-2}, M_5]]' - \frac{1}{2}[\tilde{L}_{4k}, M_3^2] \\
&\quad - \frac{1}{4}\left[M_5, \frac{1}{2}\theta M_3 \tilde{L}_{4k-2} + \frac{1}{2}\theta \tilde{L}_{4k-2} M_3 + \frac{1}{4}[\tilde{L}_{4k-4}, M_5]\right] \\
&= \left(\frac{1}{2}\theta\right)^2 [M_3^2, \tilde{L}_{4k}] - \frac{1}{2}[\tilde{L}_{4k}, M_3^2] - \left(\frac{1}{4}\right)^2 [\tilde{L}_{4k-4}, M_5^2] \\
&= \left(-\frac{1}{4}\theta^2 - \frac{1}{2}\right)[\tilde{L}_{4k}, M_3^2] - \left(\frac{1}{4}\right)^2 [\tilde{L}_{4k-4}, M_5^2].
\end{aligned}$$

(For $k = 0$, we have $[\tilde{L}_4, L_1^2] = \frac{1}{2}\theta[M_3, M_3]' = 0$.) Define $A = -\frac{1}{4}\theta^2 - \frac{1}{2}$, $B = -(\frac{1}{4})^2$; we have $[\tilde{L}_{4k+4}, L_1^2] = A[\tilde{L}_{4k}, M_3^2] + B[\tilde{L}_{4k-4}, M_5^2]$.

Similarly we have $[\tilde{L}_{4k}, M_3^2] = A[\tilde{L}_{4k-4}, M_5^2] + B[\tilde{L}_{4k-8}, M_7^2], \dots$, and the last one is $[\tilde{L}_8, M_{2k-1}^2] = A[\tilde{L}_4, M_{2k+1}^2]$. Note that

$$[\tilde{L}_4, M_{2k+1}^2] = [[\tilde{L}_4, M_{2k+1}], M_{2k+1}] = \left[\frac{1}{2}\theta M_{2k+3} \tilde{L}_2 + \frac{1}{2}\theta \tilde{L}_2 M_{2k+3}, M_{2k+1}\right] = 0.$$

Now we know $[\tilde{L}_{4k+4}, L_1^2] = 0 = [L_{4k+4}, L_1^2]$ and moreover $[\tilde{L}_{4k+4-4i}, M_{2i+1}^2] = 0$ for $i \geq 0$. Then automatically $[\tilde{L}_{4k+5}, L_1^2] = 0 = [L_{4k+5}, L_1^2]$.

By the induction hypothesis and the method in the proof of (3.9a), we get

$$(3.17) \quad [L_2, L_{4k+4}] = \theta L_1 L'_{4k+5} - \theta L_{4k+5} L_1,$$

$$(3.18) \quad [L_2, L'_{4k+4}] = \theta L'_{4k+5} L_1 - \theta L_1 L_{4k+5}.$$

Since we know $[L_{4k+4}, L_1^2] = 0 = [\tilde{L}_{4k+4}, L_1^2]$, then taking the sum of (3.17) and (3.18) we get $[L_2, \tilde{L}_{4k+4}] = 0 = [\tilde{L}_2, \tilde{L}_{4k+4}]$.

Also using the induction hypothesis and the same technique as in the proof of (3.9a) and (3.9b), we have that (3.9c), (3.9d) hold for $k + 1$.

The last and most difficult part is to prove $[\tilde{L}_{4k+6}, L_1^2] = 0$. We know

$$(3.19) \quad [\tilde{L}_{4k+6}, L_1] = [L_2, \tilde{L}_{4k+5}] = \frac{1}{2}\theta M_3 \tilde{L}_{4k+4} + \frac{1}{2}\theta \tilde{L}_{4k+4} M_3 + \frac{1}{4}[\tilde{L}_{4k+2}, M_5].$$

Then we have the following two arguments about $[\tilde{L}_{4k+6}, L_1^2]$.

Argument 1: By (3.19),

$$\begin{aligned}
[\tilde{L}_{4k+6}, L_1^2] &= [[\tilde{L}_{4k+6}, L_1], L_1] = \frac{1}{2}\theta[M_3, [\tilde{L}_{4k+4}, L_1]]' + \frac{1}{4}[[\tilde{L}_{4k+2}, M_5], L_1] \\
&= \left(\frac{1}{2}\theta\right)^2 [M_3^2, \tilde{L}_{4k+2}] + \frac{1}{2}\theta\frac{1}{4}[M_3, [\tilde{L}_{4k}, M_5]]' \\
&\quad + \frac{1}{4}[\tilde{L}_{4k+2}, -2M_3^2] - \frac{1}{4}[M_5, [\tilde{L}_{4k+2}, L_1]]
\end{aligned}$$

$$= -\frac{1}{4}(\theta^2 + 2)[\tilde{L}_{4k+2}, M_3^2] + \frac{1}{2}\theta\frac{1}{4}[M_3, [\tilde{L}_{4k}, M_5]]' \\ - \frac{1}{4}\frac{1}{2}\theta(-1)[M_3, [M_5, \tilde{L}_{4k}]]' - \left(\frac{1}{4}\right)^2[\tilde{L}_{4k-2}, M_5^2],$$

i.e., $[\tilde{L}_{4k+6}, L_1^2] = A[\tilde{L}_{4k+2}, M_3^2] + B[\tilde{L}_{4k-2}, M_5^2]$. Similarly, we have $[\tilde{L}_{4k+2}, M_3^2] = A[\tilde{L}_{4k-2}, M_5^2] + B[\tilde{L}_{4k-6}, M_7^2] \cdots$, and the last one is $[\tilde{L}_6, M_{2k+1}^2] = -\frac{1}{4}(\theta^2 + 3)[\tilde{L}_2, M_{2k+3}^2]$. So we have

$$[\tilde{L}_{4k+6}, L_1^2] = a_{k+2}[\tilde{L}_2, M_{2k+3}^2],$$

where $\{a_n\}_{n \geq 1}$ satisfies $a_1 = 1$, $a_2 = -\frac{1}{4}(\theta^2 + 3)$, $a_{n+2} = Aa_{n+1} + Ba_n$.

Argument 2: Similar to the proof of $[L_2, L_1^2] = 0$, we have

$$(3.20) \quad (q+2)[L_{4k+6}, L_1^2] = \theta L_1 L'_{4k+7} - \theta L_{4k+7} L_1 - [L_2, L_{4k+6}],$$

$$(3.21) \quad q[L_{4k+6}, L_1^2] = \theta L'_{4k+7} L_1 - \theta L_1 L_{4k+7} - [L_2, L'_{4k+6}].$$

Then taking the sum of (3.20) and (3.21) we have

$$(3.22) \quad [\tilde{L}_{4k+6}, L_1^2] = \theta L_1 \tilde{L}_{4k+7} - \theta \tilde{L}_{4k+7} L_1 - \frac{1}{2}[L_2, [\tilde{L}_{4k+5}, L_1]].$$

We know

$$(3.23) \quad \tilde{L}_{4k+7} = \frac{1}{2}\theta L_1 \tilde{L}_{4k+6} + \frac{1}{2}\theta \tilde{L}_{4k+6} L_1 + \frac{1}{4}[\tilde{L}_{4k+4}, M_3].$$

So by (3.22) (3.23) and (3.19) we have

$$[\tilde{L}_{4k+6}, L_1^2] = \frac{\theta^2}{2}[L_1^2, \tilde{L}_{4k+6}] + \frac{\theta}{4}[L_1, [\tilde{L}_{4k+4}, M_3]]' - \frac{1}{2}([\tilde{L}_{4k+6}, L_1^2] + [\tilde{L}_{4k+5}, M_3]) \\ = -\frac{1}{2}(\theta^2 + 1)[\tilde{L}_{4k+6}, L_1^2] + \frac{\theta}{4}[L_1, [\tilde{L}_{4k+4}, M_3]]' \\ - \frac{1}{2}\left[\frac{\theta}{2}L_1 \tilde{L}_{4k+4} + \frac{\theta}{2}\tilde{L}_{4k+4}L_1 + \frac{1}{4}[\tilde{L}_{4k+2}, M_3], M_3\right],$$

i.e., $(3 + \theta^2)[\tilde{L}_{4k+6}, L_1^2] = -\frac{1}{4}[\tilde{L}_{4k+2}, M_3^2]$, i.e., $(3 + \theta^2)[\tilde{L}_{4k+6}, L_1^2] = -\frac{1}{4}a_{k+1}[\tilde{L}_2, M_{2k+3}^2]$.

In summary we have

$$[\tilde{L}_{4k+6}, L_1^2] = a_{k+2}[\tilde{L}_2, M_{2k+3}^2], \\ (3 + \theta^2)[\tilde{L}_{4k+6}, L_1^2] = -\frac{1}{4}a_{k+1}[\tilde{L}_2, M_{2k+3}^2].$$

So if

$$\left| \begin{array}{cc} 1 & a_{k+2} \\ 3 + \theta^2 & -\frac{1}{4}a_{k+1} \end{array} \right| \neq 0,$$

we have $[\tilde{L}_{4k+6}, L_1^2] = 0$ and then $[L_{4k+6}, L_1^2] = 0$. We use Lemma 3.5. Here, $a_1 = 1$, $a_2 = -\frac{1}{4}(\theta^2 + 3)$, $a_{n+2} = Aa_{n+1} + Ba_n$, where $A = -\frac{1}{4}\theta^2 - \frac{1}{2}$, $B = -(\frac{1}{4})^2$. By Lemma 3.5 we have $a_n = (-\frac{1}{4})^{n-1}[2n-1]_q$. Then

$$\left| \begin{array}{cc} 1 & a_{k+2} \\ 3 + \theta^2 - \frac{1}{4}a_{k+1} & \end{array} \right| = -\left(-\frac{1}{4}\right)^{k+1}(q + q^{-1})[2k+4]_q \neq 0,$$

when q is not a root of unity. Hence $[\tilde{L}_{4k+6}, L_1^2] = 0$ and moreover we have $[\tilde{L}_{4k+6-4i}, M_{2i+1}^2] = 0$ for $i \geq 0$. Then automatically $[\tilde{L}_{4k+7}, L_1^2] = 0 = [L_{4k+7}, L_1^2]$ and similar to the proof of $[L_2, \tilde{L}_{4k+4}] = 0 = [\tilde{L}_2, \tilde{L}_{4k+4}]$ before, we can get $[L_2, \tilde{L}_{4k+6}] = 0 = [\tilde{L}_2, \tilde{L}_{4k+6}]$. Now (3.9e) and (3.9f) hold. $\square$

Remark 3.7. Note that in the proof of (3.9c), (3.9d) and $[\tilde{L}_2, \tilde{L}_{4k+2}] = 0$, we get five formulas completely similar to (3.12), (3.15), (3.16), (3.17), (3.18) corresponding to different indices. We will not distinguish them by new labels when we refer to them later.

Corollary 3.8. We have $[M_{2m+1}, M_{2n+1}] = 0$ if $m+n$ is odd, $[M_{2m+1}, M_{2n+1}] = (-1)^{\frac{m-n}{2}} 2M_{m+n+1}^2$ if $m+n$ is even, and M_{2n+1}^2 commute with L_m , for all $m, n \geq 0$.

Proof. By an easy induction on $m+n$ we can get the formula for $[M_{2m+1}, M_{2n+1}]$. Then we can see $[L_1, M_{2n+1}^2] = 0$ for all n . In the proof of Proposition 3.6, we know M_{2n+1}^2 commutes with all $\tilde{L}_{2m}$. Then by Proposition 3.6, M_{2n+1}^2 commutes with all $\tilde{L}_m$ and this is equivalent to M_{2n+1}^2 commuting with all L_m . $\square$

Remark 3.9. Note that in the proofs of Propositions 3.2 and 3.6 and Corollary 3.8, we only use quantum Serre relations.

Corollary 3.10. The comultiplications of L_n and L'_n in $K_{\geq 1}/K_{>1}$ are as follows:

$$\begin{aligned} \Delta(L_{2n}) &= \sum_{i=0}^n \theta L_{2n-2i} \otimes L_{2i} - \sum_{i=0}^{n-1} \theta L'_{2n-1-2i} \otimes L_{1+2i}, \\ \Delta(L'_{2n}) &= \sum_{i=0}^n \theta L'_{2n-2i} \otimes L'_{2i} - \sum_{i=0}^{n-1} \theta L_{2n-1-2i} \otimes L'_{1+2i}, \\ \Delta(L_{2n+1}) &= \sum_{i=0}^n \theta L_{2n+1-2i} \otimes L_{2i} + \sum_{i=0}^n \theta L'_{2n-2i} \otimes L_{1+2i}, \\ \Delta(L'_{2n+1}) &= \sum_{i=0}^n \theta L'_{2n+1-2i} \otimes L'_{2i} + \sum_{i=0}^n \theta L_{2n-2i} \otimes L'_{1+2i}. \end{aligned}$$

Proof. By $[L_1^2, L_n] = 0$ and an easy induction, we can get $\Delta(L_{2n})$ and $\Delta(L_{2n+1})$. Then we can get the expressions for $\Delta(L'_{2n})$ and $\Delta(L'_{2n+1})$ by Lemma 3.1(b). $\square$

Remark 3.11. Compared with the formulas in Corollary 3.10, we recall that the comultiplication formula in the subquotient $K_{\geq 1}/K_{> 1}$ of L_n in $U_v^+(\widehat{\mathfrak{sl}}_2)$ is

$$\Delta(L_n) = \sum_{i=0}^n \theta L_{n-i} \otimes L_i,$$

corresponding to [CP91, Proposition 4.3].

Corollary 3.12. *The elements L_n and L'_n are not 0 and linearly independent in $B(\mathbb{V})$, for $n \geq 0$.*

Proof. By Corollary 3.10 and an easy induction. $\square$

Lemma 3.13. *We have the following results using differential operators:*

- (a) $\partial_2^R(L_1^2) = -q^{-1}\theta X_2$, $\partial_2^R(Y_{2n+1}) = \theta L'_{2n}$, $\partial_2^R(Y_{2n}) = -\theta L'_{2n-1}$, and $\partial_2^R(L_{2n}) = -\theta(X_1 L'_{2n-1} - q^{-2} L'_{2n-1} X_1)$, $\partial_2^R(L_{2n+1}) = \theta(X_1 L'_{2n} - q^{-2} L'_{2n} X_1)$.
- (b) $\partial_1^L(L_1^2) = q^{-1}\theta Y_2$, $\partial_1^L(X_{2n+1}) = \theta L_{2n}$, $\partial_1^L(X_{2n}) = \theta L'_{2n-1}$, and $\partial_1^L(L'_{2n}) = \theta(L'_{2n-1} Y_1 + q^{-2} Y_1 L'_{2n-1})$, $\partial_1^L(L_{2n+1}) = \theta(L_{2n} Y_1 - q^{-2} Y_1 L_{2n})$.

Proof. By Lemma 3.1, Proposition 3.6 and induction. $\square$

Corollary 3.14. *In terms of the Lyndon word, X_n and Y_n are the root vectors at the corresponding degrees.*

Proof. By Lemma 2.8, Lemma 3.13, Corollary 3.12. $\square$

Corollary 3.15. *As an algebra $K_{\geq 1}/K_{> 1}$ is isomorphic to the subalgebra of $B(\mathbb{V})$ generated by L_n for $n > 0$; furthermore, M_{2m+1}^2 are central elements in $K_{\geq 1}/K_{> 1}$.*

Proof. By Corollary 3.14, X_n and Y_n are real root vectors. Then the only possible imaginary root vectors are L_n for $n \geq 1$ and their iterated commutators according to Lyndon word theory. By the expression of $\Delta(Y_n)$ in Lemma 2.9, $K_{\geq 1}$ is the subalgebra generated by X_n and L_n and $K_{\geq 1} = K_{> 1} \oplus K_{=1}$ where $K_{=1}$ is the subalgebra generated by L_n and $K_{> 1}$ is the subalgebra $K_{\geq 1} \cap B_{> 1}$. Then as an algebra $K_{\geq 1}/K_{> 1}$ is isomorphic to the subalgebra in $B(\mathbb{V})$ generated by L_n . $\square$

Remark 3.16. In summary, $K_{\geq 1}/K_{> 1}$ is a braided Hopf algebra whose algebra structure is nothing new, but has a manageable coalgebra structure.

§3.3. Relations described by generating functions

In this subsection we will describe the relations in $B(\mathbb{V})$ using generating functions, which will be powerful in proofs of main theorems in this paper.

Let $\theta' := q + q^{-1}$, $a = \sqrt{q/2}$, $b = i\sqrt{q^{-1}/2}$. Let $a_1 = 1$, $a_2 = \frac{1}{2}\theta$, $a_{n+2} = \frac{1}{2}\theta a_{n+1} + \frac{1}{4}a_n$; then by Lemma 3.5 we have $a_n = \frac{(\frac{1}{2}q)^n - (-\frac{1}{2}q^{-1})^n}{\frac{1}{2}(q+q^{-1})}$.

First, we introduce three fundamental generating functions in this paper. Let

$$\begin{aligned}\widetilde{\mathbb{L}}(u) &= \sum_{n=0}^{\infty} \theta \tilde{L}_{2n} u^{2n}, \quad \mathbb{M}(u) = \sum_{n=0}^{\infty} M_{2n+1} u^{2n+1}, \\ \mathbb{Z}(u) &= \sum_{n=0}^{\infty} M_{2n+1}^2 (-1)^n u^{4n+2} = \mathbb{M}(u)^2 \quad (\text{by Corollary 3.8}).\end{aligned}$$

For convenience, we denote $\mathbb{M}(hu)$, $\mathbb{Z}(hu)$ by $\mathbb{M}(h)$, $\mathbb{Z}(h)$ respectively for $h \in \mathbb{C}^\times$. Also for convenience, we introduce

$$\begin{aligned}A(u) &:= \sum_{n=0}^{\infty} \theta a_{n+1} M_{2n+1} u^{2n+1} = \frac{2\theta}{\theta'} (a\mathbb{M}(a) - b\mathbb{M}(b)), \\ B(u) &:= \sum_{n=1}^{\infty} \theta a_n M_{2n+1} u^{2n+1} = \frac{2\theta}{\theta'} (a^{-1}\mathbb{M}(a) - b^{-1}\mathbb{M}(b)), \\ A'(u) &:= \frac{1}{i} A(iu), \quad B'(u) := iB(iu).\end{aligned}$$

Sometimes we just denote $A(u)$, $B(u)$ by A , B respectively.

Proposition 3.17. *We have the following relations in the form of series:*

$$(3.24a) \quad A(u)\widetilde{\mathbb{L}}(u) = \widetilde{\mathbb{L}}(u)A'(u) = \sum_{n=0}^{\infty} \theta \tilde{L}_{2n+1} u^{2n+1},$$

$$(3.24b) \quad B(u)\widetilde{\mathbb{L}}(u) = \widetilde{\mathbb{L}}(u)B'(u) = \sum_{n=1}^{\infty} \theta \hat{L}_{2n+1} u^{2n+1},$$

$$(3.24c) \quad [A, B] = 0 = [A', B'],$$

$$(3.24d) \quad A^2 = \frac{2\theta}{\theta'} (q\mathbb{Z}(a) + q^{-1}\mathbb{Z}(b)), \quad \frac{1}{4}B^2 = -\frac{2\theta}{\theta'} (q\mathbb{Z}(b) + q^{-1}\mathbb{Z}(a)),$$

$$(3.24e) \quad \frac{1}{2}[A(u), L_1 u] = \frac{2\theta}{\theta'} (\mathbb{Z}(a) - \mathbb{Z}(b)) = \frac{1}{\theta} \left(\frac{1}{4}B^2 + A^2 \right).$$

Proof. Equations (3.24a), (3.24b) follow from Proposition 3.6 and (3.24c), (3.24d), (3.24e) follow from Corollary 3.8. $\square$

§3.4. Relations between X_n and L_m , Y_n and L_m

In this section, we will use differential operators to get the expressions for

$$[X_1, L_{2n}], \quad [X_1, L'_{2n}], \quad [X_2, L_{2n}], \quad [X_2, L'_{2n}]$$

and

$$[L_{2n}, Y_1], \quad [L'_{2n}, Y_1], \quad [L_{2n}, Y_2], \quad [L'_{2n}, Y_2].$$

In general, by Lemma 3.1(a) and that L_1^2 lies in the center of the subalgebra generated by all L_n , one can get $[X_k, L_{2n}]$ for $k \geq 3$ and others immediately.

Define two sequences of polynomials P_n and Q_n :

$$P_0 = 1, \quad P_n = q^n + q^{n-1} + (-1)^{n-1}P_{n-1},$$

$$Q_0 = 1, \quad Q_n = q^n - q^{n-1} + (-1)^nQ_{n-1}.$$

We have

$$P_n = 2q^{2\lfloor \frac{n}{2} \rfloor} \frac{1 - (-q^{-2})^{\lfloor \frac{n}{2} \rfloor + 1}}{1 + q^{-2}} + (-1)^{n+1}q^n,$$

$$Q_n = (-1)^n 2q^{2\lfloor \frac{n}{2} \rfloor} \frac{1 - (-q^{-2})^{\lfloor \frac{n}{2} \rfloor + 1}}{1 + q^{-2}} + (-1)^{n+1}q^n,$$

$$P_{2n} = Q_{2n}.$$

The next proposition is for the proof of $[\tilde{L}_{2n}, \tilde{L}_{2m}] = 0$, for all $n, m \geq 1$ in Proposition 3.22.

Proposition 3.18. *We have the following relations in $B(\mathbb{V})$:*

$$(3.25a) \quad [X_1, L_{2n}] = \sum_{i=0}^{n-1} P_{2n-1-2i} \theta L_{2i} X_{2n+1-2i} - \sum_{i=0}^{n-1} P_{2n-2-2i} \theta L'_{2i+1} X_{2n-2i},$$

$$(3.25b) \quad [X_1, L'_{2n}] = \sum_{i=0}^{n-1} q^{2n-1-2i} \theta L'_{2i} X_{2n+1-2i} - \sum_{i=0}^{n-1} q^{2n-2-2i} \theta L_{2i+1} X_{2n-2i},$$

$$(3.25c) \quad [X_2, L_{2n}] = \sum_{i=0}^{n-1} q^{2n-1-2i} \theta L_{2i} X_{2n+2-2i} + \sum_{i=0}^{n-1} q^{2n-2-2i} \theta L'_{2i+1} X_{2n-2i+1},$$

$$[X_2, L'_{2n}] = \sum_{i=0}^{n-1} Q_{2n-1-2i} \theta L'_{2i} X_{2n+2-2i} + \sum_{i=0}^{n-1} Q_{2n-2-2i} \theta L_{2i+1} X_{2n-2i+1},$$

$$[L_{2n}, Y_1] = \sum_{i=0}^{n-1} q^{2n-1-2i} \theta Y_{2n+1-2i} L_{2i} + \sum_{i=0}^{n-1} q^{2n-2-2i} \theta Y_{2n-2i} L_{2i+1},$$

$$[L'_{2n}, Y_1] = \sum_{i=0}^{n-1} Q_{2n-1-2i} \theta Y_{2n+1-2i} L'_{2i} + \sum_{i=0}^{n-1} Q_{2n-2-2i} \theta Y_{2n-2i} L'_{2i+1},$$

$$\begin{aligned}
[L_{2n}, Y_2] &= \sum_{i=0}^{n-1} P_{2n-1-2i} \theta Y_{2n+2-2i} L_{2i} - \sum_{i=0}^{n-1} P_{2n-2-2i} \theta Y_{2n-2i+1} L_{2i+1}, \\
[L'_{2n}, Y_2] &= \sum_{i=0}^{n-1} q^{2n-1-2i} \theta Y_{2n+2-2i} L'_{2i} - \sum_{i=0}^{n-1} q^{2n-2-2i} \theta Y_{2n-2i+1} L'_{2i+1}.
\end{aligned}$$

Proof. We use induction. By Proposition 3.2 we know the proposition holds for $n = 1$. Suppose the proposition holds for n , we consider the case of $n + 1$.

We have $[L_{2n+1}, L_1^2] = 0$. Hence in $B(\mathbb{V})$, by Lemma 3.13 we have

$$\begin{aligned}
0 &= \partial_2^R([L_{2n+1}, L_1^2]) \\
&= L_{2n+1}(-q^{-1}\theta X_2) + \partial_2^R(L_{2n+1})L_1^2 - (L_1^2\partial_2^R(L_{2n+1}) + q^{-1}\theta X_2 L_{2n+1}) \\
&= -q^{-1}\theta[X_2, L_{2n+1}] + [\partial_2^R(L_{2n+1}), L_1^2],
\end{aligned}$$

i.e., $[X_2, L_{2n+1}] = q[[X_1, L'_{2n}] + q^{-1}\theta L'_{2n} X_1, L_1^2]$, again by Lemma 3.13.

Using the induction hypothesis (3.25b), we have

$$(3.26) \quad [X_2, L_{2n+1}] = \sum_{i=0}^n q^{2n-2i} \theta L'_{2i} X_{2n+3-2i} - \sum_{i=0}^{n-1} q^{2n-1-2i} \theta L_{2i+1} X_{2n-2i+2}.$$

Then by (3.26),

$$\begin{aligned}
[X_1, L'_{2n+2}] &= [X_1, [X_2, Y_{2n+1}]] = q[X_2, L_{2n+1}] - \theta L_{2n+1} X_2 \\
&= \sum_{i=0}^n q^{2n-2i+1} \theta L'_{2i} X_{2n+3-2i} - \sum_{i=0}^{n-1} q^{2n-2i} \theta L_{2i+1} X_{2n-2i+2} - \theta L_{2n+1} X_2 \\
&= \sum_{i=0}^n q^{2n-2i+1} \theta L'_{2i} X_{2n+3-2i} - \sum_{i=0}^n q^{2n-2i} \theta L_{2i+1} X_{2n-2i+2}.
\end{aligned}$$

Since $[X_1, L_{2n+2}] = [X_1, L'_{2n+2}] + [X_1, [L_{2n+1}, L_1]]$, we need to know $[X_1, L_{2n+1}]$:

$$\begin{aligned}
[X_1, L_{2n+1}] &= [X_1, L'_{2n+1}] - [X_1, [L_{2n}, L_1]] \\
&= [X_1, [X_2, Y_{2n}]] - [[X_1, L_{2n}], L_1] + [X_2, L_{2n}] \\
&= (q+1)[X_2, L_{2n}] + \theta L_{2n} X_2 - [[X_1, L_{2n}], L_1] \\
&= \sum_{i=0}^{n-1} (q^{2n-2i} + q^{2n-1-2i}) \theta L_{2i} X_{2n+2-2i} \\
&\quad + \sum_{i=0}^{n-1} (q^{2n-1-2i} + q^{2n-2-2i}) \theta L'_{2i+1} X_{2n-2i+1} + \theta L_{2n} X_2
\end{aligned}$$

$$- \left(\sum_{i=0}^{n-1} P_{2n-1-2i} \theta(L_{2i} X_{2n+2-2i} + \hat{L}_{2i+1} X_{2n+1-2i}) - \sum_{i=0}^{n-1} P_{2n-2-2i} \theta(L'_{2i+1} X_{2n-2i+1} - \hat{L}_{2i+2} X_{2n-2i}) \right),$$

by (3.25a), (3.25c). By the definition of P_n , we have

$$\begin{aligned} [X_1, L_{2n+1}] &= \sum_{i=0}^{n-1} P_{2n-2i} \theta L_{2i} X_{2n+2-2i} - \sum_{i=1}^n P_{2n-2i} \theta \hat{L}_{2i} X_{2n+2-2i} + \theta L_{2n} X_2 \\ &\quad + \sum_{i=0}^{n-1} P_{2n-1-2i} \theta L'_{2i+1} X_{2n+1-2i} - \sum_{i=0}^{n-1} P_{2n-1-2i} \theta \hat{L}_{2i+1} X_{2n+1-2i} \\ (3.27) \quad &= \sum_{i=0}^n P_{2n-2i} \theta L'_{2i} X_{2n+2-2i} + \sum_{i=0}^{n-1} P_{2n-1-2i} \theta L_{2i+1} X_{2n+1-2i}. \end{aligned}$$

Then

$$\begin{aligned} [X_1, L_{2n+2}] &= [X_1, L'_{2n+2}] + [X_1, [L_{2n+1}, L_1]] \\ &= [X_1, L'_{2n+2}] + [[X_1, L_{2n+1}], L_1] + [X_2, L_{2n+1}] \\ &= (q+1)[X_2, L_{2n+1}] - \theta L_{2n+1} X_2 + [[X_1, L_{2n+1}], L_1]. \end{aligned}$$

By (3.26) and (3.27) and similar calculation, we have

$$[X_1, L_{2n+2}] = \sum_{i=0}^n P_{2n+1-2i} \theta L_{2i} X_{2n+3-2i} - \sum_{i=0}^n P_{2n-2i} \theta L'_{2i+1} X_{2n+2-2i}.$$

The expressions for $[X_2, L_{2n+2}]$, $[X_2, L'_{2n+2}]$, $[L_{2n+2}, Y_1]$, $[L'_{2n+2}, Y_1]$, $[L_{2n+2}, Y_2]$, $[L'_{2n+2}, Y_2]$ follow from a completely similar argument. We omit the details. $\square$

Remark 3.19. Note that in Proposition 3.18, we use the abstract definition of a Nichols algebra, i.e., we use differential operators to derive certain relations. It would be very tedious if we only used quantum Serre relations. However, one can prove that these relations can also be derived only by quantum Serre relations; see the proof of Theorem 6.5.

The equation $[\tilde{L}_{2m}, \tilde{L}_{2n}] = 0$ is equivalent to $[L_{2m}, L_{2n}] + [L'_{2m}, L_{2n}] + [L_{2m}, L'_{2n}] + [L'_{2m}, L'_{2n}] = 0$. So we need to know the expressions for each summand.

Lemma 3.20. Suppose $[\tilde{L}_{2k-2}, \tilde{L}_{2N}] = 0$ for all $2 \leq k \leq n$ and for all $N > 0$; then

$$[\tilde{L}_{2k-1}, \hat{L}_{2N+1}] = [\hat{L}_{2k-1}, \tilde{L}_{2N+1}] \quad \forall 1 \leq k \leq n \text{ and } \forall N > 0.$$

Proof. We only need to prove

$$[\tilde{L}_{2n-1}, [\tilde{L}_{2N}, L_1]] = [[\tilde{L}_{2n-2}, L_1], \tilde{L}_{2N+1}].$$

By Proposition 3.6,

$$\begin{aligned} \text{LHS} &= \left[\frac{1}{2} \theta L_1 \tilde{L}_{2n-2} + \frac{1}{2} \theta \tilde{L}_{2n-2} L_1 + \frac{1}{4} [\tilde{L}_{2n-4}, M_3], [\tilde{L}_{2N}, L_1] \right] \\ &= \frac{1}{2} \theta \left[L_1, [\tilde{L}_{2n-2}, [\tilde{L}_{2N}, L_1]] \right]' + \frac{1}{4} \left[[\tilde{L}_{2n-4}, M_3], [\tilde{L}_{2N}, L_1] \right], \\ \text{RHS} &= \left[[\tilde{L}_{2n-2}, L_1], \frac{1}{2} \theta L_1 \tilde{L}_{2N} + \frac{1}{2} \theta \tilde{L}_{2N} L_1 + \frac{1}{4} [\tilde{L}_{2N-2}, M_3] \right] \\ &= -\frac{1}{2} \theta \left[L_1, [[\tilde{L}_{2n-2}, L_1], \tilde{L}_{2N}] \right]' + \frac{1}{4} \left[[\tilde{L}_{2n-2}, L_1], [\tilde{L}_{2N-2}, M_3] \right]. \end{aligned}$$

Since $[\tilde{L}_{2n-2}, \tilde{L}_{2N}] = 0$, we only need to prove

$$[[\tilde{L}_{2n-4}, M_3], [\tilde{L}_{2N}, L_1]] = [[\tilde{L}_{2n-2}, L_1], [\tilde{L}_{2N-2}, M_3]].$$

By using the expressions for $[\tilde{L}_{2N}, L_1]$ and $[\tilde{L}_{2n-2}, L_1]$ in Proposition 3.6 and similar calculation, we get that we only need to prove

$$[[\tilde{L}_{2n-4}, M_3], [\tilde{L}_{2N-4}, M_5]] = [[\tilde{L}_{2n-6}, M_5], [\tilde{L}_{2N-2}, M_3]].$$

Repeating these steps this lemma will be proved. $\square$

Lemma 3.21. Suppose $[\tilde{L}_{2k}, \tilde{L}_{2N}] = 0$ for all $k \leq n-1$ and for all $N > 0$; then $[\tilde{L}_{2n}, \tilde{L}_{2N}] = 0$ for all $N > 0$ is equivalent to

$$-[[\tilde{L}_{2n-2}, \tilde{L}_{2N+1}], L_1] + [[\tilde{L}_{2N+2}, \tilde{L}_{2n-3}], L_1] = 0 \quad \text{for all } N > 0.$$

Proof. Considering $[[X_1, L_{2n}], Y_{2N}]$, using the same technique as Proposition 3.6, we get

$$\begin{aligned} [L_{2n}, L_{2N}] &= \sum_{i=0}^{n-1} P_{2n-1-2i} \theta [L_{2n+2N-2i}, L_{2i}] \\ &\quad + \sum_{i=0}^{n-1} P_{2n-2-2i} \theta (L'_{2i+1} L'_{2n+2N-2i-1} - L_{2n+2N-2i-1} L_{2i+1}). \end{aligned}$$

Note that in this calculation we do not need to care about terms with form $Y_a L_b X_c$ or $Y_a L'_b X_c$. All terms of this form would be canceled by Corollaries 3.12 and 3.14. One can also verify this directly.

Similarly considering $[[X_1, L'_{2n}], Y_{2N}]$ we can get

$$\begin{aligned} [L'_{2n}, L_{2N}] &= \sum_{i=0}^{n-1} q^{2n-1-2i} \theta[L_{2n+2N-2i}, L'_{2i}] \\ &\quad + \sum_{i=0}^{n-1} q^{2n-2-2i} \theta(L_{2i+1} L'_{2n+2N-1-2i} - L_{2n+2N-1-2i} L'_{2i+1}). \end{aligned}$$

Similarly considering $[[X_2, L_{2n}], Y_{2N-1}]$ we can get

$$\begin{aligned} [L_{2n}, L'_{2N}] &= \sum_{i=0}^{n-1} q^{2n-1-2i} \theta[L'_{2n+2N-2i}, L_{2i}] \\ &\quad + \sum_{i=0}^{n-1} q^{2n-2-2i} \theta(L'_{2n+2N-1-2i} L_{2i+1} - L'_{2i+1} L_{2n+2N-1-2i}). \end{aligned}$$

Similarly considering $[[X_2, L'_{2n}], Y_{2N-1}]$ we can get

$$\begin{aligned} [L'_{2n}, L'_{2N}] &= \sum_{i=0}^{n-1} Q_{2n-1-2i} \theta[L'_{2n+2N-2i}, L'_{2i}] \\ &\quad + \sum_{i=0}^{n-1} Q_{2n-2-2i} \theta(L'_{2n+2N-1-2i} L'_{2i+1} - L_{2i+1} L_{2n+2N-1-2i}). \end{aligned}$$

Then by the construction of P_n , Q_n and $[\tilde{L}_{2k}, \tilde{L}_{2N}] = 0$ for all $N > 0$, $k \leq n-1$,

$$\begin{aligned} &[L_{2n}, L_{2N}] + [L'_{2n}, L_{2N}] + [L'_{2n}, L_{2N}] + [L'_{2n}, L'_{2N}] \\ &= \sum_{i=0}^{n-1} (P_{2n-2-2i} + q^{2n-2-2i}) \theta([L_{2n+2N-2i}, L_{2i}] - [L'_{2n+2N-2i}, L'_{2i}]) \\ &\quad + \sum_{i=0}^{n-1} P_{2n-2-2i} \theta([L'_{2i+1}, L'_{2n+2N-2i-1}] - [L_{2i+1}, L_{2n+2N-2i-1}]) \\ &\quad + \sum_{i=0}^{n-1} q^{2n-2-2i} \theta([L_{2i+1}, L'_{2n+2N-2i-1}] - [L'_{2i+1}, L_{2n+2N-2i-1}]). \end{aligned}$$

We have the following identities by definitions and Lemma 3.20:

$$\begin{aligned} [L_{2m}, L_{2n}] - [L'_{2m}, L'_{2n}] &= [\hat{L}_{2m}, \tilde{L}_{2n}] + [\tilde{L}_{2m}, \hat{L}_{2n}], \quad \forall m, n \geq 0 \\ [L'_{2i+1}, L'_{2n+2N-2i-1}] - [L_{2i+1}, L_{2n+2N-2i-1}] \\ &= [\tilde{L}_{2i+1}, \hat{L}_{2n+2N-2i-1}] + [\hat{L}_{2i+1}, \tilde{L}_{2n+2N-2i-1}] \\ &= 2[\tilde{L}_{2i+1}, \hat{L}_{2n+2N-2i-1}], \end{aligned}$$

$$\begin{aligned}
& [L_{2i+1}, L'_{2n+2N-2i-1}] - [L'_{2i+1}, L_{2n+2N-2i-1}] \\
& = [\tilde{L}_{2i+1}, \hat{L}_{2n+2N-2i-1}] - [\hat{L}_{2i+1}, \tilde{L}_{2n+2N-2i-1}] = 0, \quad \forall 0 \leq i \leq n-1.
\end{aligned}$$

Then by the above identities,

$$\begin{aligned}
& [L_{2n}, L_{2N}] + [L'_{2n}, L_{2N}] + [L'_{2n}, L_{2N}] + [L'_{2n}, L'_{2N}] \\
& = \sum_{i=0}^{n-1} (P_{2n-2-2i} + q^{2n-2-2i}) \theta([\hat{L}_{2n+2N-2i}, \tilde{L}_{2i}] + [\tilde{L}_{2n+2N-2i}, \hat{L}_{2i}]) \\
& \quad + 2 \sum_{i=0}^{n-1} P_{2n-2-2i} \theta[\tilde{L}_{2i+1}, \hat{L}_{2n+2N-2i-1}] \\
& = \sum_{i=0}^{n-1} (P_{2n-2-2i} + q^{2n-2-2i}) \theta(-[\tilde{L}_{2n+2N-2i-1}, \hat{L}_{2i+1}] \\
& \quad + [L_1, [\tilde{L}_{2n+2N-2i-1}, \tilde{L}_{2i}]] \\
& \quad + [[\tilde{L}_{2n+2N-2i}, \tilde{L}_{2i-1}], L_1] \\
& \quad + [\tilde{L}_{2i-1}, \hat{L}_{2n+2N-2i+1}]) \\
& \quad + 2 \sum_{i=0}^{n-1} P_{2n-2-2i} \theta[\tilde{L}_{2i+1}, \hat{L}_{2n+2N-2i-1}].
\end{aligned}$$

The definition of P_n implies $P_{2n-2-2i} = q^{2n-2-2i} - q^{2n-4-2i} - P_{2n-4-2i}$. Then

$$\begin{aligned}
& [L_{2n}, L_{2N}] + [L'_{2n}, L_{2N}] + [L'_{2n}, L_{2N}] + [L'_{2n}, L'_{2N}] \\
& = \sum_{i=0}^{n-1} (P_{2n-2-2i} + q^{2n-2-2i}) \theta([L_1, [\tilde{L}_{2n+2N-2i-1}, \tilde{L}_{2i}]] + [[\tilde{L}_{2n+2N-2i}, \tilde{L}_{2i-1}], L_1]).
\end{aligned}$$

When q is not a root of unity, it is easy to see $(P_{2n-2-2i} + q^{2n-2-2i})\theta$ is not 0. Then by the induction hypothesis we can see

$$[L_{2n}, L_{2N}] + [L'_{2n}, L_{2N}] + [L'_{2n}, L_{2N}] + [L'_{2n}, L'_{2N}] = 0$$

is equivalent to

$$-[[\tilde{L}_{2n-2}, \tilde{L}_{2N+1}], L_1] + [[\tilde{L}_{2N+2}, \tilde{L}_{2n-3}], L_1] = 0,$$

for all $N \geq 0$. □

Define $D_{2k} = -[[\tilde{L}_{2k}, \tilde{L}_{2N+1}], L_1] + [[\tilde{L}_{2N+2}, \tilde{L}_{2k-1}], L_1]$. Note that D_{2k} is related to N , but we omit it, since when we deal with D_{2k} later, N is always fixed.

Define $\bar{L}_{2n}$ by the following formal power series:

$$\theta \sum_{n=0}^{\infty} \tilde{L}_{2n} u^{2n} = \exp \left(\theta \sum_{n=1}^{\infty} \bar{L}_{2n} u^{2n} \right).$$

Define $C_k(P(u))$ by the coefficient of u^k in the power series $P(u)$.

Proposition 3.22. *There exists a series of numbers R_{2k} such that $[\bar{L}_{2k}, L_1] = R_{2k}M_{2k+1}$ and $[\tilde{L}_{2m}, \tilde{L}_{2n}] = 0$ for all $k, m, n \geq 1$. The number R_{2k} is defined by*

$$\begin{aligned} \theta R_{2k} = & C_{2k-2} \left(\frac{1}{2} \theta \right) + C_{2k-2} \left(\frac{1}{2} \theta \exp \left(\sum_{n=1}^{k-1} \theta R_{2n} u^{2n} \right) \right) \\ & + C_{2k-4} \left(\frac{1}{4} \left(\exp \left(\sum_{n=1}^{k-1} \theta R_{2n} u^{2n} \right) - 1 \right) \right) - C_{2k} \left(\exp \left(\sum_{n=1}^{k-1} \theta R_{2n} u^{2n} \right) \right). \end{aligned}$$

Moreover, we have $R_{4k} = 0$ and $R_{4k+2} = \frac{1}{2^{2k}} \frac{[2k+1]_q}{2k+1}$.

Proof. We prove by induction on n that $[\bar{L}_{2n}, L_1] = R_{2n}M_{2n+1}$ for some R_{2n} and $[\tilde{L}_{2n}, \tilde{L}_{2N}] = 0$ for all $N \geq 0$.

For $n = 1$, we know that $[\bar{L}_2, L_1] = M_3$, which means $R_2 = 1$ and coincides with the formula for R_{2k} when $k = 1$, and $[\tilde{L}_2, \tilde{L}_{2N}] = 0$ for all $N \geq 0$. Now suppose this proposition holds for $n \leq k$, i.e., we know $[\bar{L}_{2n}, L_1] = R_{2n}M_{2n+1}$ and $[\tilde{L}_{2n}, \tilde{L}_{2N}] = 0$ for $n \leq k$, and for all $N \geq 0$, we consider $n = k + 1$.

First we prove $[\bar{L}_{2k+2}, L_1] = R_{2k+2}M_{2k+3}$ for some R_{2k+2} . By Proposition 3.6,

$$\begin{aligned} \theta \sum_{n=0}^{\infty} [\tilde{L}_{2n} u^{2n}, L_1] &= \frac{1}{2} \theta M_3 \theta \sum_{n=0}^{\infty} \tilde{L}_{2n-2} u^{2n} \\ &\quad + \frac{1}{2} \theta \theta \sum_{n=0}^{\infty} \tilde{L}_{2n-2} u^{2n} M_3 + \frac{1}{4} \left[\theta \sum_{n=0}^{\infty} \tilde{L}_{2n-4} u^{2n}, M_5 \right] \\ &= \frac{1}{2} \theta u^2 M_3 \theta \sum_{n=0}^{\infty} \tilde{L}_{2n-2} u^{2n-2} \\ &\quad + \frac{1}{2} \theta u^2 \theta \sum_{n=0}^{\infty} \tilde{L}_{2n-2} u^{2n-2} M_3 + \frac{1}{4} u^4 \left[\theta \sum_{n=0}^{\infty} \tilde{L}_{2n-4} u^{2n-4}, M_5 \right], \end{aligned}$$

i.e.,

$$\begin{aligned} \left[\exp \left(\theta \sum_{n=1}^{\infty} \bar{L}_{2n} u^{2n} \right), L_1 \right] &= \frac{1}{2} \theta u^2 M_3 \exp \left(\theta \sum_{n=1}^{\infty} \bar{L}_{2n} u^{2n} \right) \\ (3.28) \quad &\quad + \frac{1}{2} \theta u^2 \exp \left(\theta \sum_{n=1}^{\infty} \bar{L}_{2n} u^{2n} \right) M_3 \\ &\quad + \frac{1}{4} u^4 \left[\exp \left(\theta \sum_{n=1}^{\infty} \bar{L}_{2n} u^{2n} \right), M_5 \right]. \end{aligned}$$

Note that $\exp(\theta \sum_{n=1}^{\infty} \bar{L}_{2n} u^{2n}) = \exp(\theta \sum_{n=2}^{\infty} \bar{L}_{2n} u^{2n}) \exp(\theta \bar{L}_2 u^2)$, then we denote $\exp(\theta \sum_{n=2}^{\infty} \bar{L}_{2n} u^{2n})$ by $\exp(2)$ and in general $\exp(\theta \sum_{n=k}^{\infty} \bar{L}_{2n} u^{2n})$ by $\exp(k)$.

It is easy to get $\exp(\theta \bar{L}_{2n} u^2) L_1 = \exp(\text{ad}_{\theta \bar{L}_{2n}} u^{2n}) \circ L_1 \exp(\theta \bar{L}_{2n} u^2)$. Denote $\text{ad}_{\theta \bar{L}_{2n}} u^{2n}$ by $\mathcal{M}_{2n}$ for further convenience, where $\text{ad}_{\theta \bar{L}_{2n}}$ means the operator $[\theta \bar{L}_{2n}, \cdot]$. Now we analyze (3.28). The left-hand side is

$$[\exp(2), L_1] \exp(\theta \bar{L}_2 u^2) + \exp(2)(\exp(\mathcal{M}_2) - 1) \circ L_1 \exp(\theta \bar{L}_2 u^2).$$

Repeating similar calculation, by the induction hypothesis, we get that the left-hand side is

$$\left([\exp(k+1), L_1] + \exp(k+1) \left(\exp \left(\sum_{n=1}^k \mathcal{M}_{2n} \right) - 1 \right) \circ L_1 \right) \prod_{n=1}^k \exp(\theta \bar{L}_{2n} u^{2n}).$$

Similarly, we get that the right-hand side is

$$\begin{aligned} & \left(\frac{1}{2} \theta u^2 M_3 \exp(k+1) + \frac{1}{2} \theta u^2 \exp(k+1) \exp \left(\sum_{n=1}^k \mathcal{M}_{2n} \right) \circ M_3 \right. \\ & \quad \left. + \frac{1}{4} u^4 \left([\exp(k+1), M_5] + \exp(k+1) \left(\exp \left(\sum_{n=1}^k \mathcal{M}_{2n} \right) - 1 \right) \circ M_5 \right) \right) \\ & \quad \times \prod_{n=1}^k \exp(\theta \bar{L}_{2n} u^{2n}). \end{aligned}$$

Canceling the ending terms of both sides and comparing the coefficient of u^{2k+2} ,

$$\begin{aligned} & \theta [\bar{L}_{2k+2}, L_1] + C_{2k+2} \left(\left(\exp \left(\sum_{n=1}^k \mathcal{M}_{2n} \right) - 1 \right) \circ L_1 \right) \\ & = C_{2k} \left(\frac{1}{2} \theta M_3 \right) + C_{2k} \left(\frac{1}{2} \theta \exp \left(\sum_{n=1}^k \mathcal{M}_{2n} \right) \circ M_3 \right) \\ & \quad + C_{2k-2} \left(\frac{1}{4} \left(\exp \left(\sum_{n=1}^k \mathcal{M}_{2n} \right) - 1 \right) \circ M_5 \right). \end{aligned}$$

Hence we have $\theta [\bar{L}_{2k+2}, L_1] = R_{2k+2} M_{2k+3}$, where

$$\begin{aligned} R_{2k+2} & = C_{2k} \left(\frac{1}{2} \theta \right) + C_{2k} \left(\frac{1}{2} \theta \exp \left(\sum_{n=1}^k \theta R_{2n} u^{2n} \right) \right) \\ & \quad + C_{2k-2} \left(\frac{1}{4} \left(\exp \left(\sum_{n=1}^k \theta R_{2n} u^{2n} \right) - 1 \right) \right) - C_{2k+2} \left(\exp \left(\sum_{n=1}^k \theta R_{2n} u^{2n} \right) \right). \end{aligned}$$

Then we need to prove $D_{2k} = 0$, for any $N \geq 0$. Fix N . Considering the formal power series

$$\theta \sum_{n=0}^{\infty} D_{2n} u^{2n},$$

we have

$$\begin{aligned}\theta \sum_{n=0}^{\infty} D_{2n} u^{2n} &= - \left[\left[\theta \sum_{n=0}^{\infty} \tilde{L}_{2n} u^{2n}, \tilde{L}_{2N+1} \right], L_1 \right] \\ &\quad + \left[\left[\tilde{L}_{2N+2}, \frac{1}{2} \theta u^2 L_1 \theta \sum_{n=0}^{\infty} \tilde{L}_{2n} u^{2n} + \frac{1}{2} \theta u^2 \theta \sum_{n=0}^{\infty} \tilde{L}_{2n} u^{2n} L_1 \right. \right. \\ &\quad \left. \left. + \frac{1}{4} u^4 \left[\theta \sum_{n=0}^{\infty} \tilde{L}_{2n} u^{2n}, M_3 \right] \right], L_1 \right] \\ &= -[A, L_1] + [[\tilde{L}_{2N+2}, B], L_1].\end{aligned}$$

We calculate A and B . The calculation is similar to the previous one. We have

$$A = A_{2k+2} \prod_{n=1}^k \exp(\theta \bar{L}_{2n} u^{2n}),$$

where

$$A_{2k+2} = [\exp(k+1), \tilde{L}_{2N+1}] + \exp(k+1) \left(\exp \left(\sum_{n=1}^k \mathcal{M}_{2n} \right) - 1 \right) \circ \tilde{L}_{2N+1}.$$

$$B = B_{2k+2} \prod_{n=1}^k \exp(\theta \bar{L}_{2n} u^{2n}),$$

where

$$\begin{aligned}B_{2k+2} &= \frac{1}{2} \theta u^2 L_1 \exp(k+1) + \frac{1}{2} \theta u^2 \exp(k+1) \exp \left(\sum_{n=1}^k \mathcal{M}_{2n} \right) \circ L_1 \\ &\quad + \frac{1}{4} u^4 \left([\exp(k+1), M_3] + \exp(k+1) \left(\exp \left(\sum_{n=1}^k \mathcal{M}_{2n} \right) - 1 \right) \circ M_3 \right).\end{aligned}$$

Then

$$\begin{aligned}\theta \sum_{n=0}^{\infty} D_{2n} u^{2n} &= - \left[A_{2k+2} \prod_{n=1}^k \exp(\theta \bar{L}_{2n} u^{2n}), L_1 \right] + \left[\left[\tilde{L}_{2N+2}, B_{2k+2} \prod_{n=1}^k \exp(\theta \bar{L}_{2n} u^{2n}) \right], L_1 \right] \\ &= - \left[A_{2k+2} \prod_{n=1}^k \exp(\theta \bar{L}_{2n} u^{2n}), L_1 \right] + \left[[\tilde{L}_{2N+2}, B_{2k+2}] \prod_{n=1}^k \exp(\theta \bar{L}_{2n} u^{2n}), L_1 \right] \\ &= \left[(-A_{2k+2} + [\tilde{L}_{2N+2}, B_{2k+2}]) \prod_{n=1}^k \exp(\theta \bar{L}_{2n} u^{2n}), L_1 \right].\end{aligned}$$

We need to prove that the coefficient of u^{2k} in $\theta \sum_{n=0}^{\infty} D_{2n} u^{2n}$ is 0. It is enough to prove $C_{2k}([\Phi \prod_{n=1}^k \exp(\theta \tilde{L}_{2n} u^{2n}), L_1]) = 0$, where

$$\begin{aligned} \Phi = & -\exp\left(\sum_{n=1}^k \mathcal{M}_{2n}\right) \circ \tilde{L}_{2N+1} \\ & + \left[\tilde{L}_{2N+2}, \frac{1}{2}\theta u^2 L_1 + \frac{1}{2}\theta u^2 \exp\left(\sum_{n=1}^k \mathcal{M}_{2n}\right) \circ L_1\right. \\ & \left. + \frac{1}{4}u^4 \left(\exp\left(\sum_{n=1}^k \mathcal{M}_{2n}\right) - 1\right) \circ M_3\right]. \end{aligned}$$

We claim that $\Phi_{2i} := C_{2i}(\Phi) = 0$ for $i \leq k$.

We have $\Phi_{2i} = \phi_{2i}(\frac{1}{2}\theta M_{2i+1} \tilde{L}_{2N} + \frac{1}{2}\theta \tilde{L}_{2N} M_{2i+1} + \frac{1}{4}[\tilde{L}_{2N-2}, M_{2i+3}])$, where

$$\begin{aligned} \phi_{2i} = & C_{2i}\left(-\exp\left(\sum_{n=1}^k \theta R_{2n} u^{2n}\right) + \frac{1}{2}\theta u^2 + \frac{1}{2}\theta u^2 \exp\left(\sum_{n=1}^k \theta R_{2n} u^{2n}\right)\right. \\ & \left.+ \frac{1}{4}u^4 \left(\exp\left(\sum_{n=1}^k \theta R_{2n} u^{2n}\right) - 1\right)\right). \end{aligned}$$

Then $\Phi_{2i} = 0$ for $i \leq k$ holds by this and the induction hypothesis for R_{2i} for $i \leq k$.

It is not difficult to get the general formula for R_{2m} ; we omit the details. $\square$

Corollary 3.23. *We have the following commutation relations in $B(\mathbb{V})$:*

- (a) $[\tilde{L}_{2m}, \tilde{L}_{2n}] = 0 = [\bar{L}_{2m}, \bar{L}_{2n}]$.
- (b) $[M_{2m+1}, M_{2n+1}] = 0$ if $m+n$ is odd, $[M_{2m+1}, M_{2n+1}] = (-1)^{\frac{m-n}{2}} 2M_{m+n+1}^2$ if $m+n$ is even.
- (c) $[\bar{L}_{2m}, M_{2n+1}] = R_{2m} M_{2m+2n+1}$, where $R_{4m} = 0$ and $R_{4m+2} = \frac{1}{2^{2m}} \frac{[2m+1]_q}{2m+1}$.

By these relations, the only possible imaginary root vectors are L_{4n} at $4n\delta$, L_{4n+2} and M_{2n+1}^2 at $(4n+2)\delta$, M_{2n+1} at $(2n+1)\delta$.

Proof. These relations hold immediately by Propositions 3.22 and 3.6. Then by induction on degree we can get these possible imaginary root vectors. $\square$

Remark 3.24. Note that these commutation relations between imaginary root vectors are derived from Proposition 3.6, Corollary 3.8 and Proposition 3.18. If we prove that Proposition 3.18 can also be derived only from quantum Serre relations, then so are these commutation relations between imaginary root vectors.

§4. Comultiplication of $K_{\geq 1}/K_{>1}$

This section is about the comultiplication of $B(\mathbb{V})$'s subquotient $K_{\geq 1}/K_{>1}$, which is the most important part of this paper. The total determination of the root multiplicities and the construction of the Drinfeld second realization of $U_v^+(\mathbf{A}(0, 2)^{(4)})$ all rely on this. In this section, we also get some partial information about the commutation relations between X_1 and the distinguished imaginary root vectors with manageable comultiplication in $K_{\geq 1}/K_{>1}$.

§4.1. $\Delta(\mathbb{M}(u))$ and $\Delta(\mathbb{Z}(u))$

Proposition 4.1.

$$\begin{aligned}\Delta(\mathbb{M}(u)) &= (\mathbb{M}(u) \otimes 1 + 1 \otimes \mathbb{M}(u)) \frac{1}{1 - 2\theta \mathbb{M}(u) \otimes \mathbb{M}(u)}, \\ \Delta(\mathbb{Z}(u)) &= (\mathbb{Z}(u) \otimes 1 + 1 \otimes \mathbb{Z}(u)) \frac{1}{1 + (2\theta)^2 \mathbb{Z}(u) \otimes \mathbb{Z}(u)}.\end{aligned}$$

Proof. We have

$$(4.1) \quad \Delta(L_2) = L_2 \otimes 1 - \theta L_1 \otimes L_1 + 1 \otimes L_2, \quad \Delta(L_1) = L_1 \otimes 1 + 1 \otimes L_1.$$

Recall that $M_{2n+1} = [L_2, M_{2n-1}]$. With (4.1) and Corollary 3.8, it is not difficult to get $\Delta(\mathbb{M}(u))$. Since $\mathbb{M}(u)^2 = \mathbb{Z}(u)$, we have

$$\begin{aligned}\Delta(\mathbb{Z}(u)) &= (\mathbb{M} \otimes 1 + 1 \otimes \mathbb{M}) \frac{1}{1 - 2\theta \mathbb{M} \otimes \mathbb{M}} (\mathbb{M} \otimes 1 + 1 \otimes \mathbb{M}) \frac{1}{1 - 2\theta \mathbb{M} \otimes \mathbb{M}} \\ &= (\mathbb{M} \otimes 1 + 1 \otimes \mathbb{M})^2 \frac{1}{1 + 2\theta \mathbb{M} \otimes \mathbb{M}} \frac{1}{1 - 2\theta \mathbb{M} \otimes \mathbb{M}} \\ &= (\mathbb{M}^2 \otimes 1 + 1 \otimes \mathbb{M}^2) \frac{1}{1 + (2\theta)^2 \mathbb{M}^2 \otimes \mathbb{M}^2} \\ &= (\mathbb{Z} \otimes 1 + 1 \otimes \mathbb{Z}) \frac{1}{1 + (2\theta)^2 \mathbb{Z} \otimes \mathbb{Z}}.\end{aligned}$$

□

Define $\mathcal{Z}(u)$ by the following power series:

$$\arctan(2\theta i \mathbb{Z}(u)) = 2\theta i \mathcal{Z}(u).$$

We write $\mathcal{Z}(u)$ as $\sum_{n=0}^{\infty} Z_{4n+2} u^{4n+2}$.

Proposition 4.2. *The series $\mathcal{Z}(u)$ is primitive in the subquotient $K_{\geq 1}/K_{>1}$ and the leading term of $[X_1, Z_{4n+2}]$ is $2^{2n} \frac{[2n+1]_q}{2n+1} X_{4n+3}$, which means that Z_{4n+2} is not 0 in $B(\mathbb{V})$.*

Proof. Note that $\arctan(x) = \frac{i}{2} \log\left(\frac{i+x}{i-x}\right)$. Hence it is enough to prove that $\frac{1+2\theta\mathbb{Z}}{1-2\theta\mathbb{Z}}$ is group-like:

$$\begin{aligned}\Delta\left(\frac{1+2\theta\mathbb{Z}}{1-2\theta\mathbb{Z}}\right) &= \frac{1+2\theta\frac{\mathbb{Z}\otimes 1+1\otimes\mathbb{Z}}{1+(2\theta)^2\mathbb{Z}\otimes\mathbb{Z}}}{1-2\theta\frac{\mathbb{Z}\otimes 1+1\otimes\mathbb{Z}}{1+(2\theta)^2\mathbb{Z}\otimes\mathbb{Z}}} = \frac{(1+2\theta\mathbb{Z}\otimes 1)(1+1\otimes 2\theta\mathbb{Z})}{(1-2\theta\mathbb{Z}\otimes 1)(1-1\otimes 2\theta\mathbb{Z})} \\ &= \frac{1+2\theta\mathbb{Z}}{1-2\theta\mathbb{Z}} \otimes \frac{1+2\theta\mathbb{Z}}{1-2\theta\mathbb{Z}}.\end{aligned}$$

By induction, it is not difficult to get that the leading term of $[X_1, 2\theta(-1)^k M_{2k+1}^2]$ is $2\theta 2^{2k} X_{4k+3}$. Then the coefficient we want is

$$\frac{1}{2\theta i} C_{4n+2} \left(\arctan \left(\sum_{k=0}^{\infty} 2\theta i 2^{2k} u^{4k+2} \right) \right).$$

Let $x = 2u^2$; then we need to calculate

$$\frac{1}{2\theta i} C_{4n+2} \left(\arctan \left(\theta i \frac{x}{1-x^2} \right) \right).$$

We have

$$\frac{d}{dx} \arctan \left(\theta i \frac{x}{1-x^2} \right) = \frac{iq}{1-q^2x^2} - \frac{iq^{-1}}{1-q^{-2}x^2} = iq \sum_{n=0}^{\infty} q^{2n} x^{2n} - iq^{-1} \sum_{n=0}^{\infty} q^{-2n} x^{2n}.$$

Taking the integral we get

$$\arctan \left(\theta i \frac{x}{1-x^2} \right) = i \sum_{n=0}^{\infty} \frac{q^{2n+1}}{2n+1} x^{2n+1} - i \sum_{n=0}^{\infty} \frac{q^{-(2n+1)}}{2n+1} x^{2n+1},$$

then immediately we have that the coefficient we want is $2^{2n} \frac{[2n+1]_q}{2n+1}$. $\square$

§4.2. $\Delta(\bar{\mathbb{L}})$

Denote $\sum_{n=1}^{\infty} \theta \bar{L}_{2n} u^{2n}$ by $\bar{\mathbb{L}}$, i.e., we have $\exp(\bar{\mathbb{L}}) = \tilde{\mathbb{L}}$. Define $*(u) = \Delta(\tilde{\mathbb{L}}(u))$.

Proposition 4.3. *We have*

$$\begin{aligned}*(u) &= \left(1 - A \otimes A + \frac{1}{2} B \otimes \frac{1}{2} B \right. \\ &\quad \left. + \left(\theta^{-1} \left(\frac{1}{4} B^2 + A^2 \right) + A \cdot \frac{1}{2} B \right) \otimes \left(\theta^{-1} \left(\frac{1}{4} B^2 + A^2 \right) + A \cdot \frac{1}{2} B \right) \right) \\ &\quad \cdot \exp(\bar{\mathbb{L}}(u) \otimes 1 + 1 \otimes \bar{\mathbb{L}}(u)) \\ &:= \#(u) \cdot \exp(\bar{\mathbb{L}}(u) \otimes 1 + 1 \otimes \bar{\mathbb{L}}(u)),\end{aligned}$$

and also

$$\begin{aligned} *(u) &= \exp(\bar{\mathbb{L}}(u) \otimes 1 + 1 \otimes \bar{\mathbb{L}}(u)) \\ &\quad \cdot \left(1 - A' \otimes A' + \frac{1}{2} B' \otimes \frac{1}{2} B' \right. \\ &\quad \left. + \left(\theta^{-1} \left(\frac{1}{4} B'^2 + A'^2 \right) + A' \cdot \frac{1}{2} B' \right) \otimes \left(\theta^{-1} \left(\frac{1}{4} B'^2 + A'^2 \right) + A' \cdot \frac{1}{2} B' \right) \right) \\ &:= \exp(\bar{\mathbb{L}}(u) \otimes 1 + 1 \otimes \bar{\mathbb{L}}(u)) \cdot \#'(u). \end{aligned}$$

Proof. By Corollary 3.10, in the subquotient, we have

$$\begin{aligned} \Delta(\tilde{L}_{2n}) &= \tilde{L}_{2n} \otimes 1 - \theta \tilde{L}_{2n-1} \otimes \tilde{L}_1 + \theta \tilde{L}_{2n-2} \otimes \tilde{L}_2 - \cdots - \theta \tilde{L}_1 \otimes \tilde{L}_{2n-1} + 1 \otimes \tilde{L}_{2n} \\ &\quad + \theta \frac{1}{2} \hat{L}_{2n-2} \otimes \frac{1}{2} \hat{L}_2 + \theta \frac{1}{2} \hat{L}_{2n-3} \otimes \frac{1}{2} \hat{L}_3 + \cdots + \theta \frac{1}{2} \hat{L}_3 \otimes \frac{1}{2} \hat{L}_{2n-3} \\ &\quad + \theta \frac{1}{2} \hat{L}_2 \otimes \frac{1}{2} \hat{L}_{2n-2}, \end{aligned}$$

i.e.,

$$\begin{aligned} *(u) &= \sum_{n=0}^{\infty} \theta \tilde{L}_{2n} u^{2n} \otimes \sum_{n=0}^{\infty} \theta \tilde{L}_{2n} u^{2n} - \sum_{n=0}^{\infty} \theta \tilde{L}_{2n+1} u^{2n+1} \otimes \sum_{n=0}^{\infty} \theta \tilde{L}_{2n+1} u^{2n+1} \\ &\quad + \sum_{n=1}^{\infty} \frac{1}{2} \theta \hat{L}_{2n} u^{2n} \otimes \sum_{n=1}^{\infty} \frac{1}{2} \theta \hat{L}_{2n} u^{2n} \\ &\quad + \sum_{n=1}^{\infty} \frac{1}{2} \theta \hat{L}_{2n+1} u^{2n+1} \otimes \sum_{n=1}^{\infty} \frac{1}{2} \theta \hat{L}_{2n+1} u^{2n+1}. \end{aligned}$$

Then by (3.24a), (3.24b), (3.24e) in Proposition 3.17, it is easy to get the conclusion. $\square$

Lemma 4.4.

$$\#(u) \#'(iu) = (1 + (2\theta)^2 \mathbb{Z}(a) \otimes \mathbb{Z}(a)) \cdot (1 + (2\theta)^2 \mathbb{Z}(b) \otimes \mathbb{Z}(b)).$$

Proof. By the definitions of A , A' and B , B' ,

$$\begin{aligned} \#(u) \#'(iu) &= \left(1 - A \otimes A + \frac{1}{2} B \otimes \frac{1}{2} B \right. \\ &\quad \left. + \left(\theta^{-1} \left(\frac{1}{4} B^2 + A^2 \right) + A \cdot \frac{1}{2} B \right) \otimes \left(\theta^{-1} \left(\frac{1}{4} B^2 + A^2 \right) + A \cdot \frac{1}{2} B \right) \right) \end{aligned}$$

$$\cdot \left(1 + A \otimes A - \frac{1}{2} B \otimes \frac{1}{2} B \right. \\ \left. + \left(-\theta^{-1} \left(\frac{1}{4} B^2 + A^2 \right) + A \cdot \frac{1}{2} B \right) \otimes \left(-\theta^{-1} \left(\frac{1}{4} B^2 + A^2 \right) + A \cdot \frac{1}{2} B \right) \right).$$

Omitting a long but elementary calculation using (3.24c), (3.24d) in Proposition 3.17, we get the conclusion. $\square$

Lemma 4.5. *The leading term of $[X_1, \bar{L}_{2n}]$ is $\frac{(q^{2n} + (-1)^n)(q^n + (-1)^{n+1})}{nq^{n-1}(q^2 - 1)} X_{2n+1}$.*

Proof. We have $\log(1 + \theta \sum_{n=1}^{\infty} \tilde{L}_{2n} u^{2n}) = \theta \sum_{n=1}^{\infty} \bar{L}_{2n} u^{2n}$. By Proposition 3.18, we get that the leading term of $[X_1, \theta \tilde{L}_{2n} u^{2n}]$ is

$$\theta \left(q^{2n-2} \frac{1 - (-q^{-2})^n}{1 + q^{-2}} + q^{2n-1} \right) X_{2n+1} u^{2n} \\ = \frac{\theta}{1 + q^{-2}} \sum_{n=1}^{\infty} ((q^{-1} + q^{-2} + q^{-3})(q^2 u^2)^n - q^{-2}(-u^2)^n) X_{2n+1}.$$

Then we have that the leading term of $[X_1, \bar{L}_{2n}]$ is

$$\theta^{-1} C_{2n} \left(\log \left(1 + \frac{\theta}{1 + q^{-2}} \sum_{n=1}^{\infty} ((q^{-1} + q^{-2} + q^{-3})(q^2 u^2)^n - q^{-2}(-u^2)^n) \right) \right) X_{2n+1},$$

i.e.,

$$\theta^{-1} C_{2n} \left(\log \left(1 + \frac{\theta}{1 + q^{-2}} \left(\frac{(q^{-1} + q^{-2} + q^{-3})q^2 u^2}{1 - q^2 u^2} - \frac{q^{-2}(-u^2)}{1 + u^2} \right) \right) \right) X_{2n+1}.$$

Then by the same technique as in Proposition 4.2 we get that the leading term of $[X_1, \bar{L}_{2n}]$ is

$$\frac{(q^{2n} + (-1)^n)(q^n + (-1)^{n+1})}{nq^{n-1}(q^2 - 1)} X_{2n+1}. \quad \square$$

Define $\mathring{L}_{4n}$ by the following, recalling that $a = \sqrt{q/2}$, $b = i\sqrt{q^{-1}/2}$:

$$\sum_{n=1}^{\infty} 2\theta \mathring{L}_{4n} u^{4n} = \sum_{n=1}^{\infty} 2\theta \bar{L}_{4n} u^{4n} + \frac{1}{2} \log(1 - (2\theta)^2 \mathbb{Z}(a)^2) + \frac{1}{2} \log(1 - (2\theta)^2 \mathbb{Z}(b)^2).$$

Proposition 4.6. *The element $\mathring{L}_{4n}$ is primitive in the subquotient $K_{\geq 1}/K_{> 1}$. The leading term of $[X_1, \mathring{L}_{4n}]$ is $\frac{[4n]_q}{4n} X_{4n+1}$, which means $\mathring{L}_{4n}$ is not 0 in $B(\mathbb{V})$ when q is not a root of unity.*

Proof. Recall $\Delta(\tilde{L}(u)) = *(u)$, $\Delta(\tilde{L}(iu)) = *(iu)$, so we get $\Delta(\sum_{n=1}^{\infty} 2\theta \bar{L}_{4n} u^{4n}) = \log(*(u) * (iu))$. We have

$$\begin{aligned} *(u) * (iu) &= \#(u) \exp(\bar{L}(u) \otimes 1 + 1 \otimes \bar{L}(u)) \exp(\bar{L}(iu) \otimes 1 + 1 \otimes \bar{L}(iu)) \#'(iu) \\ &= \#(u) \exp\left(\sum_{n=1}^{\infty} 2\theta \bar{L}_{4n} u^{4n} \otimes 1 + 1 \otimes \sum_{n=1}^{\infty} 2\theta \bar{L}_{4n} u^{4n}\right) \#'(iu). \end{aligned}$$

Then

$$\log(*(u) * (iu)) = \sum_{n=1}^{\infty} 2\theta \bar{L}_{4n} u^{4n} \otimes 1 + 1 \otimes \sum_{n=1}^{\infty} 2\theta \bar{L}_{4n} u^{4n} + \log(\#(u) \#'(iu)),$$

since $\bar{L}_{4n}$ commutes with all M_{2m+1} by Corollary 3.23. Then by Lemma 4.4 and Proposition 4.1 we can get that $\bar{L}_{4n}$ is primitive immediately.

For the leading term of $[X_1, \bar{L}_{4n}]$, by Lemma 4.5 we have that the leading term of $[X_1, \bar{L}_{2n}]$ is

$$\frac{(q^{2n} + (-1)^n)(q^n + (-1)^{n+1})}{nq^{n-1}(q^2 - 1)} X_{2n+1}.$$

Recall that the leading term of $[X_1, M_{2n+1}^2]$ is $(-1)^n 2^{2n} X_{4n+3}$. Then we have that the leading term of $[X_1, \bar{L}_{4n}]$ is

$$\begin{aligned} &\frac{1}{2\theta} \left(2\theta \frac{(q^{4n} + 1)(q^{2n} - 1)}{2nq^{2n-1}(q^2 - 1)} + C_{4n} \left(\frac{1}{2} \log \left(1 - \left(2\theta \frac{a^2 u^2}{1 - 4a^4 u^4} \right)^2 \right) \right) \right. \\ &\quad \left. + C_{4n} \left(\frac{1}{2} \log \left(1 - \left(2\theta \frac{b^2 u^2}{1 - 4b^4 u^4} \right)^2 \right) \right) \right) X_{4n+1}. \end{aligned}$$

Then by the same technique as in Proposition 4.2 we can prove that the leading term of $[X_1, \bar{L}_{4n}]$ is $\frac{[4n]_q}{4n} X_{4n+1}$. $\square$

However, for the comultiplication of $\bar{L}_{4n+2}$, a similar calculation does not work since $\bar{L}_{4n}$ is central in $K_{\geq 1}/K_{> 1}$ but $\bar{L}_{4n+2}$ is not. We introduce new generating functions. Define

$$\mathbb{M}_I(u) = \sum_{n=0}^{\infty} M_{4n+1} u^{4n+1}, \quad \mathbb{M}_{II}(u) = \sum_{n=0}^{\infty} M_{4n+3} u^{4n+3}.$$

Sometimes we just use $\mathbb{M}_I$ and $\mathbb{M}_{II}$ for convenience. Define

$$\bar{\mathcal{L}} = \sum_{n=0}^{\infty} \theta(4n+2) \bar{L}_{4n+2} u^{4n+2}.$$

Proposition 4.7. *In the subquotient $K_{\geq 1}/K_{> 1}$, we have*

$$\Delta(\bar{\mathcal{L}}) = \bar{\mathcal{L}} \otimes 1 + 1 \otimes \bar{\mathcal{L}} + 2(\mathfrak{M}(au) + \mathfrak{M}(bu)),$$

where

$$\mathfrak{M}(u) = \frac{(-2\theta\mathbb{M}_I \otimes \mathbb{M}_I + 2\theta\mathbb{M}_{II} \otimes \mathbb{M}_{II})(1 + 2\theta\mathbb{M}_I \otimes \mathbb{M}_{II} + 2\theta\mathbb{M}_{II} \otimes \mathbb{M}_I)}{1 + (2\theta)^2 \mathbb{Z} \otimes \mathbb{Z}}.$$

Proof. We know $\bar{\mathbb{L}} = \log(\tilde{\mathbb{L}})$. Since $[\tilde{L}_{2n}, \tilde{L}_{2m}] = 0$ for any $m, n \in \mathbb{N}$, using $u \cdot \frac{d}{du}$ to act on it, we have

$$\sum_{n=1}^{\infty} \theta 2n \bar{L}_{2n} u^{2n} = \frac{u \cdot \frac{d}{du} \tilde{\mathbb{L}}}{\tilde{\mathbb{L}}}.$$

Then

$$\Delta\left(u \cdot \frac{d}{du} \bar{\mathbb{L}}\right) = \sum_{n=1}^{\infty} \theta 2n \Delta(\bar{L}_{2n}) u^{2n} = \frac{u \cdot \frac{d}{du} \Delta(\tilde{\mathbb{L}})}{\Delta(\tilde{\mathbb{L}})}.$$

We view $\Delta(\tilde{\mathbb{L}})^{-1}$ as right inverse. We know $\Delta(\tilde{\mathbb{L}}) = \#(u) \exp \bar{\mathbb{L}} \otimes \exp \bar{\mathbb{L}}$ by Proposition 4.3. Then

$$\begin{aligned} & \Delta\left(u \cdot \frac{d}{du} \bar{\mathbb{L}}\right) \\ &= \left(u \cdot \frac{d}{du} \#(u) \exp \bar{\mathbb{L}} \otimes \exp \bar{\mathbb{L}} + \#(u) \exp \bar{\mathbb{L}} \otimes \exp \bar{\mathbb{L}} \left(u \cdot \frac{d}{du} \bar{\mathbb{L}} \otimes 1 + 1 \otimes u \cdot \frac{d}{du} \bar{\mathbb{L}}\right)\right) \\ & \quad \cdot \exp(-\bar{\mathbb{L}}) \otimes \exp(-\bar{\mathbb{L}}) \#(u)^{-1} \\ &= u \cdot \frac{d}{du} \#(u) \cdot \#(u)^{-1} + \#(u) \left(u \cdot \frac{d}{du} \bar{\mathbb{L}} \otimes 1 + 1 \otimes u \cdot \frac{d}{du} \bar{\mathbb{L}}\right) \#(u)^{-1} \\ &= \left(u \cdot \frac{d}{du} \bar{\mathbb{L}} \otimes 1 + 1 \otimes u \cdot \frac{d}{du} \bar{\mathbb{L}}\right) + u \cdot \frac{d}{du} \#(u) \cdot \#(u)^{-1} \\ & \quad - \left[\left(u \cdot \frac{d}{du} \bar{\mathbb{L}} \otimes 1 + 1 \otimes u \cdot \frac{d}{du} \bar{\mathbb{L}}\right), \#(u)\right] \#(u)^{-1}. \end{aligned}$$

Let Ω be the operator $u \cdot \frac{d}{du} - [u \cdot \frac{d}{du} \bar{\mathbb{L}} \otimes 1 + 1 \otimes u \cdot \frac{d}{du} \bar{\mathbb{L}}, \cdot]$. Then we just need to calculate

$$(\diamond) \quad (\Omega \circ \#(u)) \cdot \#(u)^{-1}.$$

The calculation of $(\Omega \circ \#(u)) \cdot \#(u)^{-1}$ is very long but elementary, using the commutation relations between different $\mathbb{M}$. The result of $\Delta(\tilde{\mathfrak{L}})$ is a part of $\Delta(u \cdot \frac{d}{du} \bar{\mathbb{L}})$ by degree. We give a proof in Appendix A. $\square$

§5. A PBW basis of $B(\mathbb{V})$

In this section we determine the multiplicities of all the roots and give a PBW basis of $B(\mathbb{V})$. The main tool is the subquotient $K_{\geq 1}/K_{> 1}$. In detail, the key step is using the primitive element at certain degree in $K_{\geq 1}/K_{> 1}$. In the case of $U_v(\widehat{\mathfrak{sl}}_2)$, Drinfeld generators of the imaginary roots are exactly these primitive elements.

§5.1. $(2n + 1)\delta$

By Corollary 3.23, the only possible root vector at $(2n + 1)\delta$ is M_{2n+1} .

Proposition 5.1. *When q is not a root of unity, M_{2n+1} are root vectors.*

Proof. Let $\mathbb{O}_{2n+1}$ be the subalgebra generated by $\{M_{2i+1}, 0 \leq i \leq n\}$. By Proposition 4.1 we can get $\Delta(M_{2n+1}) \in$

$$M_{2n+1} \otimes 1 + 1 \otimes M_{2n+1} + 2\theta L_1^2 \otimes M_{2n-1} - 2\theta M_{2n-1} \otimes L_1^2 + \mathbb{O}_{2n-3} \otimes \mathbb{O}_{2n-3}.$$

We use Theorem 2.7. We have that M_{2n+1} is a root vector if and only if M_{2n+1} is a linear combination of $L_1^{k_1} M_3^{k_2} M_5^{k_3} \cdots M_{2n-1}^{k_n}$. By Proposition 4.1, the comultiplication of M_{2n+1} has term $2\theta L_1^2 \otimes M_{2n-1} - 2\theta M_{2n-1} \otimes L_1^2$ but the comultiplication of $L_1^2 M_{2n-1}$ can only have term $L_1^2 \otimes M_{2n-1} + M_{2n-1} \otimes L_1^2$. Hence, by induction we can see that M_{2n+1} must be a root vector when q is not a root of unity. $\square$

§5.2. $(4n + 2)\delta$ and $4n\delta$

5.2.1. M_{2n+1}^2 . Since M_{2n+1} is a root vector and $\chi((2n + 1)\delta, (2n + 1)\delta) = -1$, we should consider M_{2n+1}^2 .

Proposition 5.2. *The elements M_{2n+1}^2 are root vectors for $n \geq 0$ when q is not a root of unity.*

Proof. We use Theorem 2.7. Suppose M_{2n+1}^2 is not a root vector. Then the only possibility is M_{2n+1}^2 is a linear combination of products of M_{2k+1}^2 in $B(\mathbb{V})$, where $k \leq n - 1$. Equivalently, Z_{4n+2} is a linear combination of products of Z_{4k+2} where $k \leq n - 1$. We denote this linear combination by Π . Then $Z_{4n+2} - \Pi$ is 0 in $B(\mathbb{V})$ and must be primitive in $K_{\geq 1}/K_{> 1}$. Then it is easy to see the only possibility is $\Pi = 0$, since Z_{4k+2} are primitive in $K_{\geq 1}/K_{> 1}$ and are root vectors in $B(\mathbb{V})$ for $k \leq n - 1$. Then Z_{4n+2} is 0 in $B(\mathbb{V})$, but by Proposition 4.2, we know Z_{4n+2} is not 0 in $B(\mathbb{V})$, contradiction. $\square$

5.2.2. L_{2n} .

Proposition 5.3. *The elements L_{2n} are root vectors for $n \geq 1$ when q is not a root of unity.*

Proof. We prove this by induction on n . We still use Theorem 2.7.

The element L_2 is a root vector because $[L_2, L_1] = M_3$ and M_3 is a root vector.

For L_4 , suppose L_4 is not a root vector, then $L_4 = \sum k_{i_1, i_2, j_1} L_1^{i_1} M_3^{i_2} L_2^{j_1}$, or equivalently $\bar{L}_4 = \sum k'_{i_1, i_2, j_1} L_1^{i_1} M_3^{i_2} \bar{L}_2^{j_1}$. By $\Delta(\bar{L}_4)$ in the proof of Proposition 4.6

and L_2 is a root vector, $\bar{L}_2^2$ and $\bar{L}_1^2 \bar{L}_2$ cannot appear in this linear combination. Then $\bar{L}_4$ can only be a linear combination of $L_1 M_3$ and L_1^4 . Then using $\text{ad}_{\bar{L}_2}$ to act on $\bar{L}_4$, we get $M_3^2 + L_1 M_5 = 0$, a contradiction. Then $\bar{L}_4$ can only equal $k L_1^4$, or equivalently $\bar{L}_4 = k(L_1^2)^2$. Since $\bar{L}_4$ and L_1^2 are primitive in $K_{\geq 1}/K_{>1}$, then the only possibility is $\bar{L}_4 = 0$, but according to Proposition 4.6, $\bar{L}_4$ is not 0. Then L_4 must be a root vector.

Suppose it holds for n ; we consider $n+1$.

If n is even, then $2(n+1) = 4m+2$ for $m = \frac{n}{2}$. Suppose L_{4m+2} is not a root vector; then

$$L_{4m+2} = \sum k_{i_1, i_2, \dots, i_{2m+1}, j_1, \dots, j_{2m}} L_1^{i_1} M_3^{i_2} \cdots M_{4m+1}^{i_{2m+1}} L_2^{j_1} \cdots L_{4m}^{j_{2m}},$$

or equivalently

$$\bar{L}_{4m+2} = \sum k'_{i_1, i_2, \dots, i_{2m+1}, j_1, \dots, j_{2m}} L_1^{i_1} M_3^{i_2} \cdots M_{4m+1}^{i_{2m+1}} \bar{L}_2^{j_1} \cdots \bar{L}_{4m}^{j_{2m}}.$$

By Proposition 4.7 and the induction hypothesis, $\bar{L}_2, \dots, \bar{L}_{4m}$ cannot appear. Then

$$\bar{L}_{4m+2} = \sum k''_{i_1, i_2, \dots, i_{2m+1}} L_1^{i_1} M_3^{i_2} \cdots M_{4m+1}^{i_{2m+1}}.$$

By Corollary 3.23(c), $[\bar{L}_{4m+2}, L_1] = R_{4m+2} M_{4m+3}$. When q is not a root of unity, we know that R_{4m+2} is not 0 and M_{4m+3} is a root vector, contradiction. Then L_{4m+2} must be a root vector.

If n is odd, then $2(n+1) = 4m$ for $m = \frac{n+1}{2}$. Suppose L_{4m} is not a root vector; then

$$L_{4m} = \sum k_{i_1, i_2, \dots, i_{2m}, j_1, \dots, j_{2m-1}} L_1^{i_1} M_3^{i_2} \cdots M_{4m-1}^{i_{2m}} L_2^{j_1} \cdots L_{4m-2}^{j_{2m-1}}$$

or equivalently

$$\bar{L}_{4m} = \sum k'_{i_1, i_2, \dots, i_{2m}, j_1, \dots, j_{2m-1}} L_1^{i_1} M_3^{i_2} \cdots M_{4m-1}^{i_{2m}} \bar{L}_2^{j_1} \cdots \bar{L}_{4m-2}^{j_{2m-1}}.$$

For the same reason as the last paragraph, $\bar{L}_2, \dots, \bar{L}_{4m-2}$ cannot appear. Using $\text{ad}_{\bar{L}_2}$ to act on $\bar{L}_{4m}$, similar to the case of L_4 we get

$$\bar{L}_{4m} = \sum k''_{i_1, i_2, \dots, i_m} L_1^{i_1} M_3^{i_2} \cdots M_{2m-1}^{i_m},$$

where $i_1, i_2, \dots, i_m$ must be even. Then equivalently

$$\bar{L}_{4m} = \sum k'''_{i_1, i_2, \dots, i_m} (Z_2)^{i_1/2} (Z_6)^{i_2/2} \cdots (Z_{4m-2})^{i_m/2}.$$

Since $\bar{L}_{4m}, Z_2, Z_6, \dots, Z_{4m-2}$ are primitive in $K_{\geq 1}/K_{>1}$, the only possibility is $\bar{L}_{4m} = 0$, but by Proposition 4.6, $\bar{L}_{4m}$ is not 0. Then L_{4m} must be a root vector. $\square$

Now we have the following theorem immediately.

Theorem 5.4. *Suppose q is not a root of unity, the multiplicities of $4n\delta$ and $(2n+1)\delta$ are 1, of $(4n+2)\delta$ is 2, which coincide with the root multiplicities of the Lie superalgebra $A(0, 2)^{(4)}$. The set of decreasing ordered products of elements in $\{X_n, Y_n, L_{2n}, M_{2n-1}, n \in \mathbb{N}\}$ forms a PBW basis of $B(\mathbb{V})$. The order in $\{X_n, Y_n, L_{2n}, M_{2n-1}, n \in \mathbb{N}\}$ is*

$$\begin{aligned} X_1 &< X_2 < \cdots < X_n < \cdots \\ &< \cdots < L_{2n} < L_{2n-2} < L_{2n-4} < \cdots < L_2 \\ &< \cdots < M_{2n-1} < M_{2n-3} < \cdots < M_5 < M_3 < M_1 \\ &< \cdots < Y_n < \cdots < Y_2 < Y_1. \end{aligned}$$

§6. The defining relations of $B(\mathbb{V})$ satisfied by the Chevalley generators

We first consider the commutation relations in $B(\mathbb{V})$ between real root vectors. By definition, $[X_m, Y_n]$ is just L_{m+n-1} or L'_{m+n-1} , so we do not need to care about this case. We only care about the relations between X_n and X_m, Y_n and Y_m . First, we recall the $U_v^+(\widehat{\mathfrak{sl}}_2)$ case because they have some common properties.

Proposition 6.1. *In the case of $U_v^+(\widehat{\mathfrak{sl}}_2)$ when v is not a root of unity, we have $[X_n, X_{n+r}] = -[X_{n+r-1}, X_{n+1}]$ and $[Y_{n+r}, Y_n] = -[Y_{n+1}, Y_{n+r-1}]$ for $n, r > 0$. Moreover, these two relations can be derived from quantum Serre relations.*

Proof. They are part of the relations in the Drinfeld second realization; see [CP91]. The second conclusion can be derived from [HK08], or follows from a similar proof to Lemma 6.2 below. $\square$

Lemma 6.2. *We have the following relations about X_n and Y_n in $B(\mathbb{V})$:*

$$[X_n, X_{n+1}] = 0 \quad \text{and} \quad [Y_{n+1}, Y_n] = 0,$$

for $n \geq 1$. Moreover, these relations are derived by quantum Serre relations.

Proof. We deal with the first one; the second one is similar. In fact, we claim

$$(6.1a) \quad [X_n, X_{n+2}] = (-1)^{n+1} [X_{n+1}, X_{n+1}],$$

$$(6.1b) \quad [X_n, X_{n+3}] = -[X_{n+2}, X_{n+1}],$$

$$(6.1c) \quad [X_{n+1}, X_{n+2}] = 0,$$

for $n > 0$. We prove this claim by induction on n . First, $[X_1, X_2] = 0$ holds, which is one of the quantum Serre relations. When $n = 1$,

$$0 = [[X_1, X_2], L_1] = [X_1, X_3] - [X_2, X_2].$$

Then (6.1a) holds. Then

$$0 = [[X_1, X_3] - [X_2, X_2], L_1] = [X_1, X_4] + [X_2, X_3] - ([X_2, X_3] - [X_3, X_2]),$$

i.e.,

$$[X_1, X_4] = -[X_3, X_2],$$

which is (6.1b). We also have

$$\begin{aligned} 0 &= [[X_1, X_2], L_2] = [X_1, qX_4 + \theta L_1 X_3] + [(q+2)X_3 - \theta L_1 X_2, X_2] \\ &= q[X_1, X_4] + \theta(X_2 X_3 + L_1[X_1, X_3]) + (q+2)[X_3, X_2] \\ &\quad - \theta(L_1[X_2, X_2] - qX_3 X_2), \end{aligned}$$

which implies $[X_2, X_3] = 0$, which is (6.1c).

For the general case, the induction step follows from a completely similar argument to the case of $n = 1$, using the facts that

$$[X_{2k+1}, L_2] = (q+2)X_{2k+3} - \theta L_1 X_{2k+2}$$

and

$$[X_{2k+2}, L_2] = qX_{2k+4} + \theta L_1 X_{2k+3}.$$

Note that all of these formulas come from quantum Serre relations. $\square$

Then the following corollary holds immediately.

Corollary 6.3. *All $[X_n, X_m]$ and $[Y_n, Y_m]$ can be expressed as a linear combination of decreasing products of root vectors by quantum Serre relations.*

Remark 6.4. Note that the commutation relations between X_n and Y_m come from the definition of L_{m+n-1} and L'_{m+n-1} and Proposition 3.6, which is derived by quantum Serre relations. Then we get that the commutation relations between real root vectors all come from quantum Serre relations.

Theorem 6.5. *Suppose q is not a root of unity. The defining relations of $B(\mathbb{V})$ satisfied by the Chevalley generators are the quantum Serre relations, which means that the Drinfeld–Jimbo-type realization is obtained for $U_v(\mathcal{A}(0, 2)^{(4)})$.*

Proof. We determined all the root vectors of $B(\mathbb{V})$ in Theorem 5.4. Then it is equivalent to prove that all the commutation relations between these root vectors can be derived from quantum Serre relations. All of these relations are in Section 3 and at the beginning of Section 6. As stated in Remarks 3.9, 3.19, 3.24, 6.4, we get these commutation relations almost completely from quantum Serre relations, except Proposition 3.18.

We prove that all the commutation relations in Proposition 3.18 can be derived from quantum Serre relations. We only prove the first half because the other half is completely similar. We need to show that the expressions for $[X_2, L_{2n}]$ and $[X_2, L_{2n+1}]$ for $n \geq 1$ in Proposition 3.18, which are obtained by applying differential operators, can be derived only by quantum Serre relations. Note that relations in Proposition 3.6 all come from quantum Serre relations, and we know (3.15):

$$2L'_{2n+1} = \theta L'_{2n} L_1 + \theta L_1 L_{2n} + [L_2, L'_{2n-1}]$$

holds. Applying ad_{X_1} to this equation, we will get the result of $(q + q^{-1})[X_2, L_{2n}]$ and since q is not a root of unity, we will get the expression of $[X_2, L_{2n}]$ in terms of ordered products of root vectors, repeatedly using the induction hypothesis and the following known relations derived by quantum Serre relations: (3.12), (3.15), (3.16), (3.17), (3.18) and Corollary 3.8. On the other hand, also by these relations, it is enough to get the expression for $[X_2, L_{2n}]$ in terms of ordered products of root vectors from

$$[X_2, L_{2n}] = q^{2n-1} X_{2n+2} + q^{2n-2} \theta L'_1 X_{2n+1} + \cdots + q \theta L_{2n-2} X_4 + \theta L'_{2n-1} X_3.$$

Then these two expressions must be the same. This means the expression for $[X_2, L_{2n}]$ in Proposition 3.18 also comes from quantum Serre relations. The case of $[X_2, L_{2n+1}]$ is completely similar. $\square$

§7. The Drinfeld second realization of $U_v^+(A(0, 2)^{(4)})$

We give the Drinfeld second realization of $B(\mathbb{V})$ or $U_v^+(A(0, 2)^{(4)})$, i.e., choosing distinguished root vectors and writing all the commutation relations. The following three subsections are titled by “I&I”, “I&R”, “R&R”, where “I”, “R” mean distinguished imaginary root vectors and real root vectors, respectively.

§7.1. I&I

Proposition 7.1. *The following commutation relations hold in $B(\mathbb{V})$:*

- (a) $\bar{L}_{4n+4}$ and Z_{4n+2} commute with all imaginary root vectors, $[\bar{L}_{4m+2}, \bar{L}_{4n+2}] = 0$, for all $m, n \geq 0$.
- (b) $[M_{2m+1}, M_{2n+1}] = 0$ if $m + n$ is odd, $[M_{2m+1}, M_{2n+1}] = (-1)^{\frac{m-n}{2}} 2M_{m+n+1}^2$ if $m + n$ is even.
- (c) $[(4m+2)\bar{L}_{4m+2}, M_{2n+1}] = \frac{1}{2^{2m-1}} [2m+1]_q M_{4m+2n+3}$.

Proof. This holds immediately by Corollary 3.23. $\square$

§7.2. I&R

First, we show that the comultiplication in $K_{\geq 1}/K_{> 1}$ is closely related to these relations.

Denote the subalgebra generated by L_n , $n \geq 1$ in $B(\mathbb{V})$ by $\mathcal{L}$. We identify $\mathcal{L}$ with the subquotient $K_{\geq 1}/K_{> 1}$. Denote the subalgebras generated by X_n , $n \geq 1$, Y_n , $n \geq 1$ by $\mathcal{X}$ and $\mathcal{Y}$. We have $B(\mathbb{V}) \cong \mathcal{Y} \otimes \mathcal{L} \otimes \mathcal{X}$, as vector spaces.

Let $T \in \mathcal{L}$ and suppose that T is homogeneous. Define $d(T)$ by $\deg T = d(T)\delta$. Define $f_{X_k}(1) = X_k$ and $f_{X_k}(T)$, $T \in \mathcal{L}$ by the leading term of $[X_k, T]$, i.e., some scale of $X_{k+d(T)}$. Define $g_{Y_k}(1) = Y_k$ and $g_{Y_k}(T)$, $T \in \mathcal{L}$ by the leading term of $[T, Y_k]$, i.e., some scale of $Y_{k+d(T)}$.

Remark 7.2. It is very easy to determine f_{X_k} , g_{Y_k} precisely for any $k \geq 1$ by Proposition 3.18, Lemma 3.1(a) and L_1^2 lies in the center of $\mathcal{L}$.

Lemma 7.3. *We have $\Delta(X_n) \in X_n \otimes 1 + 1 \otimes X_n + B(\mathbb{V}) \otimes \mathcal{Y}\mathcal{L}$, i.e., no X_i appears in the right tensor factor for $i < n$.*

Proof. By the comultiplication of the super-letters, i.e., Proposition 2.4. □

Lemma 7.4. *Let $T \in \mathcal{L}$. We have $\Delta(T) \in T_{(1)} \otimes T_{(2)}$ in the subquotient $+\mathcal{L}\mathcal{X} \otimes \mathcal{Y}\mathcal{L}$.*

Proof. This also follows from Proposition 2.4. □

Proposition 7.5. *Let $T \in \mathcal{L}$. We have*

$$X_k T = T_{(1)} (-1)^{d(T_{(1)}) \cdot (k-1)} f_{X_k}(T_{(2)})$$

and

$$T Y_k = (-1)^{k \cdot d(T_{(2)})} g_{Y_k}(T_{(1)}) T_{(2)}.$$

Proof. We prove the first one. It is easy to see for $T \in \mathcal{L}$,

$$(7.1) \quad X_k T = \sum_{i=1}^{k+d(T)} k_i T_i X_i, \quad \text{for some } T_i \in \mathcal{L}.$$

Consider the terms of type $B(\mathbb{V}) \otimes X_i$ for some i , in the comultiplication of (7.1).

By Lemmas 7.3, 7.4, for the left-hand side, all terms of this type are $T_{(1)} (-1)^{d(T_{(1)}) \cdot (k-1)} \otimes f_{X_k}(T_{(2)})$. For the right-hand side, all terms of this type are $k_i T_i \otimes X_i$. Then we have

$$X_k T = T_{(1)} (-1)^{d(T_{(1)}) \cdot (k-1)} f_{X_k}(T_{(2)}).$$

The second one is similar. □

For M_{2n+1} , it is more convenient to write the relations in the language of generating functions. Recall that $\mathbb{M}(u) = \sum_{n=0}^{\infty} M_{2n+1} u^{2n+1}$.

Proposition 7.6. *We have*

$$X_k \mathbb{M} = \mathbb{M}_{(1)} (-1)^{d(\mathbb{M}_{(1)}) \cdot (k-1)} f_{X_k}(\mathbb{M}_{(2)})$$

and

$$\mathbb{M} Y_k = (-1)^{k \cdot d(\mathbb{M}_{(2)})} g_{Y_k}(\mathbb{M}_{(1)}) \mathbb{M}_{(2)},$$

for any $k \geq 1$, where

$$\mathbb{M}_{(1)} \otimes \mathbb{M}_{(2)} = (\mathbb{M} \otimes 1 + 1 \otimes \mathbb{M}) \frac{1}{1 - 2\theta \mathbb{M} \otimes \mathbb{M}}.$$

Proof. By Propositions 7.5 and 4.1. □

Recall that

$$\arctan(2\theta i \mathbb{Z}(u)) = 2\theta i \mathcal{Z}(u).$$

We write $\mathcal{Z}(u)$ as $\sum_{n=0}^{\infty} Z_{4n+2} u^{4n+2}$.

Proposition 7.7. *In $B(\mathbb{V})$ we have the following formulas for all $k \geq 1$, $n \geq 0$:*

$$[X_k, Z_{4n+2}] = 2^{2n} \frac{[2n+1]_q}{2n+1} X_{4n+k+2}, \quad [Z_{4n+2}, Y_k] = 2^{2n} \frac{[2n+1]_q}{2n+1} Y_{4n+k+2}.$$

Proof. The first formula follows immediately from Propositions 4.2 and 7.5, and that Z_{4n+2} lie in the center of $K_{\geq 1}/K_{>1}$. The second is completely similar. □

Proposition 7.8. *In $B(\mathbb{V})$, we have the following relations:*

$$[X_k, \mathring{L}_{4n}] = \frac{[4n]_q}{4n} X_{4n+k}, \quad [\mathring{L}_{4n}, Y_k] = \frac{[4n]_q}{4n} Y_{4n+k}.$$

Proof. This follows immediately from Propositions 4.6 and 7.5. □

For $(4n+2)\bar{L}_{4n+2}$, it is more convenient to write the relations in the language of generating functions. Recall that

$$\begin{aligned} \bar{\mathcal{L}} &= \sum_{n=0}^{\infty} \theta(4n+2) \bar{L}_{4n+2} u^{4n+2}, \\ \mathbb{M}_{\text{I}}(u) &= \sum_{n=0}^{\infty} M_{4n+1} u^{4n+1}, \quad \mathbb{M}_{\text{II}}(u) = \sum_{n=0}^{\infty} M_{4n+3} u^{4n+3}, \\ \mathcal{Z}(u) &= \sum_{n=0}^{\infty} M_{2n+1}^2 (-1)^n u^{4n+2} = \mathbb{M}(u)^2. \end{aligned}$$

Proposition 7.9. *We have*

$$X_k \bar{\mathfrak{L}} = \bar{\mathfrak{L}}_{(1)} (-1)^{d(\bar{\mathfrak{L}}_{(1)}) \cdot (k-1)} f_{X_k}(\bar{\mathfrak{L}}_{(2)})$$

and

$$\bar{\mathfrak{L}} Y_k = (-1)^{k \cdot d(\bar{\mathfrak{L}}_{(2)})} g_{Y_k}(\bar{\mathfrak{L}}_{(1)}) \bar{\mathfrak{L}}_{(2)},$$

where

$$\bar{\mathfrak{L}}_{(1)} \otimes \bar{\mathfrak{L}}_{(2)} = \Delta(\bar{\mathfrak{L}}) = \bar{\mathfrak{L}} \otimes 1 + 1 \otimes \bar{\mathfrak{L}} + 2(\mathfrak{M}(au) + \mathfrak{M}(bu))$$

and

$$\mathfrak{M}(u) = \frac{(-2\theta \mathbb{M}_I \otimes \mathbb{M}_I + 2\theta \mathbb{M}_{II} \otimes \mathbb{M}_{II})(1 + 2\theta \mathbb{M}_I \otimes \mathbb{M}_{II} + 2\theta \mathbb{M}_{II} \otimes \mathbb{M}_I)}{1 + (2\theta)^2 \mathbb{Z} \otimes \mathbb{Z}}.$$

Proof. This follows from Propositions 7.5 and 4.7. $\square$

§7.3. R&R

7.3.1. Relations between X_n , and between Y_n . Define a sequence of numbers $s_0 = 1$, $s_1 = -1$, $s_2 = -1$, $s_3 = 1$ and $s_n = s_{n+4}$ for $n \geq 0$. Define a map $P: \mathbb{N} \rightarrow \{0, 1\} \subset \mathbb{N}$ by $P(n) = 0$ if n is even, $P(n) = 1$ if n is odd.

Proposition 7.10. *In $B(\mathbb{V})$ the following relations hold for $n, k \geq 1$:*

$$\begin{aligned} (\dagger) \quad & [X_n, X_{n+2k-1}] = -[X_{n+2k-2}, X_{n+1}], \\ & [X_n, X_{n+2k}] = s_{P(n+1)} [X_{n+2k-1}, X_{n+1}] + 2 \sum_{i=2}^k s_{P(n+1)+i-1} [X_{n+2k-i}, X_{n+i}], \\ & [Y_{n+2k-1}, Y_n] = -[Y_{n+1}, Y_{n+2k-2}], \\ & [Y_{n+2k}, Y_n] = s_{P(n)} [Y_{n+1}, Y_{n+2k-1}] + 2 \sum_{i=2}^k s_{P(n)+i-1} [Y_{n+i}, Y_{n+2k-i}]. \end{aligned}$$

Proof. By applying ad_{L_1} to the formulas in Lemma 6.2 repeatedly. $\square$

7.3.2. The relations between X_n and Y_m . First we have

$$\begin{aligned} [X_{2i+1}, Y_{2n+2-(2i+1)}] &= L_{2n+1}, & [X_{2i}, Y_{2n+2-2i}] &= L'_{2n+1}, \\ [X_{2i+1}, Y_{2n+3-(2i+1)}] &= L_{2n+2}, & [X_{2i}, Y_{2n+3-2i}] &= L'_{2n+2}. \end{aligned}$$

However, we do not use L_n or L'_n as generators, so we should use our generators to generate L_n or L'_n .

We have

$$\begin{aligned}\sum_{n=1}^{\infty} 2\theta \mathring{L}_{4n} u^{4n} &= \sum_{n=1}^{\infty} 2\theta \bar{L}_{4n} u^{4n} + \frac{1}{2} \log(1 - (2\theta)^2 \mathbb{Z}(a)^2) + \frac{1}{2} \log(1 - (2\theta)^2 \mathbb{Z}(b)^2), \\ &= \sum_{n=1}^{\infty} 2\theta \bar{L}_{4n} u^{4n} - \log \circ \cos(2\theta i \mathcal{Z}(a)) - \log \circ \cos(2\theta i \mathcal{Z}(b)),\end{aligned}$$

and $\mathring{L}_{4n}$, $\mathcal{Z}(u)$, $\bar{L}_{4n+2}$ are the generators we choose; then we get $\bar{\mathbb{L}}(u)$ from our generators. By $\tilde{\mathbb{L}}(u) = \exp(\bar{\mathbb{L}}(u))$, we can get $\tilde{\mathbb{L}}(u)$ from our generators.

Proposition 7.11. *In $B(\mathbb{V})$ we have*

$$\begin{aligned}(\ddagger) \quad \sum_{n=0}^{\infty} \theta L_{2n+1} &= \frac{\theta}{\theta'} ((2a - a^{-1})\mathbb{M}(a) - (2b - b^{-1})\mathbb{M}(b)) \tilde{\mathbb{L}}(u), \\ \sum_{n=0}^{\infty} \theta L'_{2n+1} &= \frac{\theta}{\theta'} ((2a + a^{-1})\mathbb{M}(a) - (2b + b^{-1})\mathbb{M}(b)) \tilde{\mathbb{L}}(u), \\ \sum_{n=0}^{\infty} \theta L_{2n} &= \left(1 - \frac{4\theta}{\theta'} \mathbb{Z}(b) - \frac{2\theta^2}{\theta'} i\mathbb{M}(b)\mathbb{M}(a)\right) \tilde{\mathbb{L}}(u), \\ \sum_{n=0}^{\infty} \theta L'_{2n} &= \left(1 + \frac{4\theta}{\theta'} \mathbb{Z}(b) + \frac{2\theta^2}{\theta'} i\mathbb{M}(b)\mathbb{M}(a)\right) \tilde{\mathbb{L}}(u).\end{aligned}$$

Proof. By Lemma 3.1 we get

$$\begin{aligned}\sum_{n=0}^{\infty} \theta L_{2n+1} &= \sum_{n=0}^{\infty} \theta \tilde{L}_{2n+1} - \frac{1}{2} \sum_{n=0}^{\infty} \theta \hat{L}_{2n+1}, \\ \sum_{n=0}^{\infty} \theta L'_{2n+1} &= \sum_{n=0}^{\infty} \theta \tilde{L}_{2n+1} + \frac{1}{2} \sum_{n=0}^{\infty} \theta \hat{L}_{2n+1}, \\ \sum_{n=1}^{\infty} \theta L_{2n} &= \sum_{n=1}^{\infty} \theta \tilde{L}_{2n} + \frac{1}{2} \sum_{n=1}^{\infty} \theta \hat{L}_{2n}, \\ \sum_{n=1}^{\infty} \theta L'_{2n} &= \sum_{n=1}^{\infty} \theta \tilde{L}_{2n} - \frac{1}{2} \sum_{n=1}^{\infty} \theta \hat{L}_{2n}.\end{aligned}$$

Meanwhile, by Lemma A.1 in Appendix A, we have

$$[\mathbb{M}(a), \mathbb{M}(b)] = \frac{-2}{\theta i} (\mathbb{Z}(a) + \mathbb{Z}(b)).$$

Then the proposition holds by direct computation using Proposition 3.17. □

§7.4. The Drinfeld second realization

We summarize all the above results as a theorem.

Let $\theta = q - q^{-1}$, $\theta' = q + q^{-1}$, $a = \sqrt{q/2}$, $b = i\sqrt{q^{-1}/2}$.

We choose X_n , Y_n for $n > 0$, M_{2n+1} , Z_{4n+2} , $(4n+2)\bar{L}_{4n+2}$, $(4n+4)\mathring{L}_{4n+4}$ for $n \geq 0$ as root vectors at the corresponding root. For simplicity of notation, we denote $(4n+2)\bar{L}_{4n+2}$ by $\mathcal{L}_{4n+2}$, $4n\mathring{L}_{4n}$ by $\mathcal{L}_{4n}$. We introduce the following generating functions with root vectors as coefficients:

$$\begin{aligned}\mathbb{M}(u) &:= \sum_{n=0}^{\infty} M_{2n+1} u^{2n+1}, \quad \mathbb{M}_I(u) = \sum_{n=0}^{\infty} M_{4n+1} u^{4n+1}, \quad \mathbb{M}_{II}(u) = \sum_{n=0}^{\infty} M_{4n+3} u^{4n+3}, \\ \mathcal{Z} &:= \sum_{n=0}^{\infty} Z_{4n+2} u^{4n+2}, \quad \bar{\mathcal{L}} := \sum_{n=0}^{\infty} \theta \mathcal{L}_{4n+2} u^{4n+2}, \quad \mathring{\mathcal{L}} := \sum_{n=1}^{\infty} \theta \mathcal{L}_{4n} u^{4n}.\end{aligned}$$

We use $\int u^{-1}$ to denote the inverse operator of $u \cdot \frac{d}{du}$.

We define two new generating functions as follows:

$$\begin{aligned}\mathbb{Z}(u) &:= \sum_{n=0}^{\infty} M_{2n+1}^2 (-1)^n u^{4n+2}, \\ \tilde{\mathbb{L}}(u) &:= \exp \left(\int u^{-1} \bar{\mathcal{L}} + \int u^{-1} \mathring{\mathcal{L}} + \frac{1}{2} \log \circ \cos(2\theta i \mathcal{Z}(a)) + \frac{1}{2} \log \circ \cos(2\theta i \mathcal{Z}(b)) \right).\end{aligned}$$

Theorem 7.12. *The algebra $B(\mathbb{V})$ or equivalently $U_v^+(\mathbf{A}(0, 2)^{(4)})$ is generated by real root vectors X_n , Y_n for $n > 0$, imaginary root vectors M_{2n+1} , Z_{4n+2} , $\mathcal{L}_{4n+2}$ for $n \geq 0$, $\mathcal{L}_{4n}$ for $n > 0$, with the following defining relations:*

I&I: Z_{4n+2} , $\mathcal{L}_{4n}$ commute with all the imaginary root vectors,

$$\begin{aligned}[\mathcal{L}_{4m+2}, \mathcal{L}_{4n+2}] &= 0, \quad \forall m, n \geq 0, \\ [\mathcal{L}_{4m+2}, M_{2n+1}] &= \frac{[2m+1]_q}{2^{2m-1}} M_{4m+2n+3}, \\ [M_{2m+1}, M_{2n+1}] &= (-1)^{\frac{m-n}{2}} 2M_{m+n+1}^2 \quad \text{if } m+n \text{ is even,} \\ [M_{2n+1}, M_{2m+1}] &= 0 \quad \text{if } m+n \text{ is odd,} \\ 2\theta i \mathbb{Z}(u) &= \tan(2\theta i \mathcal{Z}(u)).\end{aligned}$$

$$\begin{aligned}\text{I\&R:} \quad X_k \mathbb{M} &= \mathbb{M}_{(1)} (-1)^{d(\mathbb{M}_{(1)}) \cdot (k-1)} f_{X_k}(\mathbb{M}_{(2)}), \\ \mathbb{M} Y_k &= (-1)^{k \cdot d(\mathbb{M}_{(2)})} g_{Y_k}(\mathbb{M}_{(1)}) \mathbb{M}_{(2)},\end{aligned}$$

where

$$\begin{aligned}\mathbb{M}_{(1)} \otimes \mathbb{M}_{(2)} &= (\mathbb{M} \otimes 1 + 1 \otimes \mathbb{M}) \frac{1}{1 - 2\theta \mathbb{M} \otimes \mathbb{M}}, \\ [X_k, Z_{4n+2}] &= 2^{2n} \frac{[2n+1]_q}{2n+1} X_{4n+k+2},\end{aligned}$$

$$[Z_{4n+2}, Y_k] = 2^{2n} \frac{[2n+1]_q}{2n+1} Y_{4n+k+2},$$

$$X_k \bar{\mathfrak{L}} = \bar{\mathfrak{L}}_{(1)} (-1)^{d(\bar{\mathfrak{L}}_{(1)}) \cdot (k-1)} f_{X_k}(\bar{\mathfrak{L}}_{(2)}),$$

and

$$\bar{\mathfrak{L}} Y_k = (-1)^{k \cdot d(\bar{\mathfrak{L}}_{(2)})} g_{Y_k}(\bar{\mathfrak{L}}_{(1)}) \bar{\mathfrak{L}}_{(2)},$$

where

$$\bar{\mathfrak{L}}_{(1)} \otimes \bar{\mathfrak{L}}_{(2)} := \Delta(\bar{\mathfrak{L}}) = \bar{\mathfrak{L}} \otimes 1 + 1 \otimes \bar{\mathfrak{L}} + 2(\mathfrak{M}(au) + \mathfrak{M}(bu))$$

and

$$\mathfrak{M}(u) = \frac{(-2\theta \mathbb{M}_I \otimes \mathbb{M}_I + 2\theta \mathbb{M}_{II} \otimes \mathbb{M}_{II})(1 + 2\theta \mathbb{M}_I \otimes \mathbb{M}_{II} + 2\theta \mathbb{M}_{II} \otimes \mathbb{M}_I)}{1 + (2\theta)^2 \mathbb{Z} \otimes \mathbb{Z}},$$

$$[X_k, \mathcal{L}_{4n}] = [4n]_q X_{4n+k},$$

$$[\mathcal{L}_{4n}, Y_k] = [4n]_q Y_{4n+k},$$

and f, g are defined in Section 7.2.

$$\text{R\&R:} \quad [X_{2i+1}, Y_{2n+2-(2i+1)}] = L_{2n+1}, \quad [X_{2i}, Y_{2n+2-2i}] = L'_{2n+1},$$

$$[X_{2i+1}, Y_{2n+3-(2i+1)}] = L_{2n+2}, \quad [X_{2i}, Y_{2n+3-2i}] = L'_{2n+2},$$

together with $(\ddagger)$ in Proposition 7.11, and $(\dagger)$ in Proposition 7.10.

Appendix A. Calculations on $(\diamond)$ in the proof of Proposition 4.7

In this appendix, we show the details of the proof of $(\diamond)$ in the proof of Proposition 4.7. We will not give all the details, but we will write down two representative parts of $(\diamond)$; see Propositions A.2 and A.3.

The main tool is Lemma A.1, in which the commutation relations of various generating functions with M_{2n+1} , $n \geq 0$ as coefficients are described. Recall that $\mathbb{M}(u) = \mathbb{M}_I(u) + \mathbb{M}_{II}(u)$, define $\mathbb{M}_I(u) - \mathbb{M}_{II}(u)$ as $\bar{\mathbb{M}}(u)$ and denote $\mathbb{M}(hu)$, $\mathbb{M}_I(hu)$, $\mathbb{M}_{II}(hu)$, $\bar{\mathbb{M}}(hu)$ by $\mathbb{M}(h)$, $\mathbb{M}_I(h)$, $\mathbb{M}_{II}(h)$, $\bar{\mathbb{M}}(h)$ for convenience, where $h \in \mathbb{C}^\times$.

Lemma A.1. *We have the following relations in $B(\mathbb{V})$:*

$$(A.1a) \quad [\mathbb{M}_I(x), \mathbb{M}_{II}(y)] = 0 \quad \text{for any } x, y \in \mathbb{C},$$

$$(A.1b) \quad [\mathbb{M}(a), \mathbb{M}(b)] = [\bar{\mathbb{M}}(a), \bar{\mathbb{M}}(b)] = \frac{-2}{\theta i} (\mathbb{Z}(a) + \mathbb{Z}(b)),$$

$$(A.1c) \quad [\mathbb{M}(a), \bar{\mathbb{M}}(b)] = [\bar{\mathbb{M}}(a), \mathbb{M}(b)] = \frac{-2}{\theta' i} (\mathbb{Z}(a) - \mathbb{Z}(b)),$$

$$(A.1d) \quad u \cdot \frac{d}{du} \mathbb{Z}(u) = 2(\mathbb{M}_I(u)^2 - \mathbb{M}_{II}(u)^2) = [\mathbb{M}(u), \bar{\mathbb{M}}(u)].$$

Proof. Equation (A.1a) follows from Corollary 3.23(b). Equations (A.1b) and (A.1c) follow from (A.1a) and direct computation using Corollary 3.23(b). Equation (A.1d) also follows from direct computation using Corollary 3.23(b). $\square$

Recall that by Lemma 4.4,

$$\#(u)\#'(iu) = (1 + (2\theta)^2\mathbb{Z}(a) \otimes \mathbb{Z}(a)) \cdot (1 + (2\theta)^2\mathbb{Z}(b) \otimes \mathbb{Z}(b)).$$

Denote $(1 + (2\theta)^2\mathbb{Z}(a) \otimes \mathbb{Z}(a)) \cdot (1 + (2\theta)^2\mathbb{Z}(b) \otimes \mathbb{Z}(b))$ by $\mathcal{C}$. Then $(\diamond)$ is

$$\frac{\Omega\#(u) \cdot \#'(iu)}{\mathcal{C}},$$

where

$$\begin{aligned} \#(u) &= \left(1 - A \otimes A + \frac{1}{2}B \otimes \frac{1}{2}B \right. \\ &\quad \left. + \left(\theta^{-1} \left(\frac{1}{4}B^2 + A^2 \right) + A \cdot \frac{1}{2}B \right) \otimes \left(\theta^{-1} \left(\frac{1}{4}B^2 + A^2 \right) + A \cdot \frac{1}{2}B \right) \right), \\ \#'(iu) &= \left(1 + A \otimes A - \frac{1}{2}B \otimes \frac{1}{2}B \right. \\ &\quad \left. + \left(\theta^{-1} \left(\frac{1}{4}B^2 + A^2 \right) - A \cdot \frac{1}{2}B \right) \otimes \left(\theta^{-1} \left(\frac{1}{4}B^2 + A^2 \right) - A \cdot \frac{1}{2}B \right) \right). \end{aligned}$$

For convenience, we denote $\theta^{-1}(\frac{1}{4}B^2 + A^2)$ by T and denote the operator $u \cdot \frac{d}{du} - [u \cdot \frac{d}{du} \bar{\mathbb{L}}(u), \cdot]$ by $\star$.

Proposition A.2. *We have*

$$\begin{aligned} \Omega \left(A \otimes A - \frac{1}{2}B \otimes \frac{1}{2}B \right) \\ = 4\theta(\mathbb{M}_I(a) \otimes \mathbb{M}_I(a) - \mathbb{M}_{II}(a) \otimes \mathbb{M}_{II}(a) + \mathbb{M}_I(b) \otimes \mathbb{M}_I(b) - \mathbb{M}_{II}(b) \otimes \mathbb{M}_{II}(b)). \end{aligned}$$

Proof. We have

$$\Omega \left(A \otimes A - \frac{1}{2}B \otimes \frac{1}{2}B \right) = \star A \otimes A + A \otimes \star A - \left(\star \frac{1}{2}B \otimes \frac{1}{2}B + \frac{1}{2}B \otimes \star \frac{1}{2}B \right).$$

First we focus on $\star A$.

Denote $\text{ad}_{L_2(xu)^2}$ by ad_x for $x \in \mathbb{C}$. By Corollary 3.23(c), we have

$$\left[u \cdot \frac{d}{du} \bar{\mathbb{L}}(u), L_1 \right] = 4 \left(\frac{\text{ad}_a}{1 - \text{ad}_a^2} + \frac{\text{ad}_b}{1 - \text{ad}_b^2} \right) \circ L_1,$$

which means

$$\left[u \cdot \frac{d}{du} \bar{\mathbb{L}}(u), A(u) \right] = 4 \left(\frac{\text{ad}_a}{1 - \text{ad}_a^2} + \frac{\text{ad}_b}{1 - \text{ad}_b^2} \right) \circ A(u).$$

Meanwhile, it is easy to get

$$u \cdot \frac{d}{du} A(u) = \frac{2\theta}{\theta'} \left(a \frac{1 + \text{ad}_a}{1 - \text{ad}_a} \circ \mathbb{M}(a) - b \frac{1 + \text{ad}_b}{1 - \text{ad}_b} \circ \mathbb{M}(b) \right).$$

Then

$$\begin{aligned} \star A &= \left(\frac{1 + \text{ad}_a}{1 - \text{ad}_a} - 4 \frac{\text{ad}_a}{1 - \text{ad}_a^2} - 4 \frac{\text{ad}_b}{1 - \text{ad}_b^2} \right) \frac{2\theta}{\theta'} a \mathbb{M}(a) \\ &\quad - \left(\frac{1 + \text{ad}_b}{1 - \text{ad}_b} - 4 \frac{\text{ad}_a}{1 - \text{ad}_a^2} - 4 \frac{\text{ad}_b}{1 - \text{ad}_b^2} \right) \frac{2\theta}{\theta'} b \mathbb{M}(b). \end{aligned}$$

We also have

$$a \mathbb{M}(a) = \frac{a^2}{1 - \text{ad}_a} \circ L_1 u, \quad b \mathbb{M}(b) = \frac{b^2}{1 - \text{ad}_b} \circ L_1 u.$$

Then we have

$$\begin{aligned} \star A(u) &= A'(u) - \frac{2\theta}{\theta'} \left(\frac{4\text{ad}_b}{1 - \text{ad}_b^2} \frac{a^2}{1 - \text{ad}_a} \circ L_1 u - \frac{4\text{ad}_a}{1 - \text{ad}_a^2} \frac{b^2}{1 - \text{ad}_b} \circ L_1 u \right) \\ &= A'(u) + \frac{4}{\theta'} (a^{-1} \mathbb{M}_I(a) - b^{-1} \mathbb{M}_I(b)). \end{aligned}$$

Similarly we have

$$\star \frac{1}{2} B(u) = -\frac{1}{2} B'(u) + \frac{8}{\theta'} (a \mathbb{M}_{II}(a) - b \mathbb{M}_{II}(b)).$$

Note that $\mathbb{M}_I(u) = \frac{\mathbb{M}(u) + \bar{\mathbb{M}}(u)}{2}$, $\mathbb{M}_{II}(u) = \frac{\mathbb{M}(u) - \bar{\mathbb{M}}(u)}{2}$. Then we have

$$\begin{aligned} \star A &= \frac{2}{\theta'} (a \theta' \bar{\mathbb{M}}(a) + b \theta' \bar{\mathbb{M}}(b) + a^{-1} \mathbb{M}(a) - b^{-1} \mathbb{M}(b)) \\ \star B &= \frac{2}{\theta'} (-a^{-1} \theta' \bar{\mathbb{M}}(a) - b^{-1} \theta' \bar{\mathbb{M}}(b) + 4a \mathbb{M}(a) - 4b \mathbb{M}(b)) \\ A &= \frac{2\theta}{\theta'} (a \mathbb{M}(a) - b \mathbb{M}(b)), \quad \frac{1}{2} B = \frac{\theta}{\theta'} (a^{-1} \mathbb{M}(a) - b^{-1} \mathbb{M}(b)). \end{aligned}$$

By direct computation with these formulas, we get $\Omega(A \otimes A - \frac{1}{2} B \otimes \frac{1}{2} B)$. $\square$

Proposition A.3. *We have*

$$\begin{aligned} &\left(\star \left(T + A \cdot \frac{1}{2} B \right) \otimes \left(T + A \cdot \frac{1}{2} B \right) + \left(T + A \cdot \frac{1}{2} B \right) \otimes \star \left(T + A \cdot \frac{1}{2} B \right) \right) \\ &\quad \cdot \left(T - A \cdot \frac{1}{2} B \right) \otimes \left(T - A \cdot \frac{1}{2} B \right) \end{aligned}$$

$$\begin{aligned}
&= (2\theta)^4 \left(\frac{1}{2}u \cdot \frac{d}{du} \mathbb{Z}(b) \otimes \mathbb{Z}(b) \cdot \mathbb{Z}(a) \otimes \mathbb{Z}(a) \right. \\
&\quad + \frac{1}{2}u \cdot \frac{d}{du} \mathbb{Z}(a) \otimes \mathbb{Z}(a) \cdot \mathbb{Z}(b) \otimes \mathbb{Z}(b) \\
&\quad + 2\mathbb{M}_I(b)\mathbb{M}_{II}(b) \otimes \mathbb{Z}(b) \cdot \mathbb{Z}(a) \otimes \mathbb{Z}(a) \\
&\quad \left. + 2\mathbb{M}_I(a)\mathbb{M}_{II}(a) \otimes \mathbb{Z}(a) \cdot \mathbb{Z}(b) \otimes \mathbb{Z}(b) \right) \\
&\quad + \text{symmetric part.}
\end{aligned}$$

Remark A.4. The terms $\frac{1}{2}u \cdot \frac{d}{du} \mathbb{Z}(b) \otimes \mathbb{Z}(b) \cdot \mathbb{Z}(a) \otimes \mathbb{Z}(a)$, $\frac{1}{2}u \cdot \frac{d}{du} \mathbb{Z}(a) \otimes \mathbb{Z}(a) \cdot \mathbb{Z}(b) \otimes \mathbb{Z}(b)$ and the symmetric part correspond to the result of $u \cdot \frac{d}{du}$ acting on $\Delta(\sum_{n=1}^{\infty} 2\theta \bar{L}_{4n} u^{4n})$, which is known from the proof of Proposition 4.6.

Proof of Proposition A.3. By Lemma A.1 and Proposition 3.17, we have the following relations:

$$\begin{aligned}
T + A \cdot \frac{1}{2}B &= -\frac{4\theta}{\theta'} \mathbb{Z}(b) - \frac{2\theta^2}{\theta'} i\mathbb{M}(b)\mathbb{M}(a), \\
T - A \cdot \frac{1}{2}B &= \frac{4\theta}{\theta'} \mathbb{Z}(a) + \frac{2\theta^2}{\theta'} i\mathbb{M}(b)\mathbb{M}(a), \\
\star \left(T + A \cdot \frac{1}{2}B \right) &= \frac{2\theta}{\theta'} \left(2\bar{\mathbb{M}}(a)\mathbb{M}(a) - 2\bar{\mathbb{M}}(b)\mathbb{M}(b) - \frac{2\theta}{\theta'} \mathbb{Z}(a) \right. \\
&\quad \left. + \frac{2\theta}{\theta'} \mathbb{Z}(b) - i\theta\mathbb{M}(b)\bar{\mathbb{M}}(a) - i\theta\bar{\mathbb{M}}(b)\mathbb{M}(a) \right), \\
T^2 - \left(A \cdot \frac{1}{2}B \right)^2 &= -4\theta^2 \mathbb{Z}(a)\mathbb{Z}(b).
\end{aligned}$$

Then this proposition follows from direct computation. $\square$

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