

Kollár’s Package for Polystable Locally Abelian Parabolic Higgs Bundles

by

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Abstract

We generalize Kollár’s package (including torsion-freeness, injectivity theorem, vanishing theorem and decomposition theorem) to polystable locally abelian parabolic Higgs bundles twisted by a multiplier ideal sheaf associated with an $\mathbb{R}$ -divisor. This gives a uniform treatment for various kinds of Kollár’s package in different topics in complex geometry. As applications, the weak positivity (in the sense of Viehweg) and the generic vanishing property for higher direct image sheaves are deduced.

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§1. Introduction

All concepts and constructions in this paper are defined over the complex number field $\mathbb{C}$. Let $f: X \rightarrow Y$ be a proper surjective morphism from a projective variety X to a complex space Y . We say that a coherent sheaf $\mathcal{F}$ on X satisfies *Kollár’s package* with respect to f if the following statements hold:

Torsion-freeness. $R^q f_*(\mathcal{F})$ is torsion-free for every $q \geq 0$ and vanishes if $q > \dim X - \dim Y$.

Injectivity theorem. If L is a semipositive (Definition 3.5) holomorphic line bundle on X so that $L^{\otimes l}$ admits a nonzero holomorphic global section s for

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some $l > 0$, then the canonical morphism

$$R^q f_*(\times s): R^q f_*(\mathcal{F} \otimes L^{\otimes k}) \rightarrow R^q f_*(\mathcal{F} \otimes L^{\otimes(k+l)})$$

is injective for every $q \geq 0$ and every $k \geq 1$.

Vanishing theorem. If Y is a projective algebraic variety and L is an ample line bundle on Y , then

$$H^q(Y, R^p f_*(\mathcal{F}) \otimes L) = 0, \quad \forall q > 0, \forall p \geq 0.$$

Decomposition theorem. $Rf_*(\mathcal{F})$ splits in $D(Y)$ the derived category of $\mathcal{O}_Y$ -modules, that is,

$$Rf_*(\mathcal{F}) \simeq \bigoplus_q R^q f_*(\mathcal{F})[-q] \in D(Y).$$

As a consequence, the spectral sequence

$$E_2^{pq}: H^p(Y, R^q f_*(\mathcal{F})) \Rightarrow H^{p+q}(X, \mathcal{F})$$

degenerates at the E_2 page.

These statements date back to Kollár [13, 14], who proved that the dualizing sheaf ω_X satisfies Kollár's package when X is smooth and Y is projective. Kollár's results have since been generalized in two directions to aim for various geometric applications.

The first direction is Kollár's package for the dualizing sheaf twisted by a $\mathbb{Q}$ -divisor, or more generally, a multiplier ideal sheaf. This particular type of Kollár's package has significant applications in various areas of research. For example, Viehweg's work on the quasi-projective moduli of polarized manifolds [36, 37], Fujino's project on the minimal model program for log-canonical varieties [7], and the Kollár–Kovács splitting criterion for du Bois singularities [15] all make use of this type of Kollár's package. Takegoshi [34] has also established a proof of Kollár's package for the dualizing sheaf twisted by a Nakano semipositive vector bundle. In addition, Matsumura [17] and Fujino–Matsumura [8] have investigated the injectivity theorem for the dualizing sheaf twisted by a general multiplier ideal sheaf. Recently, Cao–Păun [2, 3], Chan–Choi–Matsumura [4] (also refer to [16, 17, 5]) extended Kollár's injectivity theorem to the log canonical bundle twisted by a semipositive line bundle, thus confirming a conjecture proposed by Fujino [7]. However, a complete proof of Kollár's package, especially the decomposition theorem, for the dualizing sheaf twisted by a multiplier ideal sheaf remains elusive.

The other direction is to generalize Kollár's package to certain Hodge-theoretic objects, such as variations of Hodge structure and Hodge modules. Let us assume

that $f: X \rightarrow Y$ is a morphism between projective varieties. Suppose that $\mathbb{V}$ is an $\mathbb{R}$ -polarized variation of Hodge structure on some dense Zariski open subset $X^\circ \subset X_{\text{reg}}$ of the regular locus X_{reg} . In his work, Saito [29] constructs a coherent sheaf $S_X(\mathbb{V})$ (as the lowest Hodge piece of the Hodge module $IC_X(\mathbb{V})$) and shows that $S_X(\mathbb{V})$ satisfies Kollár's package with respect to f . When $\mathbb{V}$ is the trivial variation of Hodge structure, $S_X(\mathbb{V}) \simeq \omega_X$. Saito's work provides an affirmative answer to Kollár's conjecture [14, §5]. In addition to other deep results on Hodge modules, Kollár's package for $S_X(\mathbb{V})$ proves to be instrumental in the series of works by Popa–Schnell [26, 27, 28]. Recently, the authors of this article gave an L^2 -theoretic proof of Saito's result in [31].

The purpose of the present article is to demonstrate that Kollár's package holds for specific subsheaves of a polystable parabolic Higgs bundle, which is twisted by a multiplier ideal sheaf associated with an $\mathbb{R}$ -divisor. This approach provides a unified and systematic treatment for various versions of Kollár's package. Notably, even when considering the dualizing sheaf twisted by a multiplier ideal sheaf, this package yields novel results. The main arguments rely on the L^2 -theoretic method developed by Andreotti–Vesentini [1] and Hörmander [10], as well as the nonabelian Hodge theory developed by Simpson [33] and Mochizuki [19, 20].

§1.1. Main results

Let X be a smooth projective variety and D be a simple normal crossing divisor on X . Consider a locally abelian parabolic Higgs bundle $(H, \{_E H\}_{E \in \text{Div}_D(X)}, \theta)$ on (X, D) , which consists of the following data:

- a locally abelian parabolic vector bundle $(H, \{_E H\}_{E \in \text{Div}_D(X)})$ with parabolic structures on D , where the filtration $\{_E H\}$ is indexed by the set $\text{Div}_D(X)$ of $\mathbb{R}$ -divisors whose supports lie in D ;
- a Higgs field $\theta: H|_{X \setminus D} \rightarrow H|_{X \setminus D} \otimes \Omega_{X \setminus D}$ which has regular singularities along D , meaning that $\theta(_E H) \subset _E H \otimes \Omega_X(\log D)$ for every $E \in \text{Div}_D(X)$.

This parabolic Higgs bundle is required to have vanishing first and second parabolic Chern classes and to be polystable with respect to an ample line bundle A on X . Readers may refer to Section 2 for the detailed notation regarding parabolic Higgs bundles.

According to the nonabelian Hodge theory of Simpson [32, 33] and Mochizuki [19, 21], there exists a μ_A -polystable regular parabolic flat bundle $(V, \{_E V\}_{E \in \text{Div}_D(X)}, \nabla)$ associated with $(H, \{_E H\}_{E \in \text{Div}_D(X)}, \theta)$. Furthermore, there is an isomorphism between the C^∞ complex bundles:

$$\rho: H|_{X \setminus D} \otimes_{\mathcal{O}_{X \setminus D}} \mathcal{C}_{X \setminus D}^\infty = V|_{X \setminus D} \otimes_{\mathcal{O}_{X \setminus D}} \mathcal{C}_{X \setminus D}^\infty \quad (\text{Section 2.3}).$$

In particular, the C^∞ complex bundle associated with $H|_{X \setminus D}$ has two complex structures. One is the complex structure $\bar{\partial}$ of the Higgs bundle $H|_{X \setminus D}$, and the other is the complex structure $\nabla^{0,1}$ (the $(0,1)$ -part of ∇) of the flat bundle $V|_{X \setminus D}$.

Let $K \subset H|_{X \setminus D}$ be a subsheaf, and let $j: X \setminus D \rightarrow X$ denote the open immersion. For every $E \in \text{Div}_D(X)$, define ${}_{<E}K := {}_{<E}H \cap j_*K$, where ${}_{<E}H := {}_{E-\varepsilon D}H$ for $0 < \varepsilon \ll 1$. The main result of the present article is the following.

Theorem 1.1. *Let K be a locally free subsheaf of $H|_{X \setminus D}$ satisfying the following conditions:*

- Holomorphicity: $\nabla^{0,1}(K) = 0$, meaning that K is holomorphic with respect to both the complex structures $\bar{\partial}$ and $\nabla^{0,1}$.
- Weak transversality¹: $(\nabla - \theta)(K) \subset K \otimes \mathcal{A}_{X \setminus D}^{1,0}$.

Then ${}_{<E}K$ is a locally free sheaf on X for every $E \in \text{Div}_D(X)$. Let L be a line bundle on X such that $L \simeq_{\mathbb{R}} B + N$, where B is a semipositive $\mathbb{R}$ -divisor (see Definition 3.5) and N is an $\mathbb{R}$ -divisor on X supported on D . Let F be a Nakano semipositive vector bundle on X . Then the sheaf $\omega_X \otimes {}_{<D-N}K \otimes F \otimes L$ satisfies Kollár's package with respect to any proper surjective holomorphic morphism $X \rightarrow Y$ to a complex space Y .

The following examples demonstrate how Theorem 1.1 can be used to derive known results on Kollár's packages with coefficients in a Hodge module or a multiplier ideal sheaf (or a combination of both).

1.1.1. Example: Parabolic bundle. Let X be a smooth projective variety and $D \subset X$ a simple normal crossing divisor on X . Let $(H, \{{}_EH\}_{E \in \text{Div}_D(X)})$ be a locally abelian parabolic bundle on (X, D) with vanishing first and second parabolic Chern classes, which is polystable with respect to an ample line bundle A on X . In this case, one considers $(H, \{{}_EH\}_{E \in \text{Div}_D(X)})$ as a parabolic Higgs bundle with a vanishing Higgs field. By taking $K = H|_{X \setminus D}$ in Theorem 1.1, the holomorphicity and weak transversality conditions hold for K . Therefore, we obtain Kollár's package with coefficients in a polystable locally abelian parabolic bundle with vanishing first and second parabolic Chern classes.

Theorem 1.2. *Let L be a line bundle on X such that $L \simeq_{\mathbb{R}} B + N$, where B is a semipositive $\mathbb{R}$ -divisor and N is an $\mathbb{R}$ -divisor on X supported on D . Let F*

¹This condition is referred to as weak transversality due to Griffiths's transversality when H arises from a variation of Hodge structure with $\{F^p\}_{p \in \mathbb{Z}}$ as the Hodge filtration and $K = F^p$ for some p .

be an arbitrary Nakano semipositive vector bundle on X . Then $\omega_X \otimes_{<D-N} H \otimes F \otimes L$ satisfies Kollár's package with respect to any proper, surjective holomorphic morphism $X \rightarrow Y$ to a complex space Y .

1.1.2. Example: Twisted Kollár–Saito S -sheaf. Let $\mathbb{V}$ be a variation of Hodge structure on a regular Zariski open subset of a projective variety X . Kollár [14] introduced a coherent sheaf $S_X(\mathbb{V})$ that generalizes the dualizing sheaf. He conjectured that $S_X(\mathbb{V})$ satisfies Kollár's package. This conjecture was later proven by Saito [29] using the theory of mixed Hodge modules. In [31], the authors provide a new proof of Kollár's conjecture using the L^2 -method. Theorem 1.1 allows us to extend Kollár's conjecture to $S_X(\mathbb{V})$ twisted by a multiplier ideal sheaf.

Let X be a projective variety, and let $\mathbb{V} = (\mathcal{V}, \nabla, \{\mathcal{V}^{p,q}\}, Q)$ be a polarized complex variation of Hodge structure (Definition 4.1) on a dense regular Zariski open subset X° of X . For an $\mathbb{R}$ -Cartier divisor N on X , we define a coherent sheaf $S_X(\mathbb{V}, -N)$ on X (see Section 4.1.2) with the following properties:

- (1) $S_X(\mathbb{V}, 0)$ is canonically isomorphic to Kollár–Saito's $S_X(\mathbb{V})$ (see [29]). For an $\mathbb{R}$ -Cartier divisor $N \geq 0$ on X , $S_X(\mathbb{V}, -N)$ is a combination of $S_X(\mathbb{V})$ and the multiplier ideal sheaf associated with N .
- (2) When X is smooth and $X \setminus X^\circ$ is a simple normal crossing divisor such that $\text{supp}(N) \subset X \setminus X^\circ$, we have $S_X(\mathbb{V}, -N) \simeq \omega_X \otimes_{<D-N} S(\mathbb{V})$, where $(H, \{_E H\}_{E \in \text{Div}_D(X)}, \theta)$ is the parabolic Higgs bundle associated with $\mathbb{V}$ and $S(\mathbb{V})$ is the top indexed nonzero Hodge bundle.

As a consequence of Theorem 1.1, we obtain the following.

Theorem 1.3. *Let $f: X \rightarrow Y$ be a proper surjective holomorphic morphism to a complex space Y . Let L be a line bundle on X such that $L \simeq_{\mathbb{R}} B + N$, where B is a semipositive $\mathbb{R}$ -Cartier divisor and N is an $\mathbb{R}$ -Cartier divisor on X . Let F be an arbitrary Nakano semipositive vector bundle on X . Then $S_X(\mathbb{V}, -N) \otimes F \otimes L$ satisfies Kollár's package with respect to f .*

By setting $F = L = N = \mathcal{O}_X$, one can establish a proof for Kollár's conjecture [14, §5].

Remark 1.4. The projectivity condition of X in Theorem 1.1 can be relaxed to the condition of the existence of a tame harmonic metric on $H|_{X \setminus D}$, as proven in Theorem 2.9. Therefore, when X is a compact Kähler space, Theorem 1.3 remains valid. This is because a polarized variation of Hodge structure allows for a tame harmonic metric, also known as the Hodge metric.

1.1.3. Example: Multiplier Grauert–Riemenschneider sheaf. When $\mathbb{V} = \mathbb{C}_{X_{\text{reg}}}$ represents the trivial variation of Hodge structure and N is an $\mathbb{R}$ -Cartier divisor on X , the sheaf $\mathcal{K}_X(-N) := S_X(\mathbb{C}_{X_{\text{reg}}}, -N)$ ² is the Grauert–Riemenschneider sheaf twisted by the multiplier ideal sheaf (see Section 4.2) associated with N when $N \geq 0$. In fact, when $N = 0$, $\mathcal{K}_X(0)$ is the Grauert–Riemenschneider sheaf of X . If X is smooth and $N \geq 0$, then $\mathcal{K}_X(-N) \simeq \omega_X \otimes \mathcal{I}(-N)$, where $\mathcal{I}(-N)$ is the multiplier ideal sheaf associated with N .

According to Theorem 1.3, we have the following theorem.

Theorem 1.5. *Let $f: X \rightarrow Y$ be a proper, surjective holomorphic morphism from a projective variety X to a complex space Y . Let L be a line bundle on X such that $L \simeq_{\mathbb{R}} B + N$, where B is a semipositive $\mathbb{R}$ -Cartier divisor, and N is an $\mathbb{R}$ -Cartier divisor on X . Let F be an arbitrary Nakano semipositive vector bundle on X . Then $\mathcal{K}_X(-N) \otimes F \otimes L$ satisfies Kollár’s package with respect to f .*

Theorem 1.5 is applicable to Kollár’s package of pluricanonical bundles.

Corollary 1.6. *Let $f: X \rightarrow Y$ be a proper surjective holomorphic morphism from a smooth projective variety X to a complex space Y . Let A be a semipositive line bundle on X and $V \subset H^0(X, \omega_X^{\otimes km} \otimes A^{-1})$ a linear series for some positive integers k and m . Let F be an arbitrary Nakano semipositive vector bundle on X . Then $\omega_X^{\otimes k+1} \otimes \mathcal{I}(\frac{1}{m}|V|) \otimes F$ satisfies Kollár’s package with respect to f .*

Remark 1.7. For the same reason as in Remark 1.4, Theorem 1.5 holds when X is a compact Kähler space, and Corollary 1.6 holds when X is a compact Kähler manifold.

§1.2. Applications

1.2.1. Weak positivity of higher direct images. By applying Viehweg’s trick, we can deduce the following result from Theorem 1.1.

Theorem 1.8. *With notation as in Theorem 1.1, let $f: X \rightarrow Y$ be a surjective morphism between smooth projective varieties. Then $R^q f_*(\omega_{X/Y} \otimes_{D-N} K \otimes F \otimes L)$ is torsion-free and weakly positive.*

It is worth noting that the weak positivity of $R^q f_*(\omega_{X/Y} \otimes F)$ has already been established by Mourougane–Takayama [24] using an alternative method.

²The sheaf $\mathcal{K}_X(-N)$ has been mentioned in the Nadel vanishing theorem on complex spaces [6]. When X is smooth, it is referred to as the multiplier ideal by Viehweg [36, 37].

1.2.2. Generic vanishing theorem. Together with Hacon's criterion ([9], see also [25, 30]), one can deduce a generic vanishing theorem associated with a polystable parabolic Higgs bundle with vanishing first and second parabolic Chern classes.

Theorem 1.9. *Notation as in Theorem 1.1. Let $f: X \rightarrow A$ be a surjective morphism to an abelian variety. Then $R^q f_*(\omega_X \otimes_{<D-N} K \otimes F \otimes L)$ is a GV-sheaf in the sense of Pareschi–Popa [25] (Definition 4.10). As a consequence,*

$$\begin{aligned} \operatorname{codim}_{\operatorname{Pic}^0(A)} \{M \in \operatorname{Pic}^0(A) \mid H^i(X, \omega_X \otimes_{<D-N} K \otimes F \otimes L \otimes f^* M) \neq 0\} \\ \geq i - (\dim X - \dim f(X)), \end{aligned}$$

for all i .

As a result, according to Theorem 1.3, $R^q f_*(\omega_X \otimes S_X(\mathbb{V}, -N) \otimes F \otimes L)$ is a GV-sheaf. By setting $\mathbb{V} = \mathbb{C}$ as the trivial variation of Hodge structure, we can conclude that $R^q f_*(\mathcal{K}_X(-N) \otimes F \otimes L)$ is also a GV-sheaf. It is worth noting that Fujino–Matsumura [8] have demonstrated that $R^q f_*(\omega_X \otimes L \otimes \mathcal{I}(h))$ is a GV-sheaf for a line bundle L equipped with a singular Hermitian metric h with positive curvature current.

§1.3. Organization of the article

In Section 2 we review the nonabelian Hodge theory established by Simpson [33] and Mochizuki [19, 20, 21, 22]. This theory serves as the primary connection between algebraic objects like polystable parabolic Higgs bundles and transcendental objects like tame harmonic bundles. It allows for the investigation of polystable parabolic Higgs bundles using the L^2 -method. The proof of Theorem 1.1 is presented in Section 3. The main technical tool is the meta Kollár's package (see Section 3.1) developed by the authors in [31] using L^2 -methods. Section 4 showcases examples and applications of Theorem 1.1.

Notation.

- Let α and β be functions, metrics or $(1, 1)$ -forms. We write $\alpha \lesssim \beta$ if $\alpha \leq C\beta$ for some $C \in \mathbb{R}_{>0}$. We say that α and β are quasi-isometric if $\alpha \lesssim \beta$ and $\beta \lesssim \alpha$. We denote it by $\alpha \sim \beta$.
- We denote by $\Delta_\varepsilon := \{z \in \mathbb{C} \mid |z| < \varepsilon\}$ the open disk of radius $\varepsilon > 0$ in the complex plane, and for simplicity, we set $\Delta := \Delta_1$.

§2. Nonabelian Hodge theory on a smooth quasi-projective variety

In this section we will review the knowledge on nonabelian Hodge theory over a smooth quasi-projective variety, as established by Simpson [33] and Mochizuki [19, 20, 21, 22].

Throughout this section, let X be a smooth projective variety and $D = \sum_{i=1}^l D_i$ a reduced simple normal crossing divisor on X . The sheaf of rational functions on X that are regular on $X \setminus D$ is denoted as $\mathcal{O}_X[*D]$. Additionally, we let $\text{Div}_D(X)$ be the $\mathbb{R}$ -vector space of $\mathbb{R}$ -divisors E on X where $\text{supp}(E) \subset D$. Let $A = \sum_{i=1}^l a_i D_i$ and $B = \sum_{i=1}^l b_i D_i$. We use the notation $A \leq (<) B$ to indicate that $a_i \leq (<) b_i$ for each $i = 1, \dots, l$.

§2.1. Parabolic Higgs bundle

We follow the definition given by Iyer–Simpson [11], although the notation may differ slightly. A *parabolic sheaf* $*H$ on (X, D) is a torsion-free $\mathcal{O}_X[*D]$ -module H with a collection of torsion-free coherent $\mathcal{O}_X$ -submodules $\{ {}_E H \mid E \in \text{Div}_D(X) \}$, satisfying the following conditions:

- $H = \bigcup_{E \in \text{Div}_D(X)} {}_E H$,
- ${}_{E_1} H \subset {}_{E_2} H$ if $E_1 \leq E_2$,
- ${}_{E+\varepsilon D_i} H = {}_E H$ for any $E \in \text{Div}_D(X)$, any $i = 1, \dots, l$ and any constant $0 < \varepsilon \ll 1$,
- ${}_{E+D_i} H = {}_E H \otimes \mathcal{O}_X(D_i)$ for any $E \in \text{Div}_D(X)$ and every $i = 1, \dots, l$.

Consequently, ${}_E H|_{X \setminus D} = H|_{X \setminus D}$ for every $E \in \text{Div}_D(X)$. For every $E \in \text{Div}_D(X)$, we define

$${}_{<E} H := \bigcup_{E' < E} {}_{E'} H = {}_{E-\varepsilon D} H, \quad \forall 0 < \varepsilon \ll 1.$$

Definition 2.1. A parabolic sheaf $*H = (H, \{ {}_E H \}_{E \in \text{Div}_D(X)})$ on (X, D) is called a *parabolic bundle* (resp. *parabolic line bundle*) if each ${}_E H$ is a vector bundle (resp. line bundle). A parabolic bundle $*H$ on (X, D) is called *locally abelian* if there exists an isomorphism between $*H$ and a direct sum of parabolic line bundles in a Zariski neighborhood of any point $x \in X$.

Definition 2.2. A *Higgs bundle* (H, θ) on $X \setminus D$ consists of a holomorphic vector bundle H on $X \setminus D$, along with an $\mathcal{O}_{X \setminus D}$ -linear map $\theta: H \rightarrow H \otimes \Omega_{X \setminus D}$, where θ is called a Higgs field on H , satisfying $\theta \wedge \theta = 0$.

A *logarithmic Higgs bundle* (H, θ) on (X, D) consists of a holomorphic vector bundle H on X , along with an $\mathcal{O}_X$ -linear map $\theta: H \rightarrow H \otimes \Omega_X(\log D)$, where θ is called a Higgs field on H , satisfying $\theta \wedge \theta = 0$.

Definition 2.3. A locally abelian parabolic Higgs bundle $(*_H, \theta)$ on (X, D) consists of a locally abelian parabolic bundle $*H = (H, \{_E H\}_{E \in \text{Div}_D(X)})$ on (X, D) and a Higgs field $\theta: H|_{X \setminus D} \rightarrow H|_{X \setminus D} \otimes \Omega_{X \setminus D}$ such that θ extends to a logarithmic Higgs field

$$_E H \rightarrow _E H \otimes \Omega_X(\log D)$$

for every $E \in \text{Div}_D(X)$.

§2.2. Tame harmonic bundle

Let (H, θ, h) be a Higgs bundle on $X \setminus D$ with a Hermitian metric h . Let $\bar{\theta}$ be the adjoint of θ with respect to h and let ∂ be the unique $(1, 0)$ -connection such that $\partial + \bar{\partial}$ is compatible with h .

Definition 2.4. The Higgs bundle (H, θ, h) is called a *harmonic bundle* if $(\partial + \bar{\partial} + \theta + \bar{\theta})^2 = 0$. In this case, the Hermitian metric h is called a *harmonic metric*.

Let ∇_h be the Chern connection on the harmonic bundle H with respect to h and let $\Theta_h(H) = \nabla_h^2$ be its Chern curvature form. Then we have the self-dual equation

$$(2.1) \quad \Theta_h(H) + \theta \wedge \bar{\theta} + \bar{\theta} \wedge \theta = 0.$$

For the purpose of the present article, we are focusing on tame harmonic bundles in the sense of Simpson [33] and Mochizuki [20, 21].

Definition 2.5. A harmonic bundle (H, θ, h) on $X \setminus D$ is said to be *tame* if, for any point $x \in D$, there exist holomorphic local coordinates $(z_1, \dots, z_n)$ centered at $x = (0, \dots, 0)$ such that $D = \{z_1 \cdots z_r = 0\}$, the Higgs field admits a local expression of the form

$$\theta = \sum_{i=1}^r f_i \frac{dz_i}{z_i} + \sum_{i=r+1}^n f_i dz_i,$$

and the characteristic polynomials $\det(\lambda I - f_i)$, for $i = 1, \dots, n$, extend holomorphically to a neighborhood of x .

There are several equivalent characterizations of the tameness of harmonic bundles; readers are referred to [18, 20] for further details.

2.2.1. Analytic prolongations of tame harmonic bundles.

Definition 2.6 (Analytic prolongation ([18, Definition 4.2])). Let (H, h) be a Hermitian vector bundle on $X \setminus D$. Let $E \in \text{Div}_D(X)$, U be an open subset of X and $s \in \Gamma(U \setminus D, H)$ be a holomorphic section. If $|s|_h = O(\prod_{i=1}^r |z_i|^{-a_i - \varepsilon})$ for any positive number ε , where $z_1, \dots, z_n$ are holomorphic local coordinates such

that $E = \sum_{i=1}^r a_i \{z_i = 0\}$, we denote $(s) \leq_h -E$. The $\mathcal{O}_X$ -module ${}_E H_h$ is defined as

$$\Gamma(U, {}_E H_h) := \{s \in \Gamma(U \setminus D, H) \mid (s) \leq_h -E\}$$

for any open subset $U \subset X$.

Now let us recall the profound result that the set of analytic prolongations forms a locally abelian parabolic Higgs bundle (see Simpson [33, Theorem 3] and Mochizuki [20]).

Theorem 2.7. *If (H, θ, h) is a tame harmonic bundle on (X, D) , then the bundle $*H_h = (H[*D], \{{}_E H_h\}_{E \in \text{Div}_D(X)}, \theta)$ is a locally abelian parabolic Higgs bundle. Here, $H[*D] = \bigcup_{E \in \text{Div}_D(X)} {}_E H_h$ denotes the sheaf of meromorphic sections of H whose poles lie in D .*

2.2.2. Norm estimate. Consider a tame harmonic bundle (H, θ, h) on (X, D) . Let $(H[*D], \{{}_E H_h\}_{E \in \text{Div}_D(X)}, \theta)$ denote the locally abelian parabolic Higgs bundle constructed via analytic prolongation (as established in Theorem 2.7). In this subsection, we revisit a fundamental norm estimate for h in a neighborhood of a point $x \in D$, originally established by Simpson [33] in dimension one and generalized by Mochizuki [20] to the general case. For simplicity, we may assume that $X = \Delta_R^r \times \Delta_R^{n-r}$ ($0 < R < 1$) is a polydisk equipped with standard coordinates $z_1, \dots, z_n$, and that $D = \bigcup_{i=1}^r D_i$, where each $D_i := \{z_i = 0\} \subset X$ for $i = 1, \dots, r$.

Let $E = \sum_{i=1}^r a_i D_i \in \text{Div}_D(X)$. For each $i = 1, \dots, r$, let N_i denote the nilpotent part of the residue map $\text{Res}_{D_i}(\theta) \in \text{End}({}_E H_h / {}_{E-\varepsilon D_i} H_h)$ for $0 < \varepsilon \ll 1$. By [20, Lemma 8.2], the eigenvalues of $\text{Res}_{D_i}(\theta)$ are constant along D_i . Let $\{W_m^{(k)}\}_{m \in \mathbb{Z}}$ ($1 \leq k \leq r$) be the filtration on ${}_E H_h$ that induces the monodromy weight filtration on the quotient ${}_E H_h / \sum_{i=1}^k {}_{E-\varepsilon D_i} H_h$ for $0 < \varepsilon \ll 1$, associated with the nilpotent operator $N_1 + \dots + N_k$. Specifically, $W_m^{(k)}$ is determined by

$$\sum_{i=1}^k {}_{E-\varepsilon D_i} H_h \subset W_m^{(k)} \subset {}_E H_h, \quad W_m^{(k)} / \sum_{i=1}^k {}_{E-\varepsilon D_i} H_h = W_m(N_1 + \dots + N_k), \quad \forall m \in \mathbb{Z}.$$

Theorem 2.8 ([20, §13.3]). *Let $v \in {}_E H_h$ be a holomorphic local section at $\mathbf{0} = (0, \dots, 0)$. Assume that the following conditions hold:*

- $v \notin {}_{E'} H_h$ for every $E' \leq E$, $E' \neq E$;
- $v \in \bigcap_{k=1}^r W_{l_k}^{(k)}$ for some $l_1, \dots, l_r \in \mathbb{Z}$ and $0 \neq [v] \in \text{Gr}_{l_r}^{W^{(r)}} \cdots \text{Gr}_{l_1}^{W^{(1)}}({}_E H_h)$.

Then

$$|v|_h^2 \sim |z_1|^{-2a_1} \cdots |z_r|^{-2a_r} \left(\frac{\log |z_1|}{\log |z_2|} \right)^{l_1} \cdots (-\log |z_r|)^{l_r}$$

over any region of the form

$$\{(z_1, \dots, z_n) \in (\Delta_R^*)^r \times \Delta_R^{n-r} \mid \frac{\log |z_1|}{\log |z_2|} > \varepsilon, \dots, -\log |z_r| > \varepsilon, (z_{r+1}, \dots, z_n) \in M\}$$

for any $\varepsilon > 0$ and an arbitrary compact subset $M \subset \Delta_R^{n-r}$.

§2.3. Nonabelian Hodge theory of Simpson–Mochizuki

In the context of parabolic bundles, we have the concept of parabolic Chern classes. Consequently, we can define the notion of μ_A -semistable (or μ_A -stable, μ_A -polystable) parabolic Higgs bundles where A is an ample line bundle on X . For a detailed explanation of these definitions, please refer to [19, Chapter 3] and [11, Definition 2.7].

Let us recall a significant result that is essential to the nonabelian Hodge theory on smooth quasi-projective varieties (see Simpson [33] and Mochizuki [19]).

Theorem 2.9 ([33, 19]). *Let $*H = (H, \{_E H\}_{E \in \text{Div}_D(X)}, \theta)$ be a locally abelian parabolic Higgs bundle on (X, D) and A be an ample line bundle on X . Then the following two statements are equivalent:*

- (1) *$*H$ is μ_A -polystable with vanishing first and second parabolic Chern classes.*
- (2) *There exists a tame harmonic metric h on $(H|_{X \setminus D}, \theta)$ such that $*H$ is isomorphic to $(H[*D], \{_E H_h\}_{E \in \text{Div}_D(X)}, \theta)$ as a parabolic Higgs bundle.*

Using the tame harmonic metric, one can establish a profound relation between the following categories:

- $\mathcal{D}(X, D)$: the category of μ_A -polystable locally abelian parabolic flat bundles on (X, D) with vanishing first and second parabolic Chern classes;
- $\mathcal{H}(X, D)$: the category of μ_A -polystable locally abelian parabolic Higgs bundles on (X, D) with vanishing first and second parabolic Chern classes.

For the purpose of this article, let us briefly explain the equivalence functor $\mathcal{SM}: \mathcal{H}(X, D) \rightarrow \mathcal{D}(X, D)$. Let $*H = (H, \{_E H\}_{E \in \text{Div}_D(X)}, \theta) \in \mathcal{H}(X, D)$, and let h be the tame harmonic metric as in Theorem 2.9. Let $\bar{\theta}$ be the adjoint of θ , and let ∂ be the unique $(1, 0)$ -connection such that $\partial + \bar{\partial}$ is compatible with h . Then

$$(H|_{X \setminus D} \otimes_{\mathcal{O}_{X \setminus D}} \mathcal{A}_{X \setminus D}^0, \nabla := \partial + \bar{\partial} + \theta + \bar{\theta})$$

is a meromorphic flat connection that is regular along the divisor D , where $\mathcal{A}_{X \setminus D}^0$ is the sheaf of C^∞ -functions on $X \setminus D$. Let $\nabla = \nabla^{1,0} + \nabla^{0,1}$ be the decomposition according to the bidegree. Then

$$V_0 := \ker(\nabla^{0,1} : H|_{X \setminus D} \otimes_{\mathcal{O}_{X \setminus D}} \mathcal{A}_{X \setminus D}^0 \rightarrow H|_{X \setminus D} \otimes_{\mathcal{O}_{X \setminus D}} \mathcal{A}_{X \setminus D}^{0,1})$$

is a holomorphic vector bundle endowed with the flat holomorphic connection ∇ and the harmonic metric h . The analytic prolongations of (V_0, h) (Definition 2.6) give rise to a μ_A -polystable parabolic flat connection

$$(V := V_0[*D], \{ {}_E V \}_{E \in \text{Div}_D(X)}, \nabla)$$

with trivial parabolic characteristic numbers. Then we have $\mathcal{SM}(*H) = (V, \{ {}_E V \}_{E \in \text{Div}_D(X)}, \nabla)$.

Theorem 2.10 ([32, 33, 19, 22]). *The functor $\mathcal{SM}: \mathcal{H}(X, D) \rightarrow \mathcal{D}(X, D)$ is an equivalence of categories. Furthermore, if $(H, \{ {}_E H \}_{E \in \text{Div}_D(X)}, \theta) \in \mathcal{H}(X, D)$ and*

$$\mathcal{SM}((H, \{ {}_E H \}_{E \in \text{Div}_D(X)}, \theta)) = (V, \{ {}_E V \}_{E \in \text{Div}_D(X)}, \nabla),$$

then there exists an identity between C^∞ complex bundles:

$$H|_{X \setminus D} \otimes_{\mathcal{O}_{X \setminus D}} \mathcal{A}_{X \setminus D}^0 = V|_{X \setminus D} \otimes_{\mathcal{O}_{X \setminus D}} \mathcal{A}_{X \setminus D}^0.$$

§2.4. Weighted L^2 -prolongation

Suppose A is an ample line bundle on X and $(H, \{ {}_E H \}_{E \in \text{Div}_D(X)}, \theta)$ is a μ_A -polystable locally abelian parabolic Higgs bundle on (X, D) with vanishing first and second parabolic Chern classes. We define the coherent sheaf $P_{E, (2)}(H)$ which is determined by the following conditions:

- (1) ${}_{E-\varepsilon D} H \subset P_{E, (2)}(H) \subset {}_E H$ for $0 < \varepsilon \ll 1$.
- (2) Take x to be a point on D , and let $(U; z_1, \dots, z_n)$ be holomorphic local coordinates on an open neighborhood U of x in X , such that $D = \{z_1 \cdots z_r = 0\}$. Let $D_i = \{z_i = 0\}$, $i = 1, \dots, r$. Let N_i be the nilpotent part of the residue map $\text{Res}_{D_i}(\theta) \in \text{End}({}_E H / {}_{E-\varepsilon D_i} H)$ for $0 < \varepsilon \ll 1$. For any subset $I \subset \{1, \dots, r\}$, let $\{W_m^I\}_{m \in \mathbb{Z}}$ be the filtration on ${}_E H$ that induces the monodromy weight filtration on the quotient ${}_E H / \sum_{i \in I} {}_{E-\varepsilon D_i} H$ for $0 < \varepsilon \ll 1$, associated with the nilpotent operator $\sum_{i \in I} N_i$. Specifically, W_m^I is determined by

$$\sum_{i \in I} {}_{E-\varepsilon D_i} H \subset W_m^I \subset {}_E H, \quad W_m^I / \sum_{i \in I} {}_{E-\varepsilon D_i} H = W_m \left(\sum_{i \in I} N_i \right), \quad \forall m \in \mathbb{Z}.$$

Then we have

$$P_{E, (2)}(H) = \sum_{\emptyset \neq I \subset \{1, \dots, r\}} {}_{E-\varepsilon \sum_{i \notin I} D_i} H \cap \bigcap_{J \subset I} W_{-\#(J)-1}^I$$

on U , for $0 < \varepsilon \ll 1$.

When $E = D$, $P_{D, (2)}(H)$ is the sheaf of L^2 -holomorphic sections with coefficients in H (see Proposition 3.12). Originally introduced by Zucker [38] on algebraic curves,

this kind of construction involves Higgs bundles H that arise from a variation of Hodge structure, making it a significant subject of study in L^2 -cohomology of a variation of Hodge structure. The construction by Zucker has been extended to higher dimensions by Kashiwara and Kawai [12]. Recently, Mochizuki [23] provided a further generalization of the characterization of the L^2 de Rham complex to λ -connections (or twistor $\mathcal{D}$ -modules). This advancement represents a significant step in establishing the hard Lefschetz theorem for the pushforward of a twistor $\mathcal{D}$ -module within the framework of Kähler geometry. It is noteworthy that our construction $P_{D,(2)}(H)$ diverges from those presented in [38, 12, 23], as we take into account the L^2 -sections associated with a Hermitian metric on X , while other authors focus on Poincaré-type metrics.

To generalize Zucker's construction [38] to higher-dimensional bases and noncanonical indexed extensions, we have the sheaf $P_{E,(2)}(H)$. In particular, $P_{D-E,(2)}(H)$ combines elements of both $P_{D,(2)}(H)$ and the multiplier ideal sheaf associated with E when $E \geq 0$. This aspect makes $P_{E,(2)}(H)$ more convenient in applications when $E \neq D$.

§3. Proof of the main theorem

§3.1. A meta Kollár's package

In this section we recall Kollár's package established in [31]. Let X be an irreducible complex space of dimension n , and let $X^\circ \subset X_{\text{reg}}$ be a dense Zariski open subset. Consider a Hermitian vector bundle (E, h) on X° . The $\mathcal{O}_X$ -module $S_X(E, h)$ can be defined as follows. Let $U \subset X$ be an open subset. Then $S_X(E, h)(U)$ is the space of holomorphic E -valued $(n, 0)$ -forms α on $U \cap X^\circ$ such that for every point $x \in U$, there exists a neighborhood V_x of x such that

$$\int_{V_x \cap X^\circ} \alpha \wedge_h \bar{\alpha} < \infty.$$

Lemma 3.1 (Functoriality [31, Proposition 2.5]). *If $\pi: X' \rightarrow X$ is a proper holomorphic map between complex spaces which is biholomorphic over X° , then*

$$\pi_* S_{X'}(\pi^* E, \pi^* h) = S_X(E, h).$$

Lemma 3.2 ([31, Lemma 2.6]). *If (F, h_F) is a Hermitian vector bundle on X (in particular h_F is smooth on X), then*

$$S_X(E, h) \otimes F \simeq S_X(E \otimes F|_{X^\circ}, h \otimes h_F|_{X^\circ}).$$

Definition 3.3. A Hermitian vector bundle (E, h) is called *tame* on X if, for every point $x \in X$, there is an open neighborhood U containing x , a proper

bimeromorphic morphism $\pi: \tilde{U} \rightarrow U$ which is biholomorphic over $U \cap X^o$, and a Hermitian vector bundle (Q, h_Q) on $\tilde{U}$ such that

- (1) $\pi^*E|_{\pi^{-1}(X^o \cap U)} \subset Q|_{\pi^{-1}(X^o \cap U)}$ as a subsheaf;
- (2) there is a Hermitian metric h'_Q on $Q|_{\pi^{-1}(X^o \cap U)}$ so that $h'_Q|_{\pi^*E} \sim \pi^*h$ on $\pi^{-1}(X^o \cap U)$ and

$$(3.1) \quad \left(\sum_{i=1}^r \|f_i\|^2 \right)^c h_Q \lesssim h'_Q$$

for some $c \in \mathbb{R}$. Here $\{f_1, \dots, f_r\}$ is an arbitrary set of local generators of the ideal sheaf defining $\tilde{U} \setminus \pi^{-1}(X^o) \subset \tilde{U}$.

The tameness condition (3.1) is independent of the choice of the set of local generators. In this article, we construct a tame Hermitian vector bundle (E, h) as a subsheaf of a tame harmonic bundle. In this case, condition (3.1) is derived from the asymptotic behavior of the harmonic metric (Theorem 2.8).

Theorem 3.4. *Let $f: X \rightarrow Y$ be a proper, locally Kähler morphism between irreducible complex spaces. Let $X^o \subset X_{\text{reg}}$ be a dense Zariski open subset, and (E, h) be a Hermitian vector bundle on X^o with Nakano semipositive curvature. If (E, h) is tame on X , then the sheaf $S_X(E, h)$ is a coherent sheaf and satisfies Kollár's package with respect to $f: X \rightarrow Y$.*

Proof. See [31, Proposition 2.9 and §4]. □

§3.2. Preliminary for $\mathbb{R}$ -divisors

In this section we will clarify some concepts regarding $\mathbb{R}$ -divisors that should be familiar to experts. Let X be a projective variety. An $\mathbb{R}$ -Cartier divisor is a formal sum $A = \sum_{i=1}^r a_i A_i$, where $A_1, \dots, A_r$ are integral Cartier divisors and $a_1, \dots, a_r \in \mathbb{R}$. Given two $\mathbb{R}$ -Cartier divisors, A and B , we say that they are $\mathbb{R}$ -linearly equivalent, denoted by $A \simeq_{\mathbb{R}} B$, if there exist rational functions $f_1, \dots, f_m$ and real numbers $r_1, \dots, r_m$ such that $A - B = \sum_{i=1}^m r_i (f_i)$, where (f_i) represents the principal divisor defined by f_i . Now, let L be a line bundle and A be an $\mathbb{R}$ -Cartier divisor. We also use the notation $L \simeq_{\mathbb{R}} A$ when $(s) \simeq_{\mathbb{R}} A$ for some global rational section s of L .

Definition 3.5. Let $A = \sum_{i=1}^r a_i A_i$ be an $\mathbb{R}$ -Cartier divisor on X . A C^∞ Hermitian metric $h = (h_1, \dots, h_r)$ on A consists of a C^∞ Hermitian metric h_i on $\mathcal{O}_X(A_i)$ for each $i = 1, \dots, r$. The curvature of h is defined as

$$\Theta_h(A) := \sum_{i=1}^r a_i \Theta_{h_i}(A_i).$$

The $\mathbb{R}$ -divisor A is called *semipositive* if there is a C^∞ Hermitian metric h on A such that $\sqrt{-1}\Theta_h(A)$ is semipositive.

The semipositivity of $\mathbb{R}$ -Cartier divisors is preserved under $\mathbb{R}$ -linear equivalence.

Proposition 3.6. *Let A and B be two $\mathbb{R}$ -Cartier divisors on a smooth projective variety X such that $A \simeq_{\mathbb{R}} B$. If A is semipositive, then so is B .*

Proof. Assuming that $A = \sum_{i=1}^r a_i A_i$, we have h_i as C^∞ Hermitian metrics on $\mathcal{O}_X(A_i)$ for $i = 1, \dots, r$ such that the curvature $\sqrt{-1}\Theta_h(A) := \sqrt{-1} \sum_{i=1}^r a_i \Theta_{h_i}(A_i)$ is semipositive. Now, let $B = \sum_{i=1}^s b_i B_i$ and h'_i be an arbitrary C^∞ Hermitian metric on $\mathcal{O}_X(B_i)$ for $i = 1, \dots, s$. Since $A \simeq_{\mathbb{R}} B$, it follows that

$$[\sqrt{-1}\Theta_h(A)] = [\sqrt{-1}\Theta_h(B)] \in H^{1,1}(X, \mathbb{C}),$$

where $\sqrt{-1}\Theta_h(B) = \sqrt{-1} \sum_{i=1}^s b_i \Theta_{h'_i}(B_i)$. According to the $\partial\bar{\partial}$ -lemma, there exists a C^∞ function φ on X such that $\Theta_h(A) = \Theta_h(B) + \partial\bar{\partial}(\varphi)$. By defining $h_{B_1} := e^{-\varphi} h'_1$ and $h_{B_i} := h'_i$ for $i = 2, \dots, s$, we can show that B is semipositive. $\square$

A direct consequence is as follows: Let L be a line bundle and s be a global rational section of L . Then L is semipositive if and only if (s) is semipositive.

Now, consider $A = \sum_{i=1}^r a_i A_i$, an $\mathbb{R}$ -Cartier divisor on X . For each i , let h_i be an arbitrary C^∞ Hermitian metric on $\mathcal{O}_X(A_i)$. We define

$$\varphi_A := \sum_{i=1}^r a_i \log |s_i|_{h_i},$$

where $s_i \in H^0(X, \mathcal{O}_X(A_i))$ is a defining section of A_i . Such an φ_A is called a *weight function* associated with A . Given different C^∞ Hermitian metrics h'_i and defining sections s'_i , there exists a constant $C > 0$ such that

$$\sum_{i=1}^r a_i \log |s'_i|_{h'_i} - C \leq \sum_{i=1}^r a_i \log |s_i|_{h_i} \leq \sum_{i=1}^r a_i \log |s'_i|_{h'_i} + C.$$

It follows that the quasi-isometric class of $\exp(-2\varphi_A)$ is independent of the choice of h_i and s_i .

At the end of this section we discuss singular Hermitian metrics associated with an $\mathbb{R}$ -divisor.

Lemma 3.7. *Let A and B be two $\mathbb{R}$ -Cartier divisors on a smooth projective variety X . Let L be a line bundle on X such that $L \simeq_{\mathbb{R}} A + B$. Let φ_A be a weight*

function associated with A and let h_B be a C^∞ Hermitian metric on B . Then, there exists a singular Hermitian metric h on L that satisfies the following conditions:

- (1) h is smooth over $X \setminus \text{supp}(A)$;
- (2) $\Theta_h(L|_{X \setminus \text{supp}(A)}) = \Theta_{h_B}(B)|_{X \setminus \text{supp}(A)}$;
- (3) $|e|_h \sim \exp(-\varphi_A)$ for any local generator e of L .

Proof. Let $A = \sum_{i=1}^r a_i A_i$ and $h_A = (h_1, \dots, h_r)$ be a smooth Hermitian metric on A . Let $s_i \in H^0(X, \mathcal{O}_X(A_i))$ be the defining section of A_i . We can assume that $\varphi_A = \sum_{i=1}^r a_i \log |s_i|_{h_i}^2$. Let h_0 be a smooth Hermitian metric on L . Since $L \simeq_{\mathbb{R}} A + B$, we have

$$[\sqrt{-1}\Theta_{h_0}(L)] = [\sqrt{-1}\Theta_{h_A}(A) + \sqrt{-1}\Theta_{h_B}(B)] \in H^{1,1}(X, \mathbb{C}).$$

According to the $\partial\bar{\partial}$ -lemma, there exists a smooth function ψ on X such that

$$\Theta_{h_0}(L) = \Theta_{h_A}(A) + \Theta_{h_B}(B) + \partial\bar{\partial}(\psi).$$

Then the metric $h = \exp(\psi - \varphi_A)h_0$ satisfies all the required conditions in the lemma. $\square$

§3.3. Proof of Theorem 1.1

The main strategy in proving Theorem 1.1 is to give an L^2 interpretation of $\omega_X \otimes \langle_{D-N} K \otimes F \otimes L$ (Corollary 3.13 and Proposition 3.15) and use Theorem 3.4. The remaining section will provide the details of the proof.

3.3.1. Tameness and semipositivity. Let X be a smooth projective variety and D a reduced simple normal crossing divisor on X . Let $(H, \{ {}_E H \}_{E \in \text{Div}_D(X)}, \theta)$ be a locally abelian parabolic Higgs bundle on (X, D) with trivial first and second parabolic Chern classes, which is polystable with respect to an ample line bundle A on X . Let h be a tame harmonic metric on $H|_{X \setminus D}$, which is compatible with the parabolic structure. The existence of such a metric is ensured by Theorem 2.9. Let $\bar{\theta}$ be the adjoint of θ , and let ∂ be the unique $(1, 0)$ -connection such that $\partial + \bar{\partial}$ is compatible with h . Then

$$(H|_{X \setminus D} \otimes_{\mathcal{O}_{X \setminus D}} \mathcal{A}_{X \setminus D}^0, \nabla := \partial + \bar{\partial} + \theta + \bar{\theta})$$

determines a meromorphic flat connection, which is regular along the divisor D . Let $\nabla = \nabla^{1,0} + \nabla^{0,1}$ be the decomposition with respect to the bi-degree. Notice that $\nabla^{1,0} = \partial + \theta$ and $\nabla^{0,1} = \bar{\partial} + \bar{\theta}$. Let $K \subset H|_{X \setminus D}$ be a locally free subsheaf such that $\nabla^{0,1}(K) = 0$ and $(\nabla - \theta)(K) \subset K \otimes \mathcal{A}_{X \setminus D}^{1,0}$.

Lemma 3.8. *The curvature tensor $\sqrt{-1}\Theta_h(K)$ is Nakano semipositive.*

Proof. The assumptions on K imply that

$$\partial(K) = (\nabla - \theta)(K) \subset K \otimes \mathcal{A}_{X \setminus D}^{1,0}.$$

Therefore, the second fundamental form of the subbundle $K \subset H|_{X \setminus D}$ vanishes identically. It follows that $\Theta_h(K) = \Theta_h(H|_{X \setminus D})|_K$. Furthermore, since $\bar{\theta}(K) = \nabla^{0,1}(K) - \bar{\partial}(K) = 0$, we may apply the curvature formula (2.1) to deduce

$$\sqrt{-1}\Theta_h(K) = -\sqrt{-1}\theta \wedge \bar{\theta}|_K - \sqrt{-1}\bar{\theta} \wedge \theta|_K = -\sqrt{-1}\bar{\theta} \wedge \theta|_K,$$

which is Nakano semipositive. $\square$

Let (F, h_F) be an arbitrary Nakano semipositive Hermitian vector bundle on X . Let L be a line bundle on X such that $L \simeq_{\mathbb{R}} B + N$, where B is a semipositive $\mathbb{R}$ -divisor and N is an $\mathbb{R}$ -divisor on X which is supported on D . Let φ_N be a weight function associated with N . By Lemma 3.7, there is a singular Hermitian metric h_L on L such that the following conditions hold:

- (1) h_L is smooth over $X \setminus \text{supp}(N)$;
- (2) the curvature of h_L satisfies

$$(3.2) \quad \sqrt{-1}\Theta_{h_L}(L|_{X \setminus \text{supp}(N)}) = \sqrt{-1}\Theta_{h_B}(B|_{X \setminus \text{supp}(N)}) \geq 0;$$

- (3) for any local generator e of L , its norm satisfies

$$(3.3) \quad |e|_{h_L} \sim \exp(-\varphi_N).$$

Notice that the metrics h and h_L are C^∞ over $X \setminus D$.

Lemma 3.9. *The Hermitian vector bundle $(K \otimes F|_{X \setminus D} \otimes L|_{X \setminus D}, hh_F h_L)$ is Nakano semipositive on $X \setminus D$ and tame on X .*

Proof. The first claim follows from Lemma 3.8, (3.2), and the fact that (F, h_F) is Nakano semipositive. To prove the second claim, we embed $K \otimes F|_{X \setminus D} \otimes L|_{X \setminus D}$ into ${}^E H|_{X \setminus D} \otimes F|_{X \setminus D} \otimes L|_{X \setminus D}$ for some $E \in \text{Div}_D(X)$. We still need to demonstrate that

$$|z_1 \cdots z_r|^c h h_F h_L \lesssim h_0$$

for some $c > 0$ and some C^∞ Hermitian metric h_0 on ${}^E H \otimes F \otimes L$. Here, $z_1, \dots, z_n$ are holomorphic local coordinates on X such that $D = \{z_1 \cdots z_r = 0\}$. This is a direct consequence of Theorem 2.8 and (3.3). $\square$

3.3.2. L^2 interpretation of $P_{D-E, (2)}(H)$. To illustrate the asymptotic behavior of a section of $P_{D-E, (2)}(H)$, we require the following lemma.

Lemma 3.10. *Let $\varepsilon > 0$ and let*

$$S_\varepsilon = \left\{ (z_1, \dots, z_r) \in (\Delta_{\frac{1}{2}}^*)^r \mid \frac{\log |z_1|}{\log |z_2|} > \varepsilon, \dots, -\log |z_r| > \varepsilon \right\}.$$

Let $\text{vol} = dz_1 \wedge d\bar{z}_1 \wedge \dots \wedge dz_r \wedge d\bar{z}_r$. Let $a_1, \dots, a_r \in \mathbb{R}$ and $l_1, \dots, l_r \in \mathbb{Z}$. Let

$$r_0 = \begin{cases} \max\{i = 1, \dots, r \mid a_1 = \dots = a_i = 1\} & \text{if } a_1 = 1, \\ 0 & \text{if } a_1 < 1. \end{cases}$$

Then the integration

$$\int_{S_\varepsilon} |z_1|^{-2a_1} \dots |z_r|^{-2a_r} \left(\frac{\log |z_1|}{\log |z_2|} \right)^{l_1} \dots (-\log |z_r|)^{l_r} \text{vol}$$

is finite for every $\varepsilon > 0$ if and only if $a_1, \dots, a_r \leq 1$ and $l_i \leq -i - 1$ for any $1 \leq i \leq r_0$.

Proof. Step 1: Let $z_i = \rho_i e^{\sqrt{-1}\theta_i}$, $i = 1, \dots, r$. Then

$$\text{vol} = \rho_1 \dots \rho_r d\rho_1 \wedge d\theta_1 \wedge \dots \wedge d\rho_r \wedge d\theta_r$$

and

$$\begin{aligned} & \int_{S_\varepsilon} |z_1|^{-2a_1} \dots |z_r|^{-2a_r} \left(\frac{\log |z_1|}{\log |z_2|} \right)^{l_1} \dots (-\log |z_r|)^{l_r} \text{vol} \\ &= (2\pi)^r \int_{S'_\varepsilon} \rho_1^{1-2a_1} \dots \rho_r^{1-2a_r} \left(\frac{\log \rho_1}{\log \rho_2} \right)^{l_1} \dots (-\log \rho_r)^{l_r} d\rho_1 \wedge \dots \wedge d\rho_r, \end{aligned}$$

where

$$S'_\varepsilon = \left\{ (\rho_1, \dots, \rho_r) \in (0, \frac{1}{2})^r \mid \frac{\log \rho_1}{\log \rho_2} > \varepsilon, \dots, -\log \rho_r > \varepsilon \right\}.$$

Let

$$t_i = \frac{\log \rho_i}{\log \rho_{i+1}}, \quad i = 1, \dots, r-1 \quad \text{and} \quad t_r = -\log \rho_r.$$

Then

$$\rho_i = e^{-t_i \dots t_r}, \quad i = 1, \dots, r,$$

and

$$0 < \rho_i = e^{-t_i \dots t_r} < \frac{1}{2} \Leftrightarrow \log 2 < t_i \dots t_r < +\infty.$$

Therefore,

$$S'_\varepsilon = \left\{ (t_1, \dots, t_r) \in \mathbb{R}^r \mid t_i > \varepsilon, \quad t_i \dots t_r > \log 2, \forall i = 1, \dots, r \right\}.$$

The direct computation

$$\begin{aligned}
 d\rho_1 \wedge d\rho_2 \wedge \cdots \wedge d\rho_r \\
 &= de^{-t_1 \cdots t_r} \wedge de^{-t_2 \cdots t_r} \wedge \cdots \wedge de^{-t_r} \\
 &= \frac{\partial}{\partial t_1}(e^{-t_1 \cdots t_r}) \frac{\partial}{\partial t_2}(e^{-t_2 \cdots t_r}) \cdots \frac{\partial}{\partial t_r}(e^{-t_r}) dt_1 \wedge \cdots \wedge dt_r \\
 &= (-1)^r t_2 t_3^2 \cdots t_r^{r-1} e^{-t_1 \cdots t_r} e^{-t_2 \cdots t_r} \cdots e^{-t_r} dt_1 \wedge \cdots \wedge dt_r
 \end{aligned}$$

yields that

$$\begin{aligned}
 &\int_{S'_\varepsilon} \rho_1^{1-2a_1} \cdots \rho_r^{1-2a_r} \left(\frac{\log \rho_1}{\log \rho_2} \right)^{l_1} \cdots (-\log \rho_r)^{l_r} d\rho_1 \wedge \cdots \wedge d\rho_r \\
 (3.4) \quad &= (-1)^r \int_{S'_\varepsilon} \prod_{i=1}^r e^{(2a_i-2)t_i \cdots t_r} t_i^{l_i+i-1} dt_1 \wedge \cdots \wedge dt_r.
 \end{aligned}$$

Step 2: In this step, we show that $a_1, \dots, a_r \leq 1$ whenever the integral (3.4) is finite for every $\varepsilon > 0$. We will proceed by contradiction. Specifically, we assume that there exists some index $m \in \{1, \dots, r\}$ such that $a_m > 1$ (and we fix such m). Take $0 < \varepsilon < 1$ so that

$$c(m, \varepsilon) := 2a_m - 2 - \sum_{j=1}^{m-1} |2a_j - 2|(2\varepsilon)^{m-j} > 0.$$

Consider the subset

$$\begin{aligned}
 L_{m, \varepsilon} &= \{(t_1, \dots, t_r) \in \mathbb{R}^r \mid t_1, \dots, t_{m-1} \in (\varepsilon, 2\varepsilon), t_{m+1}, \dots, t_r \in [1, 2], t_m > \varepsilon^{1-m}\} \\
 &\subset S'_\varepsilon.
 \end{aligned}$$

Then the inequality

$$\begin{aligned}
 &\sum_{i=1}^r (2a_i - 2)t_i \cdots t_r \\
 &= \left((2a_m - 2) + \sum_{j=1}^{m-1} (2a_j - 2)t_j \cdots t_{m-1} \right) t_m \cdots t_r + \sum_{j=m+1}^r (2a_j - 2)t_j \cdots t_r \\
 &\geq c(m, \varepsilon)t_m - 2^{r-m} \sum_{j=m+1}^r |2a_j - 2|
 \end{aligned}$$

holds on $L_{m,\varepsilon}$. As a consequence, one has

$$\begin{aligned} & \int_{S'_\varepsilon} \prod_{i=1}^r e^{(2a_i-2)t_i \cdots t_r t_i^{l_i+i-1}} dt_1 \wedge \cdots \wedge dt_r \\ & \geq \int_{L_{m,\varepsilon}} \prod_{i=1}^r e^{(2a_i-2)t_i \cdots t_r t_i^{l_i+i-1}} dt_1 \wedge \cdots \wedge dt_r \\ & \gtrsim \int_{\varepsilon^{1-m}}^\infty e^{c(m,\varepsilon)t_m} t_m^{l_m+m-1} dt_m = +\infty \end{aligned}$$

because $c(m, \varepsilon) > 0$.

Step 3: For the converse, we assume that $a_1, \dots, a_r \leq 1$. We aim to demonstrate that the integral (3.4) is finite for every $\varepsilon > 0$ if $l_i \leq -i - 1$ for all $1 \leq i \leq r_0$. Given the assumption that $a_1, \dots, a_r \leq 1$ and $a_{r_0+1} < 1$, we can infer from

$$t_{r_0+1}^{l_{r_0+1}+r_0} \cdots t_r^{l_r+r-1} \lesssim e^{(1-a_{r_0+1})t_{r_0+1} \cdots t_r},$$

where $t_i > \varepsilon$, $i = r_0 + 1, \dots, r$, that

$$(3.5) \quad \prod_{i=r_0+1}^r e^{(2a_i-2)t_i \cdots t_r t_i^{l_i+i-1}} \lesssim e^{(a_{r_0+1}-1)t_{r_0+1} \cdots t_r},$$

where $t_i > \varepsilon$, $i = r_0 + 1, \dots, r$. As a consequence, we obtain that

$$\begin{aligned} & \int_{S'_\varepsilon} \prod_{i=1}^r e^{(2a_i-2)t_i \cdots t_r t_i^{l_i+i-1}} dt_1 \wedge \cdots \wedge dt_r \\ & \leq \int_\varepsilon^\infty \cdots \int_\varepsilon^\infty \prod_{i=1}^r e^{(2a_i-2)t_i \cdots t_r t_i^{l_i+i-1}} dt_1 \cdots dt_r \\ & = \int_\varepsilon^\infty t_1^{l_1} dt_1 \cdots \int_\varepsilon^\infty t_{r_0}^{l_{r_0}+r_0-1} dt_{r_0} \int_\varepsilon^\infty \cdots \\ & \quad \cdot \int_\varepsilon^\infty \prod_{i=r_0+1}^r e^{(2a_i-2)t_i \cdots t_r t_i^{l_i+i-1}} dt_{r_0+1} \cdots dt_r \\ & \lesssim \int_\varepsilon^\infty t_1^{l_1} dt_1 \cdots \int_\varepsilon^\infty t_{r_0}^{l_{r_0}+r_0-1} dt_{r_0} \int_\varepsilon^\infty \cdots \\ (3.6) \quad & \cdot \int_\varepsilon^\infty e^{(a_{r_0+1}-1)t_{r_0+1} \cdots t_r} dt_{r_0+1} \cdots dt_r \quad \text{from (3.5)}. \end{aligned}$$

Since $l_i + i - 1 \leq -2$ for every $1 \leq i \leq r_0$, it follows that

$$\int_\varepsilon^\infty t_i^{l_i+i-1} dt_i < \infty, \quad 1 \leq i \leq r_0.$$

Then the finiteness of the right-hand side of (3.6) is a consequence of Lemma 3.11 below.

Step 4: We assert that if $a_1, \dots, a_r \leq 1$ and $l_m > -m - 1$ for some $1 \leq m \leq r_0$, then the integral (3.4) is infinite for any $0 < \varepsilon < 1$. Let

$$L'_{m,\varepsilon} = \{(t_1, \dots, t_r) \in \mathbb{R}^r \mid t_i \in (1, 2), \forall i \neq m, t_m > 1\} \subset S'_\varepsilon.$$

Then we have

$$\begin{aligned} & \int_{S'_\varepsilon} \prod_{i=1}^r e^{(2a_i-2)t_i \cdots t_r} t_i^{l_i+i-1} dt_1 \wedge \cdots \wedge dt_r \\ & \geq \int_{L'_{m,\varepsilon}} \prod_{i=1}^r e^{(2a_i-2)t_i \cdots t_r} t_i^{l_i+i-1} dt_1 \wedge \cdots \wedge dt_r \\ & = \int_1^2 t_1^{l_1} dt_1 \cdots \int_1^\infty t_m^{l_m+m-1} dt_m \cdots \int_1^2 t_{r_0}^{l_{r_0}+r_0-1} dt_{r_0} \\ & \quad \cdot \int_1^2 \cdots \int_1^2 \prod_{i=r_0+1}^r e^{(2a_i-2)t_i \cdots t_r} t_i^{l_i+i-1} dt_{r_0+1} \cdots dt_r \\ & \gtrsim \int_1^\infty t_m^{l_m+m-1} dt_m, \end{aligned}$$

which is infinite since $l_m + m - 1 \geq -1$. □

Lemma 3.11. *Let $\varepsilon > 0$ and $c > 0$ be constants. Then*

$$\int_\varepsilon^\infty \cdots \int_\varepsilon^\infty e^{-ct_1 \cdots t_r} dt_1 \cdots dt_r < \infty.$$

Proof. We will show that

$$(3.7) \quad \int_\varepsilon^\infty \cdots \int_\varepsilon^\infty e^{-ct_1 \cdots t_k} dt_1 \cdots dt_k < \infty$$

holds by induction on k . The initial case $k = 1$ follows from

$$\int_\varepsilon^\infty e^{-ct_1} dt_1 = -\frac{1}{c} e^{-ct_1} \Big|_\varepsilon^\infty = \frac{1}{c} e^{-c\varepsilon}.$$

Assuming that (3.7) is valid for $k = r - 1$ and any $c > 0$, we proceed to show it holds for $k = r$:

$$\begin{aligned} & \int_\varepsilon^\infty \cdots \int_\varepsilon^\infty e^{-ct_1 \cdots t_r} dt_1 \cdots dt_r \\ & = \int_\varepsilon^\infty \cdots \int_\varepsilon^\infty \left(\int_\varepsilon^\infty e^{-ct_1 \cdots t_r} dt_r \right) dt_1 \cdots dt_{r-1} \end{aligned}$$

$$\begin{aligned}
&= \int_{\varepsilon}^{\infty} \cdots \int_{\varepsilon}^{\infty} \frac{1}{ct_1 \cdots t_{r-1}} e^{-c\varepsilon t_1 \cdots t_{r-1}} dt_1 \cdots dt_{r-1} \\
&\lesssim \int_{\varepsilon}^{\infty} \cdots \int_{\varepsilon}^{\infty} e^{-\frac{1}{2}c\varepsilon t_1 \cdots t_{r-1}} dt_1 \cdots dt_{r-1} \\
&< \infty.
\end{aligned}$$

Thus, the proof is concluded. $\square$

Proposition 3.12. *Let ds^2 be a Hermitian metric on X . Let $E \in \text{Div}_D(X)$ with φ_E as its weight function. Then a holomorphic section s of H lies in $P_{D-E, (2)}(H)$ if and only if $|s|_{e^{-\varphi_E h}}^2 \text{vol}_{ds^2}$ is locally integrable at every point of D .*

Proof. Since the problem is local, let us assume that $X = \Delta_R^r \times \Delta_R^{n-r}$ for some $0 \leq r \leq n$ and $0 < R < 1$. We have $z_1, \dots, z_n$ as its standard holomorphic coordinates, and $D = \bigcup_{i=1}^r D_i$, where $D_i := \{z_i = 0\} \subset X$ for $i = 1, \dots, r$. Let $E = \sum_{i=1}^r k_i D_i$ with $k_1, \dots, k_r \in \mathbb{R}$. We also have $E_0 = \sum_{i=1}^r a_i D_i \in \text{Div}_D(X)$ and $v \in E_0 H_h$ as a holomorphic local section at $\mathbf{0} = (0, \dots, 0)$.

Let N_i be the nilpotent part of the residue map $\text{Res}_{D_i}(\theta) \in \text{End}(E_0 H_h / E_{0-\varepsilon D_i} H_h)$ for $0 < \varepsilon \ll 1$. Let $\{W_m^{(k)}\}_{m \in \mathbb{Z}}$ ($1 \leq k \leq r$) be the filtration on $E_0 H_h$ that induces the monodromy weight filtration on the quotient $E_0 H_h / \sum_{i=1}^k E_{0-\varepsilon D_i} H_h$ for $0 < \varepsilon \ll 1$, associated with the nilpotent operator $N_1 + \cdots + N_k$. Assume the following conditions are met:

- $v \notin E' H_h$ for every $E' \leq E_0$, $E' \neq E_0$;
- $v \in \bigcap_{k=1}^r W_{l_k}^{(k)}$ for some $l_1, \dots, l_r \in \mathbb{Z}$ and $0 \neq [v] \in \text{Gr}_{l_r}^{W^{(r)}} \cdots \text{Gr}_{l_1}^{W^{(1)}}(E_0 H_h)$.

By Theorem 2.8, we can estimate

$$|v|_{e^{-\varphi_E h}}^2 \sim |z_1|^{-2k_1-2a_1} \cdots |z_r|^{-2k_r-2a_r} \left(\frac{\log |z_1|}{\log |z_2|} \right)^{l_1} \cdots (-\log |z_r|)^{l_r}$$

over any region of the form

$$S_{(z_1, \dots, z_r), \varepsilon} = \left\{ (z_1, \dots, z_n) \in (\Delta_R^*)^r \times \Delta_R^{n-r} \mid \frac{\log |z_1|}{\log |z_2|} > \varepsilon, \dots, -\log |z_r| > \varepsilon, (z_{r+1}, \dots, z_n) \in M \right\}$$

for any $\varepsilon > 0$ and an arbitrary compact subset $M \subset \Delta_R^{n-r}$. Let

$$r_0 = \begin{cases} \max\{i = 1, \dots, r \mid k_1 + a_1 = \cdots = k_i + a_i = 1\} & \text{if } k_1 + a_1 = 1, \\ 0 & \text{if } k_1 + a_1 < 1. \end{cases}$$

According to Lemma 3.10, the integral $\int_{S_{(z_1, \dots, z_r), \varepsilon}} |v|_{e^{-\varphi_E h}}^2 \text{vol}_{ds^2}$ is finite near $\mathbf{0}$ for every $\varepsilon > 0$ if and only if

$$k_1 + a_1, \dots, k_r + a_r \leq 1$$

and $l_i \leq -i - 1$ for every $1 \leq i \leq r_0$. Since this statement holds on every sector $S_{(z_{\sigma(1)}, \dots, z_{\sigma(r)}), \varepsilon}$ ($\sigma \in S_r$), it follows that $|v|_{e^{-\varphi_E h}}^2$ is integral near $\mathbf{0}$ if and only if $v \in P_{D-E, (2)}(H)$. This proves the proposition. $\square$

We denote the immersion by $j: X \setminus D \rightarrow X$.

Corollary 3.13.

$$\omega_X \otimes (P_{D-N, (2)}(H) \cap j_* K) \otimes F \otimes L \simeq S_X(K \otimes F|_{X \setminus D} \otimes L|_{X \setminus D}, hh_F h_L).$$

Proof. According to Proposition 3.12, we have a natural isomorphism:

$$\omega_X \otimes (P_{D-N, (2)}(H) \cap j_* K) \simeq S_X(K, e^{-\varphi_N} h).$$

Using Lemma 3.2, we obtain

$$\begin{aligned} \omega_X \otimes (P_{D-N, (2)}(H) \cap j_* K) \otimes F \otimes L &\simeq S_X(K, e^{-\varphi_N} h) \otimes F \otimes L \\ &\simeq S_X(K \otimes F|_{X \setminus D} \otimes L|_{X \setminus D}, hh_F h_L). \quad \square \end{aligned}$$

3.3.3. L^2 interpretation of ${}_{<E}K$. The main result of this section is the L^2 interpretation of ${}_{<E}K := {}_{<E}H \cap j_* K$ (Proposition 3.15), where $j: X \setminus D \rightarrow X$ is the open immersion. Recall that $K \subset H|_{X \setminus D}$ is a locally free subsheaf such that $\nabla^{0,1}(K) = 0$ and $(\nabla - \theta)(K) \subset K \otimes \mathcal{A}_{X \setminus D}^{1,0}$. These assumptions imply that

$$\partial(K) = (\nabla - \theta)(K) \subset K \otimes \mathcal{A}_{X \setminus D}^{1,0}.$$

Hence, the second fundamental form of $K \subset H|_{X \setminus D}$ vanishes. As a consequence, the orthogonal complement of K with respect to h , denoted $K^\perp$, is a holomorphic subbundle of $H|_{X \setminus D}$, and there exists an orthogonal decomposition

$$(3.8) \quad H|_{X \setminus D} = K \oplus K^\perp.$$

To get the L^2 interpretation of ${}_{<E}K$ one needs to study the local weight of ${}_{<E}K$. Let $x \in D$ and let $z_1, \dots, z_n$ be holomorphic local coordinates on some neighborhood U of x in X , such that $D \cap U = \bigcup_{i=1}^r D_i$ and $D_i := \{z_i = 0\} \subset U$ for $i = 1, \dots, r$. Let $E = \sum_{i=1}^r k_i D_i$ with $k_1, \dots, k_r \in \mathbb{R}$. Let N_i be the nilpotent part of the residue map $\text{Res}_{D_i}(\theta) \in \text{End}({}_E H / {}_{E-\varepsilon D_i} H|_U)$ for $0 < \varepsilon \ll 1$. Let $\{W_m^{(k)}\}_{m \in \mathbb{Z}}$ ($1 \leq k \leq r$) be the filtration on ${}_E H|_U$ that induces the monodromy weight filtration on the quotient ${}_E H / \sum_{i=1}^k {}_{E-\varepsilon D_i} H|_U$ for $0 < \varepsilon \ll 1$, associated with the nilpotent operator $N_1 + \dots + N_k$.

Lemma 3.14. *For every $1 \leq k \leq r$, one has*

$$({}_E H \cap j_* K)|_U \cap W_{-1}^{(k)} = {}_{<E} H \cap j_* K|_U.$$

Proof. Notice that (3.8) implies that

$${}_E H_h = {}_E K_h \oplus {}_E K_h^\perp.$$

Since $\bar{\theta}(K) = \nabla^{0,1}(K) - \bar{\partial}(K) = 0$, we have

$$(\theta(\alpha), \beta) = (\alpha, \bar{\theta}(\beta)) = 0, \quad \forall \alpha \in H|_{X \setminus D}, \beta \in K.$$

Therefore,

$$\theta({}_E H_h) \subset {}_E K_h^\perp \otimes \Omega_X(\log D).$$

This implies that

$$\mathrm{Im}(N_1 + \cdots + N_k) \cap \frac{{}_E H \cap j_* K}{\sum_{i=1}^k {}_E {}_{-\varepsilon D_i} H} = 0, \quad 0 < \varepsilon \ll 1.$$

Since

$$W_{-1}^{(k)} / \sum_{i=1}^k {}_E {}_{-\varepsilon D_i} H \subset \mathrm{Im}(N_1 + \cdots + N_k), \quad 0 < \varepsilon \ll 1,$$

one deduces that

$$({}_E H \cap j_* K)|_U \cap W_{-1}^{(k)} = {}_{<E} H \cap j_* K|_U. \quad \square$$

As a direct consequence of Lemma 3.14, the following results hold.

Proposition 3.15. *For every $E \in \mathrm{Div}_D(X)$, then $P_{E,(2)}(H) \cap j_* K = {}_{<E} K$ as subsheaves of ${}_E H$.*

Proposition 3.16. *For every $E \in \mathrm{Div}_D(X)$, the sheaf ${}_{<E} K$ is locally free on X .*

Proof. From the orthogonal decomposition (3.8), we have

$${}_{<E} H = {}_{<E} K \oplus ({}_{<E} H \cap j_*(K^\perp)).$$

Given that ${}_{<E} H$ is locally free (Theorem 2.7), it follows that ${}_{<E} K$ is also locally free. $\square$

3.3.4. Proof of the main theorem. Now we are ready to prove the main theorem of this article.

Theorem 3.17. *Let $f: X \rightarrow Y$ be a proper surjective holomorphic morphism to a complex space. Then $\omega_X \otimes {}_{<D-N} K \otimes F \otimes L$ satisfies Kollár's package with respect to f .*

Proof. Since the Hermitian bundle $(K \otimes F|_{X \setminus D} \otimes L|_{X \setminus D}, hh_F h_L)$ is Nakano semi-positive and tame on X (Lemma 3.9), the sheaf $S_X(K \otimes F|_{X \setminus D} \otimes L|_{X \setminus D}, hh_F h_L)$ satisfies Kollár's package with respect to the morphism f (Theorem 3.4). By Corollary 3.13 and Proposition 3.15, there is a natural isomorphism

$$S_X(K \otimes F|_{X \setminus D} \otimes L|_{X \setminus D}, hh_F h_L) \simeq \omega_X \otimes {}_{<D-N} K \otimes F \otimes L.$$

This establishes the desired result. $\square$

§4. Examples and applications

§4.1. Kollár's package for Kollár–Saito's S -sheaf twisted by an $\mathbb{R}$ -divisor

4.1.1. Complex variation of Hodge structure.

Definition 4.1 ([32, §8]). Let X° be a smooth variety. A *polarized complex variation of Hodge structure* on X° of weight k is a flat holomorphic connection $(\mathcal{V}, \nabla)$ on X° together with a decomposition $\mathcal{V} \otimes_{\mathcal{O}_{X^\circ}} \mathcal{A}_{X^\circ}^0 = \bigoplus_{p+q=k} \mathcal{V}^{p,q}$ of C^∞ bundles and a flat Hermitian form Q on $\mathcal{V}$ such that

- (1) the Hermitian form h_Q which equals $(-1)^p Q$ on $\mathcal{V}^{p,q}$ is a Hermitian metric on the C^∞ complex vector bundle $\mathcal{V} \otimes_{\mathcal{O}_{X^\circ}} \mathcal{A}_{X^\circ}^0$;
- (2) the decomposition $\mathcal{V} \otimes_{\mathcal{O}_{X^\circ}} \mathcal{A}_{X^\circ}^0 = \bigoplus_{p+q=k} \mathcal{V}^{p,q}$ is orthogonal with respect to h_Q ;
- (3) the Griffiths transversality condition

$$(4.1) \quad \nabla(\mathcal{V}^{p,q}) \subset \mathcal{A}^{0,1}(\mathcal{V}^{p+1,q-1}) \oplus \mathcal{A}^{1,0}(\mathcal{V}^{p,q}) \oplus \mathcal{A}^{0,1}(\mathcal{V}^{p,q}) \oplus \mathcal{A}^{1,0}(\mathcal{V}^{p-1,q+1})$$

holds for every p and q . Here, $\mathcal{A}^{i,j}(\mathcal{V}^{p,q})$ denotes the sheaf of smooth (i, j) -forms with values in $\mathcal{V}^{p,q}$.

Let X be a smooth projective variety and $\bigcup_{i=1}^l D_i = D := X \setminus X^\circ \subset X$ be a simple normal crossing divisor, where $D_1, \dots, D_l$ are irreducible components. Let $\mathbb{V} = (\mathcal{V}, \nabla, \{\mathcal{V}^{p,q}\}, Q)$ be a polarized complex variation of Hodge structure on $X^\circ := X \setminus D$. Take the decomposition

$$\nabla = \bar{\theta} + \partial + \bar{\partial} + \theta$$

according to (4.1). Let $H = \ker(\bar{\partial}: \mathcal{V} \otimes_{\mathcal{O}_{X^\circ}} \mathcal{A}_{X^\circ}^0 \rightarrow \mathcal{V} \otimes_{\mathcal{O}_{X^\circ}} \mathcal{A}_{X^\circ}^{0,1})$. Then triple (H, θ, h_Q) is a harmonic bundle associated with $(\mathcal{V}, \nabla, h_Q)$ through Simpson's correspondence [32, §8]. Furthermore, there is an orthogonal decomposition of holomorphic subbundles:

$$H = \bigoplus_{p+q=k} H^{p,q},$$

where $H^{p,q} = H \cap \mathcal{V}^{p,q}$. In addition, we have

$$\theta(H^{p,q}) \subset H^{p-1,q+1} \otimes \Omega_{X^\circ}.$$

Proposition 4.2 ([31, Proposition 5.4]). *The harmonic bundle (H, θ, h_Q) is tame on X .*

Combining the above proposition with Theorems 2.7 and 2.9, we can conclude that $(H_{\mathbb{V}} := \bigcup_{E \in \text{Div}_D(X)} {}_E H_{h_Q}, \{ {}_E H_{h_Q} \}_{E \in \text{Div}_D(X)}, \theta)$ is a μ_A -polystable locally abelian parabolic Higgs bundle on (X, D) . It is called the locally abelian parabolic Higgs bundle associated with $\mathbb{V}$.

Let $S(\mathbb{V}) := \ker(\nabla^{0,1}) \cap \mathcal{V}^{p_{\max}, k-p_{\max}}$, where $p_{\max} = \max\{p \mid \mathcal{V}^{p, k-p} \neq 0\}$ and $\nabla = \nabla^{1,0} + \nabla^{0,1}$ is the bi-degree decomposition. The following lemma can be easily derived from Definition 4.1.

Lemma 4.3. *The locally free subsheaf $S(V) \subset H_V|_{X \setminus D}$ satisfies $\nabla^{0,1}(S(\mathbb{V})) = 0$, $\bar{\partial}(S(\mathbb{V})) = 0$ and $(\nabla - \theta)(S(\mathbb{V})) \subset S(\mathbb{V}) \otimes \mathcal{A}_{X \setminus D}^{1,0}$.*

4.1.2. Twisted Kollár–Saito S -sheaf. In this section we introduce a construction that combines Kollár–Saito’s $S_X(\mathbb{V})$ and the multiplier ideal sheaf associated with an $\mathbb{R}$ -divisor. Additionally, we generalize Kollár–Saito’s construction to complex variations of Hodge structure, without making assumptions on the local monodromy. This is interesting from the perspective of nonabelian Hodge theory, as complex variations of Hodge structure are precisely the $\mathbb{C}^*$ fixed points on the moduli space of certain tame harmonic bundles (see [33, Theorem 8], [19, Proposition 1.9]).

Let X be a projective variety and $X^\circ \subset X_{\text{reg}}$ be a dense Zariski open subset. Let $\mathbb{V} = (\mathcal{V}, \nabla, \{\mathcal{V}^{p,q}\}, Q)$ be a polarized complex variation of Hodge structure on X° . Let h_Q be the Hodge metric associated with Q and let

$$\left(H_{\mathbb{V}} := \bigcup_{E \in \text{Div}_D(X)} {}_E H_{h_Q}, \{ {}_E H_{h_Q} \}_{E \in \text{Div}_D(X)}, \theta \right)$$

be the μ_A -polystable locally abelian parabolic Higgs bundle on (X, D) associated with $\mathbb{V}$. Let N be an $\mathbb{R}$ -Cartier divisor on X . We define the coherent sheaf $S_X(\mathbb{V}, -N)$ as follows.

Log smooth case. Assuming that X is smooth and $D := X \setminus X^\circ$ is a simple normal crossing divisor with $\text{supp}(N) \subset D$, we denote the irreducible decomposition of D as $D = \bigcup_{i=1}^l D_i$, and we write $N = \sum_{i=1}^l r_i D_i$ where $r_1, \dots, r_l \in \mathbb{R}$. For $\mathbf{r} = (r_1, \dots, r_l)$, we let $\mathcal{V}_{>\mathbf{r}-1}$ be the Deligne–Manin prolongation with indices $> \mathbf{r} - 1$. This is a locally free $\mathcal{O}_X$ -module that extends

$\mathcal{V}$ in such a way that ∇ induces a connection with logarithmic singularities given by

$$\nabla: \mathcal{V}_{>\mathbf{r}-1} \rightarrow \mathcal{V}_{>\mathbf{r}-1} \otimes \Omega_X(\log D),$$

where the real part of the eigenvalues of the residue of ∇ along D_i belongs to $(r_i - 1, r_i]$ for each i . We can now define $S_X(\mathbb{V}, -N)$ as

$$(4.2) \quad \begin{aligned} S_X(\mathbb{V}, -N) &:= \omega_X \otimes (j_* S(\mathbb{V}) \cap \mathcal{V}_{>\mathbf{r}-1}) \\ &\simeq \omega_X \otimes (j_* S(\mathbb{V}) \cap {}_{<D-N} H_{h_Q}). \end{aligned}$$

Here, $j: X^o \rightarrow X$ is the open immersion.

General case. Let $\pi: \tilde{X} \rightarrow X$ be a resolution of singularities such that $\pi^o := \pi|_{\pi^{-1}(X^o \setminus \text{supp}(N))}: \pi^{-1}(X^o \setminus \text{supp}(N)) \rightarrow X^o \setminus \text{supp}(N)$ is biholomorphic and the exceptional loci $E := \pi^{-1}((X \setminus X^o) \cup \text{supp}(N))$ is a simple normal crossing divisor. Then

$$S_X(\mathbb{V}, -N) \simeq \pi_*(S_{\tilde{X}}(\pi^{o*}\mathbb{V}, -\pi^*N)).$$

Proposition 4.4. *We have $S_X(\mathbb{V}, -N) \simeq S_X(S(\mathbb{V}), e^{-\varphi_N} h_Q)$, where φ_N is a weight function associated with N . In particular, $S_X(\mathbb{V}, -N)$ is a torsion-free coherent sheaf on X that is independent of the choice of the desingularization $\pi: \tilde{X} \rightarrow X$.*

Proof. Let $\pi: \tilde{X} \rightarrow X$ be a resolution of singularities such that

$$\pi^o := \pi|_{\pi^{-1}(X^o \setminus \text{supp}(N))}: \pi^{-1}(X^o \setminus \text{supp}(N)) \rightarrow X^o \setminus \text{supp}(N)$$

is biholomorphic and the reduced exceptional loci $E := \pi^{-1}((X \setminus X^o) \cup \text{supp}(N))$ is a simple normal crossing divisor. It follows from Proposition 3.15 and (4.2) that

$$\begin{aligned} S_{\tilde{X}}(\pi^{o*}\mathbb{V}, -\pi^*(N)) &\simeq \omega_{\tilde{X}} \otimes (P_{E-\pi^*N, (2)}(H_{\pi^{o*}\mathbb{V}}) \cap j_* S(\pi^{o*}\mathbb{V})) \\ &\simeq S_{\tilde{X}}(S(\pi^{o*}\mathbb{V}), e^{-\pi^*(\varphi_N)} \pi^{o*} h_Q) \quad (\text{Proposition 3.12}), \end{aligned}$$

where $j: \tilde{X} \setminus E \rightarrow \tilde{X}$ is an immersion. Applying π_* on both sides, we get

$$\begin{aligned} S_X(\mathbb{V}, -N) &\simeq \pi_*(S_{\tilde{X}}(\pi^{o*}\mathbb{V}, -\pi^*(N))) \\ &\simeq \pi_*(S_{\tilde{X}}(S(\pi^{o*}\mathbb{V}), e^{-\pi^*(\varphi_N)} \pi^{o*} h_Q)) \\ &\simeq S_X(S(\mathbb{V}), e^{-\varphi_N} h_Q) \quad (\text{Lemma 3.1}). \quad \square \end{aligned}$$

Theorem 4.5. *Let $f: X \rightarrow Y$ be a proper surjective holomorphic morphism from a projective variety X to a complex space Y . Let $\mathbb{V}$ be a polarized complex variation of Hodge structure on some regular dense Zariski open subset $X^o \subset X$. Let L be a line bundle on X such that $L \simeq_{\mathbb{R}} B + N$, where B is a semipositive $\mathbb{R}$ -Cartier*

divisor and N is an $\mathbb{R}$ -Cartier divisor on X . Let F be an arbitrary Nakano semi-positive vector bundle on X . Then $S_X(\mathbb{V}, -N) \otimes F \otimes L$ satisfies Kollár's package with respect to f .

Proof. Let $\pi: \tilde{X} \rightarrow X$ be a resolution of singularities such that

$$\pi^o := \pi|_{\pi^{-1}(X^o \setminus \text{supp}(N))}: \pi^{-1}(X^o \setminus \text{supp}(N)) \rightarrow X^o \setminus \text{supp}(N)$$

is biholomorphic and the reduced exceptional loci $E := \pi^{-1}((X \setminus X^o) \cup \text{supp}(N))$ is a simple normal crossing divisor. Let $j: \tilde{X} \setminus E \rightarrow \tilde{X}$ be an immersion. Equation (4.2) tells us that

$$S_{\tilde{X}}(\pi^{o*}\mathbb{V}, -\pi^*N) \simeq \omega_{\tilde{X}} \otimes_{<E-\pi^*N} S(\pi^{o*}\mathbb{V}).$$

By applying Theorem 1.1 (torsion-freeness) to $\pi: \tilde{X} \rightarrow X$, we obtain

$$\begin{aligned} S_X(\mathbb{V}, -N) \otimes F \otimes L &\simeq \pi_*(S_{\tilde{X}}(\pi^{o*}\mathbb{V}, -\pi^*N) \otimes \pi^*F \otimes \pi^*L) \\ &\simeq R\pi_*(S_{\tilde{X}}(\pi^{o*}\mathbb{V}, -\pi^*N) \otimes \pi^*F \otimes \pi^*L). \end{aligned}$$

Therefore,

$$Rf_*(S_X(\mathbb{V}, -N) \otimes F \otimes L) \simeq R(f \circ \pi)_*(\omega_{\tilde{X}} \otimes_{<E-\pi^*N} S(\pi^{o*}\mathbb{V}) \otimes \pi^*F \otimes \pi^*L).$$

By Lemma 4.3, the theorem follows by applying Theorem 1.1 to

$$\omega_{\tilde{X}} \otimes_{<E-\pi^*N} S(\pi^{o*}\mathbb{V}) \otimes \pi^*F \otimes \pi^*L$$

with respect to the morphism $f \circ \pi$. $\square$

Remark 4.6. Similar results hold for the S -sheaf twisted by a multiplier ideal sheaf associated with an ideal. Let $c > 0$ be a real number and $\mathfrak{a}$ be a coherent ideal sheaf on X . Consider a desingularization $\pi: \tilde{X} \rightarrow X$ such that $\mathfrak{a}\mathcal{O}_{\tilde{X}} = \mathcal{O}_{\tilde{X}}(-E)$, where $E \geq 0$ is a simple normal crossing divisor, and $\pi^*(X \setminus X^o)$ is a simple normal crossing divisor that intersects transversally with E . We define $S_X(\mathbb{V}, -c\mathfrak{a})$ as $\pi_*(S_{\tilde{X}}(\pi^*\mathbb{V}, -cE))$. Theorem 1.3 provides relevant results for $S_X(\mathbb{V}, -c\mathfrak{a})$.

§4.2. Multiplier Grauert–Riemenschneider sheaf

When $\mathbb{V} = \mathbb{C}_{X_{\text{reg}}}$ is the trivial variation of Hodge structure and N is an $\mathbb{R}$ -Cartier divisor on X , $S_X(\mathbb{C}_{X_{\text{reg}}}, -N)$ is precisely the Grauert–Riemenschneider sheaf twisted by the multiplier ideal sheaf associated with N when $N \geq 0$. This is referred to as the multiplier ideals by Viehweg [36, 37], and it also appears in the Nadel vanishing theorem on complex spaces [6]. Let us briefly describe its construction for the convenience of the readers.

Log smooth case. If X is smooth and $\text{supp}(N)$ is a simple normal crossing divisor, then

$$\mathcal{K}_X(-N) := \omega_X \otimes \mathcal{O}_X(-\lfloor N \rfloor).$$

General case. Let $\pi: \tilde{X} \rightarrow X$ be a proper bimeromorphic morphism such that $\pi^\circ := \pi|_{\pi^{-1}(X_{\text{reg}} \setminus \text{supp}(N))}: \pi^{-1}(X_{\text{reg}} \setminus \text{supp}(N)) \rightarrow X_{\text{reg}} \setminus \text{supp}(N)$ is biholomorphic and the exceptional loci $E := \pi^{-1}((X \setminus X_{\text{reg}}) \cup \text{supp}(N))$ is a simple normal crossing divisor. Then

$$\mathcal{K}_X(-N) := \pi_*(\mathcal{K}_{\tilde{X}}(-\pi^*N)).$$

Certainly, $\mathcal{K}_X(-N) \simeq S_X(\mathbb{C}_{X_{\text{reg}}}, -N)$. Proposition 4.4 implies that $\mathcal{K}_X(-N)$ is independent of the choice of the desingularization. When X is smooth and $N \geq 0$, we have

$$\mathcal{K}_X(-N) \simeq \omega_X \otimes \mathcal{I}(-N),$$

where $\mathcal{I}(-N)$ is the multiplier ideal sheaf associated with N . According to Theorem 1.3, we have the following result.

Theorem 4.7. *Let $f: X \rightarrow Y$ be a proper surjective holomorphic morphism from a projective variety X to a complex space Y . Let L be a line bundle on X such that $L \simeq_{\mathbb{R}} B + N$, where B is a semipositive $\mathbb{R}$ -Cartier divisor and N is an $\mathbb{R}$ -Cartier divisor on X . Let F be an arbitrary Nakano semipositive vector bundle on X . Then $\mathcal{K}_X(-N) \otimes F \otimes L$ satisfies Kollár's package with respect to f .*

Theorem 4.7 has an application in Kollár's package of pluricanonical bundles.

Corollary 4.8. *Let $f: X \rightarrow Y$ be a proper surjective holomorphic morphism from a smooth projective variety X to a complex space Y . Let K_X be the canonical divisor of X and $\omega_X = \mathcal{O}_X(K_X)$. Suppose A is a semipositive line bundle on X and $V \subset |kmK_X - A|$ is a linear series for some positive integers k and m . Let F be an arbitrary Nakano semipositive vector bundle on X . Then the tensor product $\omega_X^{\otimes(k+1)} \otimes \mathcal{I}(\frac{1}{m}|V|) \otimes F$ satisfies Kollár's package with respect to f .*

Proof. Let $\mathfrak{a}$ be the ideal sheaf of the base scheme of V . Consider a principalization $\pi: \tilde{X} \rightarrow X$ of the ideal sheaf $\mathfrak{a}$, which satisfies the condition that $\mathfrak{a}\mathcal{O}_{\tilde{X}} = \mathcal{O}_{\tilde{X}}(-E)$, where E is a π -exceptional divisor with $E \geq 0$. From this, we have $\pi^*(kmK_X - A) = B + E$, where B is a semi-ample divisor. Let $L = \pi^*(\omega_X^{\otimes k})$. Then we have $L \simeq_{\mathbb{Q}} \frac{1}{m}(\pi^*A + B) + \frac{1}{m}E$. By the functoriality of multiplier ideal sheaves, we obtain

$$\omega_X^{\otimes(k+1)} \otimes \mathcal{I}\left(\frac{1}{m}|V|\right) \otimes F \simeq \pi_*\left(\omega_{\tilde{X}} \otimes L \otimes \mathcal{I}\left(\frac{1}{m}E\right) \otimes \pi^*F\right).$$

Notice that $\frac{1}{m}(\pi^*A + B)$ is semipositive. By applying Theorem 1.5 to π , we have

$$\begin{aligned}\omega_X^{\otimes(k+1)} \otimes \mathcal{I}\left(\frac{1}{m}|V|\right) \otimes F &\simeq \pi_*\left(\omega_{\tilde{X}} \otimes L \otimes \mathcal{I}\left(\frac{1}{m}E\right) \otimes \pi^*F\right) \\ &\simeq R\pi_*\left(\omega_{\tilde{X}} \otimes L \otimes \mathcal{I}\left(\frac{1}{m}E\right) \otimes \pi^*F\right).\end{aligned}$$

Therefore, we have

$$Rf_*\left(\omega_X^{\otimes(k+1)} \otimes \mathcal{I}\left(\frac{1}{m}|V|\right) \otimes F\right) \simeq R(f \circ \pi)_*\left(\omega_{\tilde{X}} \otimes L \otimes \mathcal{I}\left(\frac{1}{m}E\right) \otimes \pi^*F\right).$$

Therefore, the corollary follows by applying Theorem 1.5 to $\omega_{\tilde{X}} \otimes L \otimes \mathcal{I}(\frac{1}{m}E) \otimes \pi^*F$ with respect to the morphism $f \circ \pi$. $\square$

§4.3. Semipositivity of higher direct images

Let Y be a smooth projective variety and F be a torsion-free sheaf on Y . We say that F is weakly positive (in the sense of Viehweg [35]) on some Zariski open subset $U \subset X$ if, for every ample line bundle A on Y and every $a \in \mathbb{Z}_{>0}$, there exists $b \in \mathbb{Z}_{>0}$ such that $S^{[ab]}F \otimes A^b$ is generated by global sections at each point of U . In this context, $S^{[ab]}F$ refers to the reflexive hull of the symmetric power $S^{ab}F$. We can say that F is *weakly positive* if it is weakly positive over some Zariski open subset. As noted in [35, Remark 1.3], it is sufficient to check this definition for a fixed line bundle A (which does not necessarily need to be ample, as seen in [36, Lemma 2.14]).

Theorem 4.9. *With notation as in Theorem 1.1, let $f: X \rightarrow Y$ be a surjective morphism between smooth projective varieties. Then $R^q f_*(\omega_{X/Y} \otimes_{<D-N} K \otimes F \otimes L)$ is torsion-free and weakly positive.*

Proof. Let us denote the sheaf $_{<D-N} K \otimes F \otimes L$ by $\mathcal{E}$ for simplicity. Additionally, let A_Y be a line bundle on Y that is very ample. Our goal is to demonstrate that, for every $m > 0$, the sheaf

$$S^{[m]}R^q f_*(\omega_{X/Y} \otimes \mathcal{E}) \otimes \omega_Y \otimes A_Y^{\dim Y + 1}$$

is generated by global sections over a Zariski open subset of Y .

Step 1: Let Δ_Y be the set of critical values of f . By performing possibly blow-ups $\rho: X' \rightarrow X$, we can assume that $\rho^*(f^*\Delta_Y) \cup \rho^*(D)$ forms a divisor with simple normal crossings. Using Lemma 3.1, Corollary 3.13 and Proposition 3.15, we have

$$\omega_X \otimes \mathcal{E} \simeq \rho_*(\omega_{X'} \otimes_{<\rho^*(D-N)} \rho^*K \otimes \rho^*F \otimes \rho^*L),$$

where $j': X' \setminus \rho^* D \rightarrow X'$ is the open immersion. By applying Theorem 1.1 (torsion-freeness) to the morphism $\rho: X' \rightarrow X$, we can conclude that

$$\omega_X \otimes \mathcal{E} \simeq R\rho_*(\omega_{X'} \otimes {}_{<\rho^*(D-N)}\rho^* K \otimes \rho^* F \otimes \rho^* L).$$

Therefore, there exists an isomorphism:

$$R^q(f \circ \rho)_*(\omega_{X'} \otimes {}_{<\rho^*(D-N)}\rho^* K \otimes \rho^* F \otimes \rho^* L) \simeq R^q f_*(\omega_X \otimes \mathcal{E})$$

for every q . As a result, we can assume that $f^*(\Delta_Y) \cup D$ forms a divisor with a simple normal crossing on X for the rest of the proof. Specifically, we assume that f is flat in codimension 1 on Y .

Step 2: Viehweg's trick. We denote the main component of the m -th fiber product $X \times_Y \cdots \times_Y X$ by $X^{[m]}$, and let $f^{[m]}: X^{[m]} \rightarrow Y$ denote the projection map. Let $D^{[m]} := \sum_{i=1}^m p_i^*(D)$, where $p_i: X^{[m]} \rightarrow X$ is the projection map to the i -th component. Consider the largest Zariski open subset $U \subset Y$, where (X, D) is log smooth over U . Then, $f^{[m]}: (X^{[m]}, D^{[m]}) \rightarrow Y$ is log smooth over U . Let $X^{[m]o} := (f^{[m]})^{-1}(U)$. Let $\pi: X^{(m)} \rightarrow X^{[m]}$ be a resolution of singularities such that $\pi^*(X^{[m]} \setminus X^{[m]o})$ is a simple normal crossing divisor, and π is an isomorphism on $X^{(m)o} := \pi^*(X^{[m]o})$.

According to Proposition 3.16, $\mathcal{E}$ is a locally free $\mathcal{O}_X$ -module. Let $hh_F h_L$ be the singular Hermitian metric on $\mathcal{E}$ as introduced in Section 3.3.1. It follows from Theorem 2.8 and Lemma 3.14 that there exists a C^∞ Hermitian metric h_0 on $\mathcal{E}$ such that $h_0 \lesssim hh_F h_L$. We define $\mathcal{E}^{[m]} := \bigotimes_{i=1}^m p_i^* \mathcal{E}$ and $\mathcal{E}^{(m)} := \pi^* \mathcal{E}^{[m]}$. Let $V \subset Y$ be a Zariski open subset such that $R^q f_*(\omega_X \otimes \mathcal{E})$ is locally free on V for every $q \geq 0$, and f is flat over V . Since $R^q f_*(\omega_X \otimes \mathcal{E})$ is torsion-free for each $q \geq 0$ (Theorem 1.1), we can assume that $Y \setminus V$ is of codimension ≥ 2 for simplicity.

Since $f: X \rightarrow Y$ is a Gorenstein morphism, we can conclude that $f^{[m]}: X^{[m]} \rightarrow Y$ is also a Gorenstein morphism. Moreover, there exists an isomorphism

$$\omega_{X^{[m]}/Y} \simeq \bigotimes_{i=1}^m p_i^* \omega_{X/Y}$$

over $f^{[m]*}(V)$, where $\omega_{X^{[m]}/Y}$ is the relative dualizing sheaf. Additionally, there is an injective morphism given by the trace map:

$$(4.3) \quad \mathrm{tr}: \rho_*(\omega_{X^{(m)}/Y}) \rightarrow \omega_{X^{[m]}/Y}.$$

By applying the flat base change theorem, we obtain an isomorphism in the derived category $D(V)$:

$$Rf_*^{[m]}(\omega_{X^{[m]}/Y} \otimes \mathcal{E}^{[m]})|_V \simeq \bigotimes_{i=1}^m Rf_*(\omega_X \otimes \mathcal{E})|_V.$$

Since every $R^q f_*(\omega_X \otimes \mathcal{E})|_V$ is locally free, it follows that $\bigotimes^m R^q f_*(\omega_X \otimes \mathcal{E})|_V$ is isomorphic to a direct summand of $R^{mq} f_*^{[m]}(\omega_{X^{[m]}/Y} \otimes \mathcal{E}^{[m]})|_V$. Hence, we have a surjective morphism

$$R^{mq} f_*^{[m]}(\omega_{X^{[m]}/Y} \otimes \mathcal{E}^{[m]})|_V \rightarrow \bigotimes^m R^q f_*(\omega_X \otimes \mathcal{E})|_V.$$

By taking reflexive hulls, we obtain a morphism:

$$(4.4) \quad \alpha: R^{mq} f_*^{[m]}(\omega_{X^{[m]}/Y} \otimes \mathcal{E}^{[m]}) \rightarrow \bigotimes^{[m]} R^q f_*(\omega_X \otimes \mathcal{E}) \rightarrow S^{[m]} R^q f_*(\omega_X \otimes \mathcal{E}),$$

which is surjective over V . Here, $\bigotimes^{[m]} R^q f_*(\omega_X \otimes \mathcal{E})$ denotes the reflexive hull of $\bigotimes^m R^q f_*(\omega_X \otimes \mathcal{E})$.

Let $h_{\mathcal{E}^{[m]}} = \bigotimes_{i=1}^m p_i^*(hh_F h_L)$, $h_0^{[m]} = \bigotimes_{i=1}^m p_i^*(h_0)$, $h_{\mathcal{E}^{(m)}} = \rho^*(h_{\mathcal{E}^{[m]}})$ and $h_0^{(m)} = \rho^*(h_0^{[m]})$. Then $h_0^{(m)}$ is a smooth Hermitian metric on $\mathcal{E}^{(m)}$, and $h_{\mathcal{E}^{(m)}}$ is a singular Hermitian metric on $\mathcal{E}^{(m)}$ satisfying $h_0^{(m)} \lesssim h_{\mathcal{E}^{(m)}}$. Therefore, we have the following inclusion:

$$S_{X^{(m)}}(\mathcal{E}^{(m)}, h_{\mathcal{E}^{(m)}}) \subset S_{X^{(m)}}(\mathcal{E}^{(m)}, h_0^{(m)}) \simeq \omega_{X^{(m)}} \otimes \mathcal{E}^{(m)}.$$

By combining this with the trace map (4.3), we obtain an injective morphism:

$$(4.5) \quad \beta: \rho_*(S_{X^{(m)}}(\mathcal{E}^{(m)}, h_{\mathcal{E}^{(m)}})) \rightarrow \omega_{X^{[m]}/Y} \otimes \mathcal{E}^{[m]} \otimes f^{[m]*}(\omega_Y).$$

Since $hh_F h_L$ is Nakano semipositive over some Zariski open subset of X and tame on X (Lemma 3.9), $h_{\mathcal{E}^{(m)}}$ is Nakano semipositive over some Zariski open subset of $X^{(m)}$ and tame on $X^{(m)}$. Applying Theorem 3.4, we have

$$\rho_*(S_{X^{(m)}}(\mathcal{E}^{(m)}, h_{\mathcal{E}^{(m)}})) \simeq R\rho_*(S_{X^{(m)}}(\mathcal{E}^{(m)}, h_{\mathcal{E}^{(m)}})).$$

Therefore, (4.5) induces a morphism:

$$(4.6) \quad \beta': R^{mq} f_*^{(m)}(S_{X^{(m)}}(\mathcal{E}^{(m)}, h_{\mathcal{E}^{(m)}})) \rightarrow R^{mq} f_*^{[m]}(\omega_{X^{[m]}/Y} \otimes \mathcal{E}^{[m]}) \otimes \omega_Y,$$

where $f^{(m)}$ denotes $f^{[m]} \circ \rho$. By applying Corollary 3.13 on $X^{[m]o}$, it can be seen that β is also an isomorphism over $X^{[m]o} = (f^{[m]})^{-1}(U)$. Therefore, β' is an isomorphism over U . By combining (4.6) with (4.4), we obtain a morphism

$$(4.7) \quad R^{mq} f_*^{(m)}(S_{X^{(m)}}(\mathcal{E}^{(m)}, h_{\mathcal{E}^{(m)}})) \rightarrow S^{[m]} R^q f_*(\omega_X \otimes \mathcal{E}) \otimes \omega_Y$$

which is surjective over $U \cap V$.

Step 3: Castelnuovo–Mumford’s criterion. Applying Theorem 3.4 to $S_{X^{(m)}}(\mathcal{E}^{(m)}, h_{\mathcal{E}^{(m)}})$ yields the following result:

$$H^i(Y, R^{mq} f_*^{(m)}(S_{X^{(m)}}(\mathcal{E}^{(m)}, h_{\mathcal{E}^{(m)}})) \otimes A_Y^{\dim Y + 1} \otimes A_Y^{-i}) = 0, \quad \forall i > 0.$$

This means that $R^{mq}f_*^{(m)}(S_{X^{(m)}}(\mathcal{E}^{(m)}, h_{\mathcal{E}^{(m)}})) \otimes A_Y^{\dim Y+1}$ is 0-regular and therefore generated by global sections. Combining this with the generic surjective map (4.7), we can conclude that $S^{[m]}R^q f_*(\omega_{X/Y} \otimes \mathcal{E}) \otimes \omega_Y \otimes A_Y^{\dim Y+1}$ is generated by global sections at every point of $V \cap U$. This demonstrates that $R^q f_*(\omega_{X/Y} \otimes \mathcal{E})$ is weakly positive. $\square$

§4.4. Generic vanishing theorem

Definition 4.10. Let A be an abelian variety. A coherent sheaf F on A is called a *GV-sheaf* if

$$\mathrm{codim}_{\mathrm{Pic}^0(A)} \{M \in \mathrm{Pic}^0(A) \mid H^i(A, F \otimes M) \neq 0\} \geq i$$

for every i .

Let us recall Hacon's criterion for a GV-sheaf [9] (see also [30, Theorem 25.5] and [25]).

Lemma 4.11. *Suppose that for every finite étale morphism $\varphi: B \rightarrow A$ of abelian varieties, and every ample line bundle L on B , one has $H^i(B, \varphi^*(F) \otimes L) = 0$ for $i > 0$. Then F is a GV-sheaf.*

Theorem 4.12. *Notation is as in Theorem 1.1. Let $f: X \rightarrow A$ be a morphism to an abelian variety. Then $R^q f_*(\omega_X \otimes {}_{<D-N}K \otimes F \otimes L)$ is a GV-sheaf for every $q \geq 0$. As a consequence,*

$$\begin{aligned} & \mathrm{codim}_{\mathrm{Pic}^0(A)} \{M \in \mathrm{Pic}^0(A) \mid H^i(X, \omega_X \otimes {}_{<D-N}K \otimes F \otimes L \otimes f^*M) \neq 0\} \\ & \geq i - (\dim X - \dim f(X)) \end{aligned}$$

for all i .

Proof. Let $\varphi: B \rightarrow A$ be a finite étale morphism between abelian varieties. We define $Z := X \times_A B$. Then we have the following commutative diagram:

$$\begin{array}{ccc} Z & \xrightarrow{\psi} & X \\ \downarrow g & & \downarrow f \\ B & \xrightarrow{\varphi} & A. \end{array}$$

Since ψ is étale, $D' := \psi^*(D)$ is a reduced simple normal crossing divisor on Z . Moreover, we have the following isomorphism:

$$\psi^*({}_{<D-N}K) \simeq {}_{<D'-\psi^*N}\psi^*K.$$

This implies that there is an isomorphism:

$$\varphi^* R^q f_*(\omega_X \otimes {}_{<D-N} K \otimes F \otimes L) \simeq R^q g_*(\omega_Z \otimes {}_{<D'-\psi^* N} \psi^* K \otimes \psi^* F \otimes \psi^* L).$$

Consequently, using Theorem 1.1, the conditions in Lemma 4.11 are satisfied.

This establishes the first claim of the theorem. The second claim can be derived by applying the first claim to the spectral sequence:

$$\begin{aligned} E_2^{p,q} &:= H^q(A, R^p f_*(\omega_X \otimes {}_{<D-N} K \otimes F \otimes L) \otimes M) \\ &\Rightarrow H^{p+q}(X, \omega_X \otimes {}_{<D-N} K \otimes F \otimes L \otimes f^* M), \end{aligned}$$

where $M \in \text{Pic}^0(A)$. It should be noted that $R^p f_*(\omega_X \otimes {}_{<D-N} K \otimes F \otimes L) = 0$ for every $p > \dim X - \dim f(X)$, as mentioned in Theorem 1.1. $\square$

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