

Mappings Preserving Quantum Rényi's Entropies in Von Neumann Algebras

by

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Abstract

We investigate the situation when a normal positive linear unital map on a semifinite von Neumann algebra leaving the trace invariant does not change fixed quantum Rényi entropy of the density of a normal state. It is also shown that such a map does not change the entropy of any density if and only if it is a Jordan $*$ -isomorphism on the algebra.

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§1. Introduction

In the paper, the question of invariance of quantum Rényi entropies of a density under the action of a normal positive linear unital map Φ is addressed in the case of a semifinite von Neumann algebra. This can be seen as a follow-up to the investigation in [3] where such a problem was analyzed for the Segal entropy. It is interesting that in both cases, Segal's entropy and any single Rényi entropy, the answer is the same, namely, Φ restricted to the algebra generated by the density is a $*$ -isomorphism. The question of invariance of Rényi's entropy for every density is also considered, and it turns out that this holds if and only if Φ is a Jordan $*$ -isomorphism on the algebra. At this point, it should be noted that interesting

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and in many respects similar results are given in [1]. Namely, the authors prove there that for a unital JB- or C*-algebra $\mathcal{A}$, and a faithful state ρ on $\mathcal{A}$ such that $\rho \circ \Phi = \rho$, the preservation of the Rényi or Segal entropy for some $0 \leq a \in \mathcal{A}$, defined in an obvious way for the state ρ , implies that Φ is a Jordan homomorphism on the algebra JB(a) or C*(a), respectively (consequently, if it holds for all $0 \leq a \in \mathcal{A}$ then Φ is a Jordan homomorphism on $\mathcal{A}$). Another interesting result shown in [1] states that if

$$\Phi(f(a)) = f(\Phi(a))$$

for a holomorphic function f , a continuous linear unital map Φ , and $a \in \mathcal{A}$ such that $f(a)$ is well defined, then Φ is again a Jordan homomorphism. Such an approach using holomorphic functions is ingenious and fruitful. Of course, considering elements of a JB- or C*-algebra deals, so to speak, with a “bounded” case, while in our setup we deal with an “unbounded” one considering densities which can be unbounded operators. For such densities a number of specific Jensen inequalities are derived. These inequalities are well known, in a more general form, for bounded operators.

It is interesting to observe that passing from the case of bounded densities to unbounded ones requires some arguments referring to the algebra of measurable operators, in particular, convergence in measure for such operators is widely exploited.

§2. Preliminaries and notation

Let $\mathcal{M}$ be a semifinite von Neumann algebra of operators acting on a Hilbert space $\mathcal{H}$ with a normal semifinite faithful trace τ , identity $\mathbb{1}$, and predual $\mathcal{M}_*$. By $\mathcal{M}^+$ we shall denote the set of positive operators in $\mathcal{M}$, and by $\mathcal{M}_*^+$ the set of positive functionals in $\mathcal{M}_*$. These functionals will sometimes be referred to as (nonnormalized) states.

The algebra of *measurable operators* $\widetilde{\mathcal{M}}$ is defined as a topological *-algebra of densely defined closed operators on $\mathcal{H}$ affiliated with $\mathcal{M}$ with strong addition $\dot{+}$ and strong multiplication $\cdot$, i.e.,

$$x \dot{+} y = \overline{x + y}, \quad x \cdot y = \overline{xy}, \quad x, y \in \widetilde{\mathcal{M}},$$

where $\overline{x + y}$ and $\overline{xy}$ are the closures of the corresponding operators defined by addition and composition respectively on the natural domains given by the intersections of the domains of the x and y and of the range of y and the domain of x respectively. In what follows, we shall omit the dot in the symbols of these operations and write simply $x + y$ and xy to denote $x \dot{+} y$ and $x \cdot y$. The translation-invariant measure topology is defined by a fundamental system of neighborhoods

of 0, $\{N(\varepsilon, \delta) : \varepsilon, \delta > 0\}$, given by

$$N(\varepsilon, \delta) = \{x \in \widetilde{\mathcal{M}} : \text{there exists a projection } p \text{ in } \mathcal{M} \text{ such that} \\ xp \in \mathcal{M}, \|xp\| \leq \varepsilon, \text{ and } \tau(\mathbb{1} - p) \leq \delta\}.$$

The following “technical” form of convergence in measure proved in [7, Proposition 2.7] is useful. Let

$$|x_n - x| = \int_0^\infty te_n(dt)$$

be the spectral representation of $|x_n - x|$ with spectral measures e_n taking values in $\mathcal{M}$ since $x_n - x$, and thus $|x_n - x|$, are affiliated with $\mathcal{M}$. Then $x_n \rightarrow x$ in measure if and only if for each $\varepsilon > 0$,

$$(1) \quad \tau(e_n([\varepsilon, +\infty))) \rightarrow 0.$$

The domain of a linear operator x on $\mathcal{H}$ will be denoted by $\mathcal{D}(x)$.

For each $\rho \in \mathcal{M}_*$, there is a measurable operator h such that

$$\rho(x) = \tau(xh) = \tau(hx), \quad x \in \mathcal{M}.$$

The space of all such operators is denoted by $L^1(\mathcal{M}, \tau)$, and the correspondence above is one to one and isometric, where the norm on $L^1(\mathcal{M}, \tau)$, denoted by $\|\cdot\|_1$, is defined as

$$\|h\|_1 = \tau(|h|), \quad h \in L^1(\mathcal{M}, \tau).$$

(In the theory of noncommutative L^p -spaces for semifinite von Neumann algebras, it is shown that τ can be extended to the h 's as above; see, e.g., [4, 6, 7] for a detailed account of this theory.) Moreover, to hermitian functionals in $\mathcal{M}_*$ correspond selfadjoint operators in $L^1(\mathcal{M}, \tau)$, and to states in $\mathcal{M}_*$, positive operators in $L^1(\mathcal{M}, \tau)$.

Chebyshev's inequality,

$$\tau(e([\varepsilon, +\infty))) \leq \frac{\tau(|x|)}{\varepsilon} = \frac{\|x\|_1}{\varepsilon},$$

where

$$|x| = \int_0^\infty te(dt)$$

is the spectral representation, together with relation (1), shows that convergence in $\|\cdot\|_1$ -norm implies convergence in measure.

For $\rho \in \mathcal{M}_*^+$, the corresponding element in $L^1(\mathcal{M}, \tau)$ is called the *density* of ρ and is denoted by h_ρ .

Let $\Phi: \mathcal{M} \rightarrow \mathcal{M}$ be a normal positive linear unital mapping such that $\tau \circ \Phi = \tau$ (such maps are called *channels*, usually with an additional assumption of complete

positivity imposed). It is easy to see that then Φ can be extended to a bounded map on $L^1(\mathcal{M}, \tau)$ (cf. [3, Lemma 8]).

Let $\alpha \in (0, 1) \cup (1, +\infty)$. The α -Rényi entropy of ρ , denoted by $S_\alpha(\rho)$, is defined as

$$S_\alpha(\rho) = \frac{1}{1-\alpha} \log \frac{\tau(h_\rho^\alpha)}{\tau(h_\rho)},$$

under the assumption $h_\rho^\alpha \in L^1(\mathcal{M}, \tau)$, i.e., for the spectral representation of h_ρ ,

$$(2) \quad h_\rho = \int_0^\infty t e(dt),$$

we have

$$S_\alpha(\rho) = \frac{1}{1-\alpha} \log \frac{\int_0^\infty t^\alpha \tau(e(dt))}{\int_0^\infty t \tau(e(dt))}.$$

Accordingly, we define α -Rényi's entropy for $0 \leq h \in L^1(\mathcal{M}, \tau)$ such that $h^\alpha \in L^1(\mathcal{M}, \tau)$, by the formula

$$S_\alpha(h) = \frac{1}{1-\alpha} \log \frac{\tau(h^\alpha)}{\tau(h)} = \frac{1}{1-\alpha} \log \frac{\int_0^\infty t^\alpha \tau(e(dt))}{\int_0^\infty t \tau(e(dt))},$$

where h has spectral representation as in (2). For a map Φ as before, we are interested in the problem when this map does not change Rényi's entropy of a density h , i.e., when the equality

$$S_\alpha(h) = S_\alpha(\Phi(h))$$

holds.

§3. Main theorem

In this section we aim to prove the following main result of the paper.

Main Theorem. *Let Φ be a normal linear unital mapping on $\mathcal{M}$ such that $\tau \circ \Phi = \tau$, and let for an arbitrary fixed $\alpha \in (0, 1) \cup (1, +\infty)$ and a density h such that $h^\alpha \in L^1(\mathcal{M}, \tau)$,*

$$S_\alpha(h) = S_\alpha(\Phi(h)).$$

Then $\Phi|W^*(h)$ is a $*$ -isomorphism for $\alpha \leq 2$, and the same holds for $\alpha > 2$ if we additionally assume that there exists $1 < \gamma \leq 2$ such that $h^\gamma \in L^1(\mathcal{M}, \tau)$ (this assumption is automatically satisfied for a finite algebra).

Let us first explain the general idea of the proof. The invariance of entropy is obviously equivalent to the equality

$$(3) \quad \tau(\Phi(h^\alpha)) = \tau(h^\alpha) = \tau(\Phi(h)^\alpha).$$

From this equality, the equality

$$(4) \quad \Phi(h^\alpha) = \Phi(h)^\alpha$$

is attempted to be derived, leading, in turn, to the final conclusion. It should be noted that this equality for $\alpha \in (0, 1) \cup (1, 2]$ implies for *bounded* h our final conclusion, which follows from [5]. Observe that in some cases the derivation of relation (4) from (3) is simple. Namely, for $0 < \alpha < 1$, the function $[0, +\infty) \ni t \mapsto t^\alpha$ is operator concave, and Jensen's inequality for *bounded* operators yields the inequality

$$\Phi(h^\alpha) \leq \Phi(h)^\alpha.$$

For $1 < \alpha \leq 2$ the function $[0, +\infty) \ni t \mapsto t^\alpha$ is operator convex, and again Jensen's inequality for *bounded* operators yields the inequality

$$\Phi(h^\alpha) \geq \Phi(h)^\alpha.$$

In both the cases above, the faithfulness of τ together with relation (3) yield equality (4). In the general case of not necessarily bounded densities h , we could attempt to derive appropriate Jensen's inequalities (which we actually do for $1 < \alpha \leq 2$); however, a simpler way is possible. Namely, it turns out that in the case $0 < \alpha < 1$, *equality (3) alone is sufficient for proving the thesis*. For $\alpha > 1$, we follow the route described above and obtain equality (4). From this we get

$$\Phi(h^\alpha)^{\frac{1}{\alpha}} = \Phi(h) = \Phi((h^\alpha)^{\frac{1}{\alpha}}),$$

which is equality (4) with h^α instead of h and $\frac{1}{\alpha} < 1$ instead of α , and thus we get equality (3) in the form

$$\tau(\Phi((h^\alpha)^{\frac{1}{\alpha}})) = \tau(\Phi(h^\alpha)^{\frac{1}{\alpha}}).$$

As pointed out above, this gives the claim upon observing that $W^*(h^\alpha) = W^*(h)$. Consequently, the proof of the theorem is divided into three parts: (1) case $0 < \alpha < 1$, in which the thesis is proved, (2) case $1 < \alpha < 2$, and (3) case $\alpha \geq 2$, in which cases the equality $\Phi(h^\gamma) = \Phi(h)^\gamma$ for some $\gamma > 1$ is obtained.

Proof for $0 < \alpha < 1$. Let $t \geq 0$. We have the formula

$$(5) \quad t^\alpha = \frac{\sin \alpha \pi}{\pi} \int_0^\infty \frac{1}{s^{1-\alpha}} \frac{t}{s+t} ds.$$

Let $0 \leq h \in L^1(\mathcal{M}, \tau)$. Since for $s > 0$, we have

$$\|h(s\mathbf{1} + h)^{-1}\|_1 \leq \|h\|_1 \|(s\mathbf{1} + h)^{-1}\|_\infty \leq \frac{\|h\|_1}{s},$$

there exists, for each $m > 0$, Bochner's integral

$$z_m(h) = \frac{\sin \alpha \pi}{\pi} \int_m^\infty \frac{1}{s^{1-\alpha}} h(s\mathbb{1} + h)^{-1} ds$$

with values in $L^1(\mathcal{M}, \tau)$. Let

$$h = \int_0^\infty te(dt)$$

be the spectral representation of h . Then

$$\tau(h(s\mathbb{1} + h)^{-1}) = \int_0^\infty \frac{t}{s+t} \tau(e(dt)),$$

and on account of the Lebesgue monotone convergence theorem, Fubini's theorem, and (5) we get for $m \searrow 0$,

$$\begin{aligned} \tau(z_m(h)) &= \tau(\Phi(z_m(h))) = \frac{\sin \alpha \pi}{\pi} \int_m^\infty \frac{1}{s^{1-\alpha}} \tau(h(s\mathbb{1} + h)^{-1}) ds \\ &= \frac{\sin \alpha \pi}{\pi} \int_m^\infty \frac{1}{s^{1-\alpha}} \int_0^\infty \frac{t}{s+t} \tau(e(dt)) ds \\ &\nearrow \frac{\sin \alpha \pi}{\pi} \int_0^\infty \frac{1}{s^{1-\alpha}} \int_0^\infty \frac{t}{s+t} \tau(e(dt)) ds \\ &= \frac{\sin \alpha \pi}{\pi} \int_0^\infty \int_0^\infty \frac{1}{s^{1-\alpha}} \frac{t}{s+t} ds \tau(e(dt)) = \int_0^\infty t^\alpha \tau(e(dt)) = \tau(h^\alpha). \end{aligned}$$

By the same token, we obtain

$$\tau(z_m(\Phi(h))) \nearrow \tau(\Phi(h)^\alpha).$$

Since

$$\frac{t}{s+t} = 1 - \frac{s}{s+t},$$

we infer by virtue of [3, Theorem 7] that

$$(6) \quad \Phi(h(s\mathbb{1} + h)^{-1}) \leq \Phi(h)(s\mathbb{1} + \Phi(h))^{-1},$$

thus

$$\begin{aligned} \tau(z_m(\Phi(h))) - \tau(z_m(h)) &= \tau(z_m(\Phi(h))) - \tau(\Phi(z_m(h))) \\ &= \frac{\sin \alpha \pi}{\pi} \int_m^\infty \frac{1}{s^{1-\alpha}} \tau(\Phi(h)(s\mathbb{1} + \Phi(h))^{-1} - \Phi(h(s\mathbb{1} + h)^{-1})) ds, \end{aligned}$$

and the function under the integral sign is nonnegative which means that $\tau(z_m(\Phi(h))) - \tau(z_m(h))$ increases as $m \searrow 0$. But

$$\tau(z_m(\Phi(h))) - \tau(z_m(h)) \nearrow \tau(\Phi(h)^\alpha) - \tau(h^\alpha) = 0,$$

consequently,

$$\int_m^\infty \frac{1}{s^{1-\alpha}} \tau(\Phi(h)(s\mathbb{1} + \Phi(h))^{-1} - \Phi(h(s\mathbb{1} + h))^{-1}) ds = 0$$

and the nonnegativity of the function under the integral sign yields

$$\tau(\Phi(h)(s\mathbb{1} + \Phi(h))^{-1} - \Phi(h(s\mathbb{1} + h))^{-1}) = 0 \quad \text{almost everywhere,}$$

which by virtue of inequality (6) gives

$$(7) \quad \Phi(h)(s\mathbb{1} + \Phi(h))^{-1} = \Phi(h(s\mathbb{1} + h))^{-1} \quad \text{almost everywhere,}$$

i.e., *everywhere* by continuity. Since

$$x(s\mathbb{1} + x)^{-1} = \mathbb{1} - s(s\mathbb{1} + x)^{-1}$$

for $x \geq 0$, equality (7) yields

$$(s\mathbb{1} + \Phi(h))^{-1} = \Phi((s\mathbb{1} + h)^{-1}),$$

and now repeating word for word the reasoning in [3, Theorem 12] we obtain the claim. $\square$

Proof for $1 < \alpha \leq 2$. Assume first that $\alpha < 2$, and let $\beta = \alpha - 1$. We have

$$t^{1+\beta} = \frac{\sin \beta \pi}{\pi} \int_0^\infty \frac{1}{s^{1-\beta}} \frac{t^2}{s+t} ds,$$

and the estimate

$$\|h^2(s\mathbb{1} + h)^{-1}\|_1 \leq \|h\|_1 \|h(s\mathbb{1} + h)^{-1}\|_\infty \leq \|h\|_1.$$

For $0 < m < M$, set

$$z_m^M(h) = \frac{\sin \beta \pi}{\pi} \int_m^M \frac{1}{s^{1-\beta}} h^2(s\mathbb{1} + h)^{-1} ds,$$

which is Bochner's integral in $L^1(\mathcal{M}, \tau)$. Assume for a moment that h is bounded, $h \in \mathcal{M}$. Then since

$$\|h^2(s\mathbb{1} + h)^{-1}\|_\infty = \frac{\|h\|_\infty^2}{s + \|h\|_\infty},$$

we infer that $z_m^M(h)$ is also Bochner's integral in $\mathcal{M}$, and for each $\xi \in \mathcal{H}$ we have, on account of Fubini's theorem,

$$\begin{aligned} \langle z_m^M(h)\xi \mid \xi \rangle &= \frac{\sin \beta \pi}{\pi} \int_m^M \frac{1}{s^{1-\beta}} \langle h^2(s\mathbb{1} + h)^{-1}\xi \mid \xi \rangle ds \\ &= \frac{\sin \beta \pi}{\pi} \int_m^M \frac{1}{s^{1-\beta}} \int_0^\infty \frac{t^2}{s+t} \|e(dt)\xi\|^2 ds \\ &\leq \frac{\sin \beta \pi}{\pi} \int_0^\infty \frac{1}{s^{1-\beta}} \int_0^\infty \frac{t^2}{s+t} \|e(dt)\xi\|^2 ds \\ &= \frac{\sin \beta \pi}{\pi} \int_0^\infty \int_0^\infty \frac{1}{s^{1-\beta}} \frac{t^2}{s+t} ds \|e(dt)\xi\|^2 \\ &= \int_0^\infty t^{1+\beta} \|e(dt)\xi\|^2 = \langle h^{1+\beta}\xi \mid \xi \rangle, \end{aligned}$$

showing that

$$(8) \quad z_m^M(h) \leq h^{1+\beta}.$$

Now we drop the assumption that h is bounded and put

$$h_n = \int_0^n te(dt).$$

We have

$$h_n^2(s\mathbb{1} + h_n)^{-1} \leq h_{n+1}^2(s\mathbb{1} + h_{n+1})^{-1} \leq h^2(s\mathbb{1} + h)^{-1},$$

and consequently,

$$z_m^M(h_n) \leq z_m^M(h_{n+1}) \leq z_m^M(h).$$

Moreover, by virtue of Fubini's theorem and the Lebesgue monotone convergence theorem, we obtain as $n \rightarrow \infty$,

$$\begin{aligned} \tau(z_m^M(h_n)) &= \frac{\sin \beta \pi}{\pi} \int_m^M \frac{1}{s^{1-\beta}} \tau(h_n^2(s\mathbb{1} + h_n)^{-1}) ds \\ &= \frac{\sin \beta \pi}{\pi} \int_m^M \frac{1}{s^{1-\beta}} \int_0^n \frac{t^2}{s+t} \tau(e(dt)) ds \\ &= \frac{\sin \beta \pi}{\pi} \int_0^n \int_m^M \frac{1}{s^{1-\beta}} \frac{t^2}{s+t} ds \tau(e(dt)) \\ &\nearrow \frac{\sin \beta \pi}{\pi} \int_0^\infty \int_m^M \frac{1}{s^{1-\beta}} \frac{t^2}{s+t} ds \tau(e(dt)) \\ &= \frac{\sin \beta \pi}{\pi} \int_m^M \frac{1}{s^{1-\beta}} \int_0^\infty \frac{t^2}{s+t} \tau(e(dt)) ds \\ &= \frac{\sin \beta \pi}{\pi} \int_m^M \frac{1}{s^{1-\beta}} \tau(h^2(s\mathbb{1} + h)^{-1}) ds = \tau(z_m^M(h)). \end{aligned}$$

This shows that for $n \rightarrow \infty$ we have

$$\begin{aligned} \|z_m^M(h) - z_m^M(h_n)\|_1 &= \tau(z_m^M(h) - z_m^M(h_n)) \\ (9) \qquad \qquad \qquad &= \tau(z_m^M(h)) - \tau(z_m^M(h_n)) \rightarrow 0. \end{aligned}$$

Further, we have on account of (8),

$$z_m^M(h_n) \leq h_n^{1+\beta} \leq h^{1+\beta},$$

which together with relation (9) shows that

$$z_m^M(h) \leq h^{1+\beta}.$$

Now we get, using the Lebesgue monotone convergence theorem with $m \searrow 0$, $M \nearrow \infty$,

$$\begin{aligned} \tau(z_m^M(h)) &= \frac{\sin \beta \pi}{\pi} \int_m^M \frac{1}{s^{1-\beta}} \tau(h^2(s\mathbb{1} + h)^{-1}) ds \\ &= \frac{\sin \beta \pi}{\pi} \int_m^M \frac{1}{s^{1-\beta}} \int_0^\infty \frac{t^2}{s+t} \tau(e(dt)) ds \\ &= \frac{\sin \beta \pi}{\pi} \int_0^\infty \int_m^M \frac{1}{s^{1-\beta}} \frac{t^2}{s+t} ds \tau(e(dt)) \\ &\nearrow \frac{\sin \beta \pi}{\pi} \int_0^\infty \int_0^\infty \frac{1}{s^{1-\beta}} \frac{t^2}{s+t} ds \tau(e(dt)) \\ &= \int_0^\infty t^{1+\beta} \tau(e(dt)) = \tau(h^{1+\beta}), \end{aligned}$$

showing that with $m \searrow 0$, $M \nearrow \infty$,

$$\|h^{1+\beta} - z_m^M(h)\|_1 = \tau(h^{1+\beta} - z_m^M(h)) = \tau(h^{1+\beta}) - \tau(z_m^M(h)) \rightarrow 0,$$

i.e.,

$$z_m^M(h) \rightarrow h^{1+\beta} \quad \text{in } \|\cdot\|_1\text{-norm.}$$

Taking, in the above reasoning, $\Phi(h)$ in place of h , we obtain the relation

$$z_m^M(\Phi(h)) \rightarrow \Phi(h)^{1+\beta} \quad \text{in } \|\cdot\|_1\text{-norm.}$$

Now we are in a position to prove the following version of Jensen's inequality:

$$\Phi(h^{1+\beta}) \geq \Phi(h)^{1+\beta}.$$

Since for $x \geq 0$,

$$x^2(s\mathbb{1} + x)^{-1} = x - s\mathbb{1} + s^2(s\mathbb{1} + x)^{-1},$$

we get by virtue of [3, Theorem 7] that

$$(10) \quad \begin{aligned} & \Phi(h^2(s\mathbb{1} + h)^{-1}) - \Phi(h)^2(s\mathbb{1} + \Phi(h))^{-1} \\ &= s^2(\Phi((s\mathbb{1} + h)^{-1}) - (s\mathbb{1} + \Phi(h))^{-1}) \geq 0. \end{aligned}$$

From inequality (10) we obtain

$$\begin{aligned} \Phi(z_m^M(h)) &= \frac{\sin \beta \pi}{\pi} \Phi \left(\int_m^M \frac{1}{s^{1-\beta}} h^2(s\mathbb{1} + h)^{-1} ds \right) \\ &= \frac{\sin \beta \pi}{\pi} \int_m^M \frac{1}{s^{1-\beta}} \Phi(h^2(s\mathbb{1} + h)^{-1}) ds \\ &\geq \frac{\sin \beta \pi}{\pi} \int_m^M \frac{1}{s^{1-\beta}} \Phi(h)^2(s\mathbb{1} + \Phi(h))^{-1} ds = z_m^M(\Phi(h)), \end{aligned}$$

and taking into account the relations

$$z_m^M(h) \rightarrow h^{1+\beta} \quad \text{in } \|\cdot\|_1\text{-norm}$$

and

$$z_m^M(\Phi(h)) \rightarrow \Phi(h)^{1+\beta} \quad \text{in } \|\cdot\|_1\text{-norm},$$

together with the continuity of Φ in the $\|\cdot\|_1$ -norm, we prove the claim.

Now let $\alpha = 2$. We want to prove that

$$\Phi(h^2) \geq \Phi(h)^2.$$

With the notation as before, we have

$$h_n = \int_0^n te(dt) \nearrow \int_0^\infty te(dt) = h \quad \text{in } \|\cdot\|_1\text{-norm}$$

and

$$h_n^2 = \int_0^n t^2 e(dt) \nearrow \int_0^\infty t^2 e(dt) = h^2 \quad \text{in } \|\cdot\|_1\text{-norm},$$

thus

$$\Phi(h_n) \rightarrow \Phi(h) \quad \text{in } \|\cdot\|_1\text{-norm}$$

and

$$\Phi(h_n^2) \rightarrow \Phi(h^2) \quad \text{in } \|\cdot\|_1\text{-norm}.$$

In particular,

$$\Phi(h_n) \rightarrow \Phi(h) \quad \text{in measure}$$

and

$$\Phi(h_n^2) \rightarrow \Phi(h^2) \quad \text{in measure}.$$

Since multiplication is jointly continuous in the measure topology, we get

$$\Phi(h_n)^2 \rightarrow \Phi(h)^2 \quad \text{in measure.}$$

Jensen's inequality for bounded operators yields

$$\Phi(h_n^2) \geq \Phi(h_n)^2,$$

and passing to the limit in measure on both sides of the above inequality proves the claim.

Putting together the considerations above, we obtain the following Jensen inequality for $1 < \alpha \leq 2$ and $h \in L^1(\mathcal{M}, \tau)$, $h^\alpha \in L^1(\mathcal{M}, \tau)$:

$$(11) \quad \Phi(h^\alpha) \geq \Phi(h)^\alpha.$$

Consequently, we have for $1 < \alpha \leq 2$,

$$\tau(\Phi(h^\alpha) - \Phi(h)^\alpha) = \tau(\Phi(h^\alpha)) - \tau(\Phi(h)^\alpha) = 0,$$

and the faithfulness of τ gives

$$\Phi(h^\alpha) = \Phi(h)^\alpha. \quad \square$$

Before passing to the case $\alpha > 2$ we need some auxiliary results.

Proposition 1. *Let $0 < \alpha < 1$, and let $0 \leq x_n \in L^1(\mathcal{M}, \tau)$ be such that $x_n \nearrow x \in L^1(\mathcal{M}, \tau)$ in $\|\cdot\|_1$ -norm. Then $x_n^\alpha \nearrow x^\alpha$ in $\|\cdot\|_1$ -norm.*

Proof. For arbitrary $0 \leq u \in L^1(\mathcal{M}, \tau)$, and $0 < m < M$, set

$$\tilde{z}_m^M(u) = \frac{\sin \alpha \pi}{\pi} \int_m^M \frac{1}{s^{1-\alpha}} u(s\mathbb{1} + u)^{-1} ds.$$

The estimates

$$\|u(s\mathbb{1} + u)^{-1}\|_1 \leq \frac{\|u\|_1}{s} \quad \text{and} \quad \|u(s\mathbb{1} + u)^{-1}\|_\infty \leq 1$$

show that the integral above is Bochner's integral in $L^1(\mathcal{M}, \tau)$ as well as in $\mathcal{M}$. For arbitrary $\xi \in \mathcal{D}(u^\alpha)$, we have, for the spectral representation

$$u = \int_0^\infty tp(dt)$$

of u , the following relation:

$$\begin{aligned}
 \langle \tilde{z}_m^M(u)\xi \mid \xi \rangle &= \frac{\sin \alpha \pi}{\pi} \int_m^M \frac{1}{s^{1-\alpha}} \langle u(s\mathbb{1} + u)^{-1}\xi \mid \xi \rangle ds \\
 &= \frac{\sin \alpha \pi}{\pi} \int_m^M \frac{1}{s^{1-\alpha}} \int_0^\infty \frac{t}{s+t} \|p(dt)\xi\|^2 ds \\
 &\leq \frac{\sin \alpha \pi}{\pi} \int_0^\infty \frac{1}{s^{1-\alpha}} \int_0^\infty \frac{t}{s+t} \|p(dt)\xi\|^2 ds \\
 &= \frac{\sin \alpha \pi}{\pi} \int_0^\infty \int_0^\infty \frac{1}{s^{1-\alpha}} \frac{t}{s+t} ds \|p(dt)\xi\|^2 \\
 &= \int_0^\infty t^\alpha \|p(dt)\xi\|^2 = \langle u^\alpha \xi \mid \xi \rangle,
 \end{aligned}$$

showing that

$$(12) \quad \tilde{z}_m^M(u) \leq u^\alpha.$$

Moreover, we obtain as $m \searrow 0$, $M \nearrow \infty$,

$$\begin{aligned}
 \tau(\tilde{z}_m^M(u)) &= \frac{\sin \alpha \pi}{\pi} \int_m^M \frac{1}{s^{1-\alpha}} \tau(u(s\mathbb{1} + u)^{-1}) ds \\
 &= \frac{\sin \alpha \pi}{\pi} \int_m^M \frac{1}{s^{1-\alpha}} \int_0^\infty \frac{t}{s+t} \tau(p(dt)) ds \\
 &\nearrow \frac{\sin \alpha \pi}{\pi} \int_0^\infty \frac{1}{s^{1-\alpha}} \int_0^\infty \frac{t}{s+t} \tau(p(dt)) ds \\
 &= \frac{\sin \alpha \pi}{\pi} \int_0^\infty \int_0^\infty \frac{1}{s^{1-\alpha}} \frac{t}{s+t} ds \tau(p(dt)) = \int_0^\infty t^\alpha \tau(p(dt)) = \tau(u^\alpha),
 \end{aligned}$$

which yields

$$\tilde{z}_m^M(u) \rightarrow u^\alpha \quad \text{in } \|\cdot\|_1\text{-norm.}$$

For $0 \leq u_1 \leq u_2$ we have $s\mathbb{1} + u_1 \leq s\mathbb{1} + u_2$, thus

$$(s\mathbb{1} + u_1)^{-1} \geq (s\mathbb{1} + u_2)^{-1},$$

so

$$u_1(s\mathbb{1} + u_1)^{-1} = \mathbb{1} - s(s\mathbb{1} + u_1)^{-1} \leq \mathbb{1} - s(s\mathbb{1} + u_2)^{-1} = u_2(s\mathbb{1} + u_2)^{-1},$$

which yields

$$\tilde{z}_m^M(u_1) \leq \tilde{z}_m^M(u_2).$$

Passing to the limit with $m \searrow 0$, $M \nearrow \infty$, we get

$$u_1^\alpha \leq u_2^\alpha,$$

which shows the operator monotonicity of the function $[0, +\infty) \ni t \mapsto t^\alpha$ for elements from $L^1(\mathcal{M}, \tau)$ (a result well known for bounded operators).

Let x_n and x be as in the assumption. Then

$$\tilde{z}_m^M(x_n) \leq \tilde{z}_m^M(x).$$

Further we have

$$\begin{aligned} x_n(s\mathbb{1} + x_n)^{-1} - x(s\mathbb{1} + x)^{-1} \\ &= (s\mathbb{1} + x_n)^{-1}(x_n(s\mathbb{1} + x) - (s\mathbb{1} + x_n)x)(s\mathbb{1} + x)^{-1} \\ &= s(s\mathbb{1} + x_n)^{-1}(x_n - x)(s\mathbb{1} + x_n)^{-1}, \end{aligned}$$

thus

$$\begin{aligned} \|x_n(s\mathbb{1} + x_n)^{-1} - x(s\mathbb{1} + x)^{-1}\|_1 \\ \leq s\|(s\mathbb{1} + x_n)^{-1}\|_\infty \|x_n - x\|_1 \|(s\mathbb{1} + x_n)^{-1}\|_\infty \leq \frac{\|x_n - x\|_1}{s}, \end{aligned}$$

which shows that

$$x_n(s\mathbb{1} + x_n)^{-1} \nearrow x(s\mathbb{1} + x)^{-1} \quad \text{in } \|\cdot\|_1\text{-norm.}$$

From the Lebesgue monotone convergence theorem, we obtain as $n \rightarrow \infty$,

$$\begin{aligned} \tau(\tilde{z}_m^M(x_n)) &= \frac{\sin \alpha \pi}{\pi} \int_m^M \frac{1}{s^{1-\alpha}} \tau(x_n(s\mathbb{1} + x_n)^{-1}) ds \\ &\nearrow \frac{\sin \alpha \pi}{\pi} \int_m^M \frac{1}{s^{1-\alpha}} \tau(x(s\mathbb{1} + x)^{-1}) ds = \tau(\tilde{z}_m^M(x)), \end{aligned}$$

i.e.,

$$\tilde{z}_m^M(x_n) \nearrow \tilde{z}_m^M(x) \quad \text{in } \|\cdot\|_1\text{-norm.}$$

From relation (12) and operator monotonicity, we obtain

$$\tau(\tilde{z}_m^M(x_n)) \leq \tau(x_n^\alpha) \leq \tau(x^\alpha),$$

which gives

$$\tau(\tilde{z}_m^M(x)) = \lim_{n \rightarrow \infty} \tau(\tilde{z}_m^M(x_n)) \leq \varliminf_{n \rightarrow \infty} \tau(x_n^\alpha) \leq \overline{\lim}_{n \rightarrow \infty} \tau(x_n^\alpha) \leq \tau(x^\alpha),$$

and passing to the limit with $m \searrow 0$, $M \nearrow \infty$, in the above inequality, we get

$$\tau(x^\alpha) = \lim_{\substack{m \rightarrow 0 \\ M \rightarrow \infty}} \tau(\tilde{z}_m^M(x)) \leq \varliminf_{n \rightarrow \infty} \tau(x_n^\alpha) \leq \overline{\lim}_{n \rightarrow \infty} \tau(x_n^\alpha) \leq \tau(x^\alpha),$$

showing that

$$\lim_{n \rightarrow \infty} \tau(x_n^\alpha) = \tau(x^\alpha),$$

which by virtue of the inequality $x_n^\alpha \leq x^\alpha$ proves the claim. $\square$

Another version of Jensen's inequality is the following.

Proposition 2. *Let $\alpha > 1$, and let $0 \leq h \in L^1(\mathcal{M}, \tau)$ be such that $h^\alpha \in L^1(\mathcal{M}, \tau)$. Then*

$$\tau(\Phi(h^\alpha)) \geq \tau(\Phi(h)^\alpha).$$

Proof. We have

$$h_n = \int_0^n te(dt) \nearrow \int_0^\infty te(dt) = h \quad \text{in } \|\cdot\|_1\text{-norm}$$

and

$$h_n^\alpha = \int_0^n t^\alpha e(dt) \nearrow \int_0^\infty t^\alpha e(dt) = h^\alpha \quad \text{in } \|\cdot\|_1\text{-norm},$$

thus

$$\Phi(h_n) \nearrow \Phi(h) \quad \text{in } \|\cdot\|_1\text{-norm}$$

and

$$\Phi(h_n^\alpha) \nearrow \Phi(h^\alpha) \quad \text{in } \|\cdot\|_1\text{-norm}.$$

In particular,

$$\Phi(h_n) \rightarrow \Phi(h) \quad \text{in measure},$$

and

$$\tau(\Phi(h_n^\alpha)) \rightarrow \tau(\Phi(h^\alpha)).$$

Let $\alpha = r + \beta$, where $r = [\alpha]$, $0 \leq \beta < 1$. On account of Proposition 1 we have

$$\Phi(h_n)^\beta \rightarrow \Phi(h)^\beta \quad \text{in } \|\cdot\|_1\text{-norm},$$

so

$$\Phi(h_n)^\beta \rightarrow \Phi(h)^\beta \quad \text{in measure},$$

yielding, since multiplication is jointly continuous in the measure topology,

$$\Phi(h_n)^\alpha = \Phi(h_n)^r \Phi(h_n)^\beta \rightarrow \Phi(h)^r \Phi(h)^\beta = \Phi(h)^\alpha \quad \text{in measure}.$$

Since the function $[0, +\infty) \ni t \mapsto t^\alpha$ is convex, Jensen's inequality for bounded operators implies

$$\tau(\Phi(h_n^\alpha)) \geq \tau(\Phi(h_n)^\alpha).$$

For every n , we have

$$\|\Phi(h_n)^\alpha\|_1 = \tau(\Phi(h_n)^\alpha) \leq \tau(\Phi(h_n^\alpha)) = \tau(h_n^\alpha) \leq \tau(h^\alpha) < +\infty,$$

and Fatou's lemma – [7, Theorem 2.9] – yields

$$\begin{aligned} \tau(\Phi(h)^\alpha) &= \|\Phi(h)^\alpha\|_1 \leq \varliminf_{n \rightarrow \infty} \|\Phi(h_n)^\alpha\|_1 \\ &= \varliminf_{n \rightarrow \infty} \tau(\Phi(h_n)^\alpha) \leq \varliminf_{n \rightarrow \infty} \tau(\Phi(h_n^\alpha)) = \tau(\Phi(h^\alpha)). \end{aligned}$$

□

Proof for $\alpha \geq 2$. Here we additionally assume that there exists $1 < \gamma \leq 2$ such that $h^\gamma \in L^1(\mathcal{M}, \tau)$. Let

$$f(t) = t^{\frac{\alpha}{\gamma}}, \quad t \in [0, +\infty).$$

By virtue of Proposition 2 (with h replaced by h^γ and α replaced by $\frac{\alpha}{\gamma}$), we have

$$\tau(\Phi(h^\alpha)) = \tau(\Phi((h^\gamma)^{\frac{\alpha}{\gamma}})) \geq \tau(\Phi(h^\gamma)^{\frac{\alpha}{\gamma}}) = \tau(f(\Phi(h^\gamma))).$$

Jensen's inequality (11) is now

$$\Phi(h^\gamma) \geq \Phi(h)^\gamma,$$

and since f is strictly increasing, continuous, convex, and $f(0) = 0$, we obtain by virtue of [2, Theorem 14],

$$(13) \quad \tau(f(\Phi(h^\gamma))) \geq \tau(f(\Phi(h)^\gamma)) = \tau(\Phi(h)^\alpha).$$

Since

$$\tau(\Phi(h^\alpha)) = \tau(\Phi(h)^\alpha),$$

we have equality in inequality (13) above which again by virtue of [2, Theorem 14] yields the equality

$$\Phi(h^\gamma) = \Phi(h)^\gamma. \quad \square$$

Remark. Observe that if Φ is a $*$ -isomorphism on the algebra $W^*(h)$ such that τ is Φ -invariant, then for every bounded continuous function f we have $\Phi(f(h)) = f(\Phi(h))$, and it is easy to prove that for $\alpha \in (0, 1) \cup (1, +\infty)$ such that $h^\alpha \in L^1(\mathcal{M}, \tau)$ we have $\Phi(h^\alpha) = \Phi(h)^\alpha$, and thus $S_\alpha(h) = S_\alpha(\Phi(h))$.

§4. Additional results

In this section we consider the situation when the map Φ does not change an arbitrary fixed entropy of every density.

Theorem 3. *Let Φ be a normal linear unital mapping on $\mathcal{M}$ such that $\tau \circ \Phi = \tau$, and for an arbitrary fixed $\alpha \in (0, 1) \cup (1, +\infty)$ let*

$$S_\alpha(h) = S_\alpha(\Phi(h))$$

for every density h such that $h^\alpha \in L^1(\mathcal{M}, \tau)$. Then Φ is a Jordan $*$ -isomorphism on $\mathcal{M}$.

Proof. Let $h \in \mathcal{M}$ be a density such that $h^\alpha \in L^1(\mathcal{M}, \tau)$ and, in the case $\alpha > 2$, $h^\gamma \in L^1(\mathcal{M}, \tau)$ for some $1 < \gamma \leq 2$. From the [Main Theorem](#) it follows that $\Phi|W(h)$ is a $*$ -isomorphism, in particular, we have

$$\Phi(h^2) = \Phi(h)^2.$$

On account of [3, Proposition 9], for all $x \in \mathcal{M}$ the following equality holds:

$$(14) \quad \Phi(x \circ h) = \Phi(x) \circ \Phi(h),$$

where for $u, v \in \mathcal{M}$,

$$u \circ v = \frac{uv + vu}{2}$$

is the Jordan product.

Since such h as above are σ -weakly dense in $\mathcal{M}^+$, and Φ is normal, it follows that equation (14) holds for all h in $\mathcal{M}^+$, and consequently we have

$$\Phi(x \circ y) = \Phi(x) \circ \Phi(y),$$

for all $x, y \in \mathcal{M}$, thus Φ is a Jordan $*$ -homomorphism.

If $\Phi(x) = 0$, then

$$0 = \Phi(x)^* \Phi(x) + \Phi(x) \Phi(x)^* = \Phi(x^*x + xx^*) = \Phi(x^*x) + \Phi(xx^*),$$

thus

$$\Phi(x^*x) = \Phi(xx^*) = 0,$$

and it follows that $\tau(x^*x) = \tau(\Phi(x^*x)) = 0$, i.e., $x = 0$, showing that Φ is an isomorphism. $\square$

Remark. If Φ is a Jordan $*$ -isomorphism, then it is obviously a $*$ -isomorphism on the abelian algebra $W^*(h)$, thus the equality $S_\alpha(h) = S_\alpha(\Phi(h))$ follows as in the previous remark.

At the end, let us comment on a general situation. In many cases the quantities of the form

$$(15) \quad H(h) = \tau(f(h)),$$

where h is a density, and f is a continuous function, are considered (typical examples are $f(t) = t \log t$ leading to Segal's entropy or $f(t) = t^\alpha$ leading to Rényi's entropy). Then a natural question is the invariance of H with respect to some map Φ , i.e., the relation

$$(16) \quad H(h) = H(\Phi(h)).$$

Putting aside technical problems (h may be unbounded), we have noticed that, in principle, if Φ is a normal Jordan $*$ -isomorphism, then $\Phi(f(h)) = f(\Phi(h))$, thus the additional assumption $\tau \circ \Phi = \tau$ yields relation (16). Consequently, the condition “ Φ is a normal Jordan $*$ -isomorphism such that $\tau \circ \Phi = \tau$ ” is *sufficient* for the invariance (16). A similar situation occurs when we deal with two densities. A typical example, which for simplicity we consider in finite dimension, is the relative entropy of two densities h and k defined as

$$D(h||k) = \begin{cases} \operatorname{tr} h(\log h - \log k) & \text{if } \operatorname{supp} h \leq \operatorname{supp} k, \\ +\infty & \text{otherwise.} \end{cases}$$

Now, if Φ is a normal Jordan $*$ -isomorphism leaving the trace “ tr ” invariant, we have $\operatorname{supp} \Phi(h) = \Phi(\operatorname{supp} h)$ and $\operatorname{supp} \Phi(k) = \Phi(\operatorname{supp} k)$, thus $\operatorname{supp} h \leq \operatorname{supp} k$ if and only if $\operatorname{supp} \Phi(h) \leq \operatorname{supp} \Phi(k)$, in which case we get the equality

$$\begin{aligned} D(\Phi(h)||\Phi(k)) &= \operatorname{tr} \Phi(h)(\log \Phi(h) - \log \Phi(k)) \\ &= \operatorname{tr}(\Phi(h) \circ (\log \Phi(h) - \log \Phi(k))) \\ &= \operatorname{tr}(\Phi(h) \circ (\Phi(\log h) - \Phi(\log k))) \\ &= \operatorname{tr}(\Phi(h) \circ \Phi(\log h - \log k)) = \operatorname{tr} \Phi(h \circ (\log h - \log k)) \\ &= \operatorname{tr}(h \circ (\log h - \log k)) = \operatorname{tr} h(\log h - \log k) = D(h||k), \end{aligned}$$

showing that the same condition, “ Φ is a normal Jordan $*$ -isomorphism such that $\tau \circ \Phi = \tau$ ”, is again *sufficient* for the invariance of the relative entropy. The above reasoning shows that some results which state for example that a map leaving the relative entropy, or the H as in (15), invariant must be of the form $\Phi(x) = u x u^*$, where u is a unitary or antiunitary or an isometry, are incorrect (a good example of such a Jordan $*$ -isomorphism which does not change the entropy but is not of the form above is transposition in $\mathbb{B}(\mathcal{H})$ with respect to a given orthonormal basis).

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