

Divisorial Mori Contractions of Submaximal Length

by

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Abstract

A result due to Cho, Miyaoka, and Shepherd-Barron [in: Higher dimensional birational geometry (2002), 1–88] and Kebekus [in: Complex geometry (2002), 147–155] provides a numerical characterization of projective spaces. More recently, Dedieu and Höring [Algebr. Geom. 4 (2017), 120–135] gave a characterization of smooth quadrics based on similar arguments. As a relative version of [in: Higher dimensional birational geometry (2002), 1–88] and Kebekus [in: Complex geometry (2002), 147–155], Höring and Novelli [Publ. Res. Inst. Math. Sci. 49 (2013), 215–228] proved that the locus covered by positive-dimensional fibers in a Mori contraction of maximal length is a projective bundle up to birational modification. We change the length hypothesis and we prove that the exceptional locus of a divisorial Mori contraction of submaximal length is birational either to a projective bundle, or to a quadric bundle.

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§1. Introduction

If X is a smooth Fano variety of dimension n , its pseudoindex is defined as

$$l(X) = \min \{ -K_X \cdot \Gamma \mid \Gamma \subset X \text{ a rational curve} \}.$$

This invariant provides much information on X .

Theorem 1.1 ([CMSB02, Ke02, Mi04, DH17]). *Let X be a smooth Fano variety of dimension n .*

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- [CMSB02, Ke02]: If for every rational curve $\Gamma \subset X$, one has $-K_X \cdot \Gamma \geq n+1$, then $X \simeq \mathbf{P}^n$. In this case, $l(X) = n+1$ and the equality $-K_X \cdot \Gamma = n+1$ holds if and only if Γ is a line.
- [Mi04, DH17]: If for every rational curve $\Gamma \subset X$, one has $-K_X \cdot \Gamma \geq n$, then X is isomorphic either to a quadric, or to a projective space. If $l(X) = n$, then X is a quadric and the equality $-K_X \cdot \Gamma = n$ holds if and only if Γ is a line.

Fano manifolds arise in the Mori program as general fibers of Mori fiber spaces. More generally, if $f: X \rightarrow Y$ is a *fibration* (i.e., a surjective morphism with connected fibers and $\dim X > \dim Y$) from a smooth variety X such that $-K_X$ is f -ample and F is a smooth fiber over a smooth point of Y , then F is a Fano manifold.

From now on, we will use the term *contraction* to designate any surjective morphism $f: X \rightarrow Y$ with connected fibers such that there exists at least one curve $C \subset X$ contracted by f .

Let us consider a smooth projective variety X and an elementary Mori contraction $f: X \rightarrow Y$. In other words, the curves contracted by f are those curves whose numerical equivalence classes all belong to a single K_X -negative extremal ray of the Mori cone of X . The length of such a contraction is defined as follows.

Definition 1.2 (Length). Let X be a smooth projective variety and $f: X \rightarrow Y$ an elementary Mori contraction. The length of f is defined as the minimal degree of $-K_X$ on the rational curves which are contracted by f :

$$l(f) = \min\{-K_X \cdot \Gamma \mid \Gamma \subset X \text{ a contracted rational curve}\}.$$

This notion can naturally be viewed as a relative version of the pseudoindex. Indeed, the pseudoindex of the general positive-dimensional fiber of f can sometimes be recovered from the adjunction formula and $l(f)$. If f is of fiber type (meaning $\dim X > \dim Y$), its general fiber F is a Fano manifold with $K_F = K_X|_F$, yielding $l(F) \geq l(f)$, and $l(F) = l(f)$ if F contains a rational curve Γ verifying

$$(1) \quad -K_X \cdot \Gamma = l(f).$$

If f is birational and its exceptional locus E is a hypersurface of X , and $F \subset E$ is a general fiber of $f|_E$ containing a rational curve Γ satisfying (1), then $K_E = (K_X + E)|_E$ and $K_F = K_E|_F = (K_X + E)|_F$, so that $l(F)$ and $l(f)$ differ by the degree of the divisor E on contracted curves. Thanks to a classical result due to P. Ionescu and J. Wiśniewski, the length of an elementary Mori contraction $X \rightarrow Y$ from a smooth variety X is bounded from above.

Theorem 1.3 ([Io86, Theorem 0.4], [Wi91, Theorem 1.1]). *Assume $f: X \rightarrow Y$ is an elementary Mori contraction from a smooth projective variety X . Let $E \subset X$ be an irreducible component of the f -exceptional locus (or $E = X$ if f is of fiber type) and $F \subset E$ an irreducible component of a fiber. Then*

$$\dim E + \dim F \geq \dim X + l(f) - 1.$$

Terminology. The length of f is *maximal* if there exist an irreducible component E of the f -exceptional locus (or $E = X$ if f is of fiber type) and F a fiber in E such that $l(f) = \dim E - \dim X + \dim F + 1$. The length of f is *submaximal* if it is not maximal and f admits an irreducible component E of $\text{Exc}(f)$ and a fiber $F \subset E$ such that $l(f) = \dim E - \dim X + \dim F$.

We refer to [HN13] for a study of elementary Mori contractions of maximal length, both of fiber type and birational. Given $f: X \rightarrow Y$ a Mori contraction of maximal length from a smooth projective variety X , their main results [HN13, Theorems 1.3 & 1.4] state that the f -exceptional locus (respectively X) is birational to a projective bundle over its image if f is birational (respectively, if f is of fiber type).

In this paper we change the assumptions of the above theorem, moving to the case of submaximal length for a birational elementary Mori contraction. We assume moreover that our contraction is *divisorial*, i.e., that the exceptional locus has codimension 1. Everything is specified in Setup 1.4 below.

Notation. If $f: X \rightarrow Y$ is a divisorial contraction and $E \subset X$ is the f -exceptional locus with $\dim E - \dim f(E) = n$, we denote by $E_{\text{eq}} \rightarrow Z_{\text{eq}}$ the *equidimensional locus* of $f|_E$, that is, the locus in E which is covered by the f -fibers of dimension n .

Terminology. If $f: X \rightarrow Y$ is an elementary Mori contraction and $\Gamma \subset X$ is a rational curve contracted by f on which $-K_X$ has the lowest possible degree, in other words

$$-K_X \cdot \Gamma = l(f),$$

then we say that Γ is an *f -minimal curve*, or sometimes just a *minimal curve* when there is no risk of confusion.

§1.1. Setup and statement of the main result

The following setup contains all the assumptions that the remainder of this paper is based on.

Setup 1.4. *Let $f: X \rightarrow Y$ be an elementary Mori contraction from a smooth projective variety X . We assume that it is birational and divisorial with E its*

exceptional divisor, which is irreducible by [Deb01, Proposition 6.10.b], and we set $Z := f(E)$. If $E_z = f^{-1}(z)$ is a general fiber of $f|_E$ then by the Ionescu–Wiśniewski inequality given in Theorem 1.3, the length of f is bounded from above by $\dim E_z$. We assume here that the length is submaximal, in other words

$$l(f) = \dim E_z - 1,$$

for E_z a fiber of the expected dimension $n := \dim E - \dim Z$. In other words, we have $l(f) = n - 1$ (in particular, n must be larger than 1).

We consider an f -minimal curve Γ in the equidimensional locus E_{eq} , the existence of which is ensured by Lemma 3.1, and the Cartier divisor $-E|_E$, which is ample by Kleiman’s criterion (see for instance [Deb01, Theorem 1.27.a]). Then we deduce information on $E_{\text{eq}} \rightarrow Z_{\text{eq}}$, according to the degree of $-E$ on Γ . Namely, our main result is the following.

Theorem 1.5. *Under the assumptions of Setup 1.4, we have either $-E \cdot \Gamma = 1$ or $-E \cdot \Gamma = 2$. Furthermore,*

- *if $-E \cdot \Gamma = 2$, then all the fibers of $E_{\text{eq}} \rightarrow Z_{\text{eq}}$ are normalized by $\mathbf{P}^n$ and the fibration $E_{\text{eq}} \rightarrow Z_{\text{eq}}$ is birational to a family of projective spaces. If moreover n is even, E_{eq} is isomorphic to the projectivization of a vector bundle over Z_{eq} ;*
- *if $-E \cdot \Gamma = 1$, then $E_{\text{eq}} \rightarrow Z_{\text{eq}}$ is birational to a quadric bundle. Moreover, each reducible fiber of $E_{\text{eq}} \rightarrow Z_{\text{eq}}$ has two irreducible components, the reductions of which are normalized by $\mathbf{P}^n$.*

In Section 4 we provide a case-by-case proof of Theorem 1.5, according to the degree of the exceptional divisor on the f -minimal curves. The proof is split into three parts: the first part is Proposition 3.3, in which we show that we have either $-E \cdot \Gamma = 1$ or $-E \cdot \Gamma = 2$. The second part is Theorem 4.1 (the case $-E \cdot \Gamma = 2$), and the last part is Theorem 4.5 (the case $-E \cdot \Gamma = 1$). The proof of Proposition 3.3 relies on a technique which consists in degenerating families of f -minimal curves in the general fiber of $E_{\text{eq}} \rightarrow Z_{\text{eq}}$ to families of f -minimal curves contained in singular fibers, and bounding the dimension of said families from below.

Restricting to the equidimensional locus $E_{\text{eq}} \rightarrow Z_{\text{eq}}$ is a necessary condition for Theorem 1.5. To illustrate this, Section 5 is dedicated to the construction of an example of a nonequidimensional divisorial elementary Mori contraction $X \rightarrow Y$, whose exceptional divisor is generically a quadric bundle over its image and contains a special fiber which is isomorphic to a projective space.

§2. Prerequisites

One of the techniques that we will use to study the fibers contained in the exceptional divisor of a divisorial Mori contraction relies on the degeneration of curves. This requires that we introduce the following notion.

Definition 2.1 (Deformation family). Consider a rational curve Γ contained in a variety X . The datum of this curve is equivalent to the datum of a morphism $\mathbf{P}^1 \rightarrow X$ birational onto its image, modulo automorphisms of $\mathbf{P}^1$. In other words, it corresponds to a point of $\text{RatCurves}^n(X)$: let $\mathcal{H}$ be an irreducible component of $\text{RatCurves}^n(X)$ containing this point. We say that $\mathcal{H}$ is a *deformation family* of Γ in X .

We refer to [Kol96, Prop. II.2.11.2] for the construction of $\text{RatCurves}^n(X)$ and that of deformation families. By construction, the points of such a deformation family $\mathcal{H}$ parametrize rational curves, i.e., irreducible and reduced 1-cycles in X . This family $\mathcal{H}$ is the normalization of a subspace $V \subset \text{Chow}(X)$; let $\bar{V}$ be the closure of V . Then one may consider $\bar{\mathcal{H}}$ the normalization of $\bar{V}$, such that $\mathcal{H}$ is dense in $\bar{\mathcal{H}}$. By construction, the points of $\bar{\mathcal{H}}$ parametrize either rational curves in X , or 1-cycles in X that are degenerations of rational curves.

Definition 2.2 (Closed deformation family – unsplit deformation family). We say that $\bar{\mathcal{H}}$ is a *closed deformation family* of Γ in X .

A closed deformation family $\bar{\mathcal{H}}$ is *unsplit* if all its members are reduced and irreducible, in other words if $\mathcal{H} = \bar{\mathcal{H}}$.

Moreover, given $\bar{\mathcal{H}}$ a closed deformation family of a rational curve in a variety X , there exists a *universal family* over $\bar{\mathcal{H}}$,

$$\mathcal{U} = \{(p, Z) \in X \times \bar{\mathcal{H}} \mid Z \in \bar{\mathcal{H}}, p \in \text{Supp}(Z)\},$$

with the two projections

$$\begin{array}{ccc} \mathcal{U} & \xrightarrow{\text{ev}} & X \\ q \downarrow & & \\ & & \bar{\mathcal{H}} \end{array}$$

In the above, the notation ev stands for “evaluation morphism”. The existence of $\mathcal{U}$ is explained in [Kol96, II.2.8, II.2.12]. Moreover, as $\bar{\mathcal{H}}$ is normal by construction, we have the following.

Lemma 2.3. *Let Γ be a rational curve contained in a variety X and $\bar{\mathcal{H}}$ a closed deformation family of Γ in X . Let L be a line bundle on X . Then L has the same degree on all the members of $\bar{\mathcal{H}}$.*

Proof. Let $\mathcal{U}$ be the universal family over $\overline{\mathcal{H}}$, with $\text{ev}: \mathcal{U} \rightarrow X$ the evaluation map. The fibration $q: \mathcal{U} \rightarrow \overline{\mathcal{H}}$ is equidimensional onto a normal variety, so the line bundle ev^*L has the same degree on all the fibers of q by [Kol96, I.3.12]. But by construction, the fibers of q are the members of the family $\overline{\mathcal{H}}$. $\square$

This means that each member of $\overline{\mathcal{H}}$ is a 1-cycle which is numerically equivalent to a rational curve. It may happen that $\mathcal{H} \neq \overline{\mathcal{H}}$, in which case the family $\overline{\mathcal{H}}$ contains nonreduced or reducible 1-cycles whose components are supported on rational curves. This may happen in certain situations, such as that of Mori's bend-and-break lemma; see for instance [Deb01, Proposition 3.5].

Corollary 2.4. *Let $f: X \rightarrow Y$ be a morphism, $\Gamma \subset X$ a rational curve contracted by f , and $\overline{\mathcal{H}}$ a closed deformation family of Γ in X . Then all the members of $\overline{\mathcal{H}}$ are contracted by f .*

When considering the normalization $\nu: X' \rightarrow X$ of a variety X with Cartier canonical divisor, we will sometimes need to investigate the divisor $\nu^*K_X - K_{X'}$, which yields some information on the nonnormal locus of X .

Lemma 2.5 (Conductor). *Let X be a variety with Cartier canonical class K_X which satisfies the S_2 condition, and $\nu: X' \rightarrow X$ its normalization. Then there exists an effective Weil divisor $\mathcal{D}$ on X' such that $K_{X'} + \mathcal{D}$ is Cartier, and*

$$\nu^*K_X \simeq K_{X'} + \mathcal{D}.$$

Moreover, $\nu(\mathcal{D})$ is the nonnormal locus of X .

In particular, if C is a projective, irreducible, reduced, Gorenstein curve, then the degree of the divisor $\mathcal{D}$ involved in its normalization is even.

Proof. A construction of the conductor is provided in [Kol13, 5.2.2]. In general, this is just a subscheme of X' , but when X is S_2 it is a hypersurface of X' . By construction, the support of the conductor is the locus where ν is not an isomorphism, i.e., the preimage of the nonnormal locus of X . The formula

$$\nu^*K_X \simeq K_{X'} + \mathcal{D}$$

follows from [Kol13, 5.7.1], which holds under the condition that K_X is Cartier and X is S_2 .

Now consider C a projective reduced and irreducible curve with K_C Cartier, and $C' \rightarrow C$ its normalization. By the Riemann–Roch formula for singular curves (see for instance [Har77, Exercise IV.1.9]), we have $\deg K_C = 2p_a(C) - 2$. Hence the degree of the conductor divisor is $\deg K_C - \deg K_{C'} = 2(p_a(C) - g(C'))$. $\square$

Lastly, we need to introduce the following notion about the pullback of the pushforward of a line bundle via a contraction.

Lemma 2.6 (Evaluation map). *Let $f: X \rightarrow Y$ be a contraction with projective fibers and L a line bundle on X . Then the $\mathcal{O}_X$ -module f^*f_*L admits a morphism*

$$f^*f_*L \rightarrow L$$

which we call the relative evaluation map, or simply the evaluation map, of L .

Proof. By [Har77, II.5], there is a canonical isomorphism of groups

$$\mathrm{Hom}_{\mathcal{O}_X}(f^*f_*L, L) \simeq \mathrm{Hom}_{\mathcal{O}_Y}(f_*L, f_*L),$$

and the evaluation map $f^*f_*L \rightarrow L$ is the element of $\mathrm{Hom}_{\mathcal{O}_X}(f^*f_*L, L)$ which corresponds to the identity $\mathrm{id}: f_*L \rightarrow f_*L$. $\square$

Furthermore, if $p \in Y$ is a point such that f_*L is locally free of rank r at p , and the fiber $X_p = f^{-1}(p)$ is such that

$$\dim H^0(X_p, L|_{X_p}) = r,$$

then the restriction of the evaluation map $f^*f_*L \rightarrow L$ to X_p is a morphism of vector bundles

$$H^0(X_p, L|_{X_p}) \otimes_{\mathcal{O}_{X_p}} L|_{X_p}.$$

Hence the evaluation map is surjective along the fiber X_p if and only if $L|_{X_p}$ is globally generated.

Theorem 2.7 ([AW93, Theorem 5.1]). *Let $f: X \rightarrow Y$ be a contraction from a projective normal variety with at worst klt singularities. Assume that f is supported by $K_X + rL$ for some line bundle L and r a rational number (in other words, $K_X + rL$ has degree zero on all contracted curves and positive degree on all other curves). Assume that L is f -ample, i.e., its restriction to every f -fiber is ample.*

Consider a fiber X_p of f . Assume moreover that

- *f is a fibration (meaning $\dim X > \dim Y$) and $\dim X_p < r + 1$, or*
- *f is birational and $\dim X_p \leq r + 1$.*

Then the evaluation map $f^*f_*L \rightarrow L$ is surjective along X_p .

Moreover, when $f^*f_*L \rightarrow L$ is surjective on all of X , the following birational modification arises as an extension of the locus where the direct image f_*L is locally free.

Lemma 2.8. *Let $f: X \rightarrow Y$ be a fibration between irreducible quasi-projective varieties and L a line bundle on X such that the evaluation map $f^*f_*L \rightarrow L$ is surjective. Let $\mathcal{U}_L$ be the locus where f_*L is locally free. Then there exists a birational modification $\mu: \mathcal{Y} \rightarrow Y$ which is an isomorphism on $\mathcal{U}_L$, together with a vector bundle $\mathcal{V}$ on $\mathcal{Y}$ which coincides with f_*L on $\mathcal{U}_L$.*

Moreover, let $\mathcal{X}$ be the irreducible component of the fiber product $X \times_Y \mathcal{Y}$ which dominates X , with the following commutative diagram:

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{\mu'} & X \\ f' \downarrow & & \downarrow f \\ \mathcal{Y} & \xrightarrow{\mu} & Y. \end{array}$$

Then there is a surjective map $f'^*\mathcal{V} \rightarrow \mu'^*L$ which coincides with the evaluation map $f^*f_*L \rightarrow L$ on $f^{-1}(\mathcal{U}_L)$.

Proof. By [Ro68, Theorem 3.5] there exist a birational modification $\mu: \mathcal{Y} \rightarrow Y$ and a vector bundle $\mathcal{V}$ on $\mathcal{Y}$ with a surjective map

$$(2) \quad \mu^*(f_*L) \rightarrow \mathcal{V}$$

whose kernel is torsion. On the locus $\mathcal{U}_L$ where f_*L is locally free, μ is an isomorphism; in particular, we have $f_*L \simeq \mathcal{V}$ on $\mathcal{U}_L$. Moreover, the pullback by μ' of the surjective map

$$f^*f_*L \rightarrow L$$

is also surjective. Let $\mathcal{K}$ denote the kernel of the map (2). Then in the following diagram, the top row is exact, and the vertical arrow is onto:

$$\begin{array}{ccccccc} f'^*\mathcal{K} & \longrightarrow & f'^*\mu^*(f_*L) & \simeq & \mu'^*(f^*f_*L) & \longrightarrow & f'^*\mathcal{V} \longrightarrow 0. \\ & & & & \downarrow & & \\ & & & & \mu'^*L & & \end{array}$$

$\searrow \alpha$

The map α is zero, since μ'^*L is a line bundle and $f'^*\mathcal{K}$ is torsion on the irreducible variety $\mathcal{X}$. This ensures the existence of a factorization $f'^*\mathcal{V} \rightarrow \mu'^*L$ which is surjective as well. $\square$

Remark. Algebraized version of Rossi's construction.

While the construction done in [Ro68, Theorem 3.5] is purely analytic, in our setting the modification $\mu: \mathcal{Y} \rightarrow Y$ is algebraic. Namely, we may consider a

sufficiently ample line bundle $\mathcal{L}$ on Y such that $(f_*L) \otimes \mathcal{L}$ is globally generated; on $\mathcal{U}_L$ we have a surjective map of vector bundles

$$\mathcal{O}^{\oplus p} \xrightarrow{\alpha} (f_*L) \otimes \mathcal{L} \rightarrow 0$$

and the morphism of algebraic varieties $\mathcal{U}_L \rightarrow \mathbf{Gr}(p-r, p)$, with $r := \text{rank}((f_*L) \otimes \mathcal{L})$, mapping x to $\ker(\alpha_x)$. Then $\mathcal{Y}$ is the closure of its graph in $Y \times \mathbf{Gr}(p-r, p)$ with μ the projection onto Y (both $\mathcal{Y}$ and μ are projective) and $\mathcal{V} \otimes \mu^*L$ is the pullback to $\mathcal{Y}$ of the universal quotient bundle of $\mathbf{Gr}(p-r, p)$.

§3. The general fiber of the exceptional locus

In this section, the aim is to identify the general fiber of $f|_E$. It requires that we state first the following result.

Lemma 3.1. *Under the assumptions of Setup 1.4, every fiber of $E \rightarrow Z$ contains an f -minimal curve.*

Proof. Let $E_z \subset E_{\text{eq}}$ be a generically reduced fiber of dimension n over a smooth point of Z_{eq} . Since E_z is locally a complete intersection in X , by the adjunction formula we have $-K_{E_z} = -K_E|_{E_z} = (-K_X - E)|_{E_z}$; in particular, $-K_{E_z}$ is ample. Now let $C \subset E_z$ be a rational curve whose degree is minimal with respect to $-K_{E_z}$ among rational curves meeting the smooth locus of E_z . Then

$$(3) \quad -K_{E_z} \cdot C = -K_X \cdot C - E \cdot C \geq n - 1 - E \cdot C.$$

By ampleness of $-E|_{E_z}$, it follows that $-K_{E_z} \cdot C \geq n$. Assume by contradiction that $-K_{E_z} \cdot C > n + 1$. By [Kol96, Proposition II.1.3], there exists a deformation family $\mathcal{H}_z$ of C in E_z such that

$$\dim \mathcal{H}_z \geq -K_{E_z} \cdot C + (n - 3) > 2n - 2,$$

and considering the universal family over $\mathcal{H}$,

$$\begin{array}{ccc} \mathcal{U}_z & \xrightarrow{\text{ev}} & E_z \\ \pi \downarrow & & \\ \mathcal{H}_z & & \end{array}$$

where π is a $\mathbf{P}^1$ -bundle, we have $\dim \mathcal{U}_z \geq 2n$. Fixing $p \in E_z$ a general point and considering $\mathcal{U}_{z,p} = \pi^{-1}(\pi(\text{ev}^{-1}(p)))$, we have $\dim \mathcal{U}_{z,p} \geq n + 1$, hence there exists a fiber of $\mathcal{U}_{z,p} \rightarrow E_z$ of positive dimension. In other words, there exists a point $q \in E_z$ and a positive-dimensional family of rational curves through p and q

(in particular, meeting the smooth locus of E_z), all of which have minimal degree among rational curves meeting $(E_z)_{\text{smooth}}$. By the bend-and-break lemma, there exists such a member which is reducible or nonreduced, which is not possible by the minimality of $-K_{E_z} \cdot C$. As a conclusion, we have either $-K_{E_z} \cdot C = n$ or $-K_{E_z} \cdot C = n + 1$.

Going back to (3), we deduce that one of the two following cases occurs:

- $-K_X \cdot C = n - 1 = l(f)$, in which case C is an f -minimal curve,
- $-K_X \cdot C = n$, in which case $-E \cdot C = 1$ and by [Deb01, Theorem 7.39.c] we have $(K_X - nE) \sim_f 0$ (in other words $-K_X$ and $-nE$ are relatively linearly equivalent).

Assume by contradiction that the second case holds. In that case, for $\Gamma \subset E$ an f -minimal curve, we have $-K_X \cdot \Gamma = n - 1$ by our length hypothesis, hence $-E \cdot \Gamma = \frac{n-1}{n}$, which is not an integer. This is a contradiction.

It follows that any generically reduced fiber of dimension n over the smooth locus of Z contains an f -minimal curve. This ensures that there exists a deformation family of f -minimal curves which dominates Z . This family is unsplit by the length hypothesis and Lemma 2.3, hence compact. Since it is proper over Z , it is surjective. Hence every fiber of $E \rightarrow Z$ contains an f -minimal curve. $\square$

Lemma 3.2. *Under the assumptions of Setup 1.4, the f -minimal curves cover the equidimensional locus E_{eq} .*

Proof. Let $\Gamma \subset E_{\text{eq}}$ be an f -minimal curve. As X is smooth, we may apply [Kol96, Theorem II.1.3], which ensures the existence of a deformation family $\mathcal{H}$ of Γ in X such that

$$\dim \mathcal{H} \geq -K_X \cdot \Gamma - 3 + \dim X = n - 4 + \dim X.$$

Since each member of $\mathcal{H}$ lies in a fiber of f by Corollary 2.4, and we have shown in Lemma 3.1 that every fiber of dimension n contains an f -minimal curve, there exists such a family $\mathcal{H}$ which is surjective over Z . In particular, it admits a fibration $\mathcal{H}_{\text{eq}} \rightarrow Z_{\text{eq}}$, where $\mathcal{H}_{\text{eq}}$ is dense in $\mathcal{H}$, such that the fiber $\mathcal{H}_z$ over a point $z \in Z_{\text{eq}}$ parametrizes deformations of Γ inside E_z . Consider such a fiber E_z of dimension $n = \dim E - \dim Z$; by the above inequality we have

$$\dim \mathcal{H}_z = \dim \mathcal{H} - \dim Z \geq n - 4 + \dim X - (\dim X - 1 - n) = 2n - 3.$$

Now let $\mathcal{U}_z$ be the universal family over $\mathcal{H}_z$:

$$\begin{array}{ccc} \mathcal{U}_z & \xrightarrow{\text{ev}} & E_z \\ \pi \downarrow & & \\ \mathcal{H}_z & & \end{array}$$

By the inequality $\dim \mathcal{H}_z \geq 2n - 3$, we have $\dim \mathcal{U}_z \geq 2n - 2$, and we assume by contradiction that ev is not surjective. Hence $\text{ev}(\mathcal{U}_z)$, which is the locus covered in E_z by the minimal curves, has dimension at most $\dim E_z - 1 = n - 1$. Hence the general fiber of ev has dimension $n - 1$ or more.

Pick a general point $x \in \text{ev}(\mathcal{U}_z)$ and denote $\mathcal{H}_{z,x} = \pi(\text{ev}^{-1}(x))$. This is the space parametrizing the minimal curves through x , and it is birational to $\text{ev}^{-1}(x)$: indeed, any member of $\mathcal{H}_{z,x}$ is an f -minimal curve γ through x , and its only preimage by $\pi|_{\text{ev}^{-1}(x)}$ is (x, γ) . Let $\mathcal{U}_{z,x} = \pi^{-1}(\mathcal{H}_{z,x})$, then we have

$$\dim \mathcal{U}_{z,x} = \dim \mathcal{H}_{z,x} + 1 \geq n.$$

Since the image of ev has dimension at most $n - 1$, the fibers of $\mathcal{U}_{z,x} \rightarrow E_z$ have dimension at least 1. Given a point $y \in \text{ev}(\mathcal{U}_z) - \{x\}$, the fiber over y inside $\mathcal{U}_{z,x}$ contains a positive-dimensional family of curves through the two fixed points x and y . By the bend-and-break lemma, this family admits a reducible member or a nonreduced member, which is not possible since $\mathcal{H}_z$ is unsplit by Lemma 2.3 and the length hypothesis $-K_X \cdot \Gamma = l(f)$.

The conclusion follows that ev is surjective onto E_z . This holds for any fiber E_z of dimension n , hence for the universal family $\mathcal{U}$ over $\mathcal{H}$,

$$\begin{array}{ccc} \mathcal{U} & \longrightarrow & E \\ \downarrow & & \\ \mathcal{H} & & \end{array}$$

the evaluation morphism $\mathcal{U} \rightarrow E$ is surjective onto the equidimensional locus E_{eq} . $\square$

Now, under the conditions of Setup 1.4, we investigate the generically reduced fibers of dimension $n = \dim E - \dim Z$.

Proposition 3.3. *Let $E_z \subset E$ be a generically reduced fiber of dimension n over a smooth point z of Z , and let $\Gamma \subset E_z$ be an f -minimal curve. Then one of the following cases occurs:*

- (i) $-E \cdot \Gamma = 2$ and $E_z \simeq \mathbf{P}^n$,
- (ii) $-E \cdot \Gamma = 1$.

Moreover, in case (ii), if E_z is irreducible then it is isomorphic to a normal quadric.

Proof. First, consider $F \subset E$ any generically reduced fiber of dimension n over a smooth point of Z and $\Gamma \subset F$ an f -minimal curve meeting the smooth locus of F . The existence of such a curve is ensured by Lemma 3.2.

Since $f(F)$ is a smooth point of Z , the fiber F is locally a complete intersection in X , and by adjunction we have $K_F = K_E|_F = (K_X + E)|_F$, hence $-K_F$ is ample. Moreover, since Γ meets the smooth locus of F , by [Kol96, Theorem II.1.3] we have the existence of a deformation family $\mathcal{H}$ of Γ in F such that

$$\dim \mathcal{H} \geq 2n - 4 - E \cdot \Gamma.$$

But $\mathcal{H}$ is unsplit by the length hypothesis and Lemma 2.3, so we know thanks to [CMSB02, Theorem 0.1] that $\dim \mathcal{H}$ is bounded from above by $2n - 2$, and that this bound is reached if and only if F is normalized by $\mathbf{P}^n$. Indeed, in that case $\mathcal{H}$ induces a complete family of rational curves over the normalization F' of F which have minimal degree with respect to the pullback of $-K_F$, and this implies $F' \simeq \mathbf{P}^n$ with $\Gamma \subset F$ the image of a line. In particular, $-E \cdot \Gamma$ can only equal 1 or 2.

- Let us examine situation (i) $-E \cdot \Gamma = 2$. By semicontinuity, for any generically reduced fiber E_z of dimension n over a smooth point of Z and $\Gamma \subset E_z$ a minimal curve meeting the normal locus of E_z , the deformations of Γ inside E_z form at least one family of dimension $2n - 2$, and therefore, by [CMSB02, Theorem 0.1], E_z is normalized by $\mathbf{P}^n$ with Γ the image of a line ℓ :

$$\nu: \mathbf{P}^n \rightarrow E_z.$$

As E_z is a local complete intersection in the smooth variety X , it is Cohen–Macaulay and Gorenstein. By Lemma 2.5 the conductor divisor $\mathcal{D}$ of this normalization is such that

$$\nu^* K_{E_z} \simeq K_{\mathbf{P}^n} + \mathcal{D}.$$

By adjunction, we have

$$-K_{E_z} \cdot \Gamma = (-K_X - E) \cdot \Gamma = n + 1.$$

Hence we have $-K_{E_z} \cdot \Gamma = n + 1 = -K_{\mathbf{P}^n} \cdot \ell$. As numerical equivalence implies linear equivalence on $\mathbf{P}^n$, the following equality of Cartier divisors holds:

$$\nu^* K_{E_z} = K_{\mathbf{P}^n}.$$

So the conductor $\mathcal{D}$ is trivial. Its support is the nonnormal locus of E_z (this is an application of Lemma 2.5, as E_z is Cohen–Macaulay and Gorenstein). In other words E_z is normal in codimension 1, and since it is Cohen–Macaulay we have $E_z \simeq \mathbf{P}^n$.

- Now let us see what happens in situation (ii) $-E \cdot \Gamma = 1$, under the hypothesis that E_z is an irreducible fiber of dimension n above a smooth point of Z .

We pick $\Gamma \subset E_z$ a minimal curve meeting the normal locus of E_z and we denote by $\mathcal{H}_z$ a deformation family of Γ inside E_z . By semicontinuity, the dimension of $\mathcal{H}_z$ can be equal to $2n-2$ or $2n-3$. Assume by contradiction that $\dim \mathcal{H}_z = 2n-2$. In particular, by [CMSB02, Theorem 0.1] the normalization of E_z is isomorphic to $\mathbf{P}^n$, with Γ the image of a line ℓ :

$$\nu: \mathbf{P}^n \rightarrow E_z.$$

From Lemma 2.5, the conductor divisor $\mathcal{D}$ is such that

$$\nu^* K_{E_z} \simeq K_{\mathbf{P}^n} + \mathcal{D}.$$

Moreover, since E_z is locally a complete intersection, it is Gorenstein, and by adjunction we have

$$-K_{E_z} \cdot \Gamma = (-K_X - E) \cdot \Gamma = n = -K_{\mathbf{P}^n} \cdot \ell - 1.$$

Hence the conductor divisor $\mathcal{D}$ on $\mathbf{P}^n$ satisfies $\mathcal{D} \cdot \ell = 1$, which tells us that $\mathcal{D}$ is a hyperplane of $\mathbf{P}^n$. In addition $\nu^*(-E|_{E_z})$ is also a hyperplane for the same reason.

The divisor $-E|_{E_z}$ being ample, we may consider an irreducible curve $\gamma \subset E_z$ given as a complete intersection of general divisors of the linear system $|-dE|_{E_z}|$ for $d \gg 0$ an odd integer. This curve transversally meets the image of $\mathcal{D}$ and if $\gamma' \subset \mathbf{P}^n$ stands for its proper transform, the restriction $\gamma' \rightarrow \gamma$ is the normalization of γ . By the adjunction formula,

$$\begin{aligned} K_{\gamma'} + \mathcal{D}|_{\gamma'} &= (K_{\mathbf{P}^n} - d(n-1)\nu^*E + \mathcal{D})|_{\gamma'} \\ &= \nu^*(K_{E_z} - d(n-1)E)|_{\gamma'} \\ &= (\nu|_{\gamma'})^* K_{\gamma}; \end{aligned}$$

in other words the conductor divisor of $\gamma' \rightarrow \gamma$ is $\mathcal{D}|_{\gamma'}$. Since $\mathcal{D}$ and $\nu^*(-E|_{E_z})$ are hyperplane divisors on $\mathbf{P}^n$, the divisor $\mathcal{D}|_{\gamma'}$ has degree d^{n-1} , which is odd. This is not possible, since the conductor divisor of a curve always has an even degree, by Lemma 2.5.

So, under the hypothesis that E_z is irreducible and $-E \cdot \Gamma = 1$, we have $\dim \mathcal{H}_z = 2n-3$ for every deformation family $\mathcal{H}_z$ of Γ inside E_z . It remains to be proven that E_z is a normal quadric. Since $K_{E_z} \cdot \Gamma = nE \cdot \Gamma$, the restriction $nE|_{E_z}$ is linearly equivalent to K_{E_z} , as a consequence of [Deb01, Theorem 7.39.c]. Moreover, $-E|_{E_z}$ is ample and the normalization $\nu: E'_z \rightarrow E_z$ is such that

$$\begin{aligned} (K_{E'_z} - n\nu^*E) \cdot (-\nu^*E)^{n-1} &= (K_{E'_z} - \nu^*K_{E_z}) \cdot (-\nu^*E)^{n-1} \\ &= -\mathcal{D} \cdot (-\nu^*E)^{n-1}. \end{aligned}$$

Assume by contradiction that E_z is nonnormal. We have $\mathcal{D} > 0$, or else E_z would be regular in codimension one and therefore normal, since it is Cohen–Macaulay. This yields

$$(K_{E'_z} - n\nu^*E) \cdot (-\nu^*E)^{n-1} < 0,$$

in which case there exists a birational morphism $E'_z \rightarrow \mathbf{P}^n$ such that $-E|_{E_z}$ is the pullback of the hyperplane polarization, by [Hö12, Proposition 2.13]. This is not possible, as $\dim \mathcal{H}_z = 2n - 3$, whereas the unique family of minimal curves (i.e., lines) in $\mathbf{P}^n$ has dimension $2n - 2$. We may thus conclude that E_z is normal, with the equality of ample Cartier divisors $-K_{E_z} = -nE|_{E_z}$. By [BS95, Theorem 3.1.6], the generalization of a result by Kobayashi and Ochiai, we may conclude that E_z is isomorphic to a quadric. $\square$

§4. Main theorem

§4.1. The case $-E \cdot \Gamma = 2$

Under the conditions of Setup 1.4, we assume that we have $-E \cdot \Gamma = 2$ for every f -minimal curve Γ . The goal is to exhibit a projective bundle as a birational model for the locus covered by the n -dimensional fibers. We recall the notation

$$E_{\text{eq}} \rightarrow Z_{\text{eq}}$$

for the equidimensional locus.

Theorem 4.1. *In Setup 1.4 and under the condition $-E \cdot \Gamma = 2$, all the fibers of $E_{\text{eq}} \rightarrow Z_{\text{eq}}$ are normalized by $\mathbf{P}^n$. The normalization E' of E_{eq} is a family of projective spaces over the normalization Z' of Z_{eq} which is locally trivial for the analytic topology, such that the following diagram is commutative:*

$$\begin{array}{ccc} E' & \longrightarrow & E_{\text{eq}} \\ \downarrow & & \downarrow \\ Z' & \longrightarrow & Z_{\text{eq}}. \end{array}$$

If moreover n is even, E_{eq} is isomorphic to the projectivization of a rank- $(n+1)$ vector bundle over Z_{eq} .

The proof is broken down into the following Lemmas 4.2, 4.3, and 4.4.

Lemma 4.2. *Let $E \rightarrow Z$ be an equidimensional fibration onto a normal variety with $n = \dim E - \dim Z$, such that the general fiber is a projective space.*

For such a general fiber, let ℓ be a line inside it. If there exists a relatively ample line bundle L on E such that for any contracted rational curve Γ we have

$$L \cdot \Gamma \geq d := L \cdot \ell,$$

then all the fibers are irreducible and generically reduced. Moreover, there exists a finite and birational morphism $E' \rightarrow E$, where E' is a family of projective spaces over Z . If E is normal, then we have $E' \simeq E$.

Proof. The proof that all fibers are irreducible and generically reduced is essentially the same as that of [HN13, Proposition 3.1], but without the assumption that E is normal. The argument is the following: we assume by contradiction that there exists a fiber E_z which is either reducible or not generically reduced:

$$E_z = m_1 D_1 + \cdots + m_s D_s,$$

where all the m_i are nonzero. Without loss of generality we may assume $m_1 \geq 2$ or $s \geq 2$. Either way, by [Kol96, I.3.12] we have

$$(4) \quad d^n = (L|_{E_z})^n > (L|_{D_1})^n.$$

Now consider $C \subset E_z$ a 1-cycle obtained as a degeneration of lines in the general fiber, so that $L \cdot C = d$. We deduce that C is irreducible and reduced from the length condition on the relatively ample line bundle L , namely $L \cdot \Gamma \geq d$ for any reduced and irreducible component Γ of C . We may assume without loss of generality that the curve C lies in D_1 . By semicontinuity, a deformation family of C inside D_1 has dimension $2n - 2$ or more, and [CMSB02, Theorem 0.1] ensures that D_1 is normalized by a projective space, with C the image of a line. Since $L \cdot C = d$, we obtain that the pullback of L by the normalization morphism is isomorphic to $\mathcal{O}_{\mathbf{P}^n}(d)$, and thus

$$(L|_{D_1})^n = d^n$$

which is a contradiction to inequality (4).

As a result, all the fibers are generically reduced and irreducible. If E_z is a fiber, and $C \subset E_z$ is a curve obtained as a degeneration of lines, we have $L \cdot C = d$ and a deformation family of C in E_z has dimension $2n - 2$ or more. By [CMSB02, Theorem 0.1] again, the normalization of E_z is isomorphic to $\mathbf{P}^n$, with C the image of a line:

$$\nu: \mathbf{P}^n \rightarrow E_z.$$

Moreover, the polarization L has degree d on the lines, so $\nu^*L \simeq \mathcal{O}_{\mathbf{P}^n}(d)$.

Now we consider the simultaneous normalization of $E \rightarrow Z$ whose existence is given by [Kol11, Theorem 12]. This is a finite birational modification $E' \rightarrow E$

such that all the fibers of $E' \rightarrow Z$ are normal; in this particular case, they are isomorphic to $\mathbf{P}^n$. Indeed, any fiber E_z of $E \rightarrow Z$ is normalized by $\mathbf{P}^n$, and if E'_z denotes the fiber of $E' \rightarrow Z$ over the same point, then the finite birational morphism $E'_z \rightarrow E_z$ factors through

$$E'_z \rightarrow \mathbf{P}^n$$

since E'_z is normal. It follows that $E'_z \simeq \mathbf{P}^n$ since a finite and birational morphism onto a normal variety is an isomorphism by Zariski's main theorem; see [Zar43].

If E is normal, then $E' \rightarrow E$ is an isomorphism as a finite and birational morphism onto a normal variety. $\square$

Under the conditions of Setup 1.4, and under the condition $-E \cdot \Gamma = 2$, we want to apply Lemma 4.2 to $E_{\text{eq}} \rightarrow Z_{\text{eq}}$. If Z_{eq} is normal, then Lemma 4.2 ensures that the normalization of E_{eq} is a family of projective spaces over Z_{eq} . If Z_{eq} is not normal, let us consider its normalization.

Lemma 4.3. *Let Z' be the normalization of Z_{eq} and E' the normalization of E_{eq} . Then the fibration $E' \rightarrow Z'$ is a family of projective spaces which is locally trivial for the analytic topology.*

In addition, the normalization of any fiber of $E_{\text{eq}} \rightarrow Z_{\text{eq}}$ is isomorphic to $\mathbf{P}^n$.

Proof. We know from Proposition 3.3 that the general fiber of $E' \rightarrow Z'$ is a projective space. Moreover, the pullback of the divisor $-E|_{E_{\text{eq}}}$ to E' satisfies the hypothesis of Lemma 4.2. Indeed, by Proposition 3.3, the minimal degree $-E|_E$ on contracted rational curves is reached on the lines on the fibers which are projective spaces. Hence all the fibers of $E' \rightarrow Z'$ are projective spaces by Lemma 4.2, since E' is normal. By [FG65], this family of projective spaces over Z' is relatively trivial in the analytic topology.

In this situation, the f -minimal curves in E_{eq} are the images of lines. Let us denote by λ the normalization morphism $E' \rightarrow E_{\text{eq}}$. The variety Z' being normal, by [Kol96, I.3.12] the degree of $\lambda^*(-E|_{E_{\text{eq}}})$ on each line contained in a fiber of $E' \rightarrow Z'$ is equal to 2.

Now we consider $E_z \subset E_{\text{eq}}$ the fiber over a point $z \in Z_{\text{eq}}$. Let us denote by $\nu: Z' \rightarrow Z_{\text{eq}}$ the normalization of Z_{eq} and pick a point $z' \in \nu^{-1}(z)$. The fiber in E' over z' is isomorphic to $\mathbf{P}^n$, and since $\lambda: E' \rightarrow E_{\text{eq}}$ is a finite morphism, it yields a finite morphism $\mathbf{P}^n \rightarrow E_z$. In this situation, the minimal curves which cover E_z (see Lemma 3.2) are the images of the lines of $\mathbf{P}^n$.

Since $-E \cdot \Gamma = 2$ and $\lambda^*(-E_{E_{\text{eq}}}) \cdot \ell = 2$ for ℓ a line contained in any fiber of $E' \rightarrow Z'$, and the restriction of λ to ℓ is a finite morphism, say of degree δ ,

$$\lambda|_{\ell}: \ell \xrightarrow{\delta:1} \lambda(\ell),$$

for $\ell \subset \mathbf{P}^n$ a general line and $\Gamma = \lambda(\ell) \subset E_z$ a minimal curve we have

$$2 = -E \cdot \Gamma = \frac{1}{\delta} \lambda^*(-E) \cdot \ell = \frac{2}{\delta},$$

therefore $\delta = 1$. The morphism $\mathbf{P}^n \rightarrow E_z$ is thus birational along the general line $\ell \subset \mathbf{P}^n$, so it is birational. Since it is finite, for $(E_z)'$ the normalization of E_z we have a factorization $\mathbf{P}^n \rightarrow (E_z)'$ which is birational and finite, hence an isomorphism. $\square$

If n is even, we may consider the Cartier divisor

$$J = -K_X + \left(\frac{n}{2} - 1\right)E$$

which has degree 1 on the f -minimal curves. By [Deb01, Theorem 7.39.c], the divisor $E + 2J$ is relatively trivial, so $J \sim_f -\frac{1}{2}E$ is relatively ample.

Lemma 4.4. *If n is even, the fibration $E_{\text{eq}} \rightarrow Z_{\text{eq}}$ is a projective bundle; in other words E_{eq} is isomorphic to the projectivization of a vector bundle over Z_{eq} .*

Proof. The relatively ample Cartier divisor

$$J = -K_X + \left(\frac{n}{2} - 1\right)E$$

has degree 1 on the f -minimal curves, and

$$K_X + (n - 1)J$$

is relatively trivial by [Deb01, Theorem 7.39.c]. Thanks to Theorem 2.7, if $E_z \subset E_{\text{eq}}$ is any fiber, the evaluation map $f^*f_*J \rightarrow J$ is surjective along E_z .

Now let $z \in Z_{\text{eq}}$ be the image of E_z and U an affine neighborhood of z in Y . The direct image $(f_*J)|_U$ being a coherent sheaf on an affine variety, it is endowed with a surjection of the following form:

$$\mathcal{O}_U^{\oplus m+1} \rightarrow f_*J|_U.$$

Then we may take the pullback of this by f and its restriction to E_z , yielding a surjection $\mathcal{O}_{E_z}^{\oplus m+1} \rightarrow (f^*f_*J)|_{E_z}$. Since $(f^*f_*J)|_{E_z} \rightarrow J|_{E_z}$ is surjective as well, the composition is onto, namely

$$\mathcal{O}_{E_z}^{\oplus m+1} \rightarrow J|_{E_z}.$$

Now let $e_0, \dots, e_m$ be the vectors of the canonical basis of $H^0(E_z, \mathcal{O}_{E_z}^{\oplus m+1})$ and $\sigma_0, \dots, \sigma_m$ their images in $H^0(E_z, J|_{E_z})$ ($1 \leq i \leq m + 1$). Then the σ_i do not vanish simultaneously, and they induce a morphism

$$\theta: E_z \rightarrow \mathbf{P}^m$$

such that $\theta^*\mathcal{O}_{\mathbf{P}^m}(1) = J|_{E_z}$.

From Lemma 4.3 we know that E_z is normalized by $\mathbf{P}^n$. Let us denote by $\lambda: \mathbf{P}^n \rightarrow E_z$ the normalization morphism, and $J' = \lambda^*(J|_{E_z})$. Then J' is a line bundle on $\mathbf{P}^n$ which has degree 1 on the lines, yielding $J' \simeq \mathcal{O}_{\mathbf{P}^n}(1)$. The composition $\theta \circ \lambda$ is a morphism

$$\mathbf{P}^n \rightarrow \mathbf{P}^m$$

such that $\lambda^*\theta^*\mathcal{O}_{\mathbf{P}^m}(1) = J' \simeq \mathcal{O}_{\mathbf{P}^n}(1)$. Since J' is ample and $(J')^n = 1$, $\theta \circ \lambda$ is a finite morphism onto a linear subspace of dimension n , hence an isomorphism.

From the above, we deduce that $\theta: E_z \rightarrow \mathbf{P}^n$ is birational and finite, hence an isomorphism. The fibration $E_{\text{eq}} \rightarrow Z_{\text{eq}}$ is thus a family of projective spaces. It is endowed with a relatively ample polarization J whose restriction to any fiber E_z is $\mathcal{O}_{\mathbf{P}^n}(1)$ via the isomorphism $E_z \simeq \mathbf{P}^n$, and the existence of this relative hyperplane polarization ensures $E_{\text{eq}} \simeq \mathbf{P}_{Z_{\text{eq}}}(f_*J)$. $\square$

§4.2. The case $-E \cdot \Gamma = 1$

Under the conditions of Setup 1.4, we assume now $-E \cdot \Gamma = 1$ for Γ any f -minimal curve. We aim to construct a quadric bundle as a birational model for $E_{\text{eq}} \rightarrow Z_{\text{eq}}$. Let us denote $L := \mathcal{O}_{E_{\text{eq}}}(-E|_{E_{\text{eq}}})$.

Theorem 4.5. *In Setup 1.4 and under the condition $-E \cdot \Gamma = 1$, there exist*

- *birational morphisms $E' \rightarrow E_{\text{eq}}$ and $Z' \rightarrow Z_{\text{eq}}$ with E' and Z' normal,*
- *a rank- $(n+2)$ vector bundle $\mathcal{E}$ over Z' ,*
- *and a quadric bundle $\mathcal{Q} \subset \mathbf{P}_{Z'}(\mathcal{E})$,*

such that E' is the normalization of $\mathcal{Q}$ and the following diagram is commutative:

$$\begin{array}{ccccc} \mathcal{Q} & \longleftarrow & E' & \longrightarrow & E_{\text{eq}} \\ & \searrow & \downarrow & & \downarrow \\ & & Z' & \longrightarrow & Z_{\text{eq}}. \end{array}$$

Moreover, each reducible fiber of $E_{\text{eq}} \rightarrow Z_{\text{eq}}$ has two irreducible components, the reductions of which are normalized by $\mathbf{P}^n$. Each irreducible and generically reduced fiber of $E' \rightarrow Z'$ is isomorphic to a quadric, and the reduction of any nonreduced fiber of $E' \rightarrow Z'$ is isomorphic to $\mathbf{P}^n$.

The proof is broken down into Lemma 4.6, Corollary 4.7 and Corollary 4.8.

Lemma 4.6. *There exist birational morphisms $Z' \rightarrow Z_{\text{eq}}$, $E' \rightarrow E_{\text{eq}}$ with E' and Z' normal, a fibration $f': E' \rightarrow Z'$ and a quadric bundle $\mathcal{Q}$ over Z' , such that E' is the normalization of $\mathcal{Q}$. Furthermore, the normalization $E' \rightarrow \mathcal{Q}$ is birational*

along the reduction of each irreducible component of any f' -fiber. The reduction of each irreducible component of any reducible or nonreduced fiber of f' is isomorphic to $\mathbf{P}^n$.

In addition, if Γ is an f -minimal curve in a fiber of $E_{\text{eq}} \rightarrow Z_{\text{eq}}$, and Γ' is any irreducible component of its preimage in E' , then the image of Γ' in $\mathcal{Q}$ is a line.

Proof. The general fiber of $E_{\text{eq}} \rightarrow Z_{\text{eq}}$ is a normal quadric by Proposition 3.3.

We consider the birational modification $\mu: \mathcal{Z} \rightarrow Z_{\text{eq}}$ given by Lemma 2.8, and the vector bundle $\mathcal{V}$ on $\mathcal{Z}$ which coincides with $\mu^* f_* L$ over the locus where $f_* L$ is locally free.

On the component $\mathcal{F}$ of the fiber product $E_{\text{eq}} \times_{Z_{\text{eq}}} \mathcal{Z}$ which dominates E_{eq} , in the commutative square

$$\begin{array}{ccc} \mathcal{F} & \xrightarrow{\mu'} & E_{\text{eq}} \\ g \downarrow & & \downarrow f \\ \mathcal{Z} & \xrightarrow{\mu} & Z_{\text{eq}}, \end{array}$$

there is a surjective map of vector bundles

$$g^* \mathcal{V} \rightarrow \mu'^* L$$

by Lemma 2.8. Now let $\eta: Z' \rightarrow \mathcal{Z}$ and $\eta': E' \rightarrow \mathcal{F}$ be the normalizations. Then the pullback of the above map $g^* \mathcal{V} \rightarrow \mu'^* L$ by η' is surjective:

$$\eta'^* g^* \mathcal{V} \rightarrow \eta'^* \mu'^* L.$$

In the commutative diagram

$$\begin{array}{ccc} E' & \xrightarrow{\eta'} & \mathcal{F} \\ f' \downarrow & & \downarrow g \\ Z' & \xrightarrow{\eta} & \mathcal{Z}, \end{array}$$

we have $\eta'^* g^* \mathcal{V} \simeq f'^* \eta^* \mathcal{V}$. Hence there is a surjective map

$$f'^* \eta^* \mathcal{V} \rightarrow \eta'^* \mu'^* L$$

which induces a factorization of f' by the universal property of projectivized bundles

$$\begin{array}{ccc} E' & \xrightarrow{\chi} & \mathbf{P}_{Z'}(\eta^* \mathcal{V}) \\ & \searrow f' & \swarrow \downarrow \\ & & Z'. \end{array}$$

We denote the image of χ by $\mathcal{Q}$. By [Kol96, I.3.12] every fiber of $\mathcal{Q} \rightarrow Z'$ is a quadric; in other words $\mathcal{Q}$ is a quadric bundle. Let us denote $\mathcal{J} = \eta'^* \mu'^* L$, then we have $\chi^* \mathcal{O}_{\mathcal{Q}}(1) = \mathcal{J}$.

Let $E'_w \subset E'$ be an irreducible fiber above a point $w \in Z'$. For a point w such that $\eta(w)$ is a smooth point of Z , E'_w is isomorphic to its image in E_{eq} via $\mu' \circ \eta'$, which is an irreducible quadric by Proposition 3.3. Under the additional condition that $\eta(w)$ is outside the μ -exceptional locus, $\mu' \circ \eta'$ is an isomorphism around E'_w , and we have $\chi(E'_w) \simeq E'_w$. Therefore, χ is birational, and the general fiber E'_w of $E' \rightarrow Z'$ is a quadric with $\mathcal{J}|_{E'_w}$ the hyperplane polarization.

For any $w \in Z'$, we set $z = \mu \circ \eta(w)$ and $E_z = \mu' \circ \eta'(E'_w)$. Then E_z is the fiber in E_{eq} over $z \in Z_{\text{eq}}$. By construction, μ' is an isomorphism onto E_z , and η' is finite, so the morphism $E'_w \rightarrow E_z$ is finite, ensuring that $\mathcal{J} = \eta'^* \mu'^* L$ is ample on E'_w . This ensures that χ is finite, and since it is birational, the conclusion follows that E' is the normalization of $\mathcal{Q}$.

Now let $\Gamma' \subset E'_w$ be a curve obtained as a degeneration of lines in the smooth fibers of $E_{\text{eq}} \rightarrow Z_{\text{eq}}$. By [Kol96, I.3.12] we get $\mathcal{J} \cdot \Gamma' = 1$. In particular, its image $\Gamma = \mu' \circ \eta'(\Gamma')$ in E_z is an f -minimal curve. Let $\ell = \chi(\Gamma')$ denote its image in the quadric $\mathcal{Q}_w = \chi(E'_w)$. If the degree of the finite map

$$\chi|_{\Gamma'}: \Gamma' \rightarrow \ell \subset \mathcal{Q}_w$$

is equal to δ , then we have

$$1 = \mathcal{J} \cdot \Gamma' = \delta \mathcal{O}_{\mathcal{Q}_w}(1) \cdot \ell \geq \delta,$$

hence $\delta = 1$, and ℓ is a line. Since $\mathcal{J} = \eta'^* \mu'^* L$ and $\mathcal{J}$ has degree 1 on the lines of the general fiber of $E' \rightarrow Z'$, any f -minimal curve $\Gamma \subset E_z$ is the image of $\Gamma' \subset E'$ a degeneration of lines, and $\chi(\Gamma')$ is a line in $\mathcal{Q}_w$.

If E'_w is irreducible and generically reduced, then $\mathcal{Q}_w = \chi(E'_w)$ is irreducible as well. Since $\mathcal{J} = \chi^* \mathcal{O}_{\mathcal{Q}}(1)$ with

$$(\mathcal{J}|_{E'_w})^n = 2 = (\mathcal{O}_{\mathcal{Q}_w}(1))^n,$$

the finite morphism $E'_w \rightarrow \mathcal{Q}_w$ is birational.

Now assume E'_w is reducible or not generically reduced, and D is the reduction of any of its irreducible components. In that case the ample polarization $\mathcal{J}|_{E'_w}$ verifies

$$(\mathcal{J}|_D)^n < (\mathcal{J}|_{E'_w})^n = 2,$$

so $(\mathcal{J}|_D)^n = 1$. The Cartier divisor $\mathcal{J}|_{E'_w}$ is basepoint-free since it is the pullback via the morphism χ of $\mathcal{O}_{\mathcal{Q}_w}(1)$, which is basepoint-free. It follows that the morphism $D \rightarrow \chi(D)$ induced by the linear system $|\mathcal{J}|_D|$ is birational and finite onto

its image. In particular, $\chi(D)$ is isomorphic to $\mathbf{P}^n$, which is normal, so we have $D \simeq \mathbf{P}^n$ by Zariski's main theorem. $\square$

Corollary 4.7. *If E'_w is a fiber of $E' \rightarrow Z'$, then it is either isomorphic to a normal quadric, or as a cycle it is one of the following:*

- *reducible and reduced, namely $E'_w = D_1 + D_2$ with $D_1 \simeq D_2 \simeq \mathbf{P}^n$,*
- *nonreduced, namely $E'_w = 2D$ with $D \simeq \mathbf{P}^n$.*

Proof. Given a fiber $\mathcal{Q}_w$ of the quadric bundle $\mathcal{Q} \rightarrow Z'$ and $E'_w = \chi^{-1}(\mathcal{Q}_w)$, there are three possibilities:

- The quadric $\mathcal{Q}_w$ is normal, in which case for D the reduction of an irreducible component of E'_w which dominates $\mathcal{Q}_w$, the morphism $D \rightarrow \mathcal{Q}_w$ is birational and finite by Lemma 4.6. Since $\mathcal{J}$ is f' -ample, by [Kol96, I.3.12] we have

$$2 = (\mathcal{J}|_{E'_w})^n \geq (\mathcal{J}|_D)^n = \mathcal{O}_{\mathcal{Q}_w}(1)^n = 2,$$

so $E'_w = D$ is irreducible and generically reduced, and the morphism $E'_w \rightarrow \mathcal{Q}_w$ is birational and finite by Lemma 4.6, hence an isomorphism.

- $\mathcal{Q}_w$ is reducible, in which case we have the equality of n -cycles $\mathcal{Q}_w = Q_1 + Q_2$ where $Q_i \simeq \mathbf{P}^n$. In this case $D_i = \chi^{-1}(Q_i)$ for $i = 1, 2$ are the reductions of two irreducible components of E'_w , and since the Q_i are normal and $D_i \rightarrow Q_i$ is birational and finite by Lemma 4.6 we have $D_i \simeq Q_i$. Moreover, $\mathcal{J}|_{D_i} \simeq \mathcal{O}_{\mathbf{P}^n}(1)$ via the isomorphism $D_i \simeq \mathbf{P}^n$, and we have the equality of cycles

$$E'_w = D_1 + D_2$$

since $(\mathcal{J}|_{E'_w})^n = 2$ by [Kol96, I.3.12] and $(\mathcal{J}|_{D_i})^n = 1$ for $i = 1, 2$.

- $\mathcal{Q}_w$ is nonreduced, in which case $\mathcal{Q}_w = 2P$ with $P \simeq \mathbf{P}^n$. By Zariski's main theorem, the reduction D of any irreducible component of E'_w is isomorphic to P since P is normal and $D \rightarrow P$ is birational and finite (Lemma 4.6). Moreover, $\mathcal{J}|_D \simeq \mathcal{O}_{\mathbf{P}^n}(1)$ via the isomorphism $D \simeq \mathbf{P}^n$ since $\mathcal{J}$ has degree 1 on the lines of D . By [Kol96, I.3.12] we have $(\mathcal{J}|_{E'_w})^n = 2$, whereas $(\mathcal{J}|_D)^n = 1$. As a consequence we have either the equality of cycles $E'_w = 2D$, or there exists another irreducible component D_2 of E'_w such that $E'_w = D + D_2$ and $(\mathcal{J}|_{D_2})^n = 1$. In this case $D_2 \simeq \mathbf{P}^n$ for the same reasons as above. $\square$

From Lemma 4.6 and Corollary 4.7, we can deduce information on the reducible fibers of $E_{\text{eq}} \rightarrow Z_{\text{eq}}$.

Corollary 4.8. *Let $z \in Z_{\text{eq}}$ be a point such that the fiber $E_z = f^{-1}(z) \subset E_{\text{eq}}$ is reducible. Then E_z has two irreducible components, and the reduction of each component is normalized by $\mathbf{P}^n$.*

Proof. Let w be a point of $(\mu \circ \eta)^{-1}(z)$ and E'_w the fiber over it, so that $E_z = \mu' \circ \eta'(E'_w)$. Since E_z is reducible, so is E'_w and by Corollary 4.7 it is of the form $D_1 + D_2$ with $D_i \simeq \mathbf{P}^n$. In particular, E_z has two components G_1 and G_2 with $G_i = \mu' \circ \eta'(D_i)$ for $i = 1, 2$.

Through the isomorphism $D_i \simeq \mathbf{P}^n$ we have a morphism from $\mathbf{P}^n$ to the n -fold G_i ,

$$\mu' \circ \eta': D_i \simeq \mathbf{P}^n \rightarrow G_i.$$

We know that a morphism from $\mathbf{P}^n$ to a variety of dimension n does not contract any curve. Hence $D_i \rightarrow G_i$ is finite. Moreover, the restriction of the line bundle $\mathcal{J}$ to D_i is the pullback of $L|_{G_i}$, so we have

$$(L|_{G_i})^n = (\mathcal{J}|_{D_i})^n = 1,$$

so $\mathbf{P}^n \rightarrow G_i$ is a birational and finite morphism. From this we deduce that G_i is normalized by $\mathbf{P}^n$ for $i = 1, 2$. $\square$

§5. An example of a nonequidimensional divisorial elementary Mori contraction of submaximal length

Consider in $\mathbf{C}^6$ with coordinates $(x_1, x_2, x_3, x_4, \lambda, \mu)$ the cubic affine cone

$$Y = \{\lambda x_1^2 + \lambda x_2^2 + \mu x_3^2 + \mu x_4^2 + x_1 x_2^2 + x_1 x_3^2 + x_1 x_4^2 = 0\}.$$

One can think of it as a family of affine cubics $Y_{(\lambda, \mu)} \subset \mathbf{C}^4_{(x_1, x_2, x_3, x_4)}$ indexed by $(\lambda, \mu) \in \mathbf{C}^2$. Let $\varepsilon: X \rightarrow Y$ be the blow-up of Y along $\Lambda = \{x_1 = x_2 = x_3 = x_4 = 0\} \simeq \mathbf{C}^2$.

Lemma 5.1. *The variety X is smooth.*

Proof. Since Y is smooth outside Λ , it is enough to prove that X is smooth along $E = \varepsilon^{-1}(\Lambda)$. We have a model for X inside $\mathbf{P}^3_{[u_1:u_2:u_3:u_4]} \times \mathbf{C}^6_{(x_1, x_2, x_3, x_4, \lambda, \mu)}$ with the following equations:

$$\begin{cases} \lambda u_1^2 + \lambda u_2^2 + \mu u_3^2 + \mu u_4^2 + x_2 u_1 u_2 + x_3 u_1 u_3 + x_4 u_1 u_4 = 0, \\ \det_2 \begin{pmatrix} u_1 & u_2 & u_3 & u_4 \\ x_1 & x_2 & x_3 & x_4 \end{pmatrix} = 0. \end{cases}$$

Since the codimension of X in $\mathbf{P}^3 \times \mathbf{C}^6$ is 4, checking that X is smooth amounts to showing that the Jacobian of the system of its equations has rank 4 everywhere

on X . Since X is the blow-up of the affine cone Y along an affine subspace which contains the singular points of Y , and E is the exceptional divisor of the blow-up, X is smooth on the complement of E . Hence we only need to check that the Jacobian of X has maximal rank along E , in other words when we specify $x_i = 0$ for all i . Without loss of generality, we may assume $u_1 = 1$ and work with (u_2, u_3, u_4) affine. A straightforward calculation shows that the partial Jacobian with respect to the variables $(x_1, x_2, x_3, x_4, \lambda, \mu)$ always has rank 4. $\square$

Consider now the blowdown morphism $\varepsilon: X \rightarrow Y$. Given a point $p = (\lambda, \mu) \in \Lambda - \{(0, 0)\}$, the fiber $E_{(\lambda, \mu)}$ over p is the projectivization of the tangent cone of $Y_{(\lambda, \mu)}$ at the origin; in other words $E_{(\lambda, \mu)}$ is the quadric of $\mathbf{P}^3_{[u_1:u_2:u_3:u_4]}$ given by the equation

$$\lambda u_1^2 + \lambda u_2^2 + \mu u_3^2 + \mu u_4^2 = 0.$$

The fiber $E_{(0,0)}$ over the origin is the whole $\mathbf{P}^3$.

The ε -exceptional divisor E is a hypersurface of $\mathbf{P}^3 \times \Lambda$ and it is generically a quadric bundle over Λ with an isolated fiber which is isomorphic to $\mathbf{P}^3$. In X , the divisor $-E$ is ε -anti ample and the $(-E)$ -minimal contracted curves are the lines contained in the fibers of $E \rightarrow \Lambda$.

Lemma 5.2. *Let $\ell \subset E$ be a line contracted by ε . Then $-E \cdot \ell = 1$.*

Proof. Let ℓ be a line contained in the general fiber $E_{(\lambda, \mu)}$. Then we may degenerate ℓ to a line ℓ_0 contained in $E_{(0,0)} \simeq \mathbf{P}^3$, or contained in a singular quadric $E_{(\lambda', \mu')}$, yielding $-E \cdot \ell = -E \cdot \ell_0$. Hence we only need to check that $-E \cdot \ell = 1$ for ℓ contained in a smooth quadric $E_{(\lambda, \mu)} \simeq \mathbf{P}^1 \times \mathbf{P}^1$.

Consider the surface complete intersection

$$\Sigma = Y \cap \{\lambda = \mu = 1, x_1 = 0\}.$$

This is the affine quadric cone in $\mathbf{C}^3$ given by the equation

$$x_2^2 + x_3^2 + x_4^2 = 0.$$

Let $\Sigma' \subset X$ be the proper transform of Σ via ε , i.e., $\Sigma' \rightarrow \Sigma$ is the resolution of the singular point of Σ . This resolution contracts a smooth rational curve $\gamma = \Sigma' \cap E$ with $\gamma^2 = 2$. Moreover, γ is by construction a hyperplane section of the quadric $E_{(1,1)}$. For any line ℓ in $E_{(1,1)}$ we have thus

$$-2E|_{E_{(1,1)}} \cdot \ell = -E|_{E_{(1,1)}} \cdot \gamma = -E|_{\Sigma'} \cdot \gamma = -\gamma^2 = 2,$$

yielding $-E \cdot \ell = 1$ in X . $\square$

Lemma 5.3. *The blowdown morphism $\varepsilon: X \rightarrow Y$ is a divisorial elementary Mori contraction of submaximal length.*

The fibration $E \rightarrow \Lambda = \varepsilon(E)$, generically a quadric bundle, admits a fiber E_0 which is isomorphic to $\mathbf{P}^3$ with $-K_X|_{E_0}$ the hyperplane polarization.

Proof. The general fiber E_p is embedded as a quadric surface in $\mathbf{P}^3$ and it satisfies

$$\mathcal{O}_{E_p}(2) = -K_{E_p} = (-K_X - E)|_{E_p},$$

and since $-E \cdot \ell = 1$ for ℓ a line contained in E_p and $-K_{E_p} \cdot \ell = 2$, we have

$$-K_X \cdot \ell = 1 = \dim E_p - 1$$

for ℓ any line in the quadric E_p .

If we degenerate the general line $\ell \in E_p$ to a line ℓ_0 in the central fiber $E_0 = E_{(0,0)} \simeq \mathbf{P}^3$, we have $-K_X \cdot \ell_0 = 1$ and via the isomorphism $E_0 \simeq \mathbf{P}^3$, the restriction $-K_X|_{E_0}$ is the polarization $\mathcal{O}_{\mathbf{P}^3}(1)$. In particular, the divisor $-K_X$ is relatively ample, hence ε is a Mori contraction.

It remains to be proven that ε is an elementary contraction. By the relative cone theorem (see for instance [Deb01, Theorem 7.51]) there exists a curve $C \subset E_0$ whose class is extremal in the relative Mori cone of $\varepsilon: X \rightarrow Y$, and an elementary contraction $\eta: X \rightarrow X'$ which contracts all the curves in the numerical equivalence class of C and fits in the following commutative diagram:

$$\begin{array}{ccccc} E \subset X & \xrightarrow{\eta} & X' & \supset & \eta(E). \\ & \searrow \varepsilon & & \swarrow \gamma & \\ & & Y & & \\ & & \cup & & \\ & \searrow & \Lambda & \swarrow & \end{array}$$

Since E_0 is isomorphic to $\mathbf{P}^3$, and η contracts a curve $C \subset E_0$, then $\eta(E_0)$ is a point of X' . The restriction of γ to $\eta(E)$ is thus a proper fibration over Λ whose central fiber is a point; by semicontinuity γ is locally an isomorphism over the origin of Y . This ensures that there exists a quadric fiber E_p for $p \neq 0$ such that $\eta(E_p)$ is a point; as a consequence both families of lines on the general fiber of $E \rightarrow \Lambda$ are contracted, and as a consequence $\eta(E) \simeq \Lambda$ and γ is an isomorphism. Therefore, ε is elementary. Moreover, it has submaximal length (Theorem 1.3) by the equality $l(\varepsilon) = -K_X \cdot \ell = 1 = \dim E_p - 1$, for ℓ a line contained in the general fiber E_p . $\square$

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