

Multicomplex Hyperfunctions and Multicomplex Microfunctions

by

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Abstract

In this paper, we develop a theory of multicomplex hyperfunctions. Applying functorial techniques, we introduce the notion of multicomplex hyperfunctions of several variables and establish their idempotent representation theorem. Additionally, we provide a microlocal study of multicomplex hyperfunctions, in which we define the notion of multicomplex microfunctions and prove some of their fundamental properties.

Mathematics Subject Classification 2020: 30G35 (primary); 32A45, 32C35 (secondary).

Keywords: multicomplex analysis, hyperfunctions, microfunctions, derived category.

§1. Introduction

As a generalization of functions, M. Sato introduced the notion of hyperfunctions in [10] and [11]. Let V be an open set in $\mathbb{R}^n$ and Ω a complex neighborhood of V in $\mathbb{C}^n$ such that $V = \mathbb{R}^n \cap \Omega$. Let $\mathcal{O}_{\mathbb{C}^n}$ denote the sheaf of complex holomorphic functions on $\mathbb{C}^n$. He showed that the relative cohomology group $H_V^p(\Omega; \mathcal{O}_{\mathbb{C}^n})$ of $\mathcal{O}_{\mathbb{C}^n}$ with supports in V vanishes for $p \neq n$. He then defined a hyperfunction as an element of the relative cohomology group

$$\mathcal{B}_{\mathbb{R}^n,1}(V) = H_V^n(\Omega; \mathcal{O}_{\mathbb{C}^n}).$$

He also showed that $\mathcal{B}_{\mathbb{R}^n,1}$ is a flabby sheaf on $\mathbb{R}^n$. It is well known that hyperfunctions are more natural and useful than distributions for studying linear partial differential equations with real analytic coefficients. In particular, the study of singularities of hyperfunctions in the cotangent bundle $T^*\mathbb{R}^n$ plays an important role. In order to realize this, the notion of microfunctions was introduced, and the

Communicated by T. Mochizuki. Received November 11, 2025. Revised March 9, 2026.

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theory of hyperfunctions and microfunctions has vastly developed into a field known as algebraic analysis [12].

Inspired by the work of Hamilton and Clifford on quaternions, Segre introduced the notion of multicomplex numbers in 1892. Let L be a positive integer and $\mathbb{C}_0 = \mathbb{R}$. The multicomplex algebra is defined inductively as follows:

$$\mathbb{C}_L = \{Z = w_1 + w_2 i_L \mid w_1, w_2 \in \mathbb{C}_{L-1}\},$$

where i_L is a new imaginary unit that is independent of and commutes with the elements $i_1, i_2, \dots, i_{L-1}$. For $L = 2$, $\mathbb{C}_2$ corresponds to the bicomplex algebra $\mathbb{BC}$. Since $\mathbb{C}_L$ is a commutative ring with non-trivial zero divisors in the case $L \geq 2$, the study of multicomplex functions is more difficult than that of complex functions. Nevertheless, we can define the notion of holomorphicity of multicomplex functions in a manner similar to that of complex functions. The study of multicomplex holomorphic functions is called multicomplex analysis. The reference [8] is a fundamental textbook on this subject. See Sections 2.1 and 2.2 for a quick review. For $L = 2$, the references [4] and [9] are also fundamental. See [6] and [7] for the author's recent works in bicomplex analysis.

In multicomplex analysis, the idempotent representation plays an important role. For $L \geq 2$, setting

$$\mathbf{e}_L = \frac{1 + i_{L-1} i_L}{2}, \quad \mathbf{e}_L^{\dagger} = \frac{1 - i_{L-1} i_L}{2},$$

$\mathbf{e}_L$ and $\mathbf{e}_L^{\dagger}$ are idempotent elements satisfying $\mathbf{e}_L \mathbf{e}_L^{\dagger} = 0$ and $\mathbf{e}_L, \mathbf{e}_L^{\dagger} \notin \mathbb{C}_{L-1}$. Then every multicomplex number $Z = w_1 + w_2 i_L \in \mathbb{C}_L$ can be uniquely represented by

$$Z = Z_{\mathbf{e}_L} \mathbf{e}_L + Z_{\mathbf{e}_L^{\dagger}} \mathbf{e}_L^{\dagger},$$

where $Z_{\mathbf{e}_L} = w_1 - w_2 i_{L-1}$, $Z_{\mathbf{e}_L^{\dagger}} = w_1 + w_2 i_{L-1} \in \mathbb{C}_{L-1}$. Furthermore, every multicomplex holomorphic function F can be locally expressed as

$$(1.1) \quad F(Z) = F_{\mathbf{e}_L}(Z_{\mathbf{e}_L}) \mathbf{e}_L + F_{\mathbf{e}_L^{\dagger}}(Z_{\mathbf{e}_L^{\dagger}}) \mathbf{e}_L^{\dagger},$$

where $F_{\mathbf{e}_L}$ and $F_{\mathbf{e}_L^{\dagger}}$ are multicomplex holomorphic functions dependent only on the variables $Z_{\mathbf{e}_L}$ and $Z_{\mathbf{e}_L^{\dagger}}$, respectively. The idempotent representation (1.1) is one of the significant properties of multicomplex holomorphic functions. From the idempotent representation (1.1), several fundamental properties of multicomplex holomorphic functions can be immediately derived inductively by applying properties of multicomplex holomorphic functions to the coefficients of $\mathbf{e}_L$ and $\mathbf{e}_L^{\dagger}$.

Inspired by the work of Colombo et al. on bicomplex hyperfunctions in [2], A. Vajiac and M. B. Vajiac introduced the notion of one-variable multicomplex

hyperfunctions in [14]. Let V be an open set in $\mathbb{R}$ and Ω a multicomplex neighborhood of V in $\mathbb{C}_L$ such that $V = \mathbb{R} \cap \Omega$. Let $\mathcal{O}_{\mathbb{C}_L}$ denote the sheaf of multicomplex holomorphic functions on $\mathbb{C}_L$. They showed that the relative cohomology group $H_V^p(\Omega; \mathcal{O}_{\mathbb{C}_L})$ of $\mathcal{O}_{\mathbb{C}_L}$ with supports in V vanishes for $p \neq 2^L - 1$. They then defined a one-variable multicomplex hyperfunction as an element of the relative cohomology group

$$\mathcal{B}_{\mathbb{R}, L}(V) = H_V^{2^L - 1}(\Omega; \mathcal{O}_{\mathbb{C}_L}).$$

They also showed that $\mathcal{B}_{\mathbb{R}, L}$ is a flabby sheaf on $\mathbb{R}$ and the duality theorem.

In this paper we establish a theory of multicomplex hyperfunctions of several variables. Let V be an open set in $\mathbb{R}^n$ and Ω a multicomplex neighborhood of V in $\mathbb{C}_L^n$ such that $V = \mathbb{R}^n \cap \Omega$. Let $\mathcal{O}_{\mathbb{C}_L^n}$ denote the sheaf of multicomplex holomorphic functions on $\mathbb{C}_L^n$. Applying functorial techniques, we first prove that the relative cohomology group $H_V^p(\Omega; \mathcal{O}_{\mathbb{C}_L^n})$ of $\mathcal{O}_{\mathbb{C}_L^n}$ with supports in V vanishes for $p \neq (2^L - 1)n$. Based on this result, we define a multicomplex hyperfunction of several variables as an element of the relative cohomology group

$$\mathcal{B}_{\mathbb{R}^n, L}(V) = H_V^{(2^L - 1)n}(\Omega; \mathcal{O}_{\mathbb{C}_L^n}).$$

We also prove the idempotent representation theorem for these hyperfunctions. By the idempotent representation, we can immediately derive some of their fundamental properties of multicomplex hyperfunctions, such as flabbiness. In [5], the essential ideas were explained and these results were previously proved only for the affine case with $L = 2$. In contrast, this paper goes further by developing our study on multicomplex manifolds and establishing a detailed microlocal theory. We begin by introducing the notion of multicomplexification of a real analytic manifold in Section 2.3 and subsequently develop a theory of multicomplex hyperfunctions on a real analytic manifold in Section 3. Furthermore, we establish a microlocal theory for these hyperfunctions, in which we define the notion of multicomplex microfunctions of several variables on a real analytic manifold and prove some of their fundamental properties in Sections 4 and 5. The argument of Sections 3, 4 and 5 is parallel to that of [5, Sections 3, 4, 5], which studied the affine case with $L = 2$. However, the results are generalized to the multicomplex manifold case. Finally, in Section 6, we study several operations on multicomplex hyperfunctions and microfunctions, such as linear differential operators, microlocal operators, conjugations, the inverse image, products and the direct image. As in the classical theory, the results of this paper will serve as a foundation for future developments in multicomplex analysis.

§2. Preliminaries

§2.1. Multicomplex numbers

In this subsection we review the definition and fundamental properties of multicomplex numbers. For more details, we refer to [8].

Let $\mathbb{R}$ be the field of real numbers, and set $\mathbb{C}_0 = \mathbb{R}$. Also let L be a positive integer. We inductively define the multicomplex algebra of level L by the quotient ring

$$\mathbb{C}_L = \mathbb{C}_{L-1}[X]/(X^2 + 1)$$

of the polynomial ring with coefficients in $\mathbb{C}_{L-1}$. Denoting the equivalence class of the indeterminate X by i_L , $\mathbb{C}_L$ is isomorphic to the commutative algebra

$$\mathbb{C}_L = \{Z = w_1 + w_2 i_L \mid w_1, w_2 \in \mathbb{C}_{L-1}\}$$

generated by $\mathbb{C}_{L-1}$ and i_L . Here, i_L is considered as a new imaginary unit that is independent of and commutes with the elements $i_1, i_2, \dots, i_{L-1}$:

$$i_L \notin \mathbb{C}_{L-1}, \quad i_\ell i_L = i_L i_\ell \quad (\ell = 1, 2, \dots, L-1), \quad i_L^2 = -1.$$

The set $\mathbb{C}_1$ (resp. $\mathbb{C}_2$) is the field of complex numbers (resp. the ring of bicomplex numbers), which is also denoted by $\mathbb{C}$ (resp. $\mathbb{BC}$). We call an element of $\mathbb{C}_L$ a multicomplex number of level L . For example, a multicomplex number of level 3 is represented by

$$\begin{aligned} Z &= w_1 + w_2 i_3 \\ &= z_1 + z_2 i_2 + z_3 i_3 + z_4 i_2 i_3 \\ &= x_1 + y_1 i_1 + x_2 i_2 + y_2 i_1 i_2 + x_3 i_3 + y_3 i_1 i_3 + x_4 i_2 i_3 + y_4 i_1 i_2 i_3, \end{aligned}$$

where $w_1, w_2 \in \mathbb{C}_2$, $z_1, z_2, z_3, z_4 \in \mathbb{C}_1$, $x_1, y_1, x_2, y_2, x_3, y_3, x_4, y_4 \in \mathbb{C}_0$. In general, for $\ell = 0, 1, 2, \dots, L-1$, a multicomplex number Z of level L is uniquely represented by

$$Z = \sum_{(\alpha_{\ell+1}, \dots, \alpha_L) \in \{0,1\}^{L-\ell}} Z_{(\alpha_{\ell+1}, \dots, \alpha_L)} \prod_{j=\ell+1}^L i_j^{\alpha_j} \quad (Z_{(\alpha_{\ell+1}, \dots, \alpha_L)} \in \mathbb{C}_\ell).$$

Thus, $\mathbb{C}_L$ is isomorphic to $\mathbb{C}_\ell^{2^{L-\ell}}$ ($\ell = 0, 1, 2, \dots, L-1$) as a ring.

For $L \geq 2$, the multicomplex algebra $\mathbb{C}_L$ of level L is neither a field nor an integral domain. Set $\mathfrak{S}_1 = \{0\}$. The set of zero divisors of $\mathbb{C}_L$ with 0 is inductively represented by

$$\mathfrak{S}_L = \{Z = w_1 + w_2 i_L \in \mathbb{C}_L \mid w_1^2 + w_2^2 \in \mathfrak{S}_{L-1}\}.$$

Note that in $\mathbb{C}_L$, being a zero divisor is equivalent to being a non-unit element.

Setting

$$\mathbf{e}_L = \frac{1 + i_{L-1}i_L}{2}, \quad \mathbf{e}_L^{\dagger L} = \frac{1 - i_{L-1}i_L}{2},$$

$\mathbf{e}_L$ and $\mathbf{e}_L^{\dagger L}$ are idempotent elements satisfying $\mathbf{e}_L \mathbf{e}_L^{\dagger L} = 0$ and $\mathbf{e}_L, \mathbf{e}_L^{\dagger L} \notin \mathbb{C}_{L-1}$. Every multicomplex number $Z = w_1 + w_2 i_L$ of level L has the idempotent representation

$$Z = Z_{\mathbf{e}_L} \mathbf{e}_L + Z_{\mathbf{e}_L^{\dagger L}} \mathbf{e}_L^{\dagger L},$$

where $Z_{\mathbf{e}_L} = w_1 - w_2 i_{L-1}$, $Z_{\mathbf{e}_L^{\dagger L}} = w_1 + w_2 i_{L-1} \in \mathbb{C}_{L-1}$. For example, a multicomplex number of level 3 is represented by

$$\begin{aligned} Z &= Z_{\mathbf{e}_3} \mathbf{e}_3 + Z_{\mathbf{e}_3^{\dagger 3}} \mathbf{e}_3^{\dagger 3} \\ &= (Z_{\mathbf{e}_3})_{\mathbf{e}_2} \mathbf{e}_2 \mathbf{e}_3 + (Z_{\mathbf{e}_3})_{\mathbf{e}_2^{\dagger 2}} \mathbf{e}_2^{\dagger 2} \mathbf{e}_3 + (Z_{\mathbf{e}_3^{\dagger 3}})_{\mathbf{e}_2} \mathbf{e}_2 \mathbf{e}_3^{\dagger 3} + (Z_{\mathbf{e}_3^{\dagger 3}})_{\mathbf{e}_2^{\dagger 2}} \mathbf{e}_2^{\dagger 2} \mathbf{e}_3^{\dagger 3}. \end{aligned}$$

In general, for $\ell = 1, 2, \dots, L-1$, we set

$$\Lambda(L, \ell) = \prod_{j=\ell+1}^L \{\emptyset, \dagger_j\}.$$

For $I = (\lambda_{\ell+1}, \dots, \lambda_L) \in \Lambda(L, \ell)$, we also set

$$\mathbf{e}_I = \mathbf{e}_{\ell+1}^{\lambda_{\ell+1}} \cdots \mathbf{e}_L^{\lambda_L}.$$

Note that $\mathbf{e}_I^2 = \mathbf{e}_I$ holds for $I \in \Lambda(L, \ell)$, $\mathbf{e}_I \mathbf{e}_J = 0$ holds for $I, J \in \Lambda(L, \ell)$ with $I \neq J$, and $\sum_{I \in \Lambda(L, \ell)} \mathbf{e}_I = 1$. Then a multicomplex number Z of level L is uniquely represented by a linear combination of $\{\mathbf{e}_I\}_{I \in \Lambda(L, \ell)}$ with coefficients in $\mathbb{C}_\ell$:

$$(2.1) \quad Z = \sum_{I \in \Lambda(L, \ell)} Z_{\mathbf{e}_I} \mathbf{e}_I \quad (Z_{\mathbf{e}_I} \in \mathbb{C}_\ell).$$

We also call (2.1) the idempotent representation of Z . By the idempotent representation (2.1), we have the following natural isomorphism as a ring:

$$\mathbb{C}_L \simeq \bigoplus_{I \in \Lambda(L, \ell)} \mathbb{C}_\ell \mathbf{e}_I.$$

For each index $I \in \Lambda(L, \ell)$, we define the surjective ring homomorphism $\Phi_{\mathbf{e}_I} : \mathbb{C}_L \rightarrow \mathbb{C}_\ell$ by $\Phi_{\mathbf{e}_I}(Z) = Z_{\mathbf{e}_I}$ of (2.1).

Let m be a positive integer. In the case of several multicomplex variables, we also define the maps $\Phi_{\mathbf{e}_I} : \mathbb{C}_L^m \rightarrow \mathbb{C}_\ell^m$ for $I \in \Lambda(L, \ell)$ by

$$\Phi_{\mathbf{e}_I}(Z) = (\Phi_{\mathbf{e}_I}(Z_1), \dots, \Phi_{\mathbf{e}_I}(Z_m)) \quad (Z = (Z_1, \dots, Z_m) \in \mathbb{C}_L^m).$$

Then $Z \in \mathbb{C}_L^m$ has the idempotent representation

$$Z = \sum_{I \in \Lambda(L, \ell)} \Phi_{\mathbf{e}_I}(Z) \mathbf{e}_I.$$

In order to emphasize components, we may identify the image of a point by $\Phi_{\mathbf{e}_I}$ with a point on the $\mathbf{e}_I$ -axis $\mathbb{C}_\ell^m \mathbf{e}_I$ in $\mathbb{C}_L^m$. Namely, we may use the notation $\Phi_{\mathbf{e}_I}(\mathbb{C}_L^m) = \mathbb{C}_\ell^m \mathbf{e}_I$ and so on. Then we have the following isomorphism as a ring:

$$(2.2) \quad \mathbb{C}_L^m \simeq \bigoplus_{I \in \Lambda(L, \ell)} \mathbb{C}_\ell^m \mathbf{e}_I.$$

We call (2.2) the nesting property of $\mathbb{C}_L^m$, which is one of the most important structures of $\mathbb{C}_L^m$. Note that the idempotent representation (2.1) and the nesting property (2.2) depend on the definition of $\mathbf{e}_j$ and $\mathbf{e}_j^{\dagger j}$ ($j = 2, \dots, L$). Although there are several possible definitions, in this paper we adopt the standard one following [8].

§2.2. Multicomplex holomorphic functions

In this subsection, we review fundamental definitions and results in multicomplex analysis. For more details, we refer to [8].

We denote the standard Euclidean norm on $\mathbb{C}$ by $\|\cdot\|_1$. For any multicomplex number $Z = w_1 + w_2 i_L \in \mathbb{C}_L$ of level L , we inductively define the norm $\|Z\|_L$ of Z by

$$\|Z\|_L = \sqrt{\|w_1\|_{L-1}^2 + \|w_2\|_{L-1}^2}.$$

This norm induces a topology on $\mathbb{C}_L$, which is isomorphic to the Euclidean space $\mathbb{C}_1^{2^{L-1}}$. In general, $\mathbb{C}_L$ is isomorphic to $\mathbb{C}_\ell^{2^{L-\ell}}$ for $\ell = 0, 1, 2, \dots, L-1$ as a topological space. Moreover, the map $\Phi_{\mathbf{e}_I}$ is continuous and open for any index $I \in \Lambda(L, \ell)$.

Let $\Omega \subset \mathbb{C}_L$ be an open set, $F: \Omega \rightarrow \mathbb{C}_L$ a multicomplex function on Ω , and $Z_0 \in \Omega$. We say that F is multicomplex differentiable at Z_0 if the limit

$$\lim_{\substack{Z \rightarrow Z_0 \\ Z - Z_0 \notin \mathfrak{S}_L}} \frac{F(Z) - F(Z_0)}{Z - Z_0}$$

exists. We also say that F is multicomplex holomorphic on Ω if F is multicomplex differentiable at any point of Ω . We denote the set of multicomplex holomorphic functions on Ω by $\mathcal{O}_{\mathbb{C}_L}(\Omega)$. Then $\mathcal{O}_{\mathbb{C}_L}$ is a sheaf on $\mathbb{C}_L$.

The idempotent representation of multicomplex holomorphic functions plays an important role in multicomplex analysis. Let F be a multicomplex function

on Ω . By the idempotent representation, we have

$$F(Z) = F_{\mathbf{e}_L}(Z)\mathbf{e}_L + F_{\mathbf{e}_L^\dagger}(Z)\mathbf{e}_L^\dagger,$$

where $F_{\mathbf{e}_L}$ and $F_{\mathbf{e}_L^\dagger}$ are $\mathbb{C}_{L-1}$ -valued functions of two variables in $\mathbb{C}_{L-1}$. The following theorem, called the Ringleb-type theorem, is one of the most important results in multicomplex analysis.

Theorem 2.1. *Assume that the fibers of $\Phi_{\mathbf{e}_L}|_\Omega$ and $\Phi_{\mathbf{e}_L^\dagger}|_\Omega$ are connected. Then F is multicomplex holomorphic on Ω if and only if $F_{\mathbf{e}_L}$ (resp. $F_{\mathbf{e}_L^\dagger}$) is multicomplex holomorphic on $\Phi_{\mathbf{e}_L}(\Omega)$ (resp. $\Phi_{\mathbf{e}_L^\dagger}(\Omega)$) of one variable with respect to $Z_{\mathbf{e}_L}$ (resp. $Z_{\mathbf{e}_L^\dagger}$). Furthermore, any multicomplex holomorphic function F on Ω is analytically continued to $\Phi_{\mathbf{e}_L}^{-1}(\Phi_{\mathbf{e}_L}(\Omega)) \cap \Phi_{\mathbf{e}_L^\dagger}^{-1}(\Phi_{\mathbf{e}_L^\dagger}(\Omega))$ and we have*

$$F(Z) = F_{\mathbf{e}_L}(Z_{\mathbf{e}_L})\mathbf{e}_L + F_{\mathbf{e}_L^\dagger}(Z_{\mathbf{e}_L^\dagger})\mathbf{e}_L^\dagger.$$

In general, let F be a multicomplex function on Ω . For $\ell = 1, 2, \dots, L-1$, by the idempotent representation, we have

$$F(Z) = \sum_{I \in \Lambda(L, \ell)} F_{\mathbf{e}_I}(Z)\mathbf{e}_I,$$

where each $F_{\mathbf{e}_I} : \Omega \rightarrow \mathbb{C}_\ell$ is considered as a $\mathbb{C}_\ell$ -valued function of $2^{L-\ell}$ variables in $\mathbb{C}_\ell$. Then we have the following general theorem.

Theorem 2.2. *Assume that the fibers of $\Phi_{\mathbf{e}_I}|_\Omega$ are connected for any index $I \in \Lambda(L, \ell)$. Then F is multicomplex holomorphic on Ω if and only if $F_{\mathbf{e}_I}$ is multicomplex holomorphic on $\Phi_{\mathbf{e}_I}(\Omega)$ of one variable with respect to $Z_{\mathbf{e}_I}$ for any index $I \in \Lambda(L, \ell)$. Furthermore, any multicomplex holomorphic function F on Ω is analytically continued to $\bigcap_{I \in \Lambda(L, \ell)} \Phi_{\mathbf{e}_I}^{-1}(\Phi_{\mathbf{e}_I}(\Omega))$ and we have*

$$F(Z) = \sum_{I \in \Lambda(L, \ell)} F_{\mathbf{e}_I}(Z_{\mathbf{e}_I})\mathbf{e}_I.$$

By Theorem 2.2, we can immediately derive fundamental properties of multicomplex holomorphic functions of one variable, such as Taylor's theorem and the principle of analytic continuation.

In the terminology of sheaves, Theorem 2.2 can be described as the following isomorphism of sheaves on $\mathbb{C}_L$:

$$\mathcal{O}_{\mathbb{C}_L} \simeq \bigoplus_{I \in \Lambda(L, \ell)} \Phi_{\mathbf{e}_I}^{-1}(\mathcal{O}_{\mathbb{C}_\ell \mathbf{e}_I})\mathbf{e}_I.$$

Let m be a positive integer. In the case of several multicomplex variables, let $\Omega \subset \mathbb{C}_L^m$ be an open set, and $F: \Omega \rightarrow \mathbb{C}_L$ a multicomplex function on Ω . We say that F is multicomplex holomorphic on Ω if and only if F is partially multicomplex holomorphic in each variable on Ω . We denote the sheaf of multicomplex holomorphic functions of several variables by $\mathcal{O}_{\mathbb{C}_L^m}$. Then we also have the following isomorphism of sheaves on $\mathbb{C}_L^m$:

$$(2.3) \quad \mathcal{O}_{\mathbb{C}_L^m} \simeq \bigoplus_{I \in \Lambda(L, \ell)} \Phi_{\mathbf{e}_I}^{-1}(\mathcal{O}_{\mathbb{C}_\ell^m \mathbf{e}_I}) \mathbf{e}_I.$$

We call (2.3) the nesting property of $\mathcal{O}_{\mathbb{C}_L^m}$, which is one of the most important structures of $\mathcal{O}_{\mathbb{C}_L^m}$ in multicomplex analysis.

§2.3. Multicomplexification of a real analytic manifold

In this subsection we introduce the notion of multicomplexification of a real analytic manifold. Since there appears to be no literature on multicomplex manifolds, we provide some naive explanations on them here. For the bicomplex case, we refer to [1].

Let L and m be positive integers. We say that a topological space X is an m -dimensional multicomplex manifold of level L if X has an atlas of charts to open sets in the multicomplex space $\mathbb{C}_L^m$ and all transition maps are multicomplex holomorphic of level L . In the case $L = 1$, a multicomplex manifold of level 1 is just a complex manifold.

Let X be an m -dimensional multicomplex manifold of level L . For any point Z_0 , there exist a neighborhood Ω of Z_0 and an injective multicomplex holomorphic map $\varphi: \Omega \rightarrow \mathbb{C}_L^m$ of level L . Let us fix $\ell \in \{1, 2, \dots, L-1\}$. For each $I \in \Lambda(L, \ell)$, we denote $\varphi^{-1}(\mathbb{C}_\ell^m \mathbf{e}_I)$ by $X_I \mathbf{e}_I$. By the Ringleb-type theorem, $X_I \mathbf{e}_I$ is an m -dimensional multicomplex manifold of level ℓ , and Ω is an open submanifold of $\prod_{I \in \Lambda(L, \ell)} X_I \mathbf{e}_I$. We denote $\prod_{I \in \Lambda(L, \ell)} X_I \mathbf{e}_I$ by $\bigoplus_{I \in \Lambda(L, \ell)} X_I \mathbf{e}_I$ corresponding to the idempotent representation (2.2). In this manner, a multicomplex manifold of level L is locally represented by a product of multicomplex manifolds of level ℓ . Note that this decomposition does not always hold globally.

Conversely, let $\ell \in \{1, 2, \dots, L-1\}$, and X_I an m -dimensional multicomplex manifold of level ℓ for each $I \in \Lambda(L, \ell)$. Then by the Ringleb-type theorem the product $X = \prod_{I \in \Lambda(L, \ell)} X_I$ is considered as a multicomplex manifold of level L , which can also be denoted by $\bigoplus_{I \in \Lambda(L, \ell)} X_I \mathbf{e}_I$. For each $I \in \Lambda(L, \ell)$, we define the map $\Phi_{\mathbf{e}_I}: X \rightarrow X_I$ by the natural projection. In order to emphasize components, we may identify the image of a point by $\Phi_{\mathbf{e}_I}$ with a point of the $\mathbf{e}_I$ -axis $X_I \mathbf{e}_I$ in X . Namely, we may use the notation of $\Phi_{\mathbf{e}_I}(X) = X_I \mathbf{e}_I$ and so on. We denote the sheaf of multicomplex holomorphic functions on X by $\mathcal{O}_X$. Then, by the Ringleb-

type theorem, we have the following isomorphism of sheaves on X :

$$(2.4) \quad \mathcal{O}_X \simeq \bigoplus_{I \in \Lambda(L, \ell)} \Phi_{\mathbf{e}_I}^{-1}(\mathcal{O}_{X_I \mathbf{e}_I}) \mathbf{e}_I.$$

We call (2.4) the nesting property of $\mathcal{O}_X$.

Next we define the notion of multicomplexification of a real analytic manifold. Let M be an m -dimensional real analytic manifold. We say that X is a multicomplexification of level L of M if X is an m -dimensional multicomplex manifold of level L , M is a real analytic submanifold of X (more precisely, the underlying real analytic manifold of X), and for any point $x \in M$ there exist a multicomplex neighborhood Ω of x and an injective multicomplex holomorphic map $\varphi: \Omega \rightarrow \mathbb{C}_L^m$ of level L such that $M \cap \Omega = \varphi^{-1}(\mathbb{R}^m)$.

We see that a multicomplexification of level L of M always exists as follows. Let us take a complexification X_0 of M , which is known to always exist and to be unique around M as a germ. Then

$$(2.5) \quad X = \bigoplus_{I \in \Lambda(L, 1)} X_0 \mathbf{e}_I$$

is a multicomplexification of level L of M . For the sake of simplicity, we hereafter assume that a multicomplexification is this standard one (2.5). Although this assumption is not essential, the standard multicomplexification has the following important properties. Let us fix ℓ in $\{1, 2, \dots, L-1\}$. Since we have $\Lambda(L, 1) = \Lambda(\ell, 1) \times \Lambda(L, \ell)$ and $\mathbf{e}_I = \mathbf{e}_{I_1} \mathbf{e}_{I_2}$ for $I = (I_1, I_2) \in \Lambda(\ell, 1) \times \Lambda(L, \ell)$, for each $I \in \Lambda(L, \ell)$,

$$X_I = \bigoplus_{J \in \Lambda(\ell, 1)} X_0 \mathbf{e}_J$$

is a multicomplexification of level ℓ of M . Then the standard multicomplexification X of level L of M , defined by (2.5), is naturally and globally represented by the product of multicomplex manifolds of level ℓ :

$$(2.6) \quad X = \bigoplus_{I \in \Lambda(L, \ell)} X_I \mathbf{e}_I.$$

We call (2.6) the nesting property of X . Note that in this case, the isomorphism (2.4) on the sheaf $\mathcal{O}_X$ also holds. Moreover, the real analytic manifold M is embedded into the diagonal of the product (2.6):

$$M \hookrightarrow \bigoplus_{I \in \Lambda(L, \ell)} M \mathbf{e}_I \hookrightarrow \bigoplus_{I \in \Lambda(L, \ell)} X_I \mathbf{e}_I = X.$$

§2.4. Functorial study of classical hyperfunctions and microfunctions

In this subsection we review a functorial study of classical hyperfunctions and microfunctions in the derived category of sheaves. For the theory of derived categories, we refer to [3] and adopt its terminology. We also consider [3] and [12] as fundamental references for the results presented here. For a more detailed review, see also [5].

Let M be a real analytic manifold of dimension m , and X_0 a complexification of M . A (complex) hyperfunction is a generalized function described as a finite formal sum of boundary values of complex holomorphic functions. In the derived category $\mathbf{D}^b(M)$, the sheaf of (complex) hyperfunctions $\mathcal{B}_{M,1}$ corresponds to the complex $R\Gamma_M(\mathcal{O}_{X_0})|_M[m] \otimes_{\text{or}_{M/X_0}}$. Here, $\mathcal{O}_{X_0}$ denotes the sheaf of complex holomorphic functions on X_0 , and or_{M/X_0} denotes the orientation sheaf. The following properties are well known.

Theorem 2.3. *We have the following properties:*

- (i) *The complex $R\Gamma_M(\mathcal{O}_{X_0})|_M$ is concentrated in degree m .*
- (ii) *The sheaf $\mathcal{B}_{M,1} = H^m(R\Gamma_M(\mathcal{O}_{X_0})|_M) \otimes_{\text{or}_{M/X_0}}$ is flabby on M .*

It is also well known that the complex $R\Gamma_{\Delta_{X_0}}(\mathcal{O}_{X_0 \times X_0}^{(0,m)})|_{\Delta_{X_0}}$ is concentrated in degree m and is quasi-isomorphic to the sheaf $\mathcal{D}_{X_0}^\infty$ of differential operators of infinite order on X . Here, $\mathcal{O}_{X_0 \times X_0}^{(0,m)}$ denotes the sheaf of m -forms in the second variables with coefficients that are holomorphic functions, and Δ_{X_0} denotes the diagonal set of $X_0 \times X_0$, which is identified with X_0 . Note that $\mathcal{B}_{M,1}$ is a left $\mathcal{D}_{X_0}^\infty|_M$ -module.

A (complex) microfunction is a microlocal object for a (complex) hyperfunction, and it is described as the singularities of the latter in the conormal bundle $\pi_{X_0}: T_M^*X_0 \rightarrow M$ to M in X_0 . Let $\dot{T}_M^*X_0 = T_M^*X_0 \setminus M$, and $\dot{\pi}_{X_0}$ denotes the natural projection $\dot{T}_M^*X_0 \rightarrow M$. In the derived category $\mathbf{D}^b(T_M^*X_0)$, the sheaf of (complex) microfunctions $\mathcal{C}_{M,1}$ corresponds to the complex $\mu_M(\mathcal{O}_{X_0})[m]$. Here, $\mu_M: \mathbf{D}^b(X_0) \rightarrow \mathbf{D}^b(T_M^*X_0)$ denotes the microlocalization functor along M . The following properties are well known.

Theorem 2.4. *We have the following properties:*

- (i) *The complex $\mu_M(\mathcal{O}_{X_0})$ is concentrated in degree m .*
- (ii) *Let $\mathcal{C}_{M,1} = H^m(\mu_M(\mathcal{O}_{X_0})) \otimes \pi_{X_0}^{-1}|_{\text{or}_{M/X_0}}$. The sheaf $\mathcal{C}_{M,1}|_{\dot{T}_M^*X_0}$ is conically flabby on $\dot{T}_M^*X_0$. That is, its direct image on $\dot{T}_M^*X_0/\mathbb{R}^+$ is flabby.*
- (iii) *There exists an exact sequence of sheaves on M :*

$$(2.7) \quad 0 \longrightarrow \mathcal{O}_{X_0}|_M \longrightarrow \mathcal{B}_{M,1} \longrightarrow \dot{\pi}_{X_0*}\mathcal{C}_{M,1} \longrightarrow 0.$$

(iv) *There exists an isomorphism of sheaves on M :*

$$\mathrm{sp}: \mathcal{B}_{M,1} \xrightarrow{\sim} \pi_{X_0*} \mathcal{C}_{M,1}.$$

The exact sequence (2.7) is called Sato's fundamental exact sequence, and the morphism sp is called the spectrum isomorphism.

It is also well known that the complex $\mu_{\Delta_{X_0}}(\mathcal{O}_{X_0 \times X_0}^{(0,m)})$ is concentrated in degree m . We denote its m -th cohomology by $\mathcal{E}_{X_0}^{\mathbb{R}}$ and call it the sheaf of microlocal operators on X_0 . Note that $\mathcal{C}_{M,1}$ is a left $\mathcal{E}_{X_0}^{\mathbb{R}}$ -module.

For a detailed study of Sato's fundamental exact sequence (2.7), we consider the normal bundle $\tau_{X_0}: T_M X_0 \rightarrow M$ to M in X_0 and the specialization functor $\nu_M: \mathbf{D}^b(X_0) \rightarrow \mathbf{D}^b(T_M X_0)$ along M . The following property is well known.

Theorem 2.5. *Let $\widetilde{\mathcal{A}}_{M,1} = H^0(\nu_M(\mathcal{O}_{X_0}))$. There exists an exact sequence of sheaves on $T_M X_0$:*

$$(2.8) \quad 0 \longrightarrow \widetilde{\mathcal{A}}_{M,1} \longrightarrow \tau_{X_0}^{-1} \mathcal{B}_{M,1} \longrightarrow (p_{X_0,1}^+)_*(p_{X_0,2}^+)^{-1} \mathcal{C}_{M,1},$$

where $p_{X_0,1}^+$ (resp. $p_{X_0,2}^+$) denotes the natural projection from

$$\left\{ (\eta, \xi) \in T_M X_0 \times_M T_M^* X_0 \mid \langle \eta, \xi \rangle > 0 \right\}$$

to $T_M X_0$ (resp. $T_M^* X_0$).

The distinguished triangle in $\mathbf{D}^b(T_M X_0)$ that generalizes (2.8) via the Fourier–Sato transformation is called Uchida's triangle. See [13] for details. We call a section of the specialization $\widetilde{\mathcal{A}}_{M,1}$ a holomorphic function on an infinitesimal wedge. The morphism $b: \widetilde{\mathcal{A}}_{M,1} \rightarrow \tau_{X_0}^{-1} \mathcal{B}_{M,1}$ of (2.8), which is induced by $\mathrm{id} \rightarrow \tau_{X_0}^! R\tau_{X_0!}$, is called the boundary value morphism. By the boundary value morphism b , a hyperfunction is described as a finite sum of boundary values of holomorphic functions on infinitesimal wedges.

§3. Multicomplex hyperfunctions

In this section we study multicomplex hyperfunctions by using functorial techniques. Let L and m be positive integers. Let M be an m -dimensional real analytic manifold, and X the standard multicomplexification of level L of M . Considering the classical case, it is natural to study the complex $R\Gamma_M(\mathcal{O}_X)$ of sections of the sheaf $\mathcal{O}_X$ supported in M in the derived category $\mathbf{D}^b(X)$. In order to study this, we consider the diagonal embedding

$$M \hookrightarrow \bigoplus_{I \in \Lambda(L, \ell)} M \mathbf{e}_I \hookrightarrow \bigoplus_{I \in \Lambda(L, \ell)} X_I \mathbf{e}_I = X$$

of M into X for $\ell = 1, 2, \dots, L-1$. The idempotent representation of $R\Gamma_M(\mathcal{O}_X)|_M$ is induced by that of multicomplex holomorphic functions (2.4) in the derived category $\mathbf{D}^b(M)$.

Theorem 3.1. *We have the isomorphism*

$$(3.1) \quad R\Gamma_M(\mathcal{O}_X)|_M \simeq \bigoplus_{I \in \Lambda(L, \ell)} (\Phi_{\mathbf{e}_I}|_M)^{-1} R\Gamma_{M\mathbf{e}_I}(\mathcal{O}_{X_I\mathbf{e}_I})|_{M\mathbf{e}_I} \mathbf{e}_I[-(2^L - 2^\ell)m]$$

in $\mathbf{D}^b(M)$.

Proof. By the idempotent representation (2.4), we have

$$R\Gamma_M(\mathcal{O}_X)|_M \simeq \bigoplus_{I \in \Lambda(L, \ell)} R\Gamma_M(\Phi_{\mathbf{e}_I}^{-1} \mathcal{O}_{X_I\mathbf{e}_I})|_{M\mathbf{e}_I}.$$

For $I \in \Lambda(L, \ell)$, we consider the following commutative diagram where the square is Cartesian:

$$\begin{array}{ccccc} M & \xrightarrow{d} & M\mathbf{e}_I \times \bigoplus_{\substack{J \in \Lambda(L, \ell) \\ J \neq I}} X_J\mathbf{e}_J & \xrightarrow{r} & X \\ & \searrow \Phi_{\mathbf{e}_I}|_M & \downarrow \Phi_{\mathbf{e}_I} & \square & \downarrow \Phi_{\mathbf{e}_I} \\ & & M\mathbf{e}_I & \xrightarrow{r|_{M\mathbf{e}_I}} & X_I\mathbf{e}_I. \end{array}$$

Here, d is the diagonal embedding, and r is the natural embedding. Note that $\Phi_{\mathbf{e}_I}|_M$ is an isomorphism. Since $\Phi_{\mathbf{e}_I}$ is a topological submersion with fiber dimension $(2^L - 2^\ell)m$, we have the isomorphisms

$$\begin{aligned} R\Gamma_M(\Phi_{\mathbf{e}_I}^{-1} \mathcal{O}_{X_I\mathbf{e}_I})|_M &\simeq (r \circ d)^! \Phi_{\mathbf{e}_I}^! \mathcal{O}_{X_I\mathbf{e}_I}[-(2^L - 2^\ell)m] \\ &\simeq (\Phi_{\mathbf{e}_I}|_M)^! (r|_{M\mathbf{e}_I})^! \mathcal{O}_{X_I\mathbf{e}_I}[-(2^L - 2^\ell)m] \\ &\simeq (\Phi_{\mathbf{e}_I}|_M)^{-1} R\Gamma_{M\mathbf{e}_I}(\mathcal{O}_{X_I\mathbf{e}_I})|_{M\mathbf{e}_I}[-(2^L - 2^\ell)m]. \end{aligned}$$

Therefore, we obtain (3.1). $\square$

By Theorems 2.3(i) and 3.1 with $\ell = 1$, we can prove a vanishing theorem for the complex $R\Gamma_M(\mathcal{O}_X)|_M$.

Theorem 3.2. *The complex $R\Gamma_M(\mathcal{O}_X)|_M$ is concentrated in degree $(2^L - 1)m$.*

By Theorem 3.2, we define the notion of multicomplex hyperfunctions of several variables.

Definition 3.3. We define the sheaf of multicomplex hyperfunctions of level L on M by

$$\mathcal{B}_{M,L} = H^{(2^L-1)m}(R\Gamma_M(\mathcal{O}_X)|_M) \otimes \text{or}_{M/X}.$$

We call a section of $\mathcal{B}_{M,L}$ a multicomplex hyperfunction of level L .

Note that in the affine and one variable case Vajiac–Vajiac proved Theorem 3.2 and defined the notion of multicomplex hyperfunctions of one variable in [14].

By Theorems 3.1 and 3.2, we obtain the idempotent representation theorem of nesting type for multicomplex hyperfunctions.

Theorem 3.4. *We have the following isomorphism of sheaves on M :*

$$(3.2) \quad \mathcal{B}_{M,L} \simeq \bigoplus_{I \in \Lambda(L,\ell)} (\Phi_{\mathbf{e}_I}|_M)^{-1} \mathcal{B}_{M\mathbf{e}_I,\ell} \mathbf{e}_I.$$

We call (3.2) the nesting property of $\mathcal{B}_{M,L}$. Theorem 3.4 implies that any multicomplex hyperfunction of level L is represented by a linear combination of $\mathbf{e}_I$ with coefficients that are multicomplex hyperfunctions of level ℓ . In particular, in the case of $\ell = 1$, any multicomplex hyperfunction of level L is represented by a linear combination of $\mathbf{e}_I$ with coefficients that are classical complex hyperfunctions. As a corollary of this principle, we can immediately derive several fundamental properties of multicomplex hyperfunctions by applying the properties of classical complex hyperfunctions to the coefficients of $\mathbf{e}_I$. For example, we can prove the flabbiness of the sheaf $\mathcal{B}_{M,L}$ by that of the sheaf $\mathcal{B}_{M,1}$.

Theorem 3.5. *The sheaf $\mathcal{B}_{M,L}$ is flabby on M .*

Furthermore, we can define several fundamental operations for multicomplex hyperfunctions, such as linear differential operators with multicomplex coefficients, conjugations, substitution, products and integration along fibers. See Section 6 for details. On the other hand, an explicit description of multicomplex hyperfunctions using the relative Čech cohomology group is outside the scope of this paper and reserved for future work.

§4. Multicomplex microfunctions

In this section we study the microlocal aspects of the theory developed in Section 3. Let L and m be positive integers. Also let M be an m -dimensional real analytic manifold, and X the standard multicomplexification of level L of M . In this section we consider X as a real analytic manifold and the conormal bundle $\pi: T_M^*X \rightarrow M$ to M in X . Let $\dot{T}_M^*X = T_M^*X \setminus M$, and $\dot{\pi}$ denotes the natural projection

$T_M^*X \rightarrow M$. Considering the classical case, it is natural to study the microlocalization $\mu_M(\mathcal{O}_X)$ of $\mathcal{O}_X$ along M in the derived category $\mathbf{D}^b(T_M^*X)$. In order to study it, we consider the diagonal embedding

$$M \hookrightarrow \bigoplus_{I \in \Lambda(L, \ell)} M \mathbf{e}_I \hookrightarrow \bigoplus_{I \in \Lambda(L, \ell)} X_I \mathbf{e}_I = X$$

of M into X for $\ell = 1, 2, \dots, L-1$, and the morphisms

$$T_M^*X \xleftarrow{{}^t\Phi'_{\mathbf{e}_I}} M \times_{M \mathbf{e}_I} T_{M \mathbf{e}_I}^* X_I \mathbf{e}_I \xrightarrow{\Phi_{\mathbf{e}_I \pi}} T_{M \mathbf{e}_I}^* X_I \mathbf{e}_I$$

induced by the map $\Phi_{\mathbf{e}_I}$ for $I \in \Lambda(L, \ell)$. Note that $\Phi_{\mathbf{e}_I \pi}$ is an isomorphism and ${}^t\Phi'_{\mathbf{e}_I}$ is a closed embedding. Then the idempotent representation of $\mu_M(\mathcal{O}_X)$ is induced by that of multicomplex holomorphic functions (2.4) in the derived category $\mathbf{D}^b(T_M^*X)$.

Theorem 4.1. *We have the isomorphism*

$$(4.1) \quad \mu_M(\mathcal{O}_X) \simeq \bigoplus_{I \in \Lambda(L, \ell)} {}^t\Phi'_{\mathbf{e}_I *} \Phi_{\mathbf{e}_I \pi}^{-1} \mu_{M \mathbf{e}_I}(\mathcal{O}_{X_I \mathbf{e}_I})[-(2^L - 2^\ell)m] \mathbf{e}_I$$

in $\mathbf{D}^b(T_M^*X)$.

Proof. By the idempotent representation (2.4), we have

$$\mu_M(\mathcal{O}_X) \simeq \bigoplus_{I \in \Lambda(L, \ell)} \mu_M(\Phi_{\mathbf{e}_I}^{-1} \mathcal{O}_{X_I \mathbf{e}_I}) \mathbf{e}_I.$$

Since $\Phi_{\mathbf{e}_I}$ is a topological submersion with fiber dimension $(2^L - 2^\ell)m$, $\Phi_{\mathbf{e}_I \pi}$ is an isomorphism and ${}^t\Phi'_{\mathbf{e}_I}$ is a closed embedding, we have the isomorphisms

$$\begin{aligned} \mu_M(\Phi_{\mathbf{e}_I}^{-1} \mathcal{O}_{X_I \mathbf{e}_I}) &\simeq \mu_M(\Phi_{\mathbf{e}_I}^! \mathcal{O}_{X_I \mathbf{e}_I}[-(2^L - 2^\ell)m]) \\ &\simeq {}^t\Phi'_{\mathbf{e}_I *} \Phi_{\mathbf{e}_I \pi}^! \mu_{M \mathbf{e}_I}(\mathcal{O}_{X_I \mathbf{e}_I})[-(2^L - 2^\ell)m] \\ &\simeq {}^t\Phi'_{\mathbf{e}_I *} \Phi_{\mathbf{e}_I \pi}^{-1} \mu_{M \mathbf{e}_I}(\mathcal{O}_{X_I \mathbf{e}_I})[-(2^L - 2^\ell)m] \end{aligned}$$

by [3, Proposition 4.3.5]. Therefore, we obtain (4.1). $\square$

By Theorems 2.4(i) and 4.1 with $\ell = 1$, we can prove a vanishing theorem for the complex $\mu_M(\mathcal{O}_X)$.

Theorem 4.2. *The complex $\mu_M(\mathcal{O}_X)$ is concentrated in degree $(2^L - 1)m$.*

By Theorem 4.2, we define the notion of multicomplex microfunctions.

Definition 4.3. We define the sheaf of multicomplex microfunctions of level L on T_M^*X by

$$\mathcal{C}_{M,L} = H^{(2^L-1)m}(\mu_M(\mathcal{O}_X)) \otimes \pi^{-1}\mathrm{or}_{M/X}.$$

We call a section of $\mathcal{C}_{M,L}$ a multicomplex microfunction of level L .

By Theorems 4.1 and 4.2, we obtain the idempotent representation theorem for multicomplex microfunctions.

Theorem 4.4. *We have the following isomorphism of sheaves on T_M^*X :*

$$(4.2) \quad \mathcal{C}_{M,L} \simeq \bigoplus_{I \in \Lambda(L,\ell)} {}^t\Phi'_{\mathbf{e}_I*} \Phi_{\mathbf{e}_I\pi}^{-1} \mathcal{C}_{M\mathbf{e}_I,\ell} \mathbf{e}_I.$$

We call (4.2) the nesting property of $\mathcal{C}_{M,L}$. Note that we also obtain (4.2) by applying the functor $R\pi_*$ to (3.2). Theorem 4.4 implies that any multicomplex microfunction of level L is represented by a linear combination of $\mathbf{e}_I$ with coefficients that are multicomplex microfunctions of level ℓ . In particular, in the case of $\ell = 1$, any multicomplex microfunction of level L is represented by a linear combination of $\mathbf{e}_I$ with coefficients that are classical complex microfunctions. As a corollary of this principle, we can immediately derive several fundamental properties of multicomplex microfunctions by applying properties of complex microfunctions to the coefficients of $\mathbf{e}_I$. For example, we can prove the flabbiness of the sheaf $\mathcal{C}_{M,L}$ by that of the sheaf $\mathcal{C}_{M,1}$. In order to state it explicitly, we note that the natural action of $\mathbb{R}^+$ on $T_M^*X_0$ induces an action of $(\mathbb{R}^+)^{2^{L-1}}$ on T_M^*X via ${}^t\Phi'_{\mathbf{e}_I}$ with $\ell = 1$.

Theorem 4.5. *The sheaf $\mathcal{C}_{M,L}|_{\dot{T}_M^*X}$ is multiconically flabby on $\dot{T}_M^*X$.*

Moreover, by Theorems 2.4 and 4.4, or functorial properties of the microlocalization functor μ_M in [3], we also obtain the Sato-type fundamental exact sequence and the spectrum isomorphism for multicomplex hyperfunctions and microfunctions.

Theorem 4.6. *We have the following properties:*

- (i) *There exists an exact sequence of sheaves on M :*

$$0 \longrightarrow \mathcal{O}_X|_M \longrightarrow \mathcal{B}_{M,L} \longrightarrow \dot{\pi}_* \mathcal{C}_{M,L} \longrightarrow 0.$$

- (ii) *There exists the spectrum isomorphism on M :*

$$(4.3) \quad \mathrm{sp}: \mathcal{B}_{M,L} \xrightarrow{\sim} \pi_* \mathcal{C}_{M,L}.$$

As a corollary of the spectrum isomorphism (4.3), we obtain a simpler description of the sheaf of multicomplex microfunctions. See [5] for details. Furthermore, we can define several fundamental operations of multicomplex microfunctions, such as microlocal operators, conjugations, substitution, products, integration along fibers. See Section 6 for details.

Finally, we define the notion of the singularity spectrum of a multicomplex hyperfunction.

Definition 4.7. Let u be a multicomplex hyperfunction of level L . We denote the support of its spectrum $\mathrm{sp}(u)$ in T_M^*X by $\mathrm{SS}(u)$ and call it the singularity spectrum of u . We say that u is micro-analytic at $(x; \Xi) \in T_M^*X$ if $(x; \Xi) \notin \mathrm{SS}(u)$.

By Theorem 4.4, an estimate of the singularity spectrum of a multicomplex hyperfunction is obtained. Note that the set $\bigcup_{I \in \Lambda(L, \ell)} {}^t\Phi'_{\mathbf{e}_I}(\Phi_{\mathbf{e}_I\pi}^{-1}(T_{M\mathbf{e}_I}^*X_I\mathbf{e}_I))$ is smallest in T_M^*X when $\ell = 1$.

Theorem 4.8. *Let u be a multicomplex hyperfunction of level L on M . The singularity spectrum $\mathrm{SS}(u)$ of u is contained in*

$$(4.4) \quad \bigcup_{I \in \Lambda(L, 1)} {}^t\Phi'_{\mathbf{e}_I}(\Phi_{\mathbf{e}_I\pi}^{-1}(T_{M\mathbf{e}_I}^*X_I\mathbf{e}_I))$$

in T_M^*X . In particular, every multicomplex hyperfunction is micro-analytic at each point of the outside of (4.4).

§5. Boundary value morphism and multicomplex hyperfunctions

In this section we explicitly describe the boundary value morphism for multicomplex holomorphic functions. Let L and m be positive integers. Also let M be an m -dimensional real analytic manifold, and X the standard multicomplexification of level L of M . In this section we also consider X as a real analytic manifold and the normal bundle $\tau: T_M X \rightarrow M$ to M in X . Considering the classical case, it is natural to study the specialization $\nu_M(\mathcal{O}_X)$ of $\mathcal{O}_X$ along M in the derived category $\mathbf{D}^b(T_M X)$. In order to study it, we consider the diagonal embedding

$$M \hookrightarrow \bigoplus_{I \in \Lambda(L, \ell)} M\mathbf{e}_I \hookrightarrow \bigoplus_{I \in \Lambda(L, \ell)} X_I\mathbf{e}_I = X$$

of M into X for $\ell = 1, 2, \dots, L-1$, and the morphism

$$T_M\Phi_{\mathbf{e}_I}: T_M X \longrightarrow T_{M\mathbf{e}_I}X_I\mathbf{e}_I$$

induced by the map $\Phi_{\mathbf{e}_I}$ for $I \in \Lambda(L, \ell)$. Note that $T_M \Phi_{\mathbf{e}_I}$ is smooth. Then the idempotent representation of $\nu_M(\mathcal{O}_X)$ is induced by that of multicomplex holomorphic functions (2.4) in the derived category $\mathbf{D}^b(T_M X)$.

Theorem 5.1. *We have the isomorphism*

$$(5.1) \quad \nu_M(\mathcal{O}_X) \simeq \bigoplus_{I \in \Lambda(L, \ell)} (T_M \Phi_{\mathbf{e}_I})^{-1} \nu_{M\mathbf{e}_I}(\mathcal{O}_{X_I \mathbf{e}_I}) \mathbf{e}_I$$

in $\mathbf{D}^b(T_M X)$.

Proof. By the idempotent representation (2.4), we have

$$\nu_M(\mathcal{O}_X) \simeq \bigoplus_{I \in \Lambda(L, \ell)} \nu_M(\Phi_{\mathbf{e}_I}^{-1} \mathcal{O}_{X_I \mathbf{e}_I}) \mathbf{e}_I.$$

Since $\Phi_{\mathbf{e}_I}$ is a topological submersion, by [3, Proposition 4.2.5], we have the isomorphism

$$\nu_M(\Phi_{\mathbf{e}_I}^{-1} \mathcal{O}_{X_I \mathbf{e}_I}) \simeq (T_M \Phi_{\mathbf{e}_I})^{-1} \nu_M(\mathcal{O}_{X_I \mathbf{e}_I}).$$

Therefore we obtain (5.1). $\square$

We define the notion of multicomplex holomorphic functions of level L on infinitesimal wedges.

Definition 5.2. We define the sheaf of the specialization of multicomplex holomorphic functions of level L by

$$\widetilde{\mathcal{A}}_{M, L} = H^0(\nu_M(\mathcal{O}_X)).$$

We call a section of $\widetilde{\mathcal{A}}_{M, L}$ a multicomplex holomorphic function of level L on an infinitesimal wedge.

By Theorem 5.1, we obtain the idempotent representation theorem for the specialization of multicomplex holomorphic functions.

Theorem 5.3. *We have the following isomorphism of sheaves on $T_M X$:*

$$(5.2) \quad \widetilde{\mathcal{A}}_{M, L} \simeq \bigoplus_{I \in \Lambda(L, \ell)} (T_M \Phi_{\mathbf{e}_I})^{-1} \widetilde{\mathcal{A}}_{M\mathbf{e}_I, \ell} \mathbf{e}_I.$$

We call (5.2) the nesting property of $\widetilde{\mathcal{A}}_{M, L}$. Theorem 5.3 implies that any multicomplex holomorphic function of level L on an infinitesimal wedge is represented by a linear combination of $\mathbf{e}_I$ with coefficients that are multicomplex holomorphic functions of level ℓ on infinitesimal wedges. In particular, in the case of $\ell = 1$, any multicomplex holomorphic function of level L on an infinitesimal

wedge is represented by a linear combination of $\mathbf{e}_I$ with coefficients that are classical complex holomorphic functions on infinitesimal wedges. As a corollary of this principle, we can immediately derive several fundamental properties of multicomplex holomorphic functions on infinitesimal wedges by applying properties of complex holomorphic functions on infinitesimal wedges to the coefficients of $\mathbf{e}_I$.

By Theorem 2.5, or the abstract Uchida triangle in [13], we obtain the Uchida-type fundamental exact sequence for the multicomplex specialization and the sheaves of multicomplex hyperfunctions and microfunctions. Let p_1^+ (resp. p_2^+) be the natural projection from

$$P^+ = \{(H, \Xi) \in T_M X \times_M T_M^* X \mid \langle H, \Xi \rangle > 0\}$$

to $T_M X$ (resp. $T_M^* X$).

Theorem 5.4. *There exists an exact sequence of sheaves on $T_M X$:*

$$(5.3) \quad 0 \longrightarrow \widetilde{\mathcal{A}_{M,L}} \longrightarrow \tau^{-1} \mathcal{B}_{M,L} \longrightarrow (p_1^+)_* (p_2^+)^{-1} \mathcal{C}_{M,L}.$$

See [5] for the outline of the proof. Note that the morphism $b: \widetilde{\mathcal{A}_{M,L}} \rightarrow \tau^{-1} \mathcal{B}_{M,L}$ of (5.3) is induced by $\text{id} \rightarrow \tau^! R\pi_!$. We call b the boundary value morphism. By the boundary value morphism b , a multicomplex hyperfunction is described as a finite sum of boundary values of multicomplex holomorphic functions on infinitesimal wedges.

§6. Several operations of multicomplex hyperfunctions and multicomplex microfunctions

In this section we establish fundamental operations for multicomplex hyperfunctions and microfunctions. These operations are a natural generalization of the corresponding ones in the theory of complex hyperfunctions and microfunctions.

§6.1. Linear differential operators

In this subsection we study the action of linear differential operators on multicomplex hyperfunctions. Let L and m be positive integers. Also let M be an m -dimensional real analytic manifold, X_0 a complexification of M , and X the standard multicomplexification of level L of M . Considering the classical case, it is natural to study the complex $R\Gamma_{\Delta_X}(\mathcal{O}_{X \times X}^{(0,m)})|_{\Delta_X}$ of sections of the sheaf $\mathcal{O}_{X \times X}^{(0,m)}$ supported in the diagonal set Δ_X of $X \times X$ in the derived category $\mathbf{D}^b(X)$. Here, $\mathcal{O}_{X \times X}^{(0,m)}$ denotes the sheaf of m -forms in the second variables with coefficients that are multicomplex holomorphic functions on $X \times X$, and we identify the diagonal

set Δ_X with X . Note that the nesting property (2.6) of X induces that of $X \times X$. In the same way as in Theorem 3.1, we obtain the isomorphism

$$(6.1) \quad R\Gamma_{\Delta_X}(\mathcal{O}_{X \times X}^{(0,m)})|_{\Delta_X} \simeq \bigoplus_{I \in \Lambda(L,\ell)} \Phi_{\mathbf{e}_I}^{-1} R\Gamma_{\Delta_{X_I}}(\mathcal{O}_{X_I \times X_I}^{(0,m)})|_{\Delta_{X_I}} \mathbf{e}_I[-(2^L - 2^\ell)m]$$

for $\ell = 1, 2, \dots, L-1$. By the classical result and (6.1) with $\ell = 1$, we prove that the complex $R\Gamma_{\Delta_X}(\mathcal{O}_{X \times X}^{(0,m)})|_{\Delta_X}$ is concentrated in degree $(2^L - 1)m$. Thus we define the sheaf $\mathcal{D}_X^\infty$ of differential operators of infinite order on X by

$$\mathcal{D}_X^\infty = H^{(2^L - 1)m}(R\Gamma_{\Delta_X}(\mathcal{O}_{X \times X}^{(0,m)})|_{\Delta_X}).$$

By (6.1) and the vanishing result, we obtain the idempotent representation

$$(6.2) \quad \mathcal{D}_X^\infty \simeq \bigoplus_{I \in \Lambda(L,\ell)} \Phi_{\mathbf{e}_I}^{-1} \mathcal{D}_{X_I \mathbf{e}_I}^\infty \mathbf{e}_I$$

of $\mathcal{D}_X^\infty$. We call (6.2) the nesting property of $\mathcal{D}_X^\infty$.

Since $\mathcal{B}_{M,1}$ is a left $\mathcal{D}_{X_0}^\infty|_M$ -module, this action induces a natural action of $\mathcal{D}_X^\infty|_M$ on $\mathcal{B}_{M,L}$ in the general case via the idempotent representation as follows:

$$\begin{aligned} \mathcal{D}_X^\infty|_M \otimes \mathcal{B}_{M,L} &\simeq \bigoplus_{I \in \Lambda(L,1)} (\Phi_{\mathbf{e}_I}^{-1} \mathcal{D}_{X_0 \mathbf{e}_I}^\infty)|_M \otimes (\Phi_{\mathbf{e}_I}|_M)^{-1} \mathcal{B}_{M \mathbf{e}_I,1} \mathbf{e}_I \\ &\simeq \bigoplus_{I \in \Lambda(L,1)} (\Phi_{\mathbf{e}_I}|_M)^{-1} (\mathcal{D}_{X_0 \mathbf{e}_I}^\infty|_{M \mathbf{e}_I} \otimes \mathcal{B}_{M \mathbf{e}_I,1}) \mathbf{e}_I \\ &\rightarrow \bigoplus_{I \in \Lambda(L,1)} (\Phi_{\mathbf{e}_I}|_M)^{-1} \mathcal{B}_{M \mathbf{e}_I,1} \mathbf{e}_I \simeq \mathcal{B}_{M,L}. \end{aligned}$$

Note that this action does not enlarge the singularity spectrum of a hyperfunction. By definition, the action of $\mathcal{D}_X^\infty|_M$ on $\mathcal{B}_{M,L}$ is compatible with the idempotent representation for $\ell = 1, 2, \dots, L-1$:

$$\begin{array}{ccc} \mathcal{D}_X^\infty|_M \otimes \mathcal{B}_{M,L} & \xrightarrow{\quad \quad \quad} & \mathcal{B}_{M,L} \\ \wr & & \wr \\ \bigoplus_{I \in \Lambda(L,\ell)} (\Phi_{\mathbf{e}_I}|_M)^{-1} (\mathcal{D}_{X_I \mathbf{e}_I}^\infty|_{M \mathbf{e}_I} \otimes \mathcal{B}_{M \mathbf{e}_I,\ell}) \mathbf{e}_I & \longrightarrow & \bigoplus_{I \in \Lambda(L,\ell)} (\Phi_{\mathbf{e}_I}|_M)^{-1} \mathcal{B}_{M \mathbf{e}_I,\ell} \mathbf{e}_I. \end{array}$$

Let $\pi: T_M^*X \rightarrow M$ (resp. $\pi_0: T_{X_0}^*X_0 \rightarrow M$) be the conormal bundle to M in X (resp. X_0). Since $\mathcal{C}_{M,1}$ is a left $\pi_0^{-1}(\mathcal{D}_{X_0}^\infty|_M)$ -module, this action induces a natural

action of $\pi^{-1}(\mathcal{D}_X^\infty|_M)$ on $\mathcal{C}_{M,L}$ in the general case via the idempotent representation as follows:

$$\begin{aligned} \pi^{-1}(\mathcal{D}_X^\infty|_M) \otimes \mathcal{C}_{M,L} &\simeq \bigoplus_{I \in \Lambda(L,1)} \pi^{-1}((\Phi_{\mathbf{e}_I}^{-1} \mathcal{D}_{X_0 \mathbf{e}_I}^\infty)|_M) \otimes {}^t\Phi'_{\mathbf{e}_I*} \Phi_{\mathbf{e}_I\pi}^{-1} \mathcal{C}_{M\mathbf{e}_I,1} \mathbf{e}_I \\ &\simeq \bigoplus_{I \in \Lambda(L,1)} {}^t\Phi'_{\mathbf{e}_I*} \Phi_{\mathbf{e}_I\pi}^{-1} (\pi_0^{-1}(\mathcal{D}_{X_0 \mathbf{e}_I}^\infty|_{M\mathbf{e}_I}) \otimes \mathcal{C}_{M\mathbf{e}_I,1}) \mathbf{e}_I \\ &\rightarrow \bigoplus_{I \in \Lambda(L,1)} {}^t\Phi'_{\mathbf{e}_I*} \Phi_{\mathbf{e}_I\pi}^{-1} \mathcal{C}_{M\mathbf{e}_I,1} \mathbf{e}_I \simeq \mathcal{C}_{M,L}. \end{aligned}$$

By definition, the action of $\pi^{-1}(\mathcal{D}_X^\infty|_M)$ on $\mathcal{C}_{M,L}$ is compatible with the idempotent representation for $\ell = 1, 2, \dots, L-1$:

$$\begin{array}{ccc} \pi^{-1}(\mathcal{D}_X^\infty|_M) \otimes \mathcal{C}_{M,L} & \xrightarrow{\quad\quad\quad} & \mathcal{C}_{M,L} \\ \wr & & \wr \\ \bigoplus_{I \in \Lambda(L,\ell)} {}^t\Phi'_{\mathbf{e}_I*} \Phi_{\mathbf{e}_I\pi}^{-1} (\pi_{X_I}^{-1}(\mathcal{D}_{X_I \mathbf{e}_I}^\infty|_{M\mathbf{e}_I}) \otimes \mathcal{C}_{M\mathbf{e}_I,\ell}) \mathbf{e}_I & \longrightarrow & \bigoplus_{I \in \Lambda(L,\ell)} {}^t\Phi'_{\mathbf{e}_I*} \Phi_{\mathbf{e}_I\pi}^{-1} \mathcal{C}_{M\mathbf{e}_I,\ell} \mathbf{e}_I. \end{array}$$

Here, $\pi_{X_I}: T_{M\mathbf{e}_I}^* X_I \mathbf{e}_I \rightarrow M\mathbf{e}_I$ denotes the conormal bundle to $M\mathbf{e}_I$ in $X_I \mathbf{e}_I$.

§6.2. Microlocal operators

In this subsection we study the action of microlocal operators on multicomplex microfunctions. We inherit the setting of Section 6.1. Considering the classical case, it is natural to study the microlocalization $\mu_{\Delta_X}(\mathcal{O}_{X \times X}^{(0,m)})|_{\Delta_X}$ of the sheaf $\mathcal{O}_{X \times X}^{(0,m)}$ along the diagonal set Δ_X of $X \times X$ in the derived category $\mathbf{D}^b(T^*X)$. In the same way as in Theorem 4.1, we obtain the isomorphism

$$(6.3) \quad \mu_{\Delta_X}(\mathcal{O}_{X \times X}^{(0,m)}) \simeq \bigoplus_{I \in \Lambda(L,\ell)} {}^t\Phi'_{\mathbf{e}_I*} \Phi_{\mathbf{e}_I}^{-1} \mu_{\Delta_{X_I}}(\mathcal{O}_{X_I \times X_I}^{(0,m)}) \mathbf{e}_I [-(2^L - 2^\ell)m]$$

for $\ell = 1, 2, \dots, L-1$. By the classical result and (6.3) with $\ell = 1$, we prove that the complex $\mu_{\Delta_X}(\mathcal{O}_{X \times X}^{(0,m)})|_{\Delta_X}$ is concentrated in degree $(2^L - 1)m$. Thus we define the sheaf $\mathcal{E}_X^{\mathbb{R}}$ of microlocal operators on X by

$$\mathcal{E}_X^{\mathbb{R}} = H^{(2^L - 1)m}(\mu_{\Delta_X}(\mathcal{O}_{X \times X}^{(0,m)})).$$

By (6.3) and the vanishing result, we obtain the idempotent representation

$$(6.4) \quad \mathcal{E}_X^{\mathbb{R}} \simeq \bigoplus_{I \in \Lambda(L,\ell)} {}^t\Phi'_{\mathbf{e}_I*} \Phi_{\mathbf{e}_I\pi}^{-1} \mathcal{E}_{X_I \mathbf{e}_I}^{\mathbb{R}} \mathbf{e}_I$$

of $\mathcal{E}_X^{\mathbb{R}}$. We call (6.4) the nesting property of $\mathcal{E}_X^{\mathbb{R}}$.

Since $\mathcal{C}_{M,1}$ is a left $\mathcal{E}_{X_0}^{\mathbb{R}}$ -module, this action induces a natural action of $\mathcal{E}_X^{\mathbb{R}}$ on $\mathcal{C}_{M,L}$ in the general case via the idempotent representation as follows:

$$\begin{aligned} \mathcal{E}_X^{\mathbb{R}} \otimes \mathcal{C}_{M,L} &\simeq \bigoplus_{I \in \Lambda(L,1)} {}^t\Phi'_{\mathbf{e}_I*} \Phi_{\mathbf{e}_I\pi}^{-1} \mathcal{E}_{X_0\mathbf{e}_I}^{\mathbb{R}} \otimes {}^t\Phi'_{\mathbf{e}_I*} \Phi_{\mathbf{e}_I\pi}^{-1} \mathcal{C}_{M\mathbf{e}_I,1} \mathbf{e}_I \\ &\simeq \bigoplus_{I \in \Lambda(L,1)} {}^t\Phi'_{\mathbf{e}_I*} \Phi_{\mathbf{e}_I\pi}^{-1} (\mathcal{E}_{X_0\mathbf{e}_I}^{\mathbb{R}} \otimes \mathcal{C}_{M\mathbf{e}_I,1}) \mathbf{e}_I \\ &\rightarrow \bigoplus_{I \in \Lambda(L,1)} {}^t\Phi'_{\mathbf{e}_I*} \Phi_{\mathbf{e}_I\pi}^{-1} \mathcal{C}_{M\mathbf{e}_I,1} \mathbf{e}_I \simeq \mathcal{C}_{M,L}. \end{aligned}$$

By definition, the action of $\mathcal{E}_X^{\mathbb{R}}$ on $\mathcal{C}_{M,L}$ is compatible with the idempotent representation for $\ell = 1, 2, \dots, L-1$:

$$\begin{array}{ccc} \mathcal{E}_X^{\mathbb{R}} \otimes \mathcal{C}_{M,L} & \xrightarrow{\quad\quad\quad} & \mathcal{C}_{M,L} \\ \wr & & \wr \\ \bigoplus_{I \in \Lambda(L,\ell)} {}^t\Phi'_{\mathbf{e}_I*} \Phi_{\mathbf{e}_I\pi}^{-1} (\mathcal{E}_{X_I\mathbf{e}_I}^{\mathbb{R}} \otimes \mathcal{C}_{M\mathbf{e}_I,\ell}) \mathbf{e}_I & \longrightarrow & \bigoplus_{I \in \Lambda(L,\ell)} {}^t\Phi'_{\mathbf{e}_I*} \Phi_{\mathbf{e}_I\pi}^{-1} \mathcal{C}_{M\mathbf{e}_I,\ell} \mathbf{e}_I. \end{array}$$

A more detailed study of microlocal operators and microdifferential equations will be presented in future work.

§6.3. Conjugations

In this subsection we study the conjugations of multicomplex hyperfunctions and microfunctions. The multicomplex algebra $\mathbb{C}_L$ admits several conjugations, but here we focus only on the conjugations $\dagger_j$ with respect to the imaginary units i_j for $j = 1, \dots, L$. We inherit the setting of Section 6.1. For each $j = 1, 2, \dots, L$, by shrinking X if necessary, we may assume that the conjugation $\dagger_j$ induces a multicomplex anti-holomorphic map $c_j: X \rightarrow X$ satisfying $c_j|_M = \text{id}_M$ and $c_j \circ c_j = \text{id}_X$, and the homomorphism

$$(6.5) \quad \dagger_j: \mathcal{O}_X \longrightarrow c_j^{-1} \mathcal{O}_X, \quad F(Z) \longmapsto F^{\dagger_j}(Z) = (F(Z^{\dagger_j}))^{\dagger_j}.$$

The map c_j is naturally lifted to the map ${}^tc'_j: T_M^*X \rightarrow T_M^*X$. By [3, Proposition 4.3.5] and the morphism $\dagger_j$ in (6.5), there exists a natural morphism

$$\mathcal{C}_{M,L} \longrightarrow {}^tc'_{j*} \mathcal{C}_{M,L}.$$

This morphism defines the notion of the conjugation $\dagger_j$ with respect to i_j for a multicomplex microfunction. Via the spectrum isomorphism, we obtain a morphism

$$\mathcal{B}_{M,L} \longrightarrow \mathcal{B}_{M,L},$$

and we can also define the conjugation $\dagger_j$ with respect to i_j for a multicomplex hyperfunction. For a multicomplex hyperfunction u of level L on M , we have the estimate

$$\mathrm{SS}(u^{\dagger_j}) = {}^t c'_j(\mathrm{SS}(u)).$$

These conjugations allow us to define the notion of special-valued hyperfunctions, which generalize classical real-valued hyperfunctions. They are expected to be useful for studying the microlocal properties of differential equations in future work.

Definition 6.1. Let u be a multicomplex hyperfunction of level L on M . For $\ell = 0, 1, \dots, L-1$, we say that u is $\mathbb{C}_\ell$ -valued if $u = u^{\dagger_{\ell+1}} = \dots = u^{\dagger_L}$. We also say that u is multihyperbolic-valued if $u = (\dots((u^{\dagger_1})^{\dagger_2})\dots)^{\dagger_L}$.

In particular, a $\mathbb{C}_0$ -valued (resp. $\mathbb{C}_1$ -valued) hyperfunction is also called a real-valued (resp. a complex-valued) hyperfunction.

Let us study the compatibility of the conjugations with the idempotent representation. Since we have

$$(F(Z^{\dagger_j}))^{\dagger_j} = \sum_{I \in \Lambda(L, \ell)} (F_{\mathbf{e}_I}(Z_{\mathbf{e}_I}^{\dagger_j}))^{\dagger_j} \mathbf{e}_I^{\dagger_j},$$

the morphism $\dagger_j$ in (6.5) is induced by the composition of the conjugation of the components $F_{\mathbf{e}_I}$ and a permutation of the idempotent basis $\{\mathbf{e}_I\}_{I \in \Lambda(L, \ell)}$. Note that, in particular, in the case $\ell = 1$, the conjugations $\dagger_2, \dots, \dagger_L$ are nothing but a permutation of the idempotent basis $\{\mathbf{e}_I\}_{I \in \Lambda(L, 1)}$. Then for each $I \in \Lambda(L, \ell)$, we have the commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{c_j} & X \\ \Phi_{\mathbf{e}_I} \downarrow & & \downarrow \Phi_{\mathbf{e}_I^{\dagger_j}} \\ X_I \mathbf{e}_I & \xrightarrow{c_j} & X_I \mathbf{e}_I^{\dagger_j}. \end{array}$$

Therefore, we have the following commutative diagram:

$$\begin{array}{ccc} \mathcal{C}_{M,L} & \xrightarrow{\quad\quad\quad} & {}^t c'_{j*} \mathcal{C}_{M,L} \\ \wr & & \wr \\ \bigoplus_{I \in \Lambda(L, \ell)} {}^t \Phi'_{\mathbf{e}_I*} \Phi_{\mathbf{e}_I\pi}^{-1} \mathcal{C}_{M\mathbf{e}_I, \ell} \mathbf{e}_I & \longrightarrow & \bigoplus_{I \in \Lambda(L, \ell)} {}^t \Phi'_{\mathbf{e}_I^{\dagger_j}*} \Phi_{\mathbf{e}_I^{\dagger_j}\pi}^{-1} {}^t c'_{j*} \mathcal{C}_{M\mathbf{e}_I^{\dagger_j}, \ell} \mathbf{e}_I^{\dagger_j}. \end{array}$$

By the spectrum isomorphism, we obtain the following commutative diagram:

$$\begin{array}{ccc}
 \mathcal{B}_{M,L} & \xrightarrow{\quad\quad\quad} & \mathcal{B}_{M,L} \\
 \wr & & \wr \\
 \bigoplus_{I \in \Lambda(L,\ell)} (\Phi_{\mathbf{e}_I}|_M)^{-1} \mathcal{B}_{M\mathbf{e}_I,\ell} \mathbf{e}_I & \longrightarrow & \bigoplus_{I \in \Lambda(L,\ell)} (\Phi_{\mathbf{e}_I^{\dagger j}}|_M)^{-1} \mathcal{B}_{M\mathbf{e}_I^{\dagger j},\ell} \mathbf{e}_I^{\dagger j}.
 \end{array}$$

§6.4. Inverse image

In this subsection we study the inverse image of multicomplex hyperfunctions and microfunctions, which is also called a substitution. Furthermore, in the case where f is an embedding, the inverse image is also called the restriction. Let L, m and n be positive integers. Let M (resp. N) be an m -dimensional (resp. n -dimensional) real analytic manifold, $f: N \rightarrow M$ a real analytic map, and X (resp. Y) the standard multicomplexification of level L of M (resp. N), chosen so that the multicomplex holomorphic extension of f is well defined. We denote this extension by the same letter $f: Y \rightarrow X$. Let $\pi_X: T_M^*X \rightarrow M$ (resp. $\pi_Y: T_N^*Y \rightarrow N$) be the conormal bundle to M in X (resp. N in Y). Let us consider the morphisms

$$T_N^*Y \xleftarrow{{}^t f'} N \times_{T_M^*X} \xrightarrow{f\pi} T_M^*X$$

induced by f . By [3, Proposition 4.3.5] and the natural morphism $f^{-1}\mathcal{O}_X \rightarrow \mathcal{O}_Y$, there exists a natural morphism

$$R^t f'_! f_\pi^{-1} \mathcal{C}_{M,L} \longrightarrow \mathcal{C}_{N,L}.$$

This morphism defines the notion of the inverse image of a multicomplex microfunction by f . More precisely, for a multicomplex microfunction u on an open set U of T_M^*X , if V is an open set of T_N^*Y such that ${}^t f'$ is proper on $f_\pi^{-1}(\text{supp}(u)) \cap ({}^t f')^{-1}(V)$, we can define the microfunction f^*u .

Via the spectrum morphism, we obtain the natural morphisms

$$R\pi_{Y*} R^t f'_! f_\pi^{-1} \pi_X^{-1} \mathcal{B}_{M,L} \longrightarrow R\pi_{Y*} R^t f'_! f_\pi^{-1} \mathcal{C}_{M,L} \longrightarrow R\pi_{Y*} \mathcal{C}_{N,L} \simeq \mathcal{B}_{N,L}.$$

The composition of these morphisms defines the notion of the inverse image of a multicomplex hyperfunction by f . In order to explain the condition, let us set $K = \text{Ker}({}^t f': N \times_{T_M^*X} \rightarrow T_N^*Y)$. We denote by q_1 (resp. q_2) the natural projection from K to N (resp. the inclusion of K into $N \times_{T_M^*X}$). Then we have the natural isomorphism

$$R\pi_{Y*} R^t f'_! f_\pi^{-1} \pi_X^{-1} \mathcal{B}_{M,L} \simeq Rq_{1!} q_2^{-1} f_\pi^{-1} \pi_X^{-1} \mathcal{B}_{M,L}.$$

Thus, for a multicomplex hyperfunction u on M satisfying that q_1 is proper on $q_2^{-1}f_\pi^{-1}(\text{SS}(u))$, namely $\text{SS}(u) \cap K$ is contained in the zero section of $N \times_M T_M^*X$, which is isomorphic to N , we can define the hyperfunction f^*u on N . In particular, in the case where f is smooth, we can always define the inverse image. In general, we have the estimate

$$\text{SS}(f^*u) \subset {}^t f'(f_\pi^{-1}(\text{SS}(u))).$$

Let us study the compatibility of the inverse image and the idempotent representation. For $\ell = 1, 2, \dots, L-1$, we consider the nesting properties $X = \bigoplus_{I \in \Lambda(L, \ell)} X_I \mathbf{e}_I$ and $Y = \bigoplus_{I \in \Lambda(L, \ell)} Y_I \mathbf{e}_I$ of X and Y , respectively. For each $I \in \Lambda(L, \ell)$, we denote the restriction of f to $Y_I \mathbf{e}_I$ by $f_I: Y_I \mathbf{e}_I \rightarrow X_I \mathbf{e}_I$. Then we have the commutative diagram

$$\begin{array}{ccc} Y & \xrightarrow{f} & X \\ \Phi_{\mathbf{e}_I} \downarrow & & \downarrow \Phi_{\mathbf{e}_I} \\ Y_I \mathbf{e}_I & \xrightarrow{f_I} & X_I \mathbf{e}_I. \end{array}$$

By the Ringleb-type theorem, considering the morphism $f^{-1}\mathcal{O}_X \rightarrow \mathcal{O}_Y$ is equivalent to considering the morphisms $f_I^{-1}\mathcal{O}_{X_I \mathbf{e}_I} \rightarrow \mathcal{O}_{Y_I \mathbf{e}_I}$ for any $I \in \Lambda(L, \ell)$. Associated with f , f_I and $\Phi_{\mathbf{e}_I}$, we have the commutative diagrams

$$\begin{array}{ccccc} T_N Y & \xrightarrow{T_N f} & T_M X & & \\ T_N \Phi_{\mathbf{e}_I} \downarrow & & \downarrow T_M \Phi_{\mathbf{e}_I} & & \\ T_{N \mathbf{e}_I} Y_I \mathbf{e}_I & \xrightarrow{T_N f_I} & T_{M \mathbf{e}_I} X_I \mathbf{e}_I, & & \\ \\ T_N^* Y & \xleftarrow{{}^t f'} & N \times_M T_M^* X_I & \xrightarrow{f_\pi} & T_M^* X_I \\ {}^t \Phi'_{\mathbf{e}_I} \uparrow & & \uparrow & \square & \uparrow {}^t \Phi'_{\mathbf{e}_I} \\ N \times_{N \mathbf{e}_I} T_{N \mathbf{e}_I}^* Y_I \mathbf{e}_I & \xleftarrow{{}^t f'_I} & N \times_{M \mathbf{e}_I} T_{M \mathbf{e}_I}^* X_I \mathbf{e}_I & \xrightarrow{f_I \pi} & M \times_{M \mathbf{e}_I} T_{M \mathbf{e}_I}^* X_I \mathbf{e}_I. \end{array}$$

Here, in the last diagram the right square is Cartesian. We also denote the natural projections $T_{M \mathbf{e}_I}^* X_I \mathbf{e}_I \rightarrow M \mathbf{e}_I$ and $T_{N \mathbf{e}_I}^* Y_I \mathbf{e}_I \rightarrow N \mathbf{e}_I$ by π_{X_I} and π_{Y_I} , respectively.

By the functorial property of the specialization in [3, Proposition 4.2.5], we have the commutative diagram

$$\begin{array}{ccc}
 (T_N f)^{-1} \nu_M(\Phi_{\mathbf{e}_I}^{-1} \mathcal{O}_{X_I \mathbf{e}_I}) & \longrightarrow & \nu_N(f^{-1} \Phi_{\mathbf{e}_I}^{-1} \mathcal{O}_{X_I \mathbf{e}_I}) \\
 \uparrow \wr & \nearrow & \parallel \\
 (T_N f)^{-1} (T_M \Phi_{\mathbf{e}_I})^{-1} \nu_{M \mathbf{e}_I}(\mathcal{O}_{X_I \mathbf{e}_I}) & & \nu_N(\Phi_{\mathbf{e}_I}^{-1} f_I^{-1} \mathcal{O}_{X_I \mathbf{e}_I}) \\
 \parallel & \nearrow & \uparrow \wr \\
 (T_N \Phi_{\mathbf{e}_I})^{-1} (T_N f_I)^{-1} \nu_{M \mathbf{e}_I}(\mathcal{O}_{X_I \mathbf{e}_I}) & \longrightarrow & (T_N \Phi_{\mathbf{e}_I})^{-1} \nu_{N \mathbf{e}_I}(f_I^{-1} \mathcal{O}_{X_I \mathbf{e}_I}).
 \end{array}$$

By the Fourier–Sato transformation, we have the commutative diagram

$$\begin{array}{ccc}
 R^t f'_! f_\pi^{-1} \mu_M(\Phi_{\mathbf{e}_I}^{-1} \mathcal{O}_{X_I \mathbf{e}_I})[(2^L - 1)m] & \longrightarrow & \mu_N(f^{-1} \Phi_{\mathbf{e}_I}^{-1} \mathcal{O}_{X_I \mathbf{e}_I})[(2^L - 1)n] \\
 \uparrow \wr & \nearrow & \parallel \\
 R^t f'_! f_\pi^{-1} {}^t \Phi'_{\mathbf{e}_I!} \mu_{M \mathbf{e}_I}(\mathcal{O}_{X_I \mathbf{e}_I})[(2^\ell - 1)m] & & \mu_N(\Phi_{\mathbf{e}_I}^{-1} f_I^{-1} \mathcal{O}_{X_I \mathbf{e}_I})[(2^L - 1)n] \\
 \parallel & \nearrow & \uparrow \wr \\
 {}^t \Phi'_{\mathbf{e}_I!} R^t f'_I f_{I\pi}^{-1} \mu_{M \mathbf{e}_I}(\mathcal{O}_{X_I \mathbf{e}_I})[(2^\ell - 1)m] & \longrightarrow & {}^t \Phi'_{\mathbf{e}_I!} \mu_{N \mathbf{e}_I}(f_I^{-1} \mathcal{O}_{X_I \mathbf{e}_I})[(2^\ell - 1)n].
 \end{array}$$

Here, we omit the morphisms $\Phi_{\mathbf{e}_I \pi}$ since they are trivial isomorphisms. By combining this diagram with the morphism $f^{-1} \mathcal{O}_X \rightarrow \mathcal{O}_Y$, we obtain the commutative diagram

$$\begin{array}{ccc}
 R^t f'_! f_\pi^{-1} \mathcal{C}_{M,L} & \longrightarrow & \mathcal{C}_{N,L} \\
 \wr & & \wr \\
 (6.6) \quad \bigoplus_{I \in \Lambda(L, \ell)} {}^t \Phi'_{\mathbf{e}_I!} R^t f'_I f_{I\pi}^{-1} \mathcal{C}_{M \mathbf{e}_I, \ell} \mathbf{e}_I & \longrightarrow & \bigoplus_{I \in \Lambda(L, \ell)} {}^t \Phi'_{\mathbf{e}_I!} \mathcal{C}_{N \mathbf{e}_I, \ell} \mathbf{e}_I.
 \end{array}$$

By the commutativity of diagram (6.6) with $\ell = 1$, we can also define the inverse image by f via the well-known inverse image of a complex microfunction by f_I .

Similarly, by applying the functor $R\pi_{Y*}$ to (6.6), we obtain the following commutative diagram (we omit the details here):

$$\begin{array}{ccc}
 R\pi_{Y*}R^t f'_! f_{\pi}^{-1} \pi_X^{-1} \mathcal{B}_{M,L} & \longrightarrow & \mathcal{B}_{N,L} \\
 \wr & & \wr \\
 \bigoplus_{I \in \Lambda(L, \ell)} (\Phi_{\mathbf{e}_I}|_N)^{-1} R\pi_{Y_I*} R^t f'_{I!} f_{I\pi}^{-1} \pi_{X_I}^{-1} \mathcal{B}_{M_{\mathbf{e}_I, \ell} \mathbf{e}_I} & \longrightarrow & \bigoplus_{I \in \Lambda(L, \ell)} (\Phi_{\mathbf{e}_I}|_N)^{-1} \mathcal{B}_{N_{\mathbf{e}_I, \ell} \mathbf{e}_I}.
 \end{array}$$

By the commutativity of this diagram with $\ell = 1$, we can also define the inverse image by f via the well-known inverse image of a complex hyperfunction by f_I .

§6.5. Products

In this subsection we study the product of multicomplex hyperfunctions and microfunctions. We inherit the setting of Section 6.4. Let $\pi_{X \times Y} : T_{X \times Y}^*(X \times Y) \rightarrow M \times N$ be the conormal bundle to $M \times N$ in $X \times Y$. By the natural morphism $\mathcal{O}_X \boxtimes \mathcal{O}_Y \rightarrow \mathcal{O}_{X \times Y}$ and [3, Proposition 4.3.6], we have the natural morphism

$$\mathcal{C}_{M,L} \boxtimes \mathcal{C}_{N,L} \longrightarrow \mathcal{C}_{M \times N, L}.$$

This morphism defines the notion of the external product of two multicomplex microfunctions. By applying the functor $R\pi_{X \times Y*}$ to this morphism, we obtain the morphisms

$$\mathcal{B}_{M,L} \boxtimes \mathcal{B}_{N,L} \longrightarrow R\pi_{X \times Y*}(\mathcal{C}_{M,L} \boxtimes \mathcal{C}_{N,L}) \longrightarrow R\pi_{X \times Y*} \mathcal{C}_{M \times N, L} \simeq \mathcal{B}_{M \times N, L}.$$

The composition of these morphisms defines the notion of the external product of two multicomplex hyperfunctions.

Let us study the compatibility of the external product and the idempotent representation. By the definition of operations on the specialization in [3, Propositions 4.2.5 and 4.2.6], we have the commutative diagram

$$\begin{array}{ccc}
 \nu_M(\Phi_{\mathbf{e}_I}^{-1} \mathcal{O}_{X_I \mathbf{e}_I}) \boxtimes \nu_N(\Phi_{\mathbf{e}_I}^{-1} \mathcal{O}_{Y_I \mathbf{e}_I}) & \longrightarrow & \nu_{M \times N}(\Phi_{\mathbf{e}_I}^{-1} \mathcal{O}_{X_I \mathbf{e}_I} \boxtimes \Phi_{\mathbf{e}_I}^{-1} \mathcal{O}_{Y_I \mathbf{e}_I}) \\
 \wr \uparrow & & \wr \uparrow \\
 (T_{M \times N} \Phi_{\mathbf{e}_I})^{-1}(\nu_{M_{\mathbf{e}_I}}(\mathcal{O}_{X_I \mathbf{e}_I}) \boxtimes \nu_{N_{\mathbf{e}_I}}(\mathcal{O}_{Y_I \mathbf{e}_I})) & \longrightarrow & (T_{M \times N} \Phi_{\mathbf{e}_I})^{-1} \nu_{(M \times N)_{\mathbf{e}_I}}(\mathcal{O}_{X_I \mathbf{e}_I} \boxtimes \mathcal{O}_{Y_I \mathbf{e}_I}).
 \end{array}$$

Here, we denote $T_{M \times N} \Phi_{\mathbf{e}_I} : T_{M \times N}(X \times Y) \rightarrow T_{(M \times N)_{\mathbf{e}_I}}(X \times Y_{\mathbf{e}_I})$. By the Fourier–Sato transformation, we have the commutative diagram

$$\begin{array}{ccc}
 \mu_M(\Phi_{\mathbf{e}_I}^{-1} \mathcal{O}_{X_I \mathbf{e}_I}) \boxtimes \mu_N(\Phi_{\mathbf{e}_I}^{-1} \mathcal{O}_{Y_I \mathbf{e}_I}) & \longrightarrow & \mu_{M \times N}(\Phi_{\mathbf{e}_I}^{-1} \mathcal{O}_{X_I \mathbf{e}_I} \boxtimes \Phi_{\mathbf{e}_I}^{-1} \mathcal{O}_{Y_I \mathbf{e}_I}) \\
 \wr \uparrow & & \wr \uparrow \\
 {}^t \Phi'_{\mathbf{e}_I*}(\mu_{M_{\mathbf{e}_I}}(\mathcal{O}_{X_I \mathbf{e}_I}) \boxtimes \mu_{N_{\mathbf{e}_I}}(\mathcal{O}_{Y_I \mathbf{e}_I})) & \longrightarrow & {}^t \Phi'_{\mathbf{e}_I*} \mu_{(M \times N)_{\mathbf{e}_I}}(\mathcal{O}_{X_I \mathbf{e}_I} \boxtimes \mathcal{O}_{Y_I \mathbf{e}_I}).
 \end{array}$$

Here, we omit the morphisms $\Phi_{\mathbf{e}_I \pi}$ since they are trivial isomorphisms. Thus, we obtain the commutative diagram

$$(6.7) \quad \begin{array}{ccc} \mathcal{C}_{M,L} \boxtimes \mathcal{C}_{N,L} & \xrightarrow{\quad\quad\quad} & \mathcal{C}_{M \times N, L} \\ \wr & & \wr \\ \bigoplus_{I \in \Lambda(L, \ell)} {}^t \Phi'_{\mathbf{e}_I *} (\mathcal{C}_{M \mathbf{e}_I, \ell} \boxtimes \mathcal{C}_{N \mathbf{e}_I, \ell}) \mathbf{e}_I & \longrightarrow & \bigoplus_{I \in \Lambda(L, \ell)} {}^t \Phi'_{\mathbf{e}_I *} \mathcal{C}_{(M \times N) \mathbf{e}_I, \ell} \mathbf{e}_I. \end{array}$$

By the commutativity of diagram (6.7) with $\ell = 1$, we can also define the external product of multicomplex microfunctions via the well-known external product of classical complex microfunctions.

Similarly, by applying the functor $R\pi_{(X \times Y)*}$ to (6.7), we obtain the following commutative diagram (we omit the details here):

$$\begin{array}{ccc} \mathcal{B}_{M,L} \boxtimes \mathcal{B}_{N,L} & \xrightarrow{\quad\quad\quad} & \mathcal{B}_{M \times N, L} \\ \wr & & \wr \\ \bigoplus_{I \in \Lambda(L, \ell)} (\Phi_{\mathbf{e}_I}|_{M \times N})^{-1} (\mathcal{B}_{M \mathbf{e}_I, \ell} \boxtimes \mathcal{B}_{N \mathbf{e}_I, \ell}) \mathbf{e}_I & \longrightarrow & \bigoplus_{I \in \Lambda(L, \ell)} (\Phi_{\mathbf{e}_I}|_{M \times N})^{-1} \mathcal{B}_{(M \times N) \mathbf{e}_I, \ell} \mathbf{e}_I. \end{array}$$

By the commutativity of this diagram with $\ell = 1$, we can also define the external product of multicomplex hyperfunctions via the well-known external product of classical complex hyperfunctions.

Hereafter, we consider the case where $N = M$ and $Y = X$. We denote by δ the diagonal embedding $\delta: M \rightarrow M \times M$ of M . We identify the image of δ with M . By combining the external product above with the restriction by δ , we define the morphisms

$$R^t \delta'_! \delta_\pi^{-1} (\mathcal{C}_{M,L} \boxtimes \mathcal{C}_{M,L}) \longrightarrow R^t \delta'_! \delta_\pi^{-1} \mathcal{C}_{M \times M, L} \longrightarrow \mathcal{C}_{M,L}.$$

This morphism defines the notion of the product of two multicomplex microfunctions.

Via the spectrum morphism, we obtain the natural morphisms

$$\begin{aligned} R\pi_{X \times X*} R^t \delta'_! \delta_\pi^{-1} (\pi_X^{-1} \mathcal{B}_{M,L} \boxtimes \pi_X^{-1} \mathcal{B}_{M,L}) &\longrightarrow R\pi_{X \times X*} R^t \delta'_! \delta_\pi^{-1} (\mathcal{C}_{M,L} \boxtimes \mathcal{C}_{M,L}) \\ &\longrightarrow R\pi_{X \times X*} \mathcal{C}_{M,L} \simeq \mathcal{B}_{M,L}. \end{aligned}$$

The composition of these morphisms defines the notion of the product of two multicomplex hyperfunctions. More precisely, for hyperfunctions u_1 and u_2 on M , if $\text{SS}(u_1) \cap \text{SS}(u_2)^a$ is contained in the zero section, we can define the product $u_1 u_2$. Here, a denotes the antipodal map of $T_M^* X$ and $\text{SS}(u_2)^a$ denotes the image of $\text{SS}(u_2)$ by a . We have the estimate

$$\text{SS}(u_1 u_2) \subset {}^t \delta' (\delta_\pi^{-1} (\text{SS}(u_1) \times \text{SS}(u_2))).$$

§6.6. Direct image

In this subsection we study the direct image of multicomplex hyperfunctions and microfunctions. In the case where f is smooth, the direct image is considered as the integration along fibers. We inherit the setting of Section 6.4.

Let $\mathcal{O}_X^{(m)}$ (resp. $\mathcal{O}_Y^{(n)}$) denote the sheaf of m -forms (resp. n -forms) with coefficients that are multicomplex holomorphic functions on X (resp. Y). Note that they have the nesting properties induced by those of $\mathcal{O}_X$ and $\mathcal{O}_Y$, respectively. First, we construct the Leray–Grothendieck-type residue morphism

$$(6.8) \quad Rf_! \mathcal{O}_Y^{(n)}[(2^L - 1)n] \longrightarrow \mathcal{O}_X^{(m)}[(2^L - 1)m]$$

as follows. Since we have the Leray–Grothendieck residue morphism

$$Rf_{I!} \mathcal{O}_{Y_0 \mathbf{e}_I}^{(n)}[n] \longrightarrow \mathcal{O}_{X_0 \mathbf{e}_I}^{(m)}[m]$$

for $I \in \Lambda(L, 1)$, where $\mathcal{O}_{X_0 \mathbf{e}_I}^{(m)}$ (resp. $\mathcal{O}_{Y_0 \mathbf{e}_I}^{(n)}$) denotes the sheaf of m -forms (resp. n -forms) with coefficients that are complex holomorphic functions on $X_0 \mathbf{e}_I$ (resp. $Y_0 \mathbf{e}_I$), we have the natural morphisms

$$\begin{aligned} Rf_! \mathcal{O}_Y^{(n)}[(2^L - 1)n] &\simeq \bigoplus_{I \in \Lambda(L, 1)} Rf_! \Phi_{\mathbf{e}_I}^{-1} \mathcal{O}_{Y_0 \mathbf{e}_I}^{(n)} \mathbf{e}_I[n + (2^L - 2)n] \\ &\rightarrow \bigoplus_{I \in \Lambda(L, 1)} \Phi_{\mathbf{e}_I}^{-1} Rf_{I!} \mathcal{O}_{Y_0 \mathbf{e}_I}^{(n)} \mathbf{e}_I[n + (2^L - 2)m] \\ &\rightarrow \bigoplus_{I \in \Lambda(L, 1)} \Phi_{\mathbf{e}_I}^{-1} \mathcal{O}_{X_0 \mathbf{e}_I}^{(m)} \mathbf{e}_I[m + (2^L - 2)m] \\ &\simeq \mathcal{O}_X^{(m)}[(2^L - 1)m] \end{aligned}$$

by the idempotent representation. Here, the second morphism is obtained as follows:

$$\begin{aligned} Rf_! \Phi_{\mathbf{e}_I}^{-1}[(2^L - 1)n] &\simeq Rf_! \Phi_{\mathbf{e}_I}^! [n] \\ &\rightarrow \Phi_{\mathbf{e}_I}^! \Phi_{\mathbf{e}_I!} Rf_! \Phi_{\mathbf{e}_I}^! [n] \\ &= \Phi_{\mathbf{e}_I}^! Rf_{I!} \Phi_{\mathbf{e}_I!} \Phi_{\mathbf{e}_I}^! [n] \\ &\simeq \Phi_{\mathbf{e}_I}^{-1} Rf_{I!} [n + (2^L - 2)m]. \end{aligned}$$

By [3, Proposition 4.3.4], there exists a natural morphism

$$Rf_{\pi!} {}^t f'^{-1} \mu_N(\mathcal{O}_Y^{(n)}) \longrightarrow \mu_M(Rf_! \mathcal{O}_Y^{(n)}).$$

By combining this morphism with (6.8), we obtain the morphism

$$(6.9) \quad Rf_{\pi!} {}^t f'^{-1} (\mathcal{C}_{N, L} \otimes_{\pi_Y^{-1} \mathcal{O}_Y|_N} \pi_Y^{-1} \mathcal{V}_{N, L}) \longrightarrow \mathcal{C}_{M, L} \otimes_{\pi_X^{-1} \mathcal{O}_X|_M} \pi_X^{-1} \mathcal{V}_{M, L},$$

where $\mathcal{V}_{N,L} = \mathcal{O}_Y^{(n)}|_N$ (resp. $\mathcal{V}_{M,L} = \mathcal{O}_X^{(n)}|_M$) denotes the sheaf of multicomplex analytic densities of level L on N (resp. M). The morphism (6.9) defines the notion of the direct image of a multicomplex microfunction by f . More precisely, for a multicomplex microfunction density v on an open set V on T_N^*Y , if U is an open set of T_M^*X such that f_π is proper on ${}^t f'^{-1}(\text{supp}(v)) \cap f_\pi^{-1}(U)$, we can define the microfunction f_*v .

Via the spectrum morphism, we obtain the natural morphisms

$$\begin{aligned} Rf_!(\mathcal{B}_{N,L} \otimes_{\mathcal{O}_Y|_N} \mathcal{V}_{N,L}) &\simeq Rf_! R\pi_{Y*}(\mathcal{C}_{N,L} \otimes_{\pi_Y^{-1}\mathcal{O}_Y|_N} \pi_Y^{-1}\mathcal{V}_{N,L}) \\ &\rightarrow R\pi_{X*} Rf_{\pi!} {}^t f'^{-1}(\mathcal{C}_{N,L} \otimes_{\pi_Y^{-1}\mathcal{O}_Y|_N} \pi_Y^{-1}\mathcal{V}_{N,L}) \\ &\rightarrow R\pi_{X*}(\mathcal{C}_{M,L} \otimes_{\pi_X^{-1}\mathcal{O}_X|_M} \pi_X^{-1}\mathcal{V}_{M,L}) \\ &\simeq \mathcal{B}_{M,L} \otimes_{\mathcal{O}_X|_M} \mathcal{V}_{M,L}. \end{aligned}$$

Thus, for a multicomplex hyperfunction density v on N satisfying that f is proper on $\text{supp}(v)$, we can define the hyperfunction f_*v on M . We have the estimate

$$\text{SS}(f_*v) \subset f_\pi({}^t f'^{-1}(\text{SS}(v))).$$

Let us study the compatibility of the direct image and the idempotent representation. By the definition of operations on the specialization in [3, Propositions 4.2.4 and 4.2.5], we have the commutative diagram

$$\begin{array}{ccc} R(T_N f)_! \nu_N(\Phi_{\mathbf{e}_I}^! \mathcal{O}_{Y_I \mathbf{e}_I}^{(n)}) & \longrightarrow & \nu_M(Rf_! \Phi_{\mathbf{e}_I}^! \mathcal{O}_{Y_I \mathbf{e}_I}^{(n)}) \\ \downarrow \wr & & \downarrow \\ R(T_N f)_!(T_N \Phi_{\mathbf{e}_I})^! \nu_{N \mathbf{e}_I}(\mathcal{O}_{Y_I \mathbf{e}_I}^{(n)}) & & \nu_M(\Phi_{\mathbf{e}_I}^! Rf_{I!} \mathcal{O}_{Y_I \mathbf{e}_I}^{(n)}) \\ \downarrow & & \downarrow \wr \\ (T_M \Phi_{\mathbf{e}_I})^! R(T_N f)_! \nu_{N \mathbf{e}_I}(\mathcal{O}_{Y_I \mathbf{e}_I}^{(n)}) & \longrightarrow & (T_M \Phi_{\mathbf{e}_I})^! \nu_{M \mathbf{e}_I}(Rf_{I!} \mathcal{O}_{Y_I \mathbf{e}_I}^{(n)}). \end{array}$$

By the Fourier–Sato transformation, we have the commutative diagram

$$\begin{array}{ccc} Rf_{\pi!} {}^t f'^{-1} \mu_N(\Phi_{\mathbf{e}_I}^! \mathcal{O}_{Y_I \mathbf{e}_I}^{(n)}) & \longrightarrow & \mu_M(Rf_! \Phi_{\mathbf{e}_I}^! \mathcal{O}_{Y_I \mathbf{e}_I}^{(n)}) \\ \downarrow & & \downarrow \\ {}^t \Phi_{\mathbf{e}_I}' Rf_{I\pi!} {}^t f'^{-1} \mu_{N \mathbf{e}_I}(\mathcal{O}_{Y_I \mathbf{e}_I}^{(n)}) & \longrightarrow & {}^t \Phi_{\mathbf{e}_I}' \mu_{M \mathbf{e}_I}(Rf_{I!} \mathcal{O}_{Y_I \mathbf{e}_I}^{(n)}). \end{array}$$

Here, we omit the morphisms $\Phi_{\mathbf{e}_I}\pi$ since they are trivial isomorphisms. By combining this diagram with (6.8), we obtain the commutative diagram

$$(6.10) \quad \begin{array}{ccc} Rf_{\pi!} {}^t f'^{-1}(\mathcal{C}_{N,L} \otimes \mathcal{V}_{N,L}) & \longrightarrow & \mathcal{C}_{M,L} \otimes \mathcal{V}_{M,L} \\ \downarrow & & \downarrow \wr \\ \bigoplus_{I \in \Lambda(L,\ell)} {}^t \Phi'_{\mathbf{e}_I*} Rf_{I\pi!} {}^t f_I'^{-1}(\mathcal{C}_{N\mathbf{e}_I,\ell} \otimes \mathcal{V}_{N\mathbf{e}_I,\ell}) \mathbf{e}_I & \longrightarrow & \bigoplus_{I \in \Lambda(L,\ell)} {}^t \Phi'_{\mathbf{e}_I*} (\mathcal{C}_{M\mathbf{e}_I,\ell} \otimes \mathcal{V}_{M\mathbf{e}_I,\ell}) \mathbf{e}_I. \end{array}$$

Here, we abbreviate $\mathcal{C}_{N,L} \otimes_{\pi_Y^{-1}\mathcal{O}_Y|_N} \pi_Y^{-1}\mathcal{V}_{N,L}$ to simply $\mathcal{C}_{N,L} \otimes \mathcal{V}_{N,L}$, and so on. By the commutativity of diagram (6.10) with $\ell = 1$, we can also define the direct image by f via the well-known direct image of a complex microfunction by f_I .

Similarly, by applying the functor $R\pi_{N*}$ to (6.10), we obtain the following commutative diagram (we omit the details here):

$$\begin{array}{ccc} Rf_!(\mathcal{B}_{N,L} \otimes \mathcal{V}_{N,L}) & \longrightarrow & \mathcal{B}_{M,L} \otimes \mathcal{V}_{M,L} \\ \downarrow & & \downarrow \wr \\ \bigoplus_{I \in \Lambda(L,\ell)} (\Phi_{\mathbf{e}_I}|_M)^{-1} Rf_{I!}(\mathcal{B}_{N\mathbf{e}_I,\ell} \otimes \mathcal{V}_{N\mathbf{e}_I,\ell}) \mathbf{e}_I & \longrightarrow & \bigoplus_{I \in \Lambda(L,\ell)} (\Phi_{\mathbf{e}_I}|_M)^{-1} (\mathcal{B}_{M\mathbf{e}_I,\ell} \otimes \mathcal{V}_{M\mathbf{e}_I,\ell}) \mathbf{e}_I. \end{array}$$

Here, we abbreviate $\mathcal{B}_{N,L} \otimes_{\mathcal{O}_Y|_N} \mathcal{V}_{N,L}$ to simply $\mathcal{B}_{N,L} \otimes \mathcal{V}_{N,L}$, and so on. By the commutativity of the last diagram with $\ell = 1$, we can also define the direct image by f via the well-known direct image of a complex hyperfunction by f_I .

Remark. In this paper, we focused on multicomplex algebra and studied multicomplex hyperfunctions and microfunctions. However, most of our results can be generalized to the setting $X = \prod_{j=1}^L X_0 \mathbf{e}_j$ and $\mathcal{O}_X = \bigoplus_{j=1}^L \Phi_{\mathbf{e}_j}^{-1} \mathcal{O}_{X_0 \mathbf{e}_j} \mathbf{e}_j$ for an arbitrary positive integer L . Here, X_0 is a complexification of a real analytic manifold M , $\{\mathbf{e}_1, \dots, \mathbf{e}_L\}$ is a set of orthogonal idempotent elements such that $\sum_{j=1}^L \mathbf{e}_j = 1$, and the map $\Phi_{\mathbf{e}_j}: X \rightarrow X_0 \mathbf{e}_j$ denotes the natural projection for each $j = 1, 2, \dots, L$.

Acknowledgments

The author would like to express sincere gratitude to Professors N. Honda, Y. Okada and the referee for their valuable comments and helpful suggestions.

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