

An Iterative Construction of Complete Kähler–Einstein Metrics

by

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Abstract

We extend Tsuji’s iterative construction of Kähler–Einstein metrics with negative scalar curvature to non-compact Kähler manifolds with bounded geometry, using Berndtsson’s method from the compact setting. Consequently, given a holomorphic surjective map $p: X \rightarrow Y$, where X is a weakly pseudoconvex Kähler manifold and Y is a complex manifold, and where the smooth fibers admit Kähler–Einstein metrics with negative scalar curvature and bounded geometry, we show that the fiberwise Kähler–Einstein metrics induce a semipositively curved metric on the relative canonical bundle $K_{X/Y}$. Moreover, our approach also applies to the plurisubharmonic variation of cusp Kähler–Einstein metrics.

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§1. Introduction

In the 1950s, Calabi conjectured that a compact Kähler manifold with negative first Chern class admits a unique Kähler–Einstein metric. This was famously shown independently by Aubin [Aub78] and Yau [Yau78], leading to various remarkable results in algebraic geometry for such manifolds. A question arises as to whether Kähler–Einstein metrics can be obtained as the limit of a sequence of algebraic metrics induced by the pluricanonical line bundle; cf. [Yau87]. Through the Tian–Yau–Zelditch asymptotic expansion of the Bergman kernel, Donaldson [Don01] iteratively constructed, on a polarized manifold (X, L) with discrete automorphism

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group, a sequence of “balanced metrics” whose limit is a Kähler form with constant scalar curvature. Tsuji [Tsu10] introduced a different approach to obtaining Kähler–Einstein metrics with negative scalar curvature by algebraic approximations.

Beyond the compact setting, it is natural to investigate the existence of complete Kähler–Einstein metrics on non-compact Kähler manifolds. Initially, Cheng–Yau [CY80] established necessary geometric conditions for the existence of complete Kähler–Einstein metrics with negative scalar curvature on non-compact manifolds. Later, this result was extended by Kobayashi [Kob84] and Tian–Yau [TY87]. At the same time, Mok and Yau [MY83] showed the existence of a complete Kähler–Einstein metric with negative scalar curvature on any bounded pseudoconvex domain. Recently, Wu and Yau [WY20] imposed an assumption on the holomorphic sectional curvature to construct a unique complete Kähler–Einstein metric with negative Ricci curvature, using a new complex Monge–Ampère equation. We also refer to [Ton18, HLT22] for other proofs of Wu–Yau’s result using the Kähler–Ricci flow.

Motivated by Tsuji’s iteration for numerically approximating canonical Kähler metrics in [Tsu10, Tsu13], we study Tsuji’s iterative procedure in the non-compact setting. We aim to obtain uniform convergence for Tsuji’s iterations by making use of the non-compact version of the asymptotic expansion of the Bergman kernels. In particular, our approach applies to general complex Monge–Ampère equations, rather than being restricted to the Kähler–Einstein case as in previous works.

To state our result, we first fix some notation and terminology. Let X be a complex manifold of dimension n . Let L be a positively curved Hermitian holomorphic line bundle on X with reference curvature form $\omega = dd^c\phi_L$, where ϕ_L denotes the weight of a Hermitian metric $h_L = e^{-\phi_L}$ on L . We consider the space $\mathcal{H}_L$ of (possibly singular) Hermitian metrics on L with positive curvature current; such metrics can be written as $h = e^{-\phi}$ with plurisubharmonic (psh) weight ϕ , and ω_ϕ denotes the associated curvature current. One may also identify a Hermitian metric $h = e^{-\phi}$ on L with the function $\varphi = \phi - \phi_L$, so that $h = h_L e^{-\varphi}$.

By $H^0(X, L)$ we denote the space of globally holomorphic sections of L . Let μ be a positive measure on X . For any positive integer k , we define a Hermitian product on $H^0(X, L^k)$ via the L^2 -norm

$$\|s\|_{k\phi, \mu}^2 := \int_X |s|^2 e^{-k\phi} d\mu.$$

Here the integrand $|s|^2 e^{-k\phi}$ may also be written as $|s|_{h_L}^2 e^{-k\varphi}$, where $\varphi = \phi - \phi_L$.

We then define the *Bergman function*

$$B_{k\phi,\mu}(x) = \sup\{|s(x)|^2 e^{-k\phi(x)} : s \in H^0(X, L^k), \|s\|_{k\phi,\mu} \leq 1\}.$$

The quantity $\mathcal{K}_{k\phi,\mu} = B_{k\phi,\mu} e^{k\phi}$ is the *Bergman kernel* associated with $(L, k\phi, \mu)$. In particular, $\frac{1}{k} \log \mathcal{K}_{k\phi,\mu}$ defines a Hermitian metric on L .

Fixing a smooth measure Ω on X , we are interested in the following complex Monge–Ampère equation:

$$(1.1) \quad \omega_{\phi_\infty}^n = e^{\phi_\infty - \phi_L} \Omega,$$

where ϕ_∞ is an unknown smooth Hermitian metric on L with positive curvature. Set $\varphi := \phi_\infty - \phi_L$ and $\omega := dd^c \phi_L$. Then the equation (1.1) can be rewritten as

$$(\omega + dd^c \varphi)^n = e^\varphi \Omega.$$

In particular, if the volume form Ω satisfies $\text{Ric}(\Omega) = -\omega$, then $\text{Ric}(\omega_{\phi_\infty}) = -\omega_{\phi_\infty}$, so ω_{ϕ_∞} is a Kähler–Einstein metric with Ricci curvature -1 .

We now describe Tsuji’s iteration associated with the equation (1.1). Assume that ϕ is a (possibly singular) Hermitian metric on L . Let $\mathcal{K}_{k\phi,\mu_\phi}$ be the Bergman kernel associated with the weight $k\phi$ and the measure $\mu_\phi := e^{\phi - \phi_L} \Omega/n!$. The quantity

$$(1.2) \quad \beta_k(\phi) := \frac{1}{k} (\log \mathcal{K}_{k\phi,\mu_\phi} - \log d_k)$$

defines again a metric on L , with

$$d_k = \left(\frac{k}{2\pi} \right)^n.$$

Starting from a metric ϕ_1 on L , we inductively obtain a sequence of Hermitian metrics on L by setting $\phi_{k+1} := \beta_k(\phi_k)$. Generalizing the works of Tsuji [Tsu10], Song–Weinkove [SW10] and Berndtsson [Ber09b], we show that ϕ_k uniformly converges to the metric ϕ_∞ which satisfies equation (1.1).

In particular, our result applies to solutions of the general complex Monge–Ampère equation (1.1), without assuming that the equation arises from Kähler–Einstein geometry, in contrast with previous works.

Theorem 1.1. *Let X be an n -dimensional Kähler manifold equipped with a positively curved line bundle $(L, e^{-\phi_L})$. Let Ω be a smooth volume form on X . Assume that there exists a smooth Hermitian metric ϕ_∞ on L satisfying the complex Monge–Ampère equation*

$$\omega_{\phi_\infty}^n = e^{\phi_\infty - \phi_L} \Omega.$$

Assume moreover that $(X, \omega_{\phi_\infty})$ is a Kähler manifold with bounded geometry of order $\ell \geq 5$. Let ϕ_1 be a continuous metric on L such that $\sup_X |\phi_1 - \phi_\infty| \leq C$. Let ϕ_k be a sequence of metrics on L defined by Tsuji's iteration as described above. Then $\sup_X |\phi_k - \phi_\infty| \rightarrow 0$ as $k \rightarrow \infty$ at the rate $\frac{\log k}{k}$.

The hypothesis of bounded geometry is satisfied, for example, by the Kähler–Einstein metric on a strongly pseudoconvex domain in $\mathbb{C}^n$ (cf. [CY80]), or by Kähler metrics with cusp singularities along some divisor (cf. [Kob84, TY87]).

Remark 1.2. In particular, when $L = K_X$ is an ample line bundle equipped with a smooth Hermitian metric $e^{-\phi_L}$, we can take $\Omega = e^{\phi_{K_X}}$ as a smooth volume form. For any (smooth) metric $e^{-\phi}$ on K_X , the L^2 -norm on $H^0(X, K_X^{(k+1)})$ is defined by

$$\|\sigma\|_{k\phi, \mu_\phi}^2 := c_n \int_X \sigma \wedge \bar{\sigma} e^{-(k+1)\phi} e^\phi = \int_X |\sigma|^2 e^{-k\phi}.$$

Similarly, we define Tsuji's iteration, which is analogous to that in [Ber09b]. The only difference is that the constant d_k is the dimension of $H^0(X, K_X^{(k+1)})$ in the compact setting. It is known that $\dim H^0(X, L^k) = O(k^n)$ for ample line bundle L , so our definition for d_k is that of Berndtsson modulo a uniform constant.

As mentioned above, one considers the special case where $L = K_X$ is ample; it follows from [CY80, WY20] that, under extra conditions, there exists a unique Kähler–Einstein metric ω_{KE} on X such that $\text{Ric}(\omega_{\text{KE}}) = -\omega_{\text{KE}}$ with

$$\omega_{\text{KE}} = dd^c \log(e^u \omega^n) = dd^c \log(\omega^n) + dd^c u$$

for a bounded smooth function u . We denote $\phi_\omega = \log \omega^n$ and $\phi_{\text{KE}} = \log(e^u \omega^n)$ as two Hermitian metrics on K_X . Then we obtain the following Monge–Ampère equation:

$$(1.3) \quad (dd^c \phi_{\text{KE}})^n = e^{\phi_{\text{KE}} - \phi_\omega} \omega^n.$$

In this case, we define Tsuji's iteration $\phi_{k+1} = \beta_k(\phi_k)$ with β_k defined as in (1.2) and

$$(1.4) \quad d_k = \left(\frac{k}{2\pi}\right)^n \max\left\{\frac{1}{2}, 1 + \frac{\text{Scal}(\omega_{\text{KE}})}{2k}\right\} = \left(\frac{k}{2\pi}\right)^n \max\left\{\frac{1}{2}, 1 - \frac{n}{2k}\right\}.$$

Then we obtain the non-compact version of Tsuji's iteration (cf. [Tsu10, SW10, Ber09b]).

Theorem 1.3. *With the assumptions in Theorem 1.1, suppose further that $L = K_X$ with $\phi_\infty = \phi_{\text{KE}}$ satisfying equation (1.3). Let ϕ_1 be a continuous metric on K_X such that $\sup_X |\phi_1 - \phi_{\text{KE}}| \leq C$. Let ϕ_k be a sequence of metrics on K_X , defined*

by Tsuji's iteration as described above. Then $\sup_X |\phi_k - \phi_{\text{KE}}| \rightarrow 0$ as $k \rightarrow \infty$ at the rate $\frac{1}{k}$.

This result also extends the uniform convergence obtained by the second author [T622] and, more recently, by Yoo [Yoo24], in the setting of pseudoconvex domains with bounded geometry.

The idea of the proof is to use the following uniform asymptotic expansion of the Bergman function:

$$(1.5) \quad \left| B_{k\phi, \omega_\phi^{[n]}} - \left(\frac{k}{2\pi} \right)^n \left(1 + \frac{b_1}{2k} \right) \right|_{C^0} \leq Ck^{n-2},$$

where $\omega_\phi^{[n]} := \frac{\omega_\phi^n}{n!}$ and $C > 0$ depends only on the bounded geometry of (X, ω_ϕ) , extending Berndtsson's work [Ber09b] in the compact setting. A more general asymptotic expansion in C^k -norms was proved by Ma–Marinescu [MM07, Problem 6.1] (see also [MM15, Theorem 6] and [Kel05]). For the reader's convenience, we provide an alternative proof of the uniform Bergman expansion (1.5) (see Theorem 3.1) using Tian's peak-section method [Tia90] (see also [Rua98, Lu00]), together with the Laplace method as in [Don09, Cha15]. We refer to the works [BdMS76, Tia90, Cat99, Zel98, Lu00, Cha03, DLM06, MM07, BBS08, MM15] for further background on the asymptotic behavior of Bergman kernels on compact Kähler manifolds.

As a consequence, we apply Theorem 1.3 to study the variation of complete Kähler–Einstein metrics, extending the results in [Tsu10, Sch12, Ber13] to the non-compact setting. We refer to [Pău17, Gue20a, CGP21] for more results in the compact case.

Theorem 1.4. *Let $p: X \rightarrow Y$ be a holomorphic surjective map from a weakly pseudoconvex Kähler manifold X to a complex manifold Y . Let $Y^\circ \subset Y$ be a set of points that are not critical values of p in Y and set $X^\circ = p^{-1}(Y^\circ)$ and $X_y = p^{-1}(\{y\})$. Assume that*

- (i) *for every $y \in Y^\circ$, X_y admits a Kähler–Einstein metric $\omega_{\text{KE},y}$ with negative Ricci curvature such that $(X_y, \omega_{\text{KE},y})$ has bounded geometry of order $\ell \geq 5$, uniformly in y on compact subsets of Y° ,*
- (ii) *$K_{X/Y}$ admits a singular Hermitian metric $h_0 = e^{-\phi_0}$ with positive curvature current and for every $y_0 \in Y^\circ$, there exists a neighborhood $U \Subset Y^\circ$ of y_0 such that $\|\phi_{0,y} - \phi_{\text{KE},y}\|_{L^\infty(X_y)} \leq C(U)$ for every $y \in U$.*

Then the Hermitian metric on $(K_{X/Y})|_{X^\circ}$ induced by the fiberwise Kähler–Einstein metrics is positively curved. Moreover, if we have the additional condition that

- (iii) the Kähler–Einstein potentials $(\phi_{\text{KE},y})_{y \in Y^\circ}$ are bounded from above near the singular fibers,

then this metric extends canonically across $X \setminus X^\circ$ to a positively curved metric on $K_{X/Y}$.

As a consequence, we recover the positivity of the variation of Kähler–Einstein metrics on pseudoconvex domains proved in [Tsu13, Cho15b, CY22], following Tsuji’s argument [Tsu13] (cf. Theorem 4.8). See also Choi–Yoo [CY21] for a similar result for strongly pseudoconvex domains contained in a Kähler manifold, using the Schumacher method [Sch12].

Furthermore, the statement of Theorem 1.4 also holds in the following setting (cf. Theorem 4.11). Let $p: \bar{X} \rightarrow Y$ be a holomorphic surjective and proper map between compact Kähler manifolds with relative dimension n . Let $X := \bar{X} \setminus D$ where $D = \sum_{i=1}^\ell D_i$ is a reduced divisor with simple normal crossings. Assume that Y° is the largest Zariski open subset of Y such that, if $\bar{X}^\circ := p^{-1}(Y^\circ)$, then every fiber $\bar{X}_y := p^{-1}(\{y\})$ is smooth, and $D_y := D \cap \bar{X}_y$ has simple normal crossings. We denote $X^\circ = \bar{X}^\circ \setminus D$. Then we will show that the current obtained by gluing the fiberwise Kähler–Einstein metrics with cusp singularities along D is positive.

Recall that ω_y is said to have *cusp singularities* (or *Poincaré singularities*) along D_y , locally, if $D_y = (z_1 \cdots z_\ell = 0)$; then ω_y is quasi-isometric to the cusp metric:

$$\omega_{\text{cusp}} := \sum_{j=1}^\ell \frac{i \, dz_j \wedge d\bar{z}_j}{|z_j|^2 \log^2 |z_j|^2} + \sum_{k=\ell+1}^n i \, dz_k \wedge d\bar{z}_k.$$

Assume that $K_{\bar{X}_y} + D_y$ is ample for every $y \in Y^\circ$. By the well-known result of Kobayashi [Kob84] and Tian–Yau [TY87], building upon Cheng–Yau’s method [CY80], for each fiber, there exists a unique complete Kähler metric ω_y on $X_y := \bar{X}_y \setminus D_y$ with cusp singularities along D_y and satisfying $\text{Ric}(\omega_y) = -\omega_y$ on X_y . The metric ω_y extends as a positive current on $\bar{X}_y$; it satisfies

$$(1.6) \quad \text{Ric}(\omega_y) = -\omega_y + [D_y]$$

in the sense of currents on $\bar{X}_y$, where $[D_y]$ is the integration current along D_y . We also refer to [Ban90, Wu08, Auv17, Gue14, BG14, DL17, GW16, Gue20b, BG22, DV25] for related results on complete Kähler–Einstein metrics with cusp or mixed singularities.

Over X° , the fiberwise Kähler–Einstein metrics ω_y with cusp singularities glue together to define a current T . Applying an analogue of Theorem 1.4 in this setting (cf. Theorem 4.11), we obtain the following.

Theorem 1.5. *With the setup above, the fiberwise cusp Kähler–Einstein metrics ω_y satisfying (1.6) glue to define a current T on $\bar{X}$ such that*

- (i) T is positive on $X^\circ = \bar{X}^\circ \setminus D$,
- (ii) T extends canonically across $(X \setminus X^\circ) \cup D$ to a closed positive current on $\bar{X}$.

We remark that a similar result for the family of twisted conic Kähler–Einstein metrics was obtained by Guenancia [Gue20a], and a result for the family of cusp Kähler–Einstein metrics was proved by Naumann [Nau22] (cf. also [Gue20a, Section 8.1]). In particular, they used Schumacher’s approach [Sch12], whereas our proof follows Tsuji’s method [Tsu10].

Organization of the paper

In Section 2 we start with preliminary material on (singular) Hermitian metrics on line bundles and local quasi-coordinates. The main Theorems 1.1 and 1.3 are proved in Section 3, where we also establish an asymptotic expansion for the Bergman kernel associated with weighted spaces. Section 4 is devoted to the positivity of the variation of complete Kähler–Einstein metrics, where Theorems 1.4 and 1.5 are proved.

§2. Notation and background

§2.1. General notation

Throughout the paper, (X, ω) denotes a Kähler manifold of dimension n equipped with a Kähler metric ω . We denote $d = \partial + \bar{\partial}$ and $d^c = \frac{i}{4}(\bar{\partial} - \partial)$ so that $dd^c = \frac{i}{2}\partial\bar{\partial}$.

Let L be a Hermitian holomorphic line bundle on X . The (singular) Hermitian metric on L will be written as $h = e^{-\phi}$, and we will refer to the function ϕ as a *weight* on L . In practice, ϕ is understood as a collection of locally integrable functions ϕ^U on trivializing open sets U , called *local weights*. If e^U is a local holomorphic frame of L over U , then $|e^U(z)|_\phi^2 := e^{-\phi^U(z)}$. We will use the weight ϕ instead of $h = e^{-\phi}$ for a Hermitian metric on L . The set of weights on L forms an affine space modeled on $L_{\text{loc}}^1(X)$. Namely, if ϕ_1, ϕ_2 are two weights on L , their difference $\phi_1 - \phi_2$ is a locally integrable function on X .

Equivalently, a weight may be viewed as a function on the total space of the dual bundle L^* satisfying the log-homogeneity property $\phi(\lambda v) = \log|\lambda| + \phi(v)$ for all non-zero $v \in L^*$, $\lambda \in \mathbb{C}$. A section $s \in H^0(X, L)$ induces a weight on L , denoted $\log|s|$, defined by $\log|s|(v) := \log|\langle s, v \rangle|$. The pointwise norm of s with respect to the weight ϕ is

$$|s|_\phi^2 := |s|^2 e^{-\phi}.$$

If ϕ is in the class $\mathcal{C}^2$, i.e., it has continuous derivatives of order two, the curvature form of the metric ϕ is the global form on X , locally expressed as $dd^c\phi$ which represents the first Chern class $c_1(L)$ in real de Rham cohomology. One should note that the notation $dd^c\phi$ is only symbolic: the form is not exact in general. The curvature form of a smooth metric is said to be *positive* if the local Hermitian matrix $(\phi_{i\bar{j}}) = (\frac{\partial^2\phi}{\partial z_i\partial\bar{z}_j})$ is positive definite.

In general, the curvature $dd^c\phi$ is still well defined as a $(1, 1)$ -current, since each local weight ϕ^U is only required to be locally integrable. The curvature current of a singular metric $e^{-\phi}$ is said to be *positive* when the local weights ϕ^U are plurisubharmonic (psh). We use the notation ω_ϕ and $dd^c\phi$ interchangeably for the curvature form/current of the weight ϕ .

Given a positive measure μ on X and a weight ϕ on L , we obtain an L^2 -norm on $H^0(X, L^k)$ defined by

$$\|s\|_{k\phi, \mu}^2 := \int_X |s|^2 e^{-k\phi} d\mu.$$

The *Bergman kernel of order k* associated with (L, ϕ, μ) is defined for any $x \in X$ by

$$\mathcal{K}_{k\phi, \mu}(x) := \sup\{|s(x)|^2 : s \in H^0(X, L^k), \|s\|_{k\phi, \mu} \leq 1\}.$$

Throughout the paper, the symbol C denotes a positive constant whose value may vary from line to line.

§2.2. Bounded geometry on non-compact Kähler manifolds

We recall the definition of bounded geometry for Kähler manifolds (cf. [Kob84, CY80, WY20]). Denote by $B_{\mathbb{C}^n}(0, r)$ the open ball centered at the origin $0 \in \mathbb{C}^n$ of radius $r > 0$ with respect to the Euclidean metric $\omega_{\mathbb{C}^n}$.

Definition 2.1. Suppose (X, ω) is an n -dimensional Kähler manifold. We say that (X, ω) has *bounded geometry* of order $k \in \mathbb{N}$ if there exists $0 < r$ such that for any $p \in X$ there is a domain U in $\mathbb{C}^n$ and a holomorphic map $\psi: U \rightarrow X$ of maximal rank everywhere, satisfying

- (1) $B_{\mathbb{C}^n}(0, r) \subset U$ and $\psi(0) = p$,
- (2) on U , there exists a constant $c > 0$ depending only on n, r such that

$$c^{-1}\omega_{\mathbb{C}^n} \leq \psi^*\omega \leq c\omega_{\mathbb{C}^n},$$

- (3) there is a constant $A_k > 0$ depending only on k such that

$$\sup_{x \in B_{\mathbb{C}^n}(0, r)} \left| \frac{\partial^{|\alpha|+|\beta|} g_{i\bar{j}}(x)}{\partial z^\alpha \partial \bar{z}^\beta} \right| \leq A_k, \quad \forall |\alpha| + |\beta| \leq k,$$

where $g_{i\bar{j}}$ is the component of $\psi^*\omega$ on U in terms of natural coordinates $(z^1, \dots, z^n)$, and α, β are the multiple indices with $|\alpha| = \alpha_1 + \dots + \alpha_n$.

The map ψ is called a *quasi-coordinate map* and the pair (U, ψ) is called a *quasi-coordinate chart* of X .

We have the following normal coordinates for a Kähler manifold of bounded geometry.

Lemma 2.2. *Suppose (X, ω) is a Kähler manifold with bounded geometry of order at least 5. Then there exists $c > 0$, $A_k > 0$, for $k = 1, 2, \dots$ such that for any point $p \in X$, there exist $0 < \varepsilon = \varepsilon(p) < r$ and a holomorphic map $\kappa: B_{\mathbb{C}^n}(0, \varepsilon) \rightarrow X$ which is biholomorphic on its image with $\kappa(0) = p$, $\kappa^*\omega = dd^c\varphi$ satisfying conditions (2), (3) in Definition 2.1, and $\varphi_{a\bar{b}}(0) = \delta_{ab}$, $\varphi_{a\bar{b}c}(0) = \varphi_{a\bar{b}\bar{c}}(0) = 0$, and $\varphi_{a\bar{b}c\bar{d}}(0) = \varphi_{abcd}(0) = 0$.*

Proof. For any $p \in X$, we fix a local quasi-coordinate chart (U, ψ) at p as in Definition 2.1. By complex linear transformation we can assume that $\psi^*\omega = dd^c\phi$ on U with $\phi_{i\bar{j}}(0) = \delta_{ij}$. Since ψ is of maximal rank everywhere, there exists $0 < \varepsilon_1 = \varepsilon_1(p) < r$ such that $\psi: B(0, \varepsilon_1) \rightarrow \psi(B(0, \varepsilon_1)) \subset X$ is biholomorphic. We can choose $\varepsilon_2 < \varepsilon_1$ depending on $A_{k\ell}^i, B_{mnp}^i$ such that the holomorphic map $\tilde{\psi}: B(0, \varepsilon_2) \rightarrow B(0, \varepsilon_1)$, $z^i = \tilde{\psi}(w)^i = w^i + A_{k\ell}^i w^k w^\ell + B_{mnp}^i w^m w^n w^p$ is biholomorphic on its image, where the coefficients $A_{k\ell}^i, B_{mnp}^i$ will be chosen hereafter.

Now we have $\kappa = \psi \circ \tilde{\psi}: B(0, \varepsilon_2) \rightarrow X$ is biholomorphic on its image, and the pull-back metric $\kappa^*\omega = dd^c\varphi$ satisfies

$$\begin{aligned} \varphi_{a\bar{b}}(w) &= \phi_{a\bar{b}}(\tilde{\psi}(w)) + 2A_{ak}^i \phi_{i\bar{b}} w^k + \overline{2A_{b\ell}^j \phi_{a\bar{j}} w^\ell} + 3B_{amn}^i w^m w^n \phi_{i\bar{b}} \\ &\quad + \overline{3B_{bpq}^j \phi_{a\bar{j}} \bar{w}^p \bar{w}^q} + 4A_{ak}^i \overline{A_{b\ell}^j \phi_{i\bar{j}} w^k \bar{w}^\ell} + O(|w|^3). \end{aligned}$$

Therefore, we have

$$\begin{aligned} \varphi_{a\bar{b}c}(0) &= \phi_{a\bar{b}c}(0) + 2A_{ac}^b, & \varphi_{a\bar{b}\bar{c}}(0) &= \phi_{a\bar{b}\bar{c}} + \overline{2A_{ac}^b}, \\ \varphi_{a\bar{b}cd}(0) &= \phi_{a\bar{b}cd}(0) + 6B_{acd}^b + C, \end{aligned}$$

where C only depends on A_{ac}^b and $\phi_{a\bar{b}c}(0)$. By choosing $A_{ac}^b = -\frac{1}{2}\phi_{a\bar{b}c}(0)$ and then $B_{acd}^b = -\frac{1}{6}\phi_{a\bar{b}cd}(0) - C$, we obtain the local normal coordinates for any $p \in X$. In particular, A_{bc}^a and B_{bcd}^a are uniformly bounded for any $p \in X$ by the definition of bounded geometry. Since $\varepsilon_2 < \varepsilon_1$ depends only on A_{bc}^a and B_{bcd}^a , we get the map $\kappa: B(0, \varepsilon_2) \rightarrow X$ as desired. $\square$

§3. Uniform convergence of Tsuji's iteration

§3.1. Asymptotic expansion of Bergman kernels

The asymptotic expansion of Bergman kernels (Theorem 3.1 below) in C^k -norms for Kähler manifolds with *bounded geometry* was established in Ma–Marinescu [MM07, Problem 6.1] (cf. also [MM15, Theorem 6]). In this section we provide an alternative proof for the asymptotic expansion of Bergman kernels in $\mathcal{C}^0$ -norm, which will be used later. Our proof is inspired by Donaldson [Don09] and Charles [Cha15] by using the Laplace approximation. We refer to [MM07, Section 6] for general results on the asymptotic expansion of the Bergman kernel on complete Kähler manifolds.

Theorem 3.1. *Let (X, ω_ϕ) be a complete Kähler manifold of dimension n , where $\omega_\phi := dd^c \phi > 0$ is the curvature of a Hermitian metric $h = e^{-\phi}$ on a line bundle L . Suppose that (X, ω_ϕ) has bounded geometry of order $\ell \geq 5$. Then*

$$\left| B_{k\phi, \omega_\phi^{[n]}} - \left(\frac{k}{2\pi} \right)^n \left(1 + \frac{b_1}{2k} \right) \right|_{\mathcal{C}^0} \leq Ck^{n-2},$$

where $\omega^{[n]} := \omega^n/n!$, b_1 is the scalar curvature of ω_ϕ , $C > 0$ depending on the bounded geometry of (X, ω_ϕ) , and $B_{k\phi, \omega_\phi^{[n]}} = \mathcal{K}_{k\phi, \omega_\phi^{[n]}}(x)e^{-k\phi(x)}$ denotes the Bergman function associated to $(k\phi, \omega_\phi^{[n]})$.

We will need the following Laplace approximation.

Lemma 3.2 ([Hör03, Theorem 7.7.5]). *Let $K \subset \mathbb{R}^m$ be a compact set, U an open neighborhood of K , and k a positive integer. If $\lambda > 0$, $u \in C^{2k}(K)$ and $f \in C^{3k+1}(U)$, $f \leq 0$ in U , $f'(x_0) = 0$, $f'(x) \neq 0$ in $K \setminus \{x_0\}$, $\det H_f(x_0) \neq 0$, then*

$$\left| \int u(x)e^{\lambda f(x)} dx - \frac{\left(\frac{2\pi}{\lambda}\right)^{\frac{m}{2}} e^{\lambda f(x_0)}}{(\det H_f(x_0))^{\frac{1}{2}}} \sum_{j < k} \lambda^{-j} L_j u \right| \leq C\lambda^{-k} \sum_{|\alpha| \leq 2k} \sup |D^\alpha u|$$

for a constant C depending on $\|f\|_{C^{3k+1}(U)}$, where H_f denotes the Hessian matrix of f ,

$$L_j u = \sum_{s-r=j} \sum_{2s \geq 3r} 2^{-s} \langle H_f(x_0)^{-1} D, D \rangle^s (g_{x_0}^r u)(x_0) / r! s!,$$

with

$$g_{x_0}(x) = f(x) - f(x_0) - \frac{1}{2} \langle H_f(x_0)(x - x_0), x - x_0 \rangle.$$

In particular, $L_0 u = u(x_0)$ and

$$\begin{aligned} L_1 u(x_0) = \frac{1}{2} \Big[& u \left\{ -f_{ik\ell} f_{jrs} \left(\frac{1}{4} f^{ij} f^{k\ell} f^{rs} + \frac{1}{6} f^{ij} f^{ks} f^{r\ell} \right) + \frac{1}{4} f^{ij} f^{kl} f_{ijk\ell} \right\} \\ & + f^{sq} f^{rp} f_{srq} u_p - \text{Tr}(H_u H_f^{-1}) \Big]_{x=x_0}. \end{aligned}$$

Proof of Theorem 3.1. We will establish lower and upper bounds for the Bergman kernel by adapting the method of peak sections for compact Kähler manifolds, as developed in [Tia90] (see also [Rua98, Lu00, SW10]), and by using the Laplace method as in [Don09, Cha15].

For any $x_0 \in X$, we take $(U, z = (z^1, \dots, z^n))$ a normal coordinate chart as in Lemma 2.2 centered at x_0 . Then L has a local holomorphic frame g defined on U on which the function $\psi = \phi - \log |g|^2 = -\log |g|_\phi^2$ has minimum at x_0 and

$$(3.1) \quad \psi(z) = \sum_{j=1}^n |z^j|^2 + \sum_{1 \leq i, j, m, \ell \leq n} \frac{1}{2!2!} \phi_{i\bar{j}m\bar{\ell}} z^i \bar{z}^j z^m \bar{z}^\ell + O(|z|^5).$$

Here, for example, we can choose $a_i, b_{ji}, c_{\ell mi}, d_{pqrs}$ such that $\log |g|^2(z) = \phi(0) + a_i z^i + b_{ji} z^j \bar{z}^i + c_{\ell mi} z^m \bar{z}^\ell z^i + d_{pqrs} z^p \bar{z}^q z^r \bar{z}^s$ and use Taylor's expansion to have (3.1).

Let $\eta: [0, \infty) \rightarrow [0, 1]$ be a cutoff function satisfying $\eta(t) = 1$ for $t \leq 1/2$, $\eta(t) = 0$ for $t \geq 1$, $|\eta'(t)| \leq 4$, and $|\eta''(t)| \leq 8$. Define an L^k -valued $(1, 0)$ form on X :

$$\alpha = \bar{\partial} \left[\eta \left(\frac{k|z|^2}{(\log k)^2} \right) \right] g^k,$$

which vanishes outside $A_k := \{z \mid (\log k)^2/(2k) \leq |z|^2 \leq (\log k)^2/k\} \subset U$. Let dV denote the Lebesgue measure on the local chart (U, z) . We define a weight function χ , supported in U , by setting

$$\chi = (n+2)\eta \left(\frac{k|z|^2}{2(\log k)^2} \right) \log \left(\frac{k|z|^2}{2(\log k)^2} \right)$$

for $z \in U$, and $\chi = 0$ outside U . We observe that $\chi \in [-2(n+2), 0]$ on A_k and can be approximated by a decreasing sequence of smooth functions $\{\chi_m\}_{m \in \mathbb{N}}$; cf. [Tia90, pp. 104–105]. It follows from the bounded geometry property that for $k > 0$ sufficiently large,

$$\text{Ric}(\omega_\phi) + k\omega_\phi + dd^c \chi_m \geq \frac{k}{C} \omega_\phi,$$

for a uniform constant $C > 0$. On the other hand, there exists a constant $C > 0$ depending on n and the bounded geometry of (X, ω_ϕ) ,

$$(3.2) \quad e^{-\chi} |\alpha|_{\omega_\phi}^2 \leq C \left| \eta' \left(\frac{k|z|^2}{(\log k)^2} \right) \right|^2 \frac{k^2}{(\log k)^4} \phi^{i\bar{j}} z^i \bar{z}^j |g|^{2k} \leq C \frac{k|g|^{2k}}{(\log k)^2},$$

on A_k , otherwise $\alpha = 0$. Since on A_k , we have $e^{-k\psi} = (1 - \sum_{j=1}^n |z^j|^2)^k + O(|z|^3)$ as follows from the choice of coordinates (3.1), hence locally

$$e^{-k\psi} \omega_\phi^n \leq \left(1 - \frac{1}{2} |z|^2 \right)^k n! dV,$$

where $dV = (\frac{i}{2})^n dz^1 \wedge d\bar{z}^1 \wedge \cdots \wedge dz^n \wedge d\bar{z}^n$. It follows from (3.2) that

$$\begin{aligned}
 \int_{A_k} |\alpha|_{\omega_\phi}^2 e^{-\chi - k\phi} \omega_\phi^{[n]} &\leq C \frac{k}{(\log k)^2} \int_{A_k} |g|^{2k} e^{-k\phi} \omega_\phi^{[n]} \\
 &= C \frac{k}{(\log k)^2} \int_{A_k} e^{-k\psi} \omega_\phi^{[n]} \\
 &\leq C \frac{k}{(\log k)^2} \left(\frac{(\log k)^2}{k} \right)^n \left(1 - \frac{(\log k)^2}{k} \right)^k \\
 (3.3) \qquad \qquad \qquad &\leq C (\log k)^{2n-2} k^{1-n-(\log k)/2} < \infty,
 \end{aligned}$$

because $(1 - (\log k)^2/k)^k = e^{k \log(1 - (\log k)^2/k)} \leq e^{-(\log k)^2/2} = k^{-(\log k)/2}$, where $C > 0$ only depends on n and the bounded geometry of (X, ω_ϕ) .

We apply Hörmander's L^2 estimate (see [Hör73, Chapter 4] or [Tia90, Proposition 1.1]) with the weight χ : there exists a smooth section u of L^k on X such that $\bar{\partial}u = \alpha$ and

$$\begin{aligned}
 \int_X |u|^2 e^{-k\phi - \chi} \omega_\phi^{[n]} &\leq \frac{C}{k} \int_X |\alpha|_{\omega_\phi}^2 e^{-k\phi - \chi} \omega_\phi^{[n]} \\
 &= \frac{C}{k} \int_{A_k} |\alpha|_{\omega_\phi}^2 e^{-k\phi - \chi} \omega_\phi^{[n]} \\
 &\leq C (\log k)^{2n-2} k^{-\log k/2-n},
 \end{aligned}$$

where the last inequality follows from (3.3). Because of the non-integrability of $e^{-\chi}$ at x_0 , one must have $u(x_0) = 0$.

Since $\chi \leq 0$ we have

$$\begin{aligned}
 \int_X |u|^2 e^{-k\phi} \omega_\phi^{[n]} &\leq \int_X |u|^2 e^{-k\phi - \chi} \omega_\phi^{[n]} \\
 &\leq C (\log k)^{2n-2} k^{-\log k/2-n}.
 \end{aligned}$$

It follows that $f := \eta g^k - u$ is a holomorphic section of L^k , satisfying $|f(x_0)|^2 = e^{k\phi(0)}$ since $u(x_0) = 0$ and

$$\|f\|_{k\phi}^2 = \int_X |f|^2 e^{-k\phi} \omega_\phi^{[n]} = \int_{U_k} |g|^{2k} e^{-k\phi} \omega_\phi^{[n]} + O(k^{-n-2}),$$

where $U_k = \{z : |z|^2 \leq (\log k)^2/k\}$. On U_k we have

$$\omega_\phi^{[n]} = (1 + \phi_{i\bar{j}\ell\bar{\ell}} z^i \bar{z}^j + o(|z|^3)) dV.$$

Applying the Laplace approximation (Lemma 3.2) with $u = 1 + \phi_{i\bar{j}\ell\bar{\ell}} z^i \bar{z}^j + o(|z|^3)$ and $f = -\psi$ where $\psi = -\log |g|_\phi^2$, we have

$$\begin{aligned} \int_{U_k} |g|^{2k} e^{-k\phi} \omega_\phi^{[n]} &= \int_{U_k} e^{-k\psi} \omega_\phi^{[n]} \\ &= \int_{U_k} e^{-k\psi} (1 + \phi_{i\bar{j}\ell\bar{\ell}} z^i \bar{z}^j + o(|z|^3)) dV \\ &= \left(\frac{2\pi}{k}\right)^n \left(1 + \frac{1}{2k} \phi_{i\bar{i}j\bar{j}} + O(k^{-2})\right) \\ &= \left(\frac{2\pi}{k}\right)^n \left(1 - \frac{1}{2k} S_{\omega_\phi}(0) + O(k^{-2})\right), \end{aligned}$$

where $O(k^{-2})$ is dominated by C/k^{-2} on U with a uniform constant C depending only on the bound of $|D^\alpha \phi|$ on U with $|\alpha| \leq 7$. Therefore,

$$\mathcal{K}_{k\phi, \omega_\phi^{[n]}}(x_0) \geq \frac{|f(x_0)|^2}{\|f\|_{k\phi}^2} \geq \left(\frac{k}{2\pi}\right)^n e^{k\phi(0)} \left(1 + \frac{1}{2k} S_{\omega_\phi}(0) + O(k^{-2})\right),$$

hence we get a lower bound for $\mathcal{K}_{k\phi, \omega_\phi^{[n]}} e^{-k\phi}$.

We look for an upper bound for $\mathcal{K}_{k\phi, \omega_\phi^{[n]}} e^{-k\phi}$ following the strategy in [Cha15]. It suffices to work on the coordinate chart (U, z) as above. Let f be a local holomorphic section of L^k on U . We claim that

$$|f(0)/g^k(0)|^2 \leq \frac{\int_{\Delta_R} |f/g^k|^2 e^{-k\psi_0} \rho dV}{\int_{\Delta_R} e^{-k\psi_0} \rho dV},$$

where $\Delta_R \subset U$ is a polydisc with radius R and

$$\psi_0 = |z^j|^2 + \phi_{i\bar{j}m\bar{\ell}} z^i \bar{z}^j z^m \bar{z}^\ell, \quad \rho = 1 + \phi_{i\bar{j}\ell\bar{\ell}} z^i \bar{z}^j.$$

Indeed, using polar coordinates, one knows that $\int_{\Delta_R} z^m \bar{z}^\ell dV = 0$ if $m \neq \ell$. Hence, for any multi-index $\alpha \in \mathbb{N}^n$ we have that

$$\int_{\Delta_R} e^{-k\psi_0} z^\alpha \rho dV = 0.$$

It follows that for any holomorphic function $h: U \rightarrow \mathbb{C}$,

$$\int_{\Delta_R} |h|^2 e^{-k\psi_0} \rho dV = \int_{\Delta_R} (|h(0)|^2 + |h - h(0)|^2) e^{-k\psi_0} \rho dV.$$

We thus apply this equality to the holomorphic function f/g^k (since g has no zeros in the polydisc Δ_R of radius R for $R > 0$ sufficiently small) to obtain our claim.

Choosing $R = k^{-2/3}$, we have

$$-k\psi_0 \leq -k\psi + Ck^{-7/3} \quad \text{and} \quad \rho dV \leq (1 + Ck^{-2}) \omega_\phi^{[n]},$$

hence

$$\begin{aligned} \int_{\Delta_R} |f/g^k|^2 e^{-k\psi_0} \rho \, dV &\leq (1 + Ck^{-2}) \int_{\Delta_R} |f/g^k|^2 e^{-k\psi} \omega_\phi^{[n]} \\ &= (1 + Ck^{-2}) \int_{\Delta_R} |f|^2 e^{-k\phi} \omega_\phi^{[n]}. \end{aligned}$$

Using the Laplace approximation again, we have

$$\int_{\Delta_R} e^{-k\psi_0} \rho \, dV = \left(\frac{2\pi}{k}\right)^n \left(1 - \frac{1}{2k} S_{\omega_\phi}(0) + O(k^{-2})\right).$$

Therefore, we get

$$\begin{aligned} |f(0)/g^k(0)|^2 &\leq \frac{\int_{\Delta_R} |f/g^k|^2 e^{-k\psi_0} \rho \, dV}{\int_{\Delta_R} e^{-k\psi_0} \rho \, dV} \\ &= \left(\frac{k}{2\pi}\right)^n \left(1 + \frac{1}{2k} S_{\omega_\phi}(0) + O(k^{-2})\right) \int_{\Delta_R} |f|^2 e^{-k\phi} \omega_\phi^{[n]}. \end{aligned}$$

Since $|g^k(0)|^2 = e^{k\phi(0)}$, we infer that

$$|f(0)|^2 \leq e^{k\phi(0)} \left(\frac{k}{2\pi}\right)^n \left(1 + \frac{1}{2k} S_{\omega_\phi}(0) + O(k^{-2})\right) \int_{\Delta_R} |f|^2 e^{-k\phi} \omega_\phi^{[n]},$$

hence we get an upper bound for $\mathcal{K}_{k\phi, \omega_\phi^{[n]}} e^{-k\phi}$. □

§3.2. Convergence of Tsuji's iteration

Recall that $L \rightarrow X$ is a positively curved line bundle on X equipped with a smooth Hermitian metric ϕ_L . Let Ω be a smooth volume form on X . For any Hermitian metric ϕ on L , we denote $\mu_\phi = \frac{1}{n!} e^{\phi - \phi_L} \Omega$, which is a well-defined positive measure on X . We prove the following lemma, which is a variant version of [Ber09b, Lemma 3].

Lemma 3.3. *Let ϕ_∞ be the solution of*

$$(dd^c \phi_\infty)^n = e^{\phi_\infty - \phi_L} \Omega, \quad \omega_{\phi_\infty} = dd^c \phi_\infty > 0$$

such that $(X, \omega_{\phi_\infty})$ has bounded geometry of order $\ell \geq 5$. Suppose that C_1, C_2 are two real numbers satisfying $C_1 \leq \phi - \phi_\infty \leq C_2$. Then we have

$$\beta_k(\phi) - \phi_\infty \geq \frac{k-1}{k} C_1 - \varepsilon_k$$

and

$$\beta_k(\phi) - \phi_\infty \leq \frac{k-1}{k} C_2 + \varepsilon_k,$$

where $\varepsilon_k = c/k^2$ with $c > 0$ only depending on the bounded geometry of ϕ_∞ . Moreover, if ω_{ϕ_∞} is a Kähler–Einstein metric with negative curvature, and d_k is defined in (1.4), then we can obtain $\varepsilon_k = c/k^3$.

Proof. Since $(dd^c\phi_\infty)^n = e^{\phi_\infty - \phi_L}\Omega = n!\mu_{\phi_\infty}$, using the first-order asymptotic of Bergman kernels for the weight ϕ_∞ (cf. Theorem 3.1), we have

$$\begin{aligned}\mathcal{K}_{k\phi_\infty, \mu_{\phi_\infty}} e^{-k\phi_\infty} &= \mathcal{K}_{k\phi_\infty, \omega_{\phi_\infty}^{[n]}} e^{-k\phi_\infty} \\ &= \left(\frac{k}{2\pi}\right)^n \left(1 + O\left(\frac{1}{k}\right)\right).\end{aligned}$$

By definition, $d_k = (\frac{k}{2\pi})^n$ we have

$$(3.4) \quad \beta_k(\phi_\infty) = \frac{1}{k} \log(\mathcal{K}_{k\phi_\infty}/d_k) = \phi_\infty + \varepsilon_k,$$

where $\varepsilon_k = O(\frac{1}{k^2})$.

Similarly, if ω_∞ is a Kähler–Einstein metric with $\text{Ric}(\omega_{\phi_\infty}) = -\omega_{\phi_\infty}$, using the second-order asymptotic behavior of Bergman kernels for the weight ϕ_∞ (cf. Theorem 3.1) we get

$$\begin{aligned}\mathcal{K}_{k\phi_\infty, \mu_{\phi_\infty}} e^{-k\phi_\infty} &= \mathcal{K}_{k\phi_\infty, \omega_{\phi_\infty}^{[n]}} e^{-k\phi_\infty} \\ &= \left(\frac{k}{2\pi}\right)^n \left(1 - \frac{n}{2k} + O\left(\frac{1}{k^2}\right)\right).\end{aligned}$$

Therefore, for $d_k = (\frac{k}{2\pi})^n \max\{\frac{1}{2}, (1 - \frac{n}{2k})\}$ and $k > n$, we have

$$(3.5) \quad \beta_k(\phi_\infty) = \frac{1}{k} \log(\mathcal{K}_{k\phi_\infty}/d_k) = \phi_\infty + \varepsilon_k,$$

where $\varepsilon_k = O(\frac{1}{k^3})$.

Next, it follows from the fact that $\phi - \phi_\infty \geq C_1$ for some constant C_1 , the extremal characterization of Bergman kernel implies that

$$(3.6) \quad \mathcal{K}_{k\phi} \geq e^{(k-1)C_1} \mathcal{K}_{k\phi_\infty}.$$

Indeed, if we pick a section $0 \neq \sigma \in H^0(X, L^k)$ with $\int_X |\sigma|^2 e^{-k\phi_\infty} \mu_{\phi_\infty} < +\infty$ then for any $x \in X$,

$$\frac{|\sigma(x)|^2}{\int_X |\sigma|^2 e^{-k\phi} d\mu_\phi} \geq e^{(k-1)C_1} \frac{|\sigma(x)|^2}{\int_X |\sigma|^2 e^{-k\phi_\infty} d\mu_{\phi_\infty}}.$$

Taking the supremum, we obtain the inequality (3.6). Therefore, combining with (3.4) (resp. (3.5)) we have

$$\beta_k(\phi) - \phi_\infty \geq \frac{k-1}{k} C_1 - \varepsilon_k$$

for some positive $\varepsilon_k = O(\frac{1}{k^2})$ (resp. $\varepsilon_k = O(\frac{1}{k^3})$). Similarly, we have

$$\beta_k(\phi) - \phi_\infty \leq \frac{k-1}{k} C_2 + \varepsilon_k$$

as required. $\square$

Proof of Theorems 1.1 and 1.3. We define by recurrence $\phi_{k+1} := \beta_k(\phi_k)$. For any $k \geq 1$, denote by C_k the optimal constant such that

$$\phi_k - \phi_\infty \geq C_k.$$

Lemma 3.3 thus implies that

$$C_{k+1} \geq \frac{k-1}{k} C_k - \varepsilon_k.$$

Since ε_k is of order $1/k^2$, it follows that kC_{k+1} is uniformly bounded from below by a quantity of order $O(\log k)$. Similarly, Lemma 3.3 yields a uniform upper bound for $k\tilde{C}_{k+1}$, where $\tilde{C}_k$ is the optimal constant such that

$$\phi_k - \phi_\infty \leq \tilde{C}_k.$$

Since $C_k \leq \tilde{C}_k$ by definition, we have $C_k = O(\frac{\log k}{k})$ and $\tilde{C}_k = O(\frac{\log k}{k})$. Hence we obtain the desired convergence rate $\frac{\log k}{k}$.

By the same argument, we obtain the uniform convergence with rate $\frac{1}{k}$ when ω_{ϕ_∞} is a Kähler–Einstein metric. $\square$

§4. Variation of Kähler–Einstein metrics

§4.1. Positivity of relative Bergman kernels

Setup 4.1. Let $p: X \rightarrow Y$ be a holomorphic surjective map from a Kähler manifold X of dimension $n+m$ to a complex manifold Y of dimension m . Let $Y^\circ \subset Y$ denote the set of points that are not critical values of p in Y and set $X^\circ = p^{-1}(Y^\circ)$.

If $p: X \rightarrow Y$ is in Setup 4.1, then the relative canonical bundle associated with p is defined by $K_{X/Y} := K_X - p^*(K_Y)$. It follows from [Ber09a, BP08] that we have an isomorphism between K_{X_y} and $(K_X - p^*K_Y)|_{X_y} = K_{X/Y}|_{X_y}$ for $y \in Y^\circ$. Let $t = (t_1, \dots, t_m)$ be local coordinates near a point in Y . Then these local coordinates define a local $(m, 0)$ -form $dt = dt_1 \wedge \dots \wedge dt_m$, which trivializes K_Y . This trivialization induces a local section $\tilde{u}$ of K_X over X_y by $\tilde{u} = u \wedge p^*(dt) = u \wedge d\tilde{t}$, where $\tilde{t}_j = t_j \circ p$, $j = 1, \dots, m$, are functions on X . Conversely, any local section $\tilde{u}$ of K_X can be written as $\tilde{u} = u \wedge d\tilde{t}$ locally. Thus

the restriction of u to the fiber X_y is well defined as a local section of K_{X_y} , which we call the trace of $\tilde{u}$ and denote by $\tau_y(\tilde{u})$. Then the isomorphism is given by

$$\begin{aligned} K_{X_y} + p^*(K_Y)|_{X_y} &\rightarrow K_X|_{X_y}, \\ u \otimes p^* dt &\mapsto u \wedge \tilde{dt} \end{aligned}$$

with inverse map $\tilde{u} \mapsto \tau_y(u) \otimes p^*(dt)$. Hence the induced isomorphism $K_{X_y} \rightarrow K_{X/Y}|_{X_y}$ is given by $u \mapsto u \wedge p^*(dt)/dt$.

Let ω be a relative Kähler metric on X such that $\omega_y := \omega|_{X_y}$ is a Kähler metric on X_y for $y \in Y^\circ$. Assume that L is a line bundle on X and $h_L = e^{-\phi_L}$ is a singular Hermitian metric on L with positive curvature current. For each $y \in Y^\circ$, we define $E_y := H^0(X_y, (K_{X_y} + L|_{X_y}) \otimes \mathcal{I}(\phi_L|_{X_y}))$. Here, $\mathcal{I}(\phi)$ denotes the *multiplier ideal sheaf* of holomorphic functions on X that are square integrable against $e^{-\phi}$. Then the (singular) Hermitian metric ϕ_L on L induces a Hilbert norm $\|\cdot\|_y$ on E_y by

$$\|u\|_y^2 := \int_{X_y} c_n u \wedge \bar{u} e^{-\phi_L} = \int_{X_y} |u|_\omega^2 e^{-\phi_L} dV_{X_y, \omega}.$$

Let $x \in X^\circ$, and $y = p(x)$. Let $z_1, \dots, z_{n+m}$ be local coordinates for a neighborhood $V \subset X^\circ$ of x and $t_1, \dots, t_m$ local coordinates near y . For $u \in H^0(X_y, K_{X_y} + L|_{X_y})$, we define u' by

$$u' dz = u \wedge p^* dt.$$

The fiberwise Bergman kernel $\mathcal{K}(x)$ of $(K_{X/Y} + L)|_{X^\circ}$ with respect to this norm is defined by

$$\mathcal{K}(x) = \sup_{u \in E_y, \|u\|_y^2 \leq 1} |u'(x)|^2.$$

We remark that this concept of the Bergman kernel coincides with the classical one. Thus, $\phi_{X/Y}(x) := \log \mathcal{K}(x)$ provides a Hermitian metric $h = e^{-\phi_{X/Y}}$ on $K_{X/Y}$.

We establish a pseudo-effectivity criterion for the relative adjoint line bundle $K_{X/Y} + L$ by proving the log-plurisubharmonicity of the fiberwise Bergman kernel. The only assumptions needed to obtain a non-trivial positively curved metric are the positivity of L together with the existence of an L^2 section of $K_{X/Y} + L$. The following theorem is a non-compact extension of [BP08, Theorem 0.1] (see also [PT18, Theorem 4.1.1], [ZZ18, Theorem 1.3], [Cao17, Theorem 3.5], and [DWZZ24, Theorem 1.4]).

Theorem 4.2. *Let X be a weakly pseudoconvex Kähler manifold of dimension $n + m$ and Y a complex manifold of dimension m . Let $p: X \rightarrow Y$ be a surjective holomorphic map. Let L be a line bundle on X equipped with a singular Hermitian metric $h_L = e^{-\phi_L}$ satisfying*

- $dd^c\phi_L \geq 0$ in the sense of currents,
- $H^0(X_{y_0}, (K_{X_{y_0}} + L|_{X_{y_0}}) \otimes \mathcal{I}(\phi_L|_{X_{y_0}})) \neq 0$ for some $y_0 \in Y^\circ$.

Denote by $\mathcal{K}_{\phi_L}$ the relative Bergman kernel of $(K_{X/Y} + L)|_{X^\circ}$. Then $\mathcal{K}_{\phi_L}$ has positive curvature current and extends across $X \setminus X^\circ$ to a metric with positive curvature current on all of X .

A consequence of Theorem 4.2 is the positivity of the relative Bergman kernel.

Corollary 4.3. *Let X be weakly pseudoconvex Kähler manifold and Y a complex manifold. Let $p: X \rightarrow Y$ be a surjective holomorphic map. Assume that $H^0(X_{y_0}, K_{X_{y_0}}) \neq 0$ for some $y_0 \in Y^\circ$. Then the relative pluricanonical line bundle $K_{X/Y}^k$ admits a metric with positive curvature current. Moreover, this metric coincides with the fiberwise k -Bergman kernel metric on the fibers of p .*

Proof. First, take $L = \mathcal{O}_X$ with the constant metric. Then Theorem 4.2 shows that $\phi_{X/Y}^{(1)} := \log \mathcal{K}$ defines a Hermitian metric on $K_{X/Y}$ with positive curvature current. Applying Theorem 4.2 inductively with $L = K_{X/Y}^k$ endowed with the Hermitian metric $\phi_{X/Y}^{(k)}$ for $k = 1, 2, \dots$, we obtain a new metric $\phi_{X/Y}^{(k+1)}$ on $K_{X/Y}^{k+1}$ with positive curvature current, given by the logarithm of the corresponding Bergman kernel. $\square$

The proof of Theorem 4.2 follows the same strategy as in the compact case, cf. [BP08, PT18, ZZ18]. We provide a brief sketch for the reader's convenience. We will use the following version of the Ohsawa–Takegoshi extension theorem, which can be deduced from [ZZ18, Theorem 1.1] (cf. also [Cao17, BL16, GZ15b, XZ24]).

Lemma 4.4. *Let (X, ω) be a weakly pseudoconvex Kähler manifold. Let $p: X \rightarrow \Delta_r = \{t: |t| < r\} \subset \mathbb{C}$ be a surjective holomorphic map. Suppose that $X_0 = p^{-1}(0)$ is a submanifold of codimension 1. Let L be a holomorphic line bundle on X equipped with a (possibly singular) Hermitian metric ϕ_L satisfying $dd^c\phi_L \geq 0$. Let $u \in H^0(X_0, K_{X_0} + L|_{X_0})$ with $\int_{X_0} |u|_\omega^2 e^{-\phi_L} dV_{X_0} < \infty$. Then there exists $U_r \in H^0(p^{-1}(\Delta_r), K_{X/Y} + L)$ such that $U_r|_{X_0} = u \wedge p^*dt$ and*

$$\frac{1}{\pi r^2} \int_{p^{-1}(\Delta_r)} |U_r|_\omega^2 e^{-\phi_L} dV_{X, \omega} \leq \int_{X_0} |u|_\omega^2 e^{-\phi_L} dV_{X_0, \omega}.$$

Proof of Theorem 4.2. We follow the same argument as in the compact case (see [BP08, PT18, ZZ18]). Recall that for any $y \in Y^\circ$, setting $\phi_y = \phi_L|_{X_y}$, we define the L^2 -norm on $u_y \in H^0(X_y, K_{X_y} + L|_{X_y})$ by

$$\|u_y\|^2 := \int_{X_y} |u_y|_\omega^2 e^{-\phi_y} dV_{X_y, \omega}.$$

Let $(t, z) = (t_1, \dots, t_m, z_1, \dots, z_n)$ be local coordinates near a point $x \in X$. Set $y = p(x)$. The Bergman kernel of $K_{X_y} + L|_{X_y}$ on X_y ($y = y(t)$) is defined by

$$(4.1) \quad \mathcal{K}_t(z) = \sup\{|u'_y(z)|^2 : u_y \in H^0(X_y, K_{X_y} + L|_{X_y}), \|u_y\| \leq 1\}$$

where $u_y = u'_y(z) dz \otimes e$ with e a local frame of L near x .

Remark that $p: X^\circ \rightarrow Y^\circ$ is a holomorphic submersion. To show that $\log \mathcal{K}_t(z)$ is psh with respect to (t, z) , one needs to prove the subharmonicity of $\log \mathcal{K}_t(z)$ along any complex line. Without loss of generality, we can assume that $Y^\circ = \Delta_R \subset \mathbb{C}$ is a disc ($m = 1$).

The first property is the upper semicontinuity of $\log \mathcal{K}_t(z)$ with respect to (t, z) . Indeed, thanks to Demailly's global regularization [Dem82, Section 9], we can find an increasing sequence of smooth metrics $(h_\varepsilon = e^{-\phi_\varepsilon})_{\varepsilon > 0}$ on L , converging to $h_L = e^{-\phi_L}$ in L^1 and satisfying $dd^c \phi_\varepsilon \geq -\lambda_\varepsilon \omega$, $dd^c \phi_\varepsilon \geq -\lambda_\varepsilon \omega$, where (λ_ε) is an increasing family of continuous functions on X with $\lambda_\varepsilon \searrow 0$ almost everywhere. The corresponding Bergman kernels $\mathcal{K}_{t,\varepsilon}$ are continuous. Arguing as in [PT18, Theorem 1.1], we obtain that $\log \mathcal{K}_{t,\varepsilon}(z)$ decreases to $\log \mathcal{K}_t(z)$ for $(t, z) \in U$ as $\varepsilon \searrow 0$. This proves the desired upper semicontinuity.

We next show that $\log \mathcal{K}$ satisfies the sub-mean inequality on every holomorphic disc in Y° , namely, for any $t_0 \in Y^\circ$,

$$\log \mathcal{K}_{t_0}(z) \leq \frac{1}{\pi r^2} \int_{\Delta_r} \log \mathcal{K}_t(z) idt \wedge d\bar{t}$$

for some $r \ll 1$ and $\Delta_r = \Delta(t_0, r) \subset Y^\circ$. Fix $u_0 \in H^0(X_{t_0}, K_{X_{t_0}} + L|_{X_{t_0}})$ such that $\|u_0\| = 1$ and $\mathcal{K}_0(z) = |u'_0(z)|^2$. By the Ohsawa–Takegoshi-type theorem (Lemma 4.4), u_0 can be extended to a holomorphic section U on the preimage of Δ_r such that

$$\frac{1}{\pi r^2} \int_{\Delta_r} \int_{X_t} |U'(t, z)|^2 e^{-\phi_L} dV_{X_t, \omega} idt \wedge d\bar{t} \leq \|u_0\|^2 = 1,$$

where we have used the Fubini theorem and $U = U' dz \wedge p^*(dt) \otimes e$. Since $U(t, z)$ is holomorphic in t , we can write

$$\log \mathcal{K}_{t_0}(z) = \log |U'(0, z)|^2 \leq \frac{1}{\pi r^2} \int_{\Delta_r} \log |U'(t, z)|^2 idt \wedge d\bar{t}.$$

Using the extremal property of the Bergman kernel, the term on the right-hand side is dominated by

$$\frac{1}{\pi r^2} \int_{\Delta_r} \left(\log \mathcal{K}_t(z) + \log \int_{X_t} |U'(t, z)|^2 e^{-\phi_L} dV_{X_t, \omega} \right) idt \wedge d\bar{t}.$$

By Jensen's inequality, we obtain

$$\begin{aligned} & \frac{1}{\pi r^2} \int_{\Delta_r} \log \int_{X_t} |U'(t, z)|^2 e^{-\phi_L} dV_{X_t, \omega} idt \wedge d\bar{t} \\ & \leq \log \left(\frac{1}{\pi r^2} \int_{\Delta_r} \int_{X_t} |U'(t, z)|^2 e^{-\phi_L} dV_{X_t, \omega} idt \wedge d\bar{t} \right) \leq 0. \end{aligned}$$

This tells us that $\log \mathcal{K}_t(z)$ is psh in the horizontal direction. On the other hand, by the convexity of $|u'_t(z)|$ and the definition of $\log \mathcal{K}$, we deduce that $\log \mathcal{K}(z)$ is also psh in the fiberwise direction. Therefore, $\log \mathcal{K}$ is psh on $p^{-1}(Y^\circ)$ and, in particular, the curvature of the induced singular Hermitian metric is positive on $p^{-1}(Y^\circ)$ in the sense of currents.

It remains to show that the curvature current of the induced Hermitian metric above can be extended over the whole of X . It suffices to show that its local potential is bounded from above near the singular fibers.

Fixing an open subset $V \subset X$, we define the function $\mathcal{K}$ on $V \cap X^\circ$ as (4.1). The problem is equivalent to proving that $\mathcal{K}$ is uniformly bounded from above on $V \cap X^\circ$. For any $x \in V \cap X^\circ$, there exists a section $\sigma \in H^0(X_{p(x)}, K_{X_{p(x)}} + L|_{X_{p(x)}})$ such that

$$\mathcal{K}(x) = |\sigma'(z)|^2, \quad \int_{X_{p(x)}} |\sigma|^2 e^{-\phi_{p(x)}} dV_{X_{p(x)}, \omega} = 1,$$

where (t, z) are local coordinates near x and $\sigma = \sigma' dz \otimes e$. Using the Ohsawa–Takegoshi extension theorem, we can find an extension $\tilde{\sigma}$ of σ on $p^{-1}(p(V))$ such that

$$\int_{p^{-1}(p(V))} |\tilde{\sigma}|^2 e^{-\phi_L} dV_{X, \omega} \leq C_V,$$

with $C_V > 0$ only depending on V . It follows from the sub-mean value inequality that $|\sigma'(z)|^2$ is dominated by a constant only depending on C_V . This completes the proof. $\square$

§4.2. Plurisubharmonic variation of Kähler–Einstein metrics

This section is devoted to studying the plurisubharmonic variation of Kähler–Einstein metrics on non-compact complex manifolds. The idea goes back to Tsuji [Tsu10] and Schumacher [Sch12], who studied holomorphic families of canonically polarized, compact Kähler manifolds using two different methods. Schumacher's method is based on a maximum principle, which moreover yields strict plurisubharmonicity when the family is not infinitesimally trivial. Tsuji's approach shows that Kähler–Einstein metrics can be obtained via an iteration scheme involving suitably normalized pluricanonical sections, as explained in this paper. In [Tsu13], Tsuji extended his method to families of Kähler–Einstein metrics on bounded

pseudoconvex domains, using Berndtsson’s results [Ber09a, Ber06]. For alternative proofs of this result on strongly pseudoconvex domains based on Schumacher’s method, see [Cho15b, Cho15a]. In this work, we follow Tsuji’s method to obtain the following.

Theorem 4.5. *Let $p: X \rightarrow Y$ be in Setup 4.1. Assume further that X is weakly pseudoconvex, and*

- (i) *for every $y \in Y^\circ$, X_y admits a Kähler–Einstein metric $\omega_{\text{KE},y}$ with negative Ricci curvature such that $(X_y, \omega_{\text{KE},y})$ has bounded geometry of order $\ell \geq 5$, locally uniformly in y , i.e., for any neighborhood $U \Subset Y^\circ$ the constants r , c , A_k in Definition 2.1 depend only on U ,*
- (ii) *$K_{X/Y}$ admits a singular Hermitian metric $h_0 = e^{-\phi_0}$ with positive curvature current and for every $y_0 \in Y^\circ$, there exists a neighborhood $U \Subset Y^\circ$ of y_0 such that $\|\phi_{0,y} - \phi_{\text{KE},y}\|_{L^\infty(X_y)} \leq C(U)$ for all $y \in U$.*

Then the Hermitian metric on $(K_{X/Y})|_{X^\circ}$ induced by the fiberwise Kähler–Einstein metrics is positively curved. Moreover, if we have the additional condition that

- (iii) *the Kähler–Einstein potentials $(\phi_{\text{KE},y})_{y \in Y^\circ}$ are bounded from above near the singular fibers,*

then this metric extends canonically across $X \setminus X^\circ$ to a positively curved metric.

Condition (ii) holds whenever one can compare the Bergman kernel of $H^0(X_y, K_{X_y})$ with the Kähler–Einstein measure $\omega_{\text{KE},y}^n$. Indeed, in this case, the fiberwise Bergman kernel for $H^0(X_y, K_{X_y})$ gives us a Hermitian metric ϕ_0 on $K_{X/Y}$ which satisfies (ii). The following corollary illustrates this using the comparison between the Bergman kernel and the Kähler–Einstein measure in Greene–Wu [GW79].

Corollary 4.6. *Let $p: X \rightarrow Y$ be in Setup 4.1. Assume further that X is weakly pseudoconvex and that*

- (i) *for every $y_0 \in Y^\circ$, there exist a neighborhood of y_0 , $U \Subset Y^\circ$ and $0 < A_U < B_U$ such that for any $y \in U$, X_y admits a Kähler metric ω_y whose holomorphic sectional curvature belongs to $(-B_U, -A_U)$,*
- (ii) *X_y is simply connected.*

Then the curvature current of the Hermitian metric on $(K_{X/Y})|_{X^\circ}$ induced by the Kähler–Einstein metrics on the fibers is positive.

Proof. It follows from [WY20, Corollary 7] that for any $y \in U$, the fiber X_y admits a unique Kähler–Einstein metric with bounded geometry, uniformly equivalent to ω_y , with equivalence constants depending only on A_U , B_U , and $\dim X_y$.

Moreover, by [GW79, p. 144, Theorem H (3)] the fiberwise Bergman kernel $\mathcal{K}_y$ satisfies $A_1\omega_y^n \leq \mathcal{K}_y \leq A_2\omega_y^n$ for some positive constants A_1, A_2 depending only on A_U, B_U , and $\dim X_y$. thus the fiberwise Bergman kernel defines a Hermitian metric ϕ_0 on $K_{X/Y}$ satisfying (ii) in Theorem 4.5. $\square$

Proof of Theorem 4.5. Fix $y \in Y^\circ$. We first note that $H^0(X_y, K_{X_y}^{k_0+1} \otimes \mathcal{I}(k_0\phi|_{X_y})) \neq 0$ for some $k_0 > 0$. By the Hörmander L^2 estimate, there exists $k_0 > 0$ depending only on the bounded geometry of $(X_y, \omega_{\text{KE},y})$ such that $H^0(X_y, K_{X_y}^{(k_0+1)} \otimes \mathcal{I}(k_0\phi_{\text{KE}}|_{X_y})) \neq 0$. By (ii) we have $\|\phi_{0,y} - \phi_{\text{KE},y}\| \leq C$, hence $H^0(X_y, K_{X_y}^{(k_0+1)} \otimes \mathcal{I}(k_0\phi|_{X_y})) \neq 0$.

We now adapt the argument in [Tsu10]. Set $L_{k_0} := K_{X/Y}^{k_0}$ and $h_{k_0} = e^{-\phi_{L_{k_0}}}$ with $\phi_{L_{k_0}} := -k_0\phi_0$, which has positive curvature current. By iteration, at step $k \geq k_0$, assume that $L_k := K_{X/Y}^k|_{X^\circ}$ admits a Hermitian metric $h_k = e^{-\phi_{L_k}}$ with $dd^c\phi_k \geq 0$. Define $\phi_{k+1} := \frac{1}{k} \log(\mathcal{K}_{k\phi_k}/d_k)$, where $\mathcal{K}_{k\phi_k}$ is the Bergman kernel of $(K_{X/Y} + L_k)|_{X^\circ}$. On each fiber, set $\phi_{k+1,y} = \beta_k(\phi_{k,y}) = \frac{1}{k}(\log \mathcal{K}_{k\phi_{k,y}}/d_k)$. By Theorem 3.1, $\|\phi_{k,y} - \phi_{\text{KE},y}\| \leq C$, hence $H^0(X_y, (K_{X_y} + L_k|_{X_y}) \otimes \mathcal{I}(\phi_{L_k}|_{X_y})) \neq 0$. Applying Theorem 4.2, we conclude that $dd^c\phi_{k+1} \geq 0$.

For any $y_0 \in Y^\circ$, choose $U \Subset Y^\circ$ containing y_0 . By the hypothesis and Theorem 1.3, $\|\phi_{k,y'} - \phi_{\text{KE},y'}\|_{L^\infty(X_{y'})} \leq C(U)/k$ for all $y' \in U$, hence $\|\phi_k - \phi_{\text{KE}}\|_{L^\infty(p^{-1}(U))} \leq C(U)/k$. By locally uniform convergence, the limit $\phi_{\text{KE}} := \lim_k \phi_k$ defines a singular Hermitian metric on $(K_{X/Y})|_{X^\circ}$ with positive curvature current $T := dd^c\phi_{\text{KE}}$. $\square$

§4.3. Families of Kähler–Einstein metrics on bounded pseudoconvex domains

We recall some properties of pseudoconvex domains (cf. [CY80]). Let Ω be a strongly pseudoconvex domain in $\mathbb{C}^n$. Let ρ be a smooth defining function for Ω , i.e., $\rho \in \mathcal{C}^\infty(\bar{\Omega})$, $\Omega = \{\rho < 0\}$, $\partial\rho \neq 0$ on $\partial\Omega$, and $(\rho_{i\bar{j}}) > 0$ in $\bar{\Omega}$. We define $\varphi_\Omega = -(n+1)\log(-\rho)$, which is a strictly plurisubharmonic function on Ω and $\varphi_\Omega(z) \rightarrow +\infty$ as $z \rightarrow \partial\Omega$. It follows from [CY80] that $(\Omega, \omega_{\varphi_\Omega})$ is a complete Kähler manifold, where $\omega_{\varphi_\Omega} = dd^c\varphi_\Omega$. Since

$$\det((\varphi_\Omega)_{i\bar{j}}) = (n+1)^n \left(-\frac{1}{\rho}\right)^{n+1} \det(\rho_{i\bar{j}})(-\rho + |\mathrm{d}\rho|^2),$$

we have the Ricci curvature

$$\text{Ric}(\omega_{\varphi_\Omega}) = -(\varphi_\Omega)_{i\bar{j}} + \partial_i\partial_{\bar{j}}F,$$

where $F = -\log[\det(\rho_{i\bar{j}})(-\rho + |\mathrm{d}\rho|^2)]$ is smooth in Ω and bounded on $\bar{\Omega}$.

It is explained in [CY80] that $(\Omega, \omega_{\varphi_\Omega})$ has bounded geometry. The following theorem is due to Cheng–Yau [CY80, MY83].

Theorem 4.7. *There exists a unique complete Kähler–Einstein metric $\omega_{\text{KE}} = dd^c \phi_{\text{KE}}$ with $-(n+1)$ scalar curvature which solves*

$$(dd^c \phi_{\text{KE}})^n = e^{(n+1)(\phi_{\text{KE}} - \phi_\Omega) + F} \omega_{\varphi_\Omega}^n,$$

and $|\nabla_{\varphi_\Omega}^k(\phi_{\text{KE}} - \phi_\Omega)| \leq C_k$ for any $k \in \mathbb{N}$. Moreover, $(\Omega, \omega_{\text{KE}})$ has bounded geometry of any order.

Let $p: D \subset \mathbb{C}^n \times \mathbb{C}^m \rightarrow B \subset \mathbb{C}^m$ be the second projection where D is a strictly pseudoconvex domain (with smooth boundary), and B is an open ball. Suppose that for each $t \in B$, $D_t := p^{-1}(t)$ is bounded strongly pseudoconvex domain with smooth boundary. It follows from [CY80, MY83] (cf. Theorem 4.7) that on each slice D_t there exists a unique complete Kähler form ω_{ϕ_t} such that

$$\text{Ric}(\omega_{\phi_t}) = -(n+1)\omega_{\phi_t}.$$

Then we have the following corollary of Theorem 4.5.

Theorem 4.8. *The metric ϕ_t induces a Hermitian metric on the relative line bundle $K_{D/B}$ with positive curvature current.*

The result was proved by Choi [Cho15b, Cho15a] using the boundary behavior of the geodesic curvature, which satisfies a certain elliptic equation, and by Tsuji [Tsu13] using his construction of Kähler–Einstein metrics.

Proof of Theorem 4.8. Let $(z, t) \in \mathbb{C}^n \times \mathbb{C}^m$ be the standard coordinates. We denote by ρ the defining function of D which satisfies

- $\rho \in \mathcal{C}^\infty(\bar{D})$ and $D = \{(z, t) : \rho(z, t) < 0\}$,
- $\partial\rho \neq 0$ on ∂D ,
- $(\rho_{i\bar{j}}(\cdot, t)) > 0$ in $\bar{D}$.

This implies that $\varphi_D(\cdot, t) = -\log(-\rho(\cdot, t))$ is a strictly smooth plurisubharmonic function in D_t for each $t \in B$. Applying Theorem 4.5, it suffices to show that $\|\phi_t - \varphi_{D_t}\| \leq C$ independent of t . Setting $u_t = \phi_t - \varphi_D(\cdot, t)$, it solves

$$\det((\varphi_D)_{i\bar{j}}(\cdot, t) + (u_t)_{i\bar{j}}) = e^{(n+1)u + F} \det((\varphi_D)_{i\bar{j}}(\cdot, t))$$

in D_t , where the function F is bounded on $\bar{D}$ (defined as above). Applying Theorem 4.5, it suffices to show that $\|u_t\|_{L^\infty(D_t)} \leq C$ uniformly in t . Indeed, it follows

from the maximum principle [CY80, Proposition 4.1] that

$$\sup_{D_t} |u_t| \leq \frac{1}{n+1} \sup_{D_t} |F| \leq C$$

for a constant $C > 0$ uniformly in t . This finishes the proof. $\square$

§4.4. Families of Kähler–Einstein metrics with cusp singularities

Let $p: \bar{X} \rightarrow Y$ be a holomorphic subjective proper map between compact Kähler manifolds with relative dimension n , $D = \sum_{i=1}^{\ell} D_i$ a reduced divisor with simple normal crossing support. Let $X := \bar{X} \setminus D$. Assume Y° is the largest Zariski open subset of Y such that if $\bar{X}^\circ := p^{-1}(Y^\circ)$ then every fiber $\bar{X}_y := p^{-1}(\{y\})$ is smooth, $D_y := D \cap \bar{X}_y$ has simple normal crossing support. We denote $X^\circ = \bar{X}^\circ \setminus D$.

Assume that $K_{\bar{X}_y} + D_y$ is ample for any $y \in Y^\circ$. By the well-known result of Kobayashi [Kob84] and Tian–Yau [TY87], building upon Cheng–Yau’s strategy [CY80], for each fiber, there exists a (unique) complete Kähler metric ω_y on $X_y := \bar{X}_y \setminus D_y$ with cusp singularities along D_y and $\text{Ric}(\omega_y) = -\omega_y$ on X_y . The metric ω_y can be extended as a positive current on $\bar{X}_y$, which satisfies

$$(4.2) \quad \text{Ric}(\omega_y) = -\omega_y + [D_y]$$

in the sense of currents on $\bar{X}_y$, where $[D_y]$ is the integration current along D_y .

Then we will see in Theorem 4.11 below that a similar statement in Theorem 4.5 also holds in this setting, hence we get the following application.

Theorem 4.9. *Assume the setup above; then the fiberwise cusp Kähler–Einstein metrics ω_y satisfying (4.2) can be glued to define a current T on $\bar{X}$ such that*

- (i) T is positive on $X^\circ = \bar{X}^\circ \setminus D$,
- (ii) T extends canonically across $(X \setminus X^\circ) \cup D$ to a closed positive current.

Proof. If one chooses a background Kähler metric ω on $\bar{X}$ then it induces a Hermitian metric $h_{\bar{X}/Y}^\omega = e^{-\phi_\omega}$ on $K_{\bar{X}/Y}$ with curvature form $\Theta_{h^\omega}(K_{\bar{X}/Y}) = dd^c \phi_\omega$; cf. [Pău17, Section 3.1]. Fix smooth Hermitian metrics $h_i = h_{D_i} = e^{-\phi_i}$ on $\mathcal{O}_{\bar{X}}(D_i)$ with curvature form $\theta_{D_i} = dd^c \phi_i \in c_1(D_i)$. Denote $\theta_{i,y} = \sum \theta_{D_i}|_{\bar{X}_y}$ for every generic point $y \in Y^\circ$. Let s_i be a holomorphic section cutting out D_i , and denote $|s_i|^2 = |s_i|_{h_i}^2$ and $|s|^2 = \prod |s_i|^2$. On each fiber $\bar{X}_y$, we set $\eta_y = \Theta_{h^\omega}(K_{X_y}) + \sum \theta_{D_i}|_{X_y}$ being chosen to be Kähler (the restriction of $\Theta_{h^\omega}(K_{\bar{X}/Y}) + \sum \theta_{D_i}$). Equation (4.2) can be written as

$$(\eta_y + dd^c \varphi_y)^n = |s|^{-2} e^{\varphi_y} \omega^n,$$

where φ_y is the unknown function. By [BG14, Theorem 4.2] (see also [DV25, Theorem 1.2]), there exists a unique $\varphi_y \in \mathcal{E}^1(\bar{X}_y, \eta_y)$, which is the space of functions

with finite energy. We can glue the functions φ_y to obtain a function φ on $\bar{X}^\circ$. We will show that φ defines a Hermitian metric $\phi_{\text{KE}} := \phi_\omega + \sum_i \phi_i + \varphi$ on $(K_{X/Y})|_{X^\circ}$, with positive curvature current $T = dd^c \phi_{\text{KE}}$, and ϕ_{KE} extends across $X \setminus X^\circ$ to a singular Hermitian metric with closed positive current by applying Theorem 4.11 below, which is a variant of Theorem 4.5.

It is convenient to work with the complete Kähler metric with Poincaré singularities $\omega_{P,y} = \eta_y - \sum_i dd^c \log(\log |s_i|^2)^2$ on $X_y = \bar{X}_y \setminus D_y$ (up to scaling the Hermitian metrics on $\mathcal{O}_{\bar{X}}(D_i)$). It indeed defines a smooth Kähler metric with Poincaré growth along D_y , having bounded geometry at any order on $X_y = \bar{X}_y \setminus D_y$ as follows from [Gue14, Section 1] (cf. also [Kob84, Lemma 2]); all of these properties are satisfied uniformly in y . Setting $\psi_y = \varphi_y + \sum_i \log(\log |s_i|^2)^2$, we have

$$(\omega_{P,y} + dd^c \psi_y)^n = e^{\psi_y + F} \omega_{P,y}^n$$

where $F = \log(\omega^n / \prod_i |s_i|^2 \log |s_i|^2 \omega_P^n)$ is known to be a bounded smooth function on $\bar{X} \setminus D$; cf. [Gue14, Lemma 1.6] or [CG72, Proposition 2.1].

For any $y_0 \in Y^\circ$, take a neighborhood $U \Subset Y^\circ$ of y_0 . By the result of Kobayashi [Kob84] and Tian–Yau [TY87], we have that $\|\psi_y\|_{L^\infty(X_y)} \leq C(U)$ for all $y \in U$. Indeed, since the bisectional curvature of $(X_y, \omega_{P,y})$ is bounded below by some constant depending on U for all $y \in U$ (cf. [Gue14, Section 4]), the Yau generalized maximum principle (see [CY80, Proposition 4.1]) yields

$$\sup_{X_y} |\psi_y| \leq \sup_{X_y} |F| \leq C(U)$$

for any $y \in U$ because F is bounded on $\bar{X} \setminus D$.

Hence, for $\phi_0 := \phi_\omega + \sum_i \phi_i - \sum_i \log(\log |s_i|^2)^2$ and $\phi_{\text{KE}} = \phi_\omega + \sum_i \phi_i + \varphi$, we have

$$\sup_{X_y} |\phi_{\text{KE},y} - \phi_{0,y}| = \sup_{X_y} |\psi_y| \leq C(U).$$

Also, the result of [Kob84, TY87] shows that the Kähler–Einstein metric $dd^c \phi_{\text{KE},y} = \eta_y + dd^c \varphi_y$ is complete and has bounded geometry on $X_y = \bar{X}_y \setminus D_y$ which is locally uniform in $y \in Y^\circ$.

With this setup, we apply Theorem 4.11 to $\phi_0 = \phi_\omega + \sum_i \phi_i - \sum_i \log(\log |s_i|^2)^2$ and $(\phi_{\text{KE},y})_{y \in Y^\circ}$, to obtain a Hermitian metric $\phi_{\text{KE}} = \phi_\omega + \sum_i \phi_i + \varphi$ on $(K_{X/Y})|_{X^\circ}$ which has a positive curvature current $T = dd^c \phi_{\text{KE}}$ on X° . Moreover, we also have that ϕ_{KE} is bounded from above near the singular fibers of $p|_X$. This can be proved following the same argument as in [Gue20a, Section 6], which is based on the approach of Păun [Pău17, Section 3.3]. Therefore, we can extend T to X as a positive current.

We conclude the proof by showing that $\phi = \phi_{\text{KE}}$ extends across D_{reg} , where D_{reg} is the regular part of D , and hence across D , following [BG14, Lemma 3.7]. Fix $x \in D_{\text{reg}}$, and let $(U, (z^1, \dots, z^{n+m}))$ be local coordinates around x , where $\dim_{\mathbb{C}} \bar{X} = n + m$. In these coordinates, we write $D = (z^1 \cdots z^\ell = 0)$. It suffices to treat the case $\ell = 1$, as the general case can be proven similarly. We have the integral

$$\int_{U \setminus D} \frac{e^\phi dV}{|z^1|^2} \leq C \int_{\substack{0 < |z^1| < 1 \\ |z^j| < 1}} \frac{idz^1 \wedge d\bar{z}^1}{|z^1|^2 \log^2(|z^1|^2)} \wedge \bigwedge_{j=2}^{n+m} idz^j \wedge d\bar{z}^j < \infty.$$

Suppose, by contradiction, that ϕ does not tend to $-\infty$ near D . Then there exists a sequence (z_k) in $U \setminus D$ converging to D such that $\phi(z_k) \geq -C$ for some constant C . Let Δ_k denote the polydisc centered at z_k with polyradius $(r_k, \delta, \dots, \delta)$ for $r_k \searrow 0$ and some fixed $\delta > 0$, so that $\Delta_k \subset U \setminus D$. It follows from the sub-mean value inequality for ϕ at z_k that

$$-C \leq \frac{1}{\text{Vol}(\Delta_k)} \int_{\Delta_k} \phi dV.$$

Using Jensen's inequality, we obtain

$$e^{-C(n)} \leq \int_{\Delta_k} \frac{e^\phi dV}{r_k^2 \delta^{2n}},$$

and on Δ_k , $|z^1| \leq 2r_k$, hence

$$C' \leq \int_{\Delta_k} \frac{e^\phi dV}{|z^1|^2}$$

with $C' = \frac{1}{4}e^{-C}\delta^{2n}$. Since the measure of Δ_k goes to zero as $k \rightarrow +\infty$ we infer that $\int_{U \setminus D} \frac{e^\phi dV}{|z^1|^2}$ is infinite, which is a contradiction. $\square$

To end the section, we establish an extension result of Ohsawa–Takegoshi type for the family of quasi-projective manifolds, which ensures that the statements in Theorems 4.2 and 4.5 hold in this case.

Lemma 4.10. *Let $(\bar{X}, \omega)$ be a Kähler manifold fibered over the disc $\Delta_r = \{t \in \mathbb{C} : |t| < r\}$ with $r > 1$, with compact fibers $\bar{X}_t$. Let D be a reduced divisor with simple normal crossing support in $\bar{X}$ not contained in $\bar{X}_0$. Let $X = \bar{X} \setminus D$ and $X_t = \bar{X}_t \setminus D$. Let L be a holomorphic line bundle on $\bar{X}$ equipped with a singular Hermitian metric $e^{-\phi}$ with semipositive curvature. Let $u \in H^0(X_0, K_{X_0} + L|_{X_0})$ such that $\int_{X_0} u \wedge \bar{u} e^{-\phi} < \infty$. Then there exists $U \in H^0(X, K_X + L|_X)$ such that*

$U = u \wedge dt$ for $t = 0$ and

$$(4.3) \quad \frac{1}{\pi r^2} \int_X U \wedge \bar{U} e^{-\phi} \leq \int_{X_0} u \wedge \bar{u} e^{-\phi}.$$

Proof. The proof is almost identical to that of [Cao17, Theorem 1.3], so we will be somewhat brief and focus on the differences. We observe that $X = \bar{X} \setminus D$ admits a complete metric ω_P with Poincaré singularity type; cf. [Kob84]. We take a covering of $\bar{X}$ by finite open sets V_α biholomorphic to the polydisc in $\mathbb{C}^n$. For each α we fix local coordinates $(t, z_\alpha^1, \dots, z_\alpha^n)$ such that $\bar{X}_0 \cap V_\alpha = (t = 0)$ and $D \cap V_\alpha = (z_\alpha^1 \cdots z_\alpha^\ell = 0)$, so that $X \cap V_\alpha \simeq (\Delta^*)^\ell \times \Delta^{n+1-\ell}$ and $X_0 \cap V_\alpha \simeq (\Delta^*)^\ell \times \Delta^{n-\ell}$, where Δ is the unit disc in $\mathbb{C}$ and $\Delta^* = \Delta \setminus \{0\}$. Locally on U_α , we can write $u = f_\alpha \otimes e_\alpha$, where f_α is a holomorphic function on $X_0 \cap V_\alpha$. We write the Hermitian metric $h = e^{-\phi} = h_0 e^{-\varphi}$ for h_0 a fixed smooth Hermitian metric and a function φ on $\bar{X}$. Thanks to Demailly's approximation theorem, there exists an increasing sequence $\{h_k\} = \{h_0 e^{-\varphi_k}\}$ of metrics with analytic singularities with curvature $i\Theta_{h_k}(L) \geq -\frac{\omega}{k}$.

Step 1. We claim that there exists a smooth section $\tilde{u}$ on $K_X + L|_X$ such that $\tilde{u}|_{X_0} = u$, $\bar{\partial}\tilde{u}|_{X_0} = 0$, and for some $\sigma > 0$,

$$\int_X \frac{|\bar{\partial}\tilde{u}|_{\omega_P, h_0}^2 e^{-(1+\sigma)\varphi}}{|t|^2 (\log |t|)^2} dV_{X, \omega_P} \leq C \int_{X_0} |u|_{\omega_P, h}^2 dV_{X_0, \omega_P}.$$

Thanks to the strong openness [GZ15a], we can find $\sigma > 0$ such that

$$\int_{X_0 \cap V_\alpha} |u|_{\omega_P, h_0}^2 e^{-(1+\sigma)\varphi} dV_{X_0, \omega_P} \leq 2 \int_{X_0 \cap V_\alpha} |u|_{\omega_P, h}^2 dV_{X_0, \omega_P}.$$

We apply the usual version of the Ohsawa–Takegoshi theorem [Dem12, Theorem 13.6] with the weight $e^{-(1+\sigma)\varphi}$, and we can find holomorphic sections $\tilde{f}_\alpha$ of $K_X + L|_X$ on U_α extending u such that

$$\int_{V_\alpha \setminus D} \frac{|\tilde{f}_\alpha|_{\omega_P, h_0}^2 e^{-(1+\sigma)\varphi}}{|t|^2 (\log |t|)^2} dV_{X, \omega_P} \leq \int_{X_0 \cap V_\alpha} |u|_{\omega_P, h}^2 dV_{X_0, \omega_P}.$$

Let $\{\chi_\alpha\}$ be a partition of unity subordinate to $\{V_\alpha\}$. We can check $\tilde{u} := \sum_\alpha \chi_\alpha \tilde{f}_\alpha$ satisfies the claim.

Step 2. Set $\Psi = \log |t|^2$ and $g_m = \bar{\partial}((1 - b'_m \circ \Psi)\tilde{u})$, where b_m is a $\mathcal{C}^1$ function defined on $\mathbb{R}$ such that

$$b_m(s) = s \text{ for } s \geq -m, \quad b''_m(s) = \mathbf{1}_{\{-m-1 \leq s \leq -m\}}(s).$$

We claim that there exists a sequence $\{a_m\} \subset \mathbb{N}$ tending to ∞ , L -valued locally L^2 -forms γ_m and β_m such that

$$\bar{\partial}\gamma_m + \left(\frac{m}{a_m}\right)^{\frac{1}{2}}\beta_m = g_m, \quad \lim_{m \rightarrow \infty} \frac{m}{a_m} = 0,$$

and for $k \geq a_m$,

$$(4.4) \quad \varlimsup_{m \rightarrow \infty} \int_X |\gamma_m|_{\omega_P, h_{a_m}}^2 dV_{X, \omega_P} \leq \pi r^2 \int_{X_0} |u|_{\omega_P, h_0}^2 e^{-\varphi_k} dV_{X_0, \omega_P}.$$

The proof of the claim follows line by line from that in [Cao17, p. 25]. We note here that if φ_k has singularities along the analytic set Z_k then $X \setminus Z_k = \bar{X} \setminus (D \cup Z_k)$ has a complete Kähler metric (cf. [Dem12, Lemma 12.9]), so we can apply the L^2 existence theorem for solutions to the $\bar{\partial}$ -equation; cf. [Dem12, Proposition 13.4, Remark 13.5], also [Cao17, Appendix]. While, in [Cao17], X is assumed to be weakly pseudoconvex, so that $X \setminus Z$ is a complete Kähler manifold for an analytic subset Z . The latter property is still true in our setting.

Step 3. Up to passing to a subsequence, we assume that $((1 - b'_m \circ \Psi)\tilde{u} - \gamma_m)_m$ converges weakly to some $U \in L^2(X, K_X + L|_X)_{\omega_P, h_0}$. It suffices to verify that U is holomorphic and satisfies (4.3). Since $\bar{\partial}((1 - b'_m \circ \Psi)\tilde{u} - \gamma_m) = (m/a_m)^{\frac{1}{2}}\beta_m$ converges to zero and $\bar{\partial}$ is a closed operator, it follows that F is holomorphic. The inequality (4.3) follows immediately from (4.4) (cf. [Cao17, p. 26]).

It remains to check that $U|_{X_0 \cap V_\alpha} = u$. We solve the $\bar{\partial}$ -equation on $V_\alpha \setminus D$: $\bar{\partial}w_m = \beta_m$ and

$$\int_{V_\alpha \setminus D} |w_m|_{\omega_P, h_{a_m}}^2 dV_{X, \omega_P} \leq C \int_{V_\alpha \setminus D} |\beta_m|_{\omega_P, h_{a_m}}^2 dV_{X, \omega_P} \leq C_1$$

for some uniform constant $C_1 > 0$. Then

$$U_m := (1 - b'_m \circ \Psi)\tilde{u} - \gamma_m - \left(\frac{m}{a_m}\right)^{\frac{1}{2}}w_m$$

is a holomorphic function on $V_\alpha \setminus D$, and $U_m \rightarrow U$ weakly. Since $U_m|_{X_0 \cap V_\alpha} = u$ we have $U|_{X_0 \cap V_\alpha} = u$. $\square$

Theorem 4.11. *Theorems 4.2 and 4.5 hold in the case when X is a complex manifold fibered over a complex manifold Y , with fiber $X_y = \bar{X}_y \setminus D$.*

Proof. The statement in Theorem 4.2 for the family $p: X = \bar{X} \setminus D \rightarrow Y$ follows immediately from the global regularization result and the Ohsawa–Takegoshi theorem above.

We then argue as in Theorem 4.5 to obtain a sequence of singular Hermitian metrics ϕ_k on $K_{X/Y}^k|_{X^\circ}$ with positive curvature current $T_k := dd^c\phi_k$. Since ϕ_k

uniformly converges to ϕ_{KE} locally, we have ϕ_{KE} is locally plurisubharmonic on X° . Hence ϕ_{KE} defines a singular Hermitian metric on $(K_{X/Y})|_{X^\circ}$, with positive curvature current $T := dd^c \phi_{\text{KE}}$. $\square$

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