

On Homomorphisms from Finite Subgroups of $SU(2)$ to Langlands Dual Pairs of Groups

by

Yuki KOJIMA and Yuji TACHIKAWA

Abstract

Let $N(\Gamma, G)$ be the number of homomorphisms from Γ to G up to conjugation by G . The physics of four-dimensional $\mathcal{N}=4$ supersymmetric gauge theories predicts that $N(\Gamma, G) = N(\Gamma, \tilde{G})$ when Γ is a finite subgroup of $SU(2)$, G is a connected compact simple Lie group and $\tilde{G}$ is its Langlands dual. This statement is known to be true when $\Gamma = \mathbb{Z}_n$, but the statement for non-Abelian Γ is new, to the knowledge of the authors. To lend credence to this conjecture, we prove this equality in a few cases, namely $(G, \tilde{G}) = (SU(n), PU(n))$ and $(Sp(n), SO(2n+1))$ for arbitrary Γ , and $(PSp(n), Spin(2n+1))$ for exceptional Γ .

A more refined version of the conjecture, together with proofs of some concrete cases, will also be presented. The authors would like to ask mathematicians to provide a more uniform proof applicable to all cases.

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§1. Introduction

§1.1. A conjecture

Let G be a connected compact simple Lie group, and T be the Cartan torus of G . Its lattice of characters, M , sits between the root lattice Q and the weight lattice P of G :

$$Q \subset M \subset P \subset \mathfrak{t}^*,$$

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Y. Kojima: Kavli Institute for the Physics and Mathematics of the Universe (WPI), The University of Tokyo, 5-1-5 Kashiwanoha, Kashiwa, Chiba 277-8583, Japan;

e-mail: yuki.kojima@ipmu.jp

Author IDs: ORCID [0009-0005-0209-130X](https://orcid.org/0009-0005-0209-130X)

Y. Tachikawa: Kavli Institute for the Physics and Mathematics of the Universe (WPI), The University of Tokyo, 5-1-5 Kashiwanoha, Kashiwa, Chiba 277-8583, Japan;

e-mail: yuji.tachikawa@ipmu.jp

Author IDs: zbMATH [tachikawa.yuji](https://zbmath.org/authors/?q=tachikawa.yuji) MR [703539](https://www.ams.org/mathscinet/msc-authors/?q=tachikawa.yuji) ORCID [0000-0002-4547-2146](https://orcid.org/0000-0002-4547-2146)

where $\mathfrak{t}$ is the Lie algebra of T . Then the lattice M^* of cocharacters (i.e., the kernel of the exponentiation map $\exp: \mathfrak{t} \rightarrow T$) is naturally dual to M and satisfies

$$P^* \subset M^* \subset Q^* \subset \mathfrak{t}.$$

Here, for a free finitely generated Abelian group A , we denote by A^* the dual free finitely generated Abelian group such that there exists a perfect pairing $A \times A^* \rightarrow \mathbb{Z}$.

Now, it is well known that there is a connected compact simple Lie group $\tilde{G}$, uniquely determined up to isomorphism, such that its root lattice is P^* , the weight lattice is Q^* , and the character lattice of its Cartan torus is M^* . The groups G and $\tilde{G}$ are said to be Langlands dual to each other. Examples include

$$(G, \tilde{G}) = (\mathrm{SU}(n), \mathrm{PU}(n)), \quad (\mathrm{Sp}(n), \mathrm{SO}(2n+1)), \quad (\mathrm{PSp}(n), \mathrm{Spin}(2n+1)).$$

The Langlands dual groups appear not only in number theory but also in the physics of four-dimensional supersymmetric quantum field theory, and in this paper we will be interested in the following conjecture coming from the latter context. To state the conjecture, we need some definitions.

Definition 1.1. Let Γ be a finite group, and G be a connected compact simple Lie group. Let $\phi(\Gamma, G)$ be the set of homomorphisms $f: \Gamma \rightarrow G$, and we define two such homomorphisms f_1 and f_2 to be equivalent, $f_1 \sim f_2$, when they are conjugate by the action of G , i.e., there is an element $g \in G$ such that

$$f_2(\gamma) = g f_1(\gamma) g^{-1}, \quad \text{for all } \gamma \in \Gamma.$$

We then let

$$\psi(\Gamma, G) = \phi(\Gamma, G) / \sim$$

and

$$N(\Gamma, G) = \#\psi(\Gamma, G),$$

i.e., the number of homomorphisms $f: \Gamma \rightarrow G$ up to conjugation by G .

Conjecture 1.2. *Let Γ be a finite subgroup of $\mathrm{SU}(2)$, and let $(G, \tilde{G})$ be a Langlands dual pair of connected compact simple Lie groups. Then*

$$N(\Gamma, G) = N(\Gamma, \tilde{G}).$$

This is known to be true when $\Gamma = \mathbb{Z}_n$, see [Djo85, Theorem 1], but this statement for non-Abelian Γ is new, to the knowledge of the authors. Our first aim of this paper is to provide evidence for this conjecture, by providing proofs of this conjecture in the following cases:

- $(G, \tilde{G}) = (\mathrm{SU}(n), \mathrm{PU}(n))$ for arbitrary Γ ,
- $(G, \tilde{G}) = (\mathrm{Sp}(n), \mathrm{SO}(2n+1))$ for arbitrary Γ and
- $(G, \tilde{G}) = (\mathrm{PSp}(n), \mathrm{Spin}(2n+1))$ for exceptional Γ .

Remark 1.3. Although the conjecture as formulated above does not seem to have been previously given in the literature, the structure of $\psi(\Gamma, G)$ for some choices of Γ and G has been studied in the past. The paper [Djo85] given above is one. The book [Kac90] has a very concrete method to describe each element of $\psi(\mathbb{Z}_n, G)$ when G is of the adjoint type using the Dynkin diagram of G , in its Section 8.6. The studies [Fre98a, Fre98b, Fre01, FR20] have descriptions of $\psi(\Gamma, G)$ when Γ is the binary icosahedral group and $G = E_6, E_7$ and E_8 . The last paper mentioned also has a description of $\psi(\Gamma, E_8)$ when Γ is some of the binary dihedral groups.

Remark 1.4. As the partial proofs of this conjecture we give below will amply show, we do *not* expect that there is a natural bijection between $\psi(\Gamma, G)$ and $\psi(\Gamma, \tilde{G})$. It seems to be more like a discrete Fourier transformation on these two sets.

§1.2. A more refined conjecture

There is actually a finer version of the conjecture. To state it, we need more preparation. Below, we use $A^\wedge$ to denote the Pontryagin dual of a finite Abelian group A . We denote the natural pairing $A^\wedge \times A \rightarrow U(1) = \{z \mid |z| = 1\}$ by $(-, -)$. Let Γ be a finite group, and let G be a group, and let $Z \subset G$ be an Abelian subgroup of the center of G . Recall that elements of $H^2(B\Gamma; Z)$ classify central extensions

$$0 \rightarrow Z \rightarrow \check{\Gamma} \xrightarrow{p} \Gamma \rightarrow 0$$

up to equivalence.

Definition 1.5. We pick and fix a particular cocycle in the class w , and write the corresponding extension as $\check{\Gamma}_{Z,w}$. Then a (Z, w) -twisted homomorphism from Γ to G is a homomorphism $f: \check{\Gamma}_{Z,w} \rightarrow G$ such that $Z \subset \check{\Gamma}_{Z,w}$ is mapped identically to $Z \subset G$. We denote the set of (Z, w) -twisted homomorphisms from Γ to G by $\phi_{Z,w}(\Gamma, G)$. We let $\psi_{Z,w}(\Gamma, G)$ be its quotient by $\sim$.

Remark 1.6. It is clear that $\phi_{Z,0}(\Gamma, G) = \phi(\Gamma, G)$ and $\psi_{Z,0}(\Gamma, G) = \psi(\Gamma, G)$.

Recall that elements of $H^1(B\Gamma; Z)$ are homomorphisms $z: \Gamma \rightarrow Z$. Given a w -twisted homomorphism $f: \check{\Gamma}_{Z,w} \rightarrow G$, we define $zf: \check{\Gamma}_{Z,w} \rightarrow G$ to be the homomorphism defined by

$$zf(\bar{\gamma}) := z(\gamma)f(\bar{\gamma}),$$

where $\gamma = p(\bar{\gamma})$. This defines an action of $H^1(B\Gamma, Z)$ on $\phi_{Z,w}(\Gamma, G)$ and $\psi_{Z,w}(\Gamma, G)$.

Definition 1.7. Let $V_{Z,w}(\Gamma, G)$ be a $\mathbb{C}$ -vector space with basis vectors $v([f])$ for $[f] \in \psi_{Z,w}(\Gamma, G)$, and

$$V_Z(\Gamma, G) := \bigoplus_{w \in H^2(B\Gamma; Z)} V_{Z,w}(\Gamma, G).$$

Let $z \in H^1(B\Gamma; Z)$ act on $V_Z(\Gamma, G)$ by

$$zv([f]) := v([zf]),$$

and let $\widehat{w} \in H^2(B\Gamma; Z)^\wedge$ act on $V_{Z,w}(\Gamma, G)$ by

$$\widehat{w}v([f]) = (\widehat{w}, w)v([f]).$$

This makes $V_Z(\Gamma, G)$ into a representation of

$$F(\Gamma; Z) := H^1(B\Gamma; Z) \times H^2(B\Gamma; Z)^\wedge.$$

Let us now specialize to the case when Γ is a finite subgroup of $SU(2)$. Indeed, Γ has a natural action on $\mathbb{C}^2$ coming from the embedding $\Gamma \subset SU(2)$. It therefore acts on the unit sphere S^3 of $\mathbb{C}^2$. Consequently, $S^3 \rightarrow S^3/\Gamma$ is a principal Γ -bundle and determines a map $S^3/\Gamma \rightarrow B\Gamma$. It can be checked that, via this map $S^3/\Gamma \rightarrow B\Gamma$, $H^1(B\Gamma; Z)$ and $H^2(B\Gamma; Z)$ pull back isomorphically to $H^1(S^3/\Gamma; Z)$ and $H^2(S^3/\Gamma; Z)$, respectively. Now, using Poincaré duality on S^3/Γ , one has $H^1(S^3/\Gamma; Z) \simeq H^2(S^3/\Gamma; Z^\wedge)^\wedge$. Combining, we conclude that there is a naturally defined isomorphism

$$\iota_Z: H^1(B\Gamma; Z) \xrightarrow{\sim} H^2(B\Gamma; Z^\wedge)^\wedge.$$

Definition 1.8. We let

$$F(\Gamma; Z) := H^1(B\Gamma; Z) \times H^2(B\Gamma; Z)^\wedge,$$

and define the *swap* isomorphism

$$s: F(\Gamma; Z) \xrightarrow{\sim} F(\Gamma; Z^\wedge)$$

by demanding that it sends an element

$$(z, \iota_Z(z')) \in F(\Gamma; Z) = H^1(B\Gamma; Z) \times H^2(B\Gamma; Z)^\wedge$$

to

$$(z', \iota_Z(z)) \in F(\Gamma; Z^\wedge) = H^1(B\Gamma; Z^\wedge) \times H^2(B\Gamma; Z^\wedge)^\wedge.$$

We can now state a refinement of Conjecture 1.2: Let G be a connected compact simple Lie group, $Z \subset G$ be an Abelian subgroup of the center of G , and

$H = G/Z$, so that we have a central extension of groups

$$0 \rightarrow Z \rightarrow G \rightarrow H \rightarrow 0.$$

Then the Langlands duals $\tilde{G}, \tilde{H}$ of G, H sit in a central extension

$$0 \rightarrow Z^\wedge \rightarrow \tilde{H} \rightarrow \tilde{G} \rightarrow 0.$$

Pick a finite subgroup Γ of $\mathrm{SU}(2)$. Then $V_Z(\Gamma, G)$ is a representation of $F(\Gamma; Z)$ and $V_{Z^\wedge}(\Gamma, \tilde{H})$ is a representation of $F(\Gamma; Z^\wedge)$.

Conjecture 1.9. *Under the setup above, $V_Z(\Gamma, G)$ as a representation of $F(\Gamma; Z)$ and $V_{Z^\wedge}(\Gamma, \tilde{H})$ as a representation of $F(\Gamma; Z^\wedge)$ are equivalent when we identify the groups $F(\Gamma; Z)$ and $F(\Gamma; Z^\wedge)$ using the swap isomorphism s .*

Remark 1.10. It is unclear to the authors whether such an equivalence is canonically defined, since it turns out that the action ρ of $F(\Gamma; Z)$ on $V_Z(\Gamma, G)$ is such that ρ and $\rho \circ \mathrm{inv}$ are equivalent, where inv is the automorphism of $F(\Gamma; Z)$ given by sending $a \mapsto a^{-1}$.

Again, to provide evidence for this conjecture, we provide proofs of Conjecture 1.9 in a few cases:

- arbitrary $(G, \tilde{G}), (H, \tilde{H}), Z$ for $\Gamma = \mathbb{Z}_n$,
- arbitrary Γ for

$$(G, \tilde{G}) = (\mathrm{SU}(n), \mathrm{PU}(n)), \quad (H, \tilde{H}) = (\mathrm{PU}(n), \mathrm{SU}(n)), \quad Z = \mathbb{Z}_n,$$

- and exceptional Γ for

$$(G, \tilde{G}) = (\mathrm{Sp}(n), \mathrm{SO}(2n+1)), \quad (H, \tilde{H}) = (\mathrm{PSp}(n), \mathrm{Spin}(2n+1)), \quad Z = \mathbb{Z}_2.$$

We note that Conjecture 1.2 follows from Conjecture 1.9 by setting $Z = \{e\}$ and taking the dimension, since $N(\Gamma, G) = \dim V_{\{e\}}(\Gamma, G)$. There is also a less obvious relation between Conjecture 1.2 and Conjecture 1.9, as can be seen from the proposition below.

Proposition 1.11. *There is a natural identification*

$$\begin{aligned} V_Z(\Gamma, G)^{H^2(B\Gamma; Z)^\wedge} &\simeq V_{\{e\}}(\Gamma, G), \\ V_Z(\Gamma, G)^{H^1(B\Gamma; Z)} &\simeq V_{\{e\}}(\Gamma, G/Z), \end{aligned}$$

where V^G for a space V with a G action means the subspace of V invariant under G .

Proof. To show the first isomorphism, note that $V_Z(\Gamma, G)^{H^2(B\Gamma; Z)^\wedge} = V_{Z,0}(\Gamma, G)$. As $(Z, 0)$ -twisted homomorphisms are just genuine homomorphisms, we have $V_{Z,0}(\Gamma, G) = V_{\{e\}}(\Gamma, G)$.

To prove the second isomorphism, we first note that, by definition, any (Z, w) -twisted homomorphism f from Γ to G descends to a genuine homomorphism from Γ to G/Z . Moreover, f and zf descend to the same homomorphism from Γ to G/Z .

Conversely, given an arbitrary homomorphism $f: \Gamma \rightarrow G/Z$, let us pick a pointwise lift $\tilde{f}(\gamma) \in G$ of $f(\gamma) \in G/Z$ for each $\gamma \in \Gamma$. We then have $\tilde{f}(\gamma)\tilde{f}(\gamma') = \tilde{w}(\gamma, \gamma')\tilde{f}(\gamma\gamma')$ for some $\tilde{w}(\gamma, \gamma') \in Z$, where $\tilde{w}: \Gamma \times \Gamma \rightarrow Z$ is a two-cocycle. Any other lift $\tilde{f}'(\gamma)$ is of the form $\tilde{f}'(\gamma) = x(\gamma)\tilde{f}(\gamma)$, where $x: \Gamma \rightarrow Z$ is a one-cochain. The two-cocycle $\tilde{w}'$ defined from $\tilde{f}'$ is then given by $\tilde{w}' = (\delta x)\tilde{w}$. This means that the class $w := [\tilde{w}] = [\tilde{w}'] \in H^2(B\Gamma; Z)$ is well defined. We now choose x so that $\tilde{w}'$ is the two-cocycle within w chosen in Definition 1.5. Then the set $Z \times \Gamma$ with the multiplication

$$(z, \gamma) \cdot (z', \gamma') = (zz'\tilde{w}'(\gamma, \gamma'), \gamma\gamma')$$

is the central extension $\check{\Gamma}_{Z,w}$ in Definition 1.5, and $\check{f}: \check{\Gamma}_{Z,w} \rightarrow G$ defined by

$$\check{f}: (z, \gamma) \mapsto z\tilde{f}(\gamma)$$

is a (Z, w) -twisted homomorphism. It is easy to check that two such (Z, w) -twisted homomorphisms $\check{f}$ and $\check{f}'$ from Γ to G come from a single $f: \Gamma \rightarrow G/Z$ if and only if there is a homomorphism $x: \Gamma \rightarrow Z$ such that $\check{f}' = x\check{f}$. $\square$

Corollary 1.12. *Conjecture 1.9 for the data $\Gamma, G, H=G/Z, \tilde{H}, \tilde{G}=\tilde{H}/Z^\wedge$ implies the equalities*

$$N(\Gamma, G) = N(\Gamma, \tilde{G}), \quad N(\Gamma, H) = N(\Gamma, \tilde{H}),$$

i.e., Conjecture 1.2 for the data $\Gamma, G, \tilde{G}$ and for the data $\Gamma, H, \tilde{H}$.

Proof. Apply Proposition 1.11 to Conjecture 1.9 and take the dimension. $\square$

§1.3. Physics background

Physics derivations of Conjectures 1.2 and 1.9 will be detailed in [CT26]; the analysis there was motivated and heavily influenced by earlier physics papers [Ju24a, Ju24b]. Roughly, for any given G , there is a four-dimensional $\mathcal{N}=4$ supersymmetric quantum field theory $T(G, g)$, where g is the coupling constant. It is believed that $T(G, g) \simeq T(\tilde{G}, 1/g)$, which is called the S-duality of these theories. The most common physics approach to quantum field theory uses Taylor expansion in the coupling constant g , and therefore this equality cannot be seen within

such a standard approach. This S-duality is a non-Abelian generalization of the electromagnetic duality of the ordinary electromagnetism, i.e., the Abelian $U(1)$ gauge theory.

In this interpretation, $V_{\{e\}}(\Gamma, G)$ is then the lowest-energy subspace of the Hilbert space of the theory $T(G, g)$ on S^3/Γ . As $T(G, g) \simeq T(\tilde{G}, 1/g)$, we should have $V_{\{e\}}(\Gamma, G) \simeq V_{\{e\}}(\Gamma, \tilde{G})$, and taking the dimension, we obtain Conjecture 1.2. The finer conjecture, Conjecture 1.9, is obtained by taking into account the action of what are called 1-form symmetries of the four-dimensional quantum field theory, and the swap isomorphism is related to the electromagnetic duality. Then Proposition 1.11 describes the process of the gauging of such 1-form symmetries.

§1.4. Organization of the paper

The rest of the paper is organized as follows.

- We begin in Section 2 by recalling the proof of Conjecture 1.2 for $\Gamma = \mathbb{Z}_n$ found in [Djo85]. We also generalize the proof in [Djo85] to prove the refined version of the conjecture, Conjecture 1.9, for $\Gamma = \mathbb{Z}_n$ in this section.
- Then in Section 3 we prove Conjecture 1.2 for arbitrary Γ and $(G, \tilde{G}) = (\mathrm{SU}(n), \mathrm{PU}(n))$. We also prove Conjecture 1.9 for $G = \tilde{H} = \mathrm{SU}(n)$, $H = \tilde{G} = \mathrm{PU}(n)$ and arbitrary Γ .
- Then in Section 4 we prove Conjecture 1.2 for arbitrary Γ for $(G, \tilde{G}) = (\mathrm{Sp}(n), \mathrm{SO}(2n+1))$.
- In the final section, Section 5, we prove Conjecture 1.9 for $(G, \tilde{G}) = (\mathrm{Sp}(n), \mathrm{SO}(2n+1))$, $(H, \tilde{H}) = (\mathrm{PSp}(n), \mathrm{Spin}(2n+1))$ and $\Gamma = \hat{\mathcal{O}}$.

Some comments are in order.

- Readers will see that the proofs in Sections 2 and 3 are somewhat conceptual and follow a similar approach. Namely, we rewrite $N(\Gamma, G)$ as the dimensions of V^K , where V is an auxiliary vector space, K is an auxiliary finite group acting on it, and V^K is the fixed point subspace. Similarly, we rewrite $N(\Gamma, \tilde{G})$ as the dimension of $\tilde{V}^{\tilde{K}}$, for a suitable choice of $\tilde{V}$ and $\tilde{K}$. Then we will show that there is a linear isomorphism $f: V \rightarrow \tilde{V}$ intertwining the actions of K and $\tilde{K}$. The map f works as a discrete Fourier transform. It is to be noted that the choice of V and K for a given (Γ, G) in Sections 2 and 3 would be different even for (Γ, G) common to both sections.
- In contrast, our proofs in Sections 4 and 5 are more combinatorial, and utilize computations using generating functions. It would be nice if the proofs here could be rephrased in a form closer to the proofs in Sections 2 and 3.

- In Section 3 and later, when we deal with general Γ , we will rely heavily on McKay correspondence between finite subgroups of $SU(2)$ and ADE Dynkin diagrams.

The authors are theoretical physicists. They would hope that some mathematicians reading this paper would get interested and eventually find a more conceptual and uniform proof of the conjectures.

§2. The case $\Gamma = \mathbb{Z}_n$, arbitrary $(G, \tilde{G})$

§2.1. The proof of the basic conjecture

We first reproduce the proof of the following theorem, originally found in [Djo85]; see Theorem 1 there. This is a subcase of our Conjecture 1.2 when $\Gamma = \mathbb{Z}_n$.

Theorem 2.1. *Conjecture 1.2 holds when $\Gamma = \mathbb{Z}_n$. That is, we have $N(\mathbb{Z}_n, G) = N(\mathbb{Z}_n, \tilde{G})$ for arbitrary Langlands dual pair, $(G, \tilde{G})$.*

Proof. Let M^* and M be the cocharacter lattice of G and $\tilde{G}$, respectively. Then clearly we have

$$\begin{aligned} N(\mathbb{Z}_n, G) &= \text{the number of } W \text{ orbits in } \frac{1}{n}M^*/M^*, \\ N(\mathbb{Z}_n, \tilde{G}) &= \text{the number of } W \text{ orbits in } \frac{1}{n}M/M, \end{aligned}$$

where W is the Weyl group, common to both G and $\tilde{G}$. As M and M^* are dual to each other, $\frac{1}{n}M^*/M^*$ and $\frac{1}{n}M/M \simeq M/nM$ are Pontryagin dual to each other. In view of this fact, let us write $A = \frac{1}{n}M^*/M^*$ and $A^\wedge = \frac{1}{n}M/M$. Let $V(A)$ be a complex vector space with basis $v(a)$ for each $a \in A$, and similarly for $A^\wedge$. We then have

$$N(\mathbb{Z}_n, G) = \dim V(A)^W, \quad N(\mathbb{Z}_n, \tilde{G}) = \dim V(A^\wedge)^W.$$

As $V(A^\wedge) = V(A)^*$ and the actions of W are conjugate, we have $N(\mathbb{Z}_n, G) = N(\mathbb{Z}_n, \tilde{G})$. $\square$

§2.2. The proof of the refined conjecture

This proof can be readily generalized for the finer version of the conjecture.

Theorem 2.2. *Conjecture 1.9 holds when $\Gamma = \mathbb{Z}_n$.*

Proof. Let us recall the setup. We have two Langlands dual pairs $(G, \tilde{G})$ and $(H, \tilde{H})$ sitting in the sequences

$$0 \rightarrow Z \rightarrow G \rightarrow H \rightarrow 0, \quad 0 \rightarrow Z^\wedge \rightarrow \tilde{H} \rightarrow \tilde{G} \rightarrow 0.$$

Denoting the Cartan torus of G by T_G , etc., we also have

$$0 \rightarrow Z \rightarrow T_G \rightarrow T_H \rightarrow 0, \quad 0 \rightarrow Z^\wedge \rightarrow T_{\tilde{H}} \rightarrow T_{\tilde{G}} \rightarrow 0.$$

Denote by M and N the character lattices of G and H , respectively. Then we have

$$Q \subset N \subset M \subset P \subset \mathfrak{t}^*, \quad P^* \subset M^* \subset N^* \subset Q^* \subset \mathfrak{t},$$

and

$$N^*/M^* \simeq Z, \quad M/N \simeq Z^\wedge.$$

Let us first study $V_Z(\mathbb{Z}_n, G)$. In this case, $H^2(B\mathbb{Z}_n, Z) \simeq Z/nZ$, and therefore a $w \in H^2(B\mathbb{Z}_n, Z)$ is specified by an element $\tilde{w} \in Z$ modulo nZ . We pick $\tilde{w}$ for each w .

A (Z, w) -twisted homomorphism from $\mathbb{Z}_n$ to G is a (Z, w) -twisted homomorphism from $\mathbb{Z}_n$ to T_G up to the action of the Weyl group W . It is specified by the image a of $1 \in \mathbb{Z}_n$ such that $na = \tilde{w} \in Z$. Therefore, using

$$(n \times): \frac{1}{n}N^*/M^* \rightarrow N^*/M^* = Z,$$

we have

$$\psi_{Z,w}(\mathbb{Z}_n, T_G) \simeq (n \times)^{-1}(\tilde{w}).$$

On this set, the group $H^1(B\mathbb{Z}_n, Z) = \text{Ker } Z \rightarrow n \times Z$ acts by the addition.

Introduce now a vector space $V(\frac{1}{n}N^*/M^*)$ with a basis $v(a)$ for each element $a \in \frac{1}{n}N^*/M^*$. Identify the dual $(Z/nZ)^\wedge$ of Z/nZ with $\text{Ker } Z^\wedge \xrightarrow{n \times} Z^\wedge$. Pick, then, an element $b \in Z^\wedge$ such that $nb = 0$, and let it act on $V(\frac{1}{n}N^*/M^*)$ via

$$bv(a) \mapsto (b, na)v(a).$$

This makes $V(\frac{1}{n}N^*/M^*)$ into a representation of $F(\mathbb{Z}_n, Z) = H^1(B\mathbb{Z}_n, Z) \times H^2(B\mathbb{Z}_n, Z)^\wedge$, and now it is routine to check that

$$V\left(\frac{1}{n}N^*/M^*\right)^W = |nZ| \text{ copies of } V_Z(\mathbb{Z}_n, G),$$

as a representation of $F(\mathbb{Z}_n, Z)$.

We can similarly show that

$$V\left(\frac{1}{n}M/N\right)^W = |nZ^\wedge| \text{ copies of } V_{Z^\wedge}(\mathbb{Z}_n, \tilde{H})$$

as a representation of $F(\mathbb{Z}_n, Z^\wedge)$. Using the Pontryagin duality between $\frac{1}{n}N^*/M^*$ and $\frac{1}{n}M/N$, it is straightforward to check that the actions of $F(\mathbb{Z}_n, Z)$ and $F(\mathbb{Z}_n, Z^\wedge)$ on $V(\frac{1}{n}N^*/M^*)$ and $V(\frac{1}{n}M/N)$, respectively, are compatible under the swap isomorphism of Definition 1.8. All that is left is to take the W -invariant parts. $\square$

§3. The case $(G, \tilde{G}) = (\mathrm{SU}(n), \mathrm{PU}(n))$, arbitrary Γ

§3.1. Properties of irreducible representations of Γ

In our discussion below, we heavily utilize the McKay correspondence concerning irreducible representations of finite subgroups of $\mathrm{SU}(2)$ and the ADE Dynkin diagram [McK80]. Before proceeding to the rest of the paper, we provide here a minimal amount of information.

Let Γ be a finite subgroup of $\mathrm{SU}(2)$. Let V be its defining two-dimensional representation, i.e., the one coming from the embedding $\Gamma \subset \mathrm{SU}(2)$. Let $\Phi = \{\rho_i\}$ be the set of isomorphism classes of irreducible representations of Γ . We now consider a graph, whose vertices are ρ_i , such that ρ_i and ρ_j are connected if and only if $\rho_i \otimes V$ contains ρ_j as an irreducible component. This is known to produce an extended Dynkin diagram of type A , D or E . The finite subgroups of $\mathrm{SU}(2)$ are then as follows:

ADE	Symbol	Order	Name	Presentation
A_{n-1}	$\mathbb{Z}_n$	n	Cyclic	$a^n = 1$
D_{n+2}	$\widehat{D}_n$	$4n$	Binary dihedral	$a^2 = b^n = (ab)^2$
E_6	$\widehat{T}$	24	Binary tetrahedral	$a^3 = b^3 = (ab)^2$
E_7	$\widehat{O}$	48	Binary octahedral	$a^4 = b^3 = (ab)^2$
E_8	$\widehat{I}$	120	Binary icosahedral	$a^5 = b^3 = (ab)^2$

We will refer to them mainly by the symbols in the second column. The presentations given above go back to [Cox40].

We note that $\dim \rho_i$ equals the comark of the extended Dynkin diagram. We will utilize various other correspondences of the properties of Γ and $\mathfrak{g}$. They will be introduced as they become needed in our discussions.

§3.2. The proof of the basic conjecture

The first objective of this section is to prove Conjecture 1.2 for arbitrary Γ in the case $(G, \tilde{G}) = (\mathrm{SU}(n), \mathrm{PU}(n))$. We start with some preparation.

Let Γ be a finite subgroup of $\mathrm{SU}(2)$. Let A be the set of characters of Γ_{ab} , the Abelianization of Γ . We have $A^\wedge = \Gamma_{ab}$.

Note $\psi(\Gamma, U(n))$ is the set of isomorphism classes of n -dimensional complex representations of Γ . Therefore, A naturally acts on $\psi(\Gamma, U(n))$ by tensor product. Let $V(\Gamma, U(n))$ be the complex vector space with basis $v([f])$ for each $[f] \in \psi(\Gamma, U(n))$. Then A acts on $V(\Gamma, U(n))$.

We now construct an action of $A^\wedge$ on $V(\Gamma, U(n))$.

Definition 3.1. We let $\det: \psi(\Gamma, U(n)) \rightarrow A$ be the map which sends a representation $\rho: \Gamma \rightarrow U(n)$ to its determinant, i.e., the composition $\Gamma \xrightarrow{\rho} U(n) \xrightarrow{\det} U(1)$.

Then we define $a^\wedge \in A^\wedge$ to act on $v([f])$ via

$$a^\wedge v([f]) = (a^\wedge, \det([f]))v([f]).$$

Remark 3.2. Note that the A action and the $A^\wedge$ action do not generally commute. Rather, there is an action of a finite Heisenberg group which is an extension of $A \times A^\wedge$.

Lemma 3.3. $N(\Gamma, \mathrm{SU}(n)) = \dim V(\Gamma, U(n))^{A^\wedge}$.

Proof. Immediate. □

Lemma 3.4. $N(\Gamma, \mathrm{PU}(n)) = \dim V(\Gamma, U(n))^A$.

Proof. Any homomorphism $f: \Gamma \rightarrow \mathrm{PU}(n)$ can be lifted to a homomorphism $\tilde{f}: \Gamma \rightarrow U(n)$, since $H^2(B\Gamma, U(1)) = 0$. Suppose now two such lifts, $\tilde{f}$ and $\tilde{f}'$, descend to the same f . Define $z(g) \in U(1)$ for $g \in \Gamma$ by the condition $\tilde{f}(g) = z(g)\tilde{f}'(g)$. Then $z \in A$. Therefore, $N(\Gamma, \mathrm{PU}(n))$ is the number of A orbits in $\psi(\Gamma, U(n))$. The statement immediately follows from this. □

To compare $\dim V(\Gamma, U(n))^{A^\wedge}$ and $\dim V(\Gamma, U(n))^A$, we use the McKay correspondence. Let $\mathfrak{g}$ be the ADE type of Γ . Then the irreducible representations ρ_i of Γ form the extended Dynkin diagram of type $\mathfrak{g}$.

An element of $\psi(\Gamma, U(n))$ specifies an n -dimensional representation ρ of Γ up to isomorphism. As such it is specified by the number n_i of the copies of the irreducible representation ρ_i it contains. These nonnegative integers n_i satisfy $\sum_i n_i \dim \rho_i = n$. This is exactly what specifies an irreducible integrable highest-weight representation λ of the affine Lie algebra $\hat{\mathfrak{g}}$ of level n .

Definition 3.5. We denote by

$$\iota: \psi(\Gamma, U(n)) \xrightarrow{\sim} R(\hat{\mathfrak{g}}_n)$$

the McKay correspondence between two finite sets described above, and call it the McKay map.

Remark 3.6. It might be worth mentioning here that a deeper connection between $\hat{\mathfrak{g}}_n$ and $U(n)$ bundles on $\mathbb{C}^2/\Gamma$ is known to exist; see e.g., [Nak02].

Note that A is the subset of Φ such that the comarks are 1. It is known that A is naturally the outer automorphism group of the affine Lie algebra $\hat{\mathfrak{g}}$, and therefore acts on $R(\hat{\mathfrak{g}}_n)$. We will need the following lemma.

Lemma 3.7. *The A action on $\psi(\Gamma, U(n))$ and the A action on $R(\hat{\mathfrak{g}}_n)$ are compatible under the McKay map ι .*

Proof. This should be standard. For an explicit case-by-case verification, see e.g., [CT26, Appendix B]. $\square$

It is also known that $A^\wedge = \Gamma_{ab}$ is naturally isomorphic to the center Z of the simply connected group G of type $\mathfrak{g}$. Using this, we make the following definition.

Definition 3.8. The irreducible decomposition of an irreducible integrable highest weight representation of $\hat{\mathfrak{g}}_n$ contains only a single type of irreducible representation of Z . We let $\det'$ be the map which associates this element in $Z^\wedge = A$ to an element in $R(\hat{\mathfrak{g}}_n)$.

Again, we will need the following lemma.

Lemma 3.9. *The map $\det: \psi(\Gamma, U(n)) \rightarrow A$ of Definition 3.1 and the map $\det': R(\hat{\mathfrak{g}}_n) \rightarrow A$ of Definition 3.8 are compatible under the McKay map ι .*

Proof. This should be standard. For an explicit case-by-case verification, see e.g., [CT26, Appendix B]. $\square$

We now introduce a vector space $V(\hat{\mathfrak{g}}_n)$ with basis $v(\lambda)$ for each $\lambda \in R(\hat{\mathfrak{g}}_n)$. Thus, A and $A^\wedge$ naturally act on $V(\hat{\mathfrak{g}}_n)$.

Lemma 3.10. *The A actions on $V(\Gamma, U(n))$ and on $V(\hat{\mathfrak{g}}_n)$ are compatible under the McKay map ι . Similarly, the $\hat{A}$ actions on $V(\Gamma, U(n))$ and on $V(\hat{\mathfrak{g}}_n)$ are compatible under the McKay map ι .*

Proof. Immediate from Lemmas 3.7 and 3.9. $\square$

Definition 3.11. We define the modular S -matrix

$$S: V(\hat{\mathfrak{g}}_n) \rightarrow V(\hat{\mathfrak{g}}_n)$$

by $Sv(\lambda) = \sum_{\mu} S_{\lambda\mu} v(\mu)$, so that

$$S\chi_{\lambda}(-1/\tau) = \sum_{\mu} S_{\lambda\mu} \chi_{\mu}(\tau).$$

Here $\chi_{\lambda}(\tau)$ is the character of the irreducible integrable highest-weight representation λ .

Proposition 3.12. *The A action on $V(\hat{\mathfrak{g}}_n)$ and the $A^\wedge$ action on $V(\hat{\mathfrak{g}}_n)$ is conjugate by the action of the modular S matrix, for a suitable identification $A \simeq A^\wedge$.*

Proof. See [DFMS97, Section 14.6.4]. $\square$

Remark 3.13. It seems to the authors that there is no canonical isomorphism $A \simeq A^\wedge$. Two isomorphisms $A \xrightarrow{\sim} A^\wedge$ differing by composing with $a \mapsto -a$ either on the side of A or on the side of $A^\wedge$ seem to give equally good isomorphisms.

Corollary 3.14. *The A action and the $A^\wedge$ action on $V(\Gamma, U(n))$ are conjugate.*

Proof. Immediate from the proposition above and Lemma 3.10. $\square$

Theorem 3.15. *We have $N(\Gamma, \mathrm{SU}(n)) = N(\Gamma, \mathrm{PU}(n))$ for arbitrary finite subgroup Γ of $\mathrm{SU}(2)$.*

Proof. This follows from Lemma 3.3, Lemma 3.4 and Corollary 3.14. $\square$

§3.3. The proof of the refined conjecture

It is also not difficult to generalize the proof above to show the finer version of the conjecture when $(G, \tilde{G}) = (\mathrm{SU}(n), \mathrm{PU}(n))$, $Z = \mathbb{Z}_n$ and $(H, \tilde{H}) = (\mathrm{PU}(n), \mathrm{SU}(n))$. As preparation, we first need to study $H^1(B\Gamma, Z)$ and $H^2(B\Gamma, Z)$ when $Z = \mathbb{Z}_n$, the center of $\mathrm{SU}(n)$. Recall that we defined $A = H^1(B\Gamma, U(1))$, i.e., the set of homomorphisms $\Gamma \rightarrow U(1)$. Then we can identify $H^1(B\Gamma, \mathbb{Z}_n) = \mathrm{Ker}(n \times): A \rightarrow A$.

Next we consider $H^2(B\Gamma, \mathbb{Z}_n)$. Take a cocycle representative $\omega(g, h) \in \mathbb{Z}_n$. As $H^2(B\Gamma, U(1)) = 0$, we can take $\nu(g) \in U(1)$ such that

$$\omega(g, h) = \nu(g)\nu(h)/\nu(gh).$$

Two such ν differ by an element of A . Note also that $\nu(g)^n$ is a homomorphism. Let us denote it by $a \in A$. This is well defined up to nA . In this manner, we have defined a map $H^2(B\Gamma, \mathbb{Z}_n) \rightarrow A/nA$. This is actually an isomorphism, as can be seen by studying the Bockstein long exact sequence.

We consider a (Z, ω) -twisted homomorphism from Γ to $\mathrm{SU}(n)$ as a projective representation ρ of Γ in $\mathrm{SU}(n)$ such that

$$\rho(gh) = \omega(g, h)\rho(g)\rho(h).$$

Now define $\tilde{\rho}(g) := \nu(g)\rho(g) \in U(n)$. Then $\tilde{\rho}: \Gamma \rightarrow U(n)$ is a genuine representation. Furthermore,

$$\det \tilde{\rho}(g) = \nu(g)^n = a(g),$$

meaning that $\det \tilde{\rho}$ is a one-dimensional representation of Γ .

Conversely, suppose we are given a genuine representation $\tilde{\rho}: \Gamma \rightarrow U(n)$ such that $\det \tilde{\rho} = a \in A$. Let $\rho(g) := \nu(g)^{-1}\tilde{\rho}(g)$. This defines a (Z, ω) -twisted homomorphism from Γ to $\mathrm{SU}(n)$.

In this manner we establish the following proposition.

Proposition 3.16. *The set $\psi_{Z,w}(\Gamma, \mathrm{SU}(n))$ can be identified with the subset $\det^{-1}(a)$ of $\psi(\Gamma, U(n))$, where $a \in A$ is a representative of w under the map $A \rightarrow A/nA \simeq H^2(B\Gamma, \mathbb{Z}_n)$.*

The action of $H^1(B\Gamma, \mathbb{Z}_n)$ on $\psi_{Z,w}(\Gamma, \mathrm{SU}(n))$ is the action of $\mathrm{Ker}(n \times): A \rightarrow A$ by the tensor product.

Using the fact that $(A/nA)^\wedge = \mathrm{Ker}(n \times): A^\wedge \rightarrow A^\wedge$, we obtain the following proposition.

Proposition 3.17. *There is a natural identification of two vector spaces,*

$$V(\Gamma, U(n)) \quad \text{and} \quad |nA| \text{ copies of } V_Z(\Gamma, \mathrm{SU}(n)),$$

as representations of $H^1(B\Gamma, \mathbb{Z}_n) \times H^2(B\Gamma, \mathbb{Z}_n)^\wedge$, where

$$H^1(B\Gamma, \mathbb{Z}_n) \simeq \mathrm{Ker}(n \times): A \rightarrow A$$

acts via the A action on $V(\Gamma, U(n))$ and

$$H^2(B\Gamma, \mathbb{Z}_n)^\wedge \simeq \mathrm{Ker}(n \times): A^\wedge \rightarrow A^\wedge$$

acts via the $A^\wedge$ action on $V(\Gamma, U(n))$.

Theorem 3.18. *Conjecture 1.9 holds when $(G, \tilde{G}) = (\mathrm{SU}(n), \mathrm{PU}(n))$, $(H, \tilde{H}) = (\mathrm{PU}(n), \mathrm{SU}(n))$ and $Z = \mathbb{Z}_n$.*

Proof. Immediate from the proposition above and Corollary 3.14. □

§4. The case $(G, \tilde{G}) = (\mathrm{Sp}(n), \mathrm{SO}(2n+1))$, arbitrary Γ

In this section we prove the following theorem.

Theorem 4.1. *Conjecture 1.2 holds for arbitrary Γ when $(G, \tilde{G}) = (\mathrm{Sp}(n), \mathrm{SO}(2n+1))$. Namely, we have $N(\Gamma, \mathrm{Sp}(n)) = N(\Gamma, \mathrm{SO}(2n+1))$ for an arbitrary finite subgroup Γ of $\mathrm{SU}(2)$.*

We need some preparation.

§4.1. Generalities

Given a finite group Γ , let us denote by ρ_i its irreducible representations over $\mathbb{C}$.

- When $\bar{\rho}_i \not\simeq \rho_i$, we call ρ_i a complex representation. We denote $\bar{\rho}_i$ by $\rho_{\bar{i}}$, and call $(\rho_i, \rho_{\bar{i}})$ a complex pair.

- When $\bar{\rho}_i \simeq \rho_i$, we call ρ_i a real representation (in a wider sense). Furthermore,
 - if ρ_i is a complexification of a representation over $\mathbb{R}$, we call ρ_i a strictly real representation;
 - if not, we call ρ_i a pseudoreal representation. In this case, ρ_i is obtained by taking a quaternionic action of Γ on a quaternionic vector space $\mathbb{H}^n$ and regarding it as an action on $\mathbb{C}^{2n}$.

Consider a homomorphism $\rho: \Gamma \rightarrow \mathrm{Sp}(n)$. We have an action of Γ on $\mathbb{H}^n$. Regarding $\mathbb{H} = \mathbb{C}^2$, we have an action of Γ on $\mathbb{C}^{2n}$, and then we can decompose it into irreducibles. Let us say ρ contains n_i copies of ρ_i , so that $2n = \sum_i n_i \dim \rho_i$. This lifts to an action on $\mathbb{H}^n$ if and only if

- n_i is even for all strictly real ρ_i , and
- $n_i = n_{\bar{i}}$ for all complex conjugate pairs ρ_i and $\rho_{\bar{i}}$.

Similarly, consider a homomorphism $\rho: \Gamma \rightarrow \mathrm{SO}(2n+1)$. This gives an action of Γ on $\mathbb{R}^{2n+1}$. After complexification, we have an action of Γ on $\mathbb{C}^{2n+1}$, and then we can decompose it into irreducibles. Let us say ρ contains n_i copies of ρ_i , so that $2n+1 = \sum_i n_i \dim \rho_i$. This lifts to an action on $\mathbb{R}^{2n+1}$ if and only if

- n_i is even for all pseudoreal ρ_i , and
- $n_i = n_{\bar{i}}$ for all complex conjugate pairs ρ_i and $\rho_{\bar{i}}$.

This only guarantees that it is a homomorphism $\rho: \Gamma \rightarrow O(2n+1)$. To guarantee that it is a homomorphism into $\mathrm{SO}(2n+1)$, we require $\det \rho$ to be the trivial representation.

Remark 4.2. Note that the roles of strictly real irreducible representations ($\mathbb{R}$) and pseudoreal irreducible representations ($\mathbb{H}$) are swapped when constructing general real representations ($\mathbb{R}$) and general pseudoreal representations ($\mathbb{H}$). These observations will have many repercussions in the generating functions we see below.

§4.2. More properties of irreducible representations of Γ

We now need to know the reality properties of complex irreducible representations of Γ . The trivial representation is strictly real, while the defining two-dimensional representation V coming from $\Gamma \subset \mathrm{SU}(2)$ is clearly pseudoreal, since $\mathrm{SU}(2) \simeq \mathrm{Sp}(1)$ naturally acts on $\mathbb{H}$. The irreducible decomposition after tensoring by V then allows us to determine the reality properties of many of the irreducible representations in a straightforward manner.

4.2.1. $\Gamma = \mathbb{Z}_n$. For $\mathbb{Z}_{2n+1}$, there are $2n+1$ irreducible representations

$$\rho_k(a) = e^{2\pi i k/(2n+1)}, \quad k = -n, -n+1, \dots, +n.$$

Here, ρ_0 is a strictly real representation, and $\rho_{\pm k}$ for $k \neq 0$ are complex pairs of representations.

For $\mathbb{Z}_{2n}$, there are $2n$ irreducible representations ρ_k for $k = -n+1, \dots, n-1, n$, given by

$$\rho_k(a) = e^{2\pi i k / (2n)}.$$

In this case, ρ_0 and ρ_n are strictly real. The rest are complex pairs, such that $\overline{\rho_k} = \rho_{-k}$.

4.2.2. $\widehat{\mathcal{D}}_m$. Let us first describe the structure of $\widehat{\mathcal{D}}_m$ common to both even m and odd m . There are $m-1$ two-dimensional representations ρ_k for $k = 1, \dots, m-1$, given by

$$\rho_{2,k}(a) = \begin{pmatrix} 0 & i^k \\ i^k & 0 \end{pmatrix}, \quad \rho_{2,k}(b) = \begin{pmatrix} \alpha^k & 0 \\ 0 & \alpha^{-k} \end{pmatrix},$$

where $\alpha = e^{\pi i / m}$. We can similarly define $\rho_{2,0}$ and $\rho_{2,m}$, but they decompose into two one-dimensional representations:

$$\rho_{2,0} = \rho_1 \oplus \rho_{1'}, \quad \rho_{2,m} = \rho_{1''} \oplus \rho_{1''' }.$$

Explicitly, we have

$$\begin{array}{c|cccc} & 1 & 1' & 1'' & 1''' \\ \hline a & 1 & -1 & i^m & -i^m \\ b & 1 & 1 & -1 & -1. \end{array}$$

These are the irreducible representations of $\widehat{\mathcal{D}}_m$. They form the extended Dynkin diagram of type D_{m+2} :

$$\begin{array}{ccccccc} & 1' & & & & & 1'' \\ & | & & & & & | \\ 1 & -2 & 1 & -2 & 2 & - \cdots & -2 & m-1 & 1''' \end{array}$$

Let us now specialize to $\widehat{\mathcal{D}}_{2n+1}$. In this case,

- ρ_1 and $\rho_{1'}$ are strictly real;
- $\rho_{1''}$ and $\rho_{1'''}$ form a complex pair;
- $\rho_{2,k}$ for odd k is pseudoreal, while $\rho_{2,k}$ for even k is strictly real. From explicit computations, $\det \rho_{2,k}$ for odd k is 1 and for even k is $1'$.

Let us next discuss $\widehat{\mathcal{D}}_{2n}$. In this case,

- $\rho_1, \rho_{1'}, \rho_{1''}$ and $\rho_{1'''}$ are all strictly real;
- $\rho_{2,k}$ for odd k is pseudoreal, while $\rho_{2,k}$ for even k is strictly real;
- from explicit computations, $\det \rho_{2,k}$ for odd k is 1 and for even k is $1'$.

4.2.3. $\Gamma = \widehat{\mathcal{T}}$. For $\Gamma = \widehat{\mathcal{T}}$ of type E_6 , the irreducible representations can be displayed according to the McKay correspondence as

$$\begin{array}{c} 1'' \\ | \\ 2'' \\ | \\ 1 - 2 - 3 - 2' - 1', \end{array}$$

where we used the dimension and additional primes if necessary to label them. Here, 1 is the trivial representation and 2 is the defining representation coming from the embedding $\Gamma \subset \mathrm{SU}(2)$. Moreover, 1 and 3 are strictly real representations, 2 is a pseudoreal representation, while $(1', 1'')$ and $(2', 2'')$ are complex pairs of irreducible representations. Finally, $\det$ maps 2, 3 to 1, $2'$ to $1''$ and $2''$ to $1'$.

4.2.4. $\Gamma = \widehat{\mathcal{O}}$. For $\Gamma = \widehat{\mathcal{O}}$ of type E_7 , the irreducible representations can be displayed according to the McKay correspondence as

$$\begin{array}{c} 2'' \\ | \\ 1 - 2 - 3 - 4 - 3' - 2' - 1'. \end{array}$$

The 1 is the trivial representation, the 2 is the two-dimensional representation coming from the defining inclusion $\Gamma_{E_7} \subset \mathrm{SU}(2)$. The 3 is the three-dimensional representation obtained by applying the projection $\mathrm{SU}(2) \rightarrow \mathrm{SO}(3)$ to 2. The $1'$ is the sign representation $\rho_{1'}(a) = -1$ and $\rho_{1'}(b) = 1$. Then $2' = 2 \otimes 1'$ and $3' = 3 \otimes 1'$. Finally, the $2''$ is given by

$$(4.3) \quad \rho_{2''}(a) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \rho_{2''}(b) = 120^\circ \text{ rotation}.$$

From these descriptions, we know that 1, $1'$, 3, $3'$ and $2''$ are strictly real representations, while 2, $2'$ and 4 are pseudoreal representations. Moreover, the determinants of all pseudoreal representations are 1.

Among strictly real representations, $\det$ maps 1 and 3 to 1, and the rest to $1'$. Checking this needs some work. We use the explicit presentation of $\widehat{\mathcal{O}}$. As $b^3 = a^4$, $\det(\rho(b))^3 = \det(\rho(a))^4$. Noting $\det \rho(a) = \pm 1$, this forces $\det(\rho(b)^3) = 1$, so $\det(\rho(b)) = 1$. As for $\det \rho(a)$, we know explicitly that $\det \rho(a) = 1$ for 1 (as it is a trivial representation) and for 3 (as it is the projection from the defining representation in $\mathrm{SU}(2)$ to $\mathrm{SO}(3)$). We know $\rho(a) = -1$ for $1'$. Therefore, $\det \rho(a) = -1$ for $3'$. For $2''$, we use the explicit representative given above to find $\det \rho(a) = -1$.

4.2.5. $\Gamma = \widehat{\mathcal{I}}$. Finally, for $\Gamma = \widehat{\mathcal{I}}$ of type E_8 , the irreducible representations can be displayed as

$$\begin{array}{c} 3' \\ | \\ 1 - 2 - 3 - 4 - 5 - 6 - 4' - 2'. \end{array}$$

Here, 1, 3, 5, 4', and 3' are strictly real representations, and 2, 4, 6, 2' are pseudoreal representations.

§4.3. The proof of the basic conjecture

From these data and the discussions above, we have the following result for the generating functions.

Proposition 4.4. *We have the following generating functions for $N(\Gamma, G)$, for $G = \mathrm{Sp}(n)$ and $G = \mathrm{SO}(2n+1)$:*

- For $\Gamma = \mathbb{Z}_m$,

$$\begin{aligned} \sum_{n=0}^{\infty} q^{2n} N(\mathbb{Z}_m, \mathrm{Sp}(n)) &= \frac{1}{(1-q^2)^{\lfloor \frac{m}{2} \rfloor + 1}}, \\ \sum_{n=0}^{\infty} q^{2n+1} N(\mathbb{Z}_m, \mathrm{SO}(2n+1)) &= \frac{1}{2} \sum_{a=0}^1 (-1)^a \frac{1}{1 - (-1)^a q} \frac{1}{(1 - (-1)^{2a} q^2)^{\lfloor \frac{m}{2} \rfloor}}. \end{aligned}$$

- For $\Gamma = \widehat{D}_{2k}$,

$$\begin{aligned} \sum_{n=0}^{\infty} q^{2n} N(\widehat{D}_{2k}, \mathrm{Sp}(n)) &= \frac{1}{(1-q^2)^{k+4}} \frac{1}{(1-q^4)^{k-1}}, \\ \sum_{n=0}^{\infty} q^{2n+1} N(\widehat{D}_{2k}, \mathrm{SO}(2n+1)) &= \frac{1}{2} \sum_{a=0}^1 \frac{1}{2^2} \sum_{b_0, b_1=0}^1 (-1)^a \frac{1}{1 - (-1)^a q} \frac{1}{1 - (-1)^{a+b_0} q} \frac{1}{1 - (-1)^{a+b_1} q} \\ &\quad \times \frac{1}{1 - (-1)^{a+b_0+b_1} q} \frac{1}{(1 - (-1)^{2a+b_0} q^2)^{k-1}} \frac{1}{(1 - (-1)^{4a} q^4)^k}. \end{aligned}$$

- For $\Gamma = \widehat{D}_{2k+1}$,

$$\sum_{n=0}^{\infty} q^{2n} N(\widehat{D}_{2k+1}, \mathrm{Sp}(n)) = \frac{1}{(1-q^2)^{k+3}} \frac{1}{(1-q^4)^k},$$

$$\begin{aligned} & \sum_{n=0}^{\infty} q^{2n+1} N(\widehat{D}_{2k+1}, \mathrm{SO}(2n+1)) \\ &= \frac{1}{2} \sum_{a=0}^1 \frac{1}{2} \sum_{b=0}^1 (-1)^a \frac{1}{1 - (-1)^a q} \frac{1}{1 - (-1)^{a+b} q} \frac{1}{1 - (-1)^{2a} q^2} \\ & \quad \times \frac{1}{(1 - (-1)^{2a+b} q^2)^k} \frac{1}{(1 - (-1)^{4a} q^4)^k}. \end{aligned}$$

- For $\Gamma = \widehat{\mathcal{T}}$,

$$\begin{aligned} & \sum_{n=0}^{\infty} q^{2n} N(\widehat{\mathcal{T}}, \mathrm{Sp}(n)) = \frac{1}{(1 - q^2)^3} \frac{1}{1 - q^4} \frac{1}{1 - q^6}, \\ & \sum_{n=0}^{\infty} q^{2n+1} N(\widehat{\mathcal{T}}, \mathrm{SO}(2n+1)) \\ &= \frac{1}{2} \sum_{a=0}^1 (-1)^a \frac{1}{1 - (-1)^a q} \frac{1}{1 - (-1)^{2a} q^2} \frac{1}{1 - (-1)^{3a} q^3} \frac{1}{(1 - (-1)^{4a} q^4)^2}, \end{aligned}$$

- For $\Gamma = \widehat{\mathcal{O}}$,

$$\begin{aligned} & \sum_{n=0}^{\infty} q^{2n} N(\widehat{\mathcal{O}}, \mathrm{Sp}(n)) = \frac{1}{(1 - q^2)^4} \frac{1}{(1 - q^4)^2} \frac{1}{(1 - q^6)^2}, \\ & \sum_{n=0}^{\infty} q^{2n+1} N(\widehat{\mathcal{O}}, \mathrm{SO}(2n+1)) \\ &= \frac{1}{2} \sum_{a=0}^1 \frac{1}{2} \sum_{b=0}^1 (-1)^a \frac{1}{1 - (-1)^a q} \frac{1}{1 - (-1)^{a+b} q} \frac{1}{1 - (-1)^{2a+b} q^2} \\ & \quad \times \frac{1}{1 - (-1)^{3a} q^3} \frac{1}{1 - (-1)^{3a+b} q^3} \frac{1}{(1 - (-1)^{4a} q^4)^2} \frac{1}{1 - (-1)^{8a} q^8}, \end{aligned}$$

- For $\Gamma = \widehat{\mathcal{I}}$,

$$\begin{aligned} & \sum_{n=0}^{\infty} q^{2n} N(\widehat{\mathcal{I}}, \mathrm{Sp}(n)) = \frac{1}{(1 - q^2)^3} \frac{1}{1 - q^4} \frac{1}{(1 - q^6)^3} \frac{1}{1 - q^8} \frac{1}{1 - q^{10}}, \\ & \sum_{n=0}^{\infty} q^{2n+1} N(\widehat{\mathcal{I}}, \mathrm{SO}(2n+1)) \\ &= \frac{1}{2} \sum_{a=0}^1 (-1)^a \frac{1}{1 - (-1)^a q} \frac{1}{(1 - (-1)^{3a} q^3)^2} \frac{1}{(1 - (-1)^{4a} q^4)^3} \\ & \quad \times \frac{1}{1 - (-1)^{5a} q^5} \frac{1}{1 - (-1)^{8a} q^8} \frac{1}{1 - (-1)^{12a} q^{12}}. \end{aligned}$$

Proof. We prove only the simplest case, $\Gamma = \mathbb{Z}_{\text{odd}}$, and the most complicated case, $\Gamma = \widehat{D}_{\text{even}}$. The other cases can be proved analogously.

- When $(\Gamma, G) = (\mathbb{Z}_{2k+1}, \text{Sp}(n))$,

$$\begin{aligned} \sum_{n=0}^{\infty} q^{2n} N(\mathbb{Z}_{2k+1}, \text{Sp}(n)) &= \sum_{n=0}^{\infty} \sum_{\substack{0 \leq l_0, \dots, l_k \\ 2(l_0 + \dots + l_k) = 2n}} q^{2n} \\ &= \sum_{0 \leq l_0, \dots, l_k} q^{2(l_0 + \dots + l_k)} = \frac{1}{(1 - q^2)^{k+1}}. \end{aligned}$$

- When $(\Gamma, G) = (\mathbb{Z}_{2k+1}, \text{SO}(2n+1))$,

$$\begin{aligned} \sum_{n=0}^{\infty} q^{2n+1} N(\mathbb{Z}_{2k+1}, \text{SO}(2n+1)) &= \frac{1}{2} \sum_{a=0}^1 (-1)^a \sum_{n=0}^{\infty} (-1)^{na} q^n N(\mathbb{Z}_{2k+1}, \text{SO}(n)) \\ &= \frac{1}{2} \sum_{a=0}^1 (-1)^a \sum_{n=0}^{\infty} \sum_{\substack{0 \leq l_0, \dots, l_k \\ l_0 + 2(l_1 + \dots + l_k) = n}} (-1)^{na} q^n \\ &= \frac{1}{2} \sum_{a=0}^1 (-1)^a \frac{1}{1 - (-1)^a q} \frac{1}{(1 - (-1)^{2a} q^2)^k}. \end{aligned}$$

- When $(\Gamma, G) = (\widehat{D}_{2k}, \text{Sp}(n))$,

$$\begin{aligned} \sum_{n=0}^{\infty} q^{2n} N(\widehat{D}_{2k}, \text{Sp}(n)) &= \sum_{n=0}^{\infty} \sum_{\substack{0 \leq l_1, \dots, l_4, m_1, \dots, m_{2k-1} \\ 2(l_1 + \dots + l_4) + 2(m_1 + \dots + m_{2k-1}) + 4(m_2 + \dots + m_{2k-2}) = 2n}} q^{2n} \\ &= \frac{1}{(1 - q^2)^4} \frac{1}{(1 - q^2)^k} \frac{1}{(1 - q^4)^{k-1}}. \end{aligned}$$

- When $(\Gamma, G) = (\widehat{D}_{2k}, \text{SO}(2n+1))$,

$$\begin{aligned} \sum_{n=0}^{\infty} q^{2n+1} N(\widehat{D}_{2k}, \text{SO}(2n+1)) &= \frac{1}{2} \sum_{a=0}^1 (-1)^a \sum_{n=0}^{\infty} (-1)^{na} q^n N(\widehat{D}_{2k}, \text{SO}(n)) \\ &= \frac{1}{2} \sum_{a=0}^1 (-1)^a \sum_{n=0}^{\infty} \sum_{\substack{0 \leq l_1, \dots, l_4, m_1, \dots, m_{2k-1} \\ l_1 + \dots + l_4 + 4(m_1 + \dots + m_{2k-1}) + 2(m_2 + \dots + m_{2k-2}) = n \\ (l_2 + l_4 + m_2 + \dots + m_{2k-2}) \text{ even}, (l_3 + l_4) \text{ even}}} (-1)^{na} q^n \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2} \sum_{a=0}^1 (-1)^a \sum_{n=0}^{\infty} \sum_{\substack{0 \leq l_1, \dots, l_4, m_1, \dots, m_{2k-1} \\ l_1 + \dots + l_4 + 4(m_1 + \dots + m_{2k-1}) + 2(m_2 + \dots + m_{2k-2}) = n}} \\
&\quad \times \frac{1}{2^2} \sum_{b_0, b_1=0}^1 (-1)^{b_0(l_2 + l_4 + m_2 + \dots + m_{2k-2}) + b_1(l_3 + l_4)} (-1)^{na} q^n \\
&= \frac{1}{2} \sum_{a=0}^1 \frac{1}{2^2} \sum_{b_0, b_1=0}^1 (-1)^a \frac{1}{1 - (-1)^a q} \frac{1}{1 - (-1)^{a+b_0} q} \frac{1}{1 - (-1)^{a+b_1} q} \\
&\quad \times \frac{1}{1 - (-1)^{a+b_0+b_1} q} \frac{1}{(1 - (-1)^{4a} q^4)^k} \frac{1}{(1 - (-1)^{2a+b_0} q^2)^{k-1}}. \quad \square
\end{aligned}$$

We now prove the following three propositions, which are a bit more general than the original statements we had to check. The first one corresponds to the case $\Gamma = \mathbb{Z}_n, \widehat{\mathcal{T}}, \widehat{\mathcal{L}}$, the second one corresponds to the case $\Gamma = \widehat{\mathcal{D}}_{\text{odd}}, \widehat{\mathcal{O}}$ and the last one corresponds to the case $\Gamma = \widehat{\mathcal{D}}_{\text{even}}$.

Proposition 4.5. *Let $l \in \mathbb{Z}_{\geq 0}$, $k_1, \dots, k_4, v_1, \dots, v_l \in \mathbb{N}$. Set*

$$\begin{cases} F_1(q) = \prod_{r=1}^1 \frac{1}{1 - q^{4k_r-2}} \prod_{i=1}^l \frac{1}{1 - q^{2v_i}}, \\ \widetilde{F}_1(q) = \prod_{r=1}^1 \frac{1}{1 - q^{2k_r-1}} \prod_{i=1}^l \frac{1}{1 - q^{2v_i}}. \end{cases}$$

For $k_1 \neq k_2$, set

$$\begin{cases} F_2(q) = \frac{1}{1 - q^{2k_2-2k_1}} \prod_{r=1}^2 \frac{1}{1 - q^{4k_r-2}} \prod_{i=1}^l \frac{1}{1 - q^{2v_i}}, \\ \widetilde{F}_2(q) = \frac{1}{1 - q^{4k_2-4k_1}} \prod_{r=1}^2 \frac{1}{1 - q^{2k_r-1}} \prod_{i=1}^l \frac{1}{1 - q^{2v_i}}. \end{cases}$$

For $k_1 + k_2 = k_3 + k_4$, $\{k_1, k_2\} \neq \{k_3, k_4\}$, set

$$\begin{cases} F_4(q) = \frac{1}{1 - q^{2k_1+2k_2-2}} \frac{1}{1 - q^{2k_3-2k_1}} \frac{1}{1 - q^{2k_4-2k_1}} \prod_{r=1}^4 \frac{1}{1 - q^{4k_r-2}} \prod_{i=1}^l \frac{1}{1 - q^{2v_i}}, \\ \widetilde{F}_4(q) = \frac{1}{1 - q^{4k_1+4k_2-4}} \frac{1}{1 - q^{4k_3-4k_1}} \frac{1}{1 - q^{4k_4-4k_1}} \prod_{r=1}^4 \frac{1}{1 - q^{2k_r-1}} \prod_{i=1}^l \frac{1}{1 - q^{2v_i}}. \end{cases}$$

Then we have

$$q^{2k_1-1} F_s(q) = \frac{1}{2} \sum_{a=0}^1 (-1)^a \widetilde{F}_s((-1)^a q)$$

for all $s = 1, 2, 4$.

Proof. It is sufficient to consider the case $l = 0$. We first note the following identities:

$$\begin{aligned}
 \text{(I)} \quad & (1 + q^{2k_1-1}) - (1 - q^{2k_1-1}) = 2q^{2k_1-1}, \\
 \text{(II)} \quad & \prod_{r=1}^2 (1 + q^{2k_r-1}) - \prod_{r=1}^2 (1 - q^{2k_r-1}) = 2q^{2k_1-1} (1 + q^{2k_2-2k_1}), \\
 & \prod_{r=1}^4 (1 + q^{2k_r-1}) - \prod_{r=1}^4 (1 - q^{2k_r-1}) = 2q^{2k_1-1} (1 + q^{2k_1+2k_2-2}) \\
 \text{(IV)} \quad & \times (1 + q^{2k_3-2k_1})(1 + q^{2k_4-2k_1}),
 \end{aligned}$$

where the last identity uses the condition $k_1 + k_2 = k_3 + k_4$. We refer to these identities as (I), (II) and (IV). Each case $s = 1, 2, 4$ follows from (I), (II) and (IV), respectively. For instance, in the case $s = 4$, we have

$$\begin{aligned}
 & \frac{1}{2} \sum_{a=0}^1 (-1)^a \tilde{F}_4((-1)^a q) \\
 &= \frac{1}{2} \frac{1}{1 - q^{4k_1+4k_2-4}} \frac{1}{1 - q^{4k_3-4k_1}} \frac{1}{1 - q^{4k_4-4k_1}} \left(\prod_{r=1}^4 \frac{1}{1 - q^{2k_r-1}} - \prod_{r=1}^4 \frac{1}{1 + q^{2k_r-1}} \right) \\
 &= \frac{1}{1 - q^{4k_1+4k_2-4}} \frac{1}{1 - q^{4k_3-4k_1}} \frac{1}{1 - q^{4k_4-4k_1}} \\
 & \quad \times \frac{q^{2k_1-1} (1 + q^{2k_3-2k_1})(1 + q^{2k_4-2k_1})(1 + q^{2k_1+2k_2-2})}{(1 - q^{4k_1-2})(1 - q^{4k_2-2})(1 - q^{4k_3-2})(1 - q^{4k_4-2})} \\
 &= q^{2k_1-1} \frac{1}{1 - q^{2k_1+2k_2-2}} \frac{1}{1 - q^{2k_3-2k_1}} \frac{1}{1 - q^{2k_4-2k_1}} \prod_{r=1}^4 \frac{1}{1 - q^{4k_r-2}} \\
 &= q^{2k_1-1} F_4(q),
 \end{aligned}$$

and the remaining cases follow in the same way. $\square$

Proposition 4.6. Let $l_0, l_1 \in \mathbb{Z}_{\geq 0}$, $k_1, k_2, v_{0,1}, \dots, v_{0,l_0}, v_{1,1}, \dots, v_{1,l_1} \in \mathbb{N}$. Set

$$\begin{cases} F_2(q) = \prod_{r=1}^1 \frac{1}{(1 - q^{4k_r-2})^2} \prod_{i=1}^{l_0} \frac{1}{1 - q^{2v_{0,i}}} \prod_{j=1}^{l_1} \frac{1}{1 - q^{2v_{1,j}}}, \\ \tilde{F}_2(q, t) = \prod_{r=1}^1 \frac{1}{(1 - q^{2k_r-1})(1 - tq^{2k_r-1})} \prod_{i=1}^{l_0} \frac{1}{1 - q^{2v_{0,i}}} \prod_{j=1}^{l_1} \frac{1}{1 - tq^{2v_{1,j}}}. \end{cases}$$

For $k_1 \neq k_2$, set

$$\left\{ \begin{array}{l} F_4(q) = \frac{1}{1 - q^{2k_1+2k_2-2}} \frac{1}{1 - q^{2k_2-2k_1}} \prod_{r=1}^2 \frac{1}{(1 - q^{4k_r-2})^2} \prod_{i=1}^{l_0} \frac{1}{1 - q^{2v_{0,i}}} \\ \quad \times \prod_{j=1}^{l_1} \frac{1}{1 - q^{2v_{1,j}}}, \\ \tilde{F}_4(q, t) = \frac{1}{1 - q^{4k_1+4k_2-4}} \frac{1}{1 - q^{4k_2-4k_1}} \prod_{r=1}^2 \frac{1}{(1 - q^{2k_r-1})(1 - tq^{2k_r-1})} \\ \quad \times \prod_{i=1}^{l_0} \frac{1}{1 - q^{2v_{0,i}}} \prod_{j=1}^{l_1} \frac{1}{1 - tq^{2v_{1,j}}}. \end{array} \right.$$

Then we have

$$q^{2k_1-1} F_s(q) = \frac{1}{2} \sum_{a=0}^1 \frac{1}{2} \sum_{b=0}^1 (-1)^a \tilde{F}_s((-1)^a q, (-1)^b)$$

for both $s = 2$ and 4 .

Proof. Each case $s = 2, 4$ follows from the identities (II) and (IV) in the proof of the previous proposition, respectively. We first consider the case $s = 4$. We easily find that

$$\tilde{F}_4(q, -1) = \tilde{F}_4(-q, -1)$$

and these terms in the sum on the right-hand side of the equality cancel each other. Therefore, only the terms with $t = 1$ remain, so that we have

$$\frac{1}{2} \sum_{a=0}^1 \frac{1}{2} \sum_{b=0}^1 (-1)^a \tilde{F}_4((-1)^a q, (-1)^b) = \frac{1}{4} \sum_{a=0}^1 (-1)^a \tilde{F}_4((-1)^a q, 1).$$

As in the last proposition, it is enough to prove the case when $l_0 = l_1 = 0$. Under these conditions, we have

$$\begin{aligned} & \frac{1}{4} \sum_{a=0}^1 (-1)^a \tilde{F}_4((-1)^a q, 1) \\ &= \frac{1}{4} \frac{1}{1 - q^{4k_1+4k_2-4}} \frac{1}{1 - q^{4k_2-4k_1}} \left(\prod_{r=1}^2 \frac{1}{(1 - q^{2k_r-1})^2} - \prod_{r=1}^2 \frac{1}{(1 + q^{2k_r-1})^2} \right) \\ &= \frac{1}{4} \frac{1}{1 - q^{4k_1+4k_2-4}} \frac{1}{1 - q^{4k_2-4k_1}} \cdot \frac{4q^{2k_1-1}(1 + q^{2k_2-2k_1})(1 + q^{2k_1+2k_2-2})}{(1 - q^{4k_1-2})^2(1 - q^{4k_2-2})^2} \end{aligned}$$

$$\begin{aligned}
&= q^{2k_1-1} \frac{1}{1-q^{2k_1+2k_2-2}} \frac{1}{1-q^{2k_2-2k_1}} \prod_{r=1}^2 \frac{1}{(1-q^{4k_r-2})^2} \\
&= q^{2k_1-1} F_4(q),
\end{aligned}$$

which is what we wanted to prove. The case $s = 2$ can be proved in the same way. $\square$

Proposition 4.7. *Let $l_{00}, l_{01}, l_{10}, l_{11} \in \mathbb{Z}_{\geq 0}$, $k, v_{00,1}, \dots, v_{00,l_{00}}, v_{01,1}, \dots, v_{11,l_{11}} \in \mathbb{N}$ and set*

$$\begin{cases} F(q) = \frac{1}{(1-q^{4k-2})^5} \prod_{p_0=0}^1 \prod_{p_1=0}^1 \prod_{i=1}^{l_{p_1 p_0}} \frac{1}{1-q^{2v_{p_1 p_0, i}}}, \\ \tilde{F}(q, t_0, t_1) = \frac{1}{1-q^{8k-4}} \prod_{p_0=0}^1 \prod_{p_1=0}^1 \left(\frac{1}{1-t_1^{p_1} t_0^{p_0} q^{2k-1}} \prod_{i=1}^{l_{p_1 p_0}} \frac{1}{1-t_1^{p_1} t_0^{p_0} q^{2v_{p_1 p_0, i}}} \right). \end{cases}$$

Then we have

$$q^{2k-1} F(q) = \frac{1}{2} \sum_{a=0}^1 \frac{1}{2} \sum_{b_0=0}^1 \frac{1}{2} \sum_{b_1=0}^1 (-1)^a \tilde{F}((-1)^a q, (-1)^{b_0}, (-1)^{b_1}).$$

Proof. We easily find that

$$\begin{aligned}
\tilde{F}(q, 1, -1) &= \tilde{F}(-q, 1, -1), \\
\tilde{F}(q, -1, 1) &= \tilde{F}(-q, -1, 1), \\
\tilde{F}(q, -1, -1) &= \tilde{F}(-q, -1, -1).
\end{aligned}$$

Hence, as before, we have

$$\frac{1}{2} \sum_{a=0}^1 \frac{1}{2} \sum_{b_0=0}^1 \frac{1}{2} \sum_{b_1=0}^1 (-1)^a \tilde{F}((-1)^a q, (-1)^{b_0}, (-1)^{b_1}) = \frac{1}{8} \sum_{a=0}^1 (-1)^a \tilde{F}((-1)^a q, 1, 1).$$

Clearly, it is sufficient to prove the case when $l_{00} = l_{01} = l_{10} = l_{11} = 0$. Under these conditions, we have

$$\begin{aligned}
\frac{1}{8} \sum_{a=0}^1 (-1)^a \tilde{F}((-1)^a q, 1, 1) &= \frac{1}{8} \frac{1}{1-q^{8k-4}} \left(\frac{1}{(1-q^{2k-1})^4} - \frac{1}{(1+q^{2k-1})^4} \right) \\
&= \frac{1}{8} \frac{1}{1-q^{8k-4}} \cdot \frac{8q^{2k-1}(1+q^{4k-2})}{(1-q^{4k-2})^4} \\
&= q^{2k-1} \frac{1}{(1-q^{4k-2})^5} \\
&= q^{2k-1} F(q),
\end{aligned}$$

where we used identity (IV) in the proof of Proposition 4.5 in the second equality. $\square$

Proof of Theorem 4.1. The generating function in each case is given in Proposition 4.4:

- For $\Gamma = \mathbb{Z}_m$, put $(s; k_1; l; v_i) = (1; 1; \lfloor \frac{m}{2} \rfloor; 1)$ in Proposition 4.5.
- For $\Gamma = \widehat{\mathcal{T}}$, put $(s; k_1, k_2; l; v_1, v_2) = (2; 1, 2; 2; 1, 2)$ in Proposition 4.5.
- For $\Gamma = \widehat{\mathcal{I}}$, put $(s; k_1, k_2, k_3, k_4; l; v_1, v_2) = (4; 1, 3, 2, 2; 2; 2, 4)$ in Proposition 4.5.
- For $\Gamma = \widehat{\mathcal{D}}_{2m+1}$, put $(s; k_1; l_0, l_1; v_{0,1}, v_{0,2}, \dots, v_{0,m+1}, v_{1,j}) = (2; 1; m+1, m; 1, 2, \dots, 2, 1)$ in Proposition 4.6.
- For $\Gamma = \widehat{\mathcal{O}}$, put $(s; k_1, k_2; l_0, l_1; v_{0,1}, v_{1,1}) = (4; 1, 2; 1, 1; 2, 1)$ in Proposition 4.6.
- Finally, for $\Gamma = \widehat{\mathcal{D}}_{2m}$, put $(k; l_{00}, l_{01}, l_{10}, l_{11}; v_{00,i}, v_{01,i}) = (1; m-1, m-1, 0, 0; 2, 1)$ in Proposition 4.7.

This completes the proof. $\square$

§5. The case $(G, \widetilde{G}) = (\mathrm{PSp}(n), \mathrm{Spin}(2n+1))$, $\Gamma = \widehat{\mathcal{O}}$

In the previous section, we treated the case when $(G, \widetilde{G}) = (\mathrm{Sp}(n), \mathrm{SO}(2n+1))$. Using that knowledge, let us additionally discuss some cases when $(G, \widetilde{G}) = (\mathrm{PSp}(n), \mathrm{Spin}(2n+1))$ and the ADE type of Γ is exceptional.

Our aim is to establish the theorem below.

Theorem 5.1. *We have $N(\Gamma, \mathrm{PSp}(n)) = N(\Gamma, \mathrm{Spin}(2n+1))$ for $\Gamma = \widehat{\mathcal{T}}, \widehat{\mathcal{O}}, \widehat{\mathcal{I}}$.*

We first deal with the easy case $\Gamma = \widehat{\mathcal{T}}$ or $\widehat{\mathcal{I}}$.

Proposition 5.2. *For $\Gamma = \widehat{\mathcal{T}}$ or $\widehat{\mathcal{I}}$, we have $N(\Gamma, \mathrm{Sp}(n)) = N(\Gamma, \mathrm{PSp}(n))$ and $N(\Gamma, \mathrm{SO}(2n+1)) = N(\Gamma, \mathrm{Spin}(2n+1))$.*

Proof. Recall that a homomorphism $f: \Gamma \rightarrow G/Z$, where Z is a central Abelian subgroup of G , defines a homomorphism $\check{f}: \check{\Gamma} \rightarrow G$, where $\check{\Gamma}$ is a central extension $0 \rightarrow Z \rightarrow \check{\Gamma} \rightarrow \Gamma \rightarrow 0$, such that $\check{f}$ maps Z identically. Let $Z = \mathbb{Z}_2$. Then $H^2(B\Gamma, \mathbb{Z}_2) = 0$ for $\Gamma = \widehat{\mathcal{T}}$ and $\Gamma = \widehat{\mathcal{I}}$. This means that any such $\check{f}: \check{\Gamma} \rightarrow G$ can be modified to a homomorphism $\Gamma \rightarrow G$. Therefore, there is a one-to-one correspondence between homomorphisms from Γ to G and homomorphisms from Γ to $G/\mathbb{Z}_2$. $\square$

We now restrict our attention to the case $\Gamma = \widehat{\mathcal{O}}$. In this case, it is no more difficult to prove the finer version of the conjecture, so we are going to establish the following theorem.

Theorem 5.3. *Conjecture 1.9 holds for the case $\Gamma = \widehat{\mathcal{O}}$, $G = \mathrm{Sp}(n)$, $H = \mathrm{PSp}(n)$, $\widetilde{H} = \mathrm{Spin}(2n+1)$, $\widetilde{G} = \mathrm{SO}(2n+1)$ and $Z = \mathbb{Z}_2$.*

The proof of this theorem will occupy the bulk of the rest of this paper.

Corollary 5.4. *We have $N(\widehat{\mathcal{O}}, \mathrm{PSp}(n)) = N(\widehat{\mathcal{O}}, \mathrm{Spin}(2n+1))$.*

Proof. Apply Corollary 1.12 to Theorem 5.3. □

Proof of Theorem 5.1. Immediate from Proposition 5.2 and Corollary 5.4. □

We can now concentrate on proving Theorem 5.3. Before doing this, we need some more general discussions, and some more properties of irreducible representations of $\Gamma = \widehat{\mathcal{O}}$.

§5.1. Even more properties of irreducible representations of $\Gamma = \widehat{\mathcal{O}}$

We let $\mathbf{G}$ be either $G = \mathrm{Sp}(n)$ or $\widetilde{H} = \mathrm{Spin}(2n+1)$. Our presentation of $\widehat{\mathcal{O}}$ is

$$\widehat{\mathcal{O}} = \langle a, b \mid a^4 = b^3 = (ab)^2 \rangle.$$

The set $\psi_{\mathbb{Z}_2, m}(\widehat{\mathcal{O}}, \mathbf{G})$ for $m \in H^2(B\widehat{\mathcal{O}}, \mathbb{Z}_2) = \mathbb{Z}_2$ consists of pairs $(\rho(a), \rho(b)) \in \mathbf{G}^2$ satisfying $\rho(a)^4 = \rho(b)^3 = (-1)^m(\rho(a)\rho(b))^2$, up to simultaneous conjugation by $\mathbf{G}$.

The only nontrivial one-dimensional representation of $\widehat{\mathcal{O}}$, which we previously denoted by $1'$ but we now denote by x , is $x(a) = -1$, $x(b) = 1$. It generates $\mathbb{Z}_2$, and acts on $\psi_{\mathbb{Z}_2, m}(\widehat{\mathcal{O}}, \mathbf{G})$ by

$$x: (\rho(a), \rho(b)) \mapsto (\rho'(a), \rho'(b)) := (-\rho(a), \rho(b)).$$

Note that ρ and ρ' can still be conjugate in $\mathbf{G}$.

We let

$$V_{\mathbb{Z}_2, m}(\widehat{\mathcal{O}}, \mathbf{G}) = V_{\mathbb{Z}_2, m}^0(\widehat{\mathcal{O}}, \mathbf{G}) \oplus V_{\mathbb{Z}_2, m}^1(\widehat{\mathcal{O}}, \mathbf{G}),$$

where $V_{\mathbb{Z}_2, m}^e(\widehat{\mathcal{O}}, \mathbf{G})$ is the subspace on which x acts by $(-1)^e$. Similarly, we let

$$\psi_{\mathbb{Z}_2, m}(\widehat{\mathcal{O}}, \mathbf{G}) = \psi_{\mathbb{Z}_2, m}^{\text{fixed}}(\widehat{\mathcal{O}}, \mathbf{G}) \sqcup \psi_{\mathbb{Z}_2, m}^{\text{not fixed}}(\widehat{\mathcal{O}}, \mathbf{G})$$

where the superscripts “fixed”, “not fixed” are with respect to the action by x .

Proposition 5.5. *We have*

$$\begin{aligned} \dim V_{\mathbb{Z}_2, m}^0(\widehat{\mathcal{O}}, \mathbf{G}) &= \#\psi_{\mathbb{Z}_2, m}^{\text{fixed}}(\widehat{\mathcal{O}}, \mathbf{G}) + \frac{1}{2}\#\psi_{\mathbb{Z}_2, m}^{\text{not fixed}}(\widehat{\mathcal{O}}, \mathbf{G}), \\ \dim V_{\mathbb{Z}_2, m}^1(\widehat{\mathcal{O}}, \mathbf{G}) &= \frac{1}{2}\#\psi_{\mathbb{Z}_2, m}^{\text{not fixed}}(\widehat{\mathcal{O}}, \mathbf{G}). \end{aligned}$$

Proof. Immediate by expanding the definitions. $\square$

To compute $\dim V_{\mathbb{Z}_2, m}^e(\widehat{\mathcal{O}}, \mathbb{G})$, we need to learn to enumerate $\psi_{\mathbb{Z}_2, m}(\widehat{\mathcal{O}}, \mathbb{G})$, and understand the action of x . We look into each of them in turn.

5.1.1. $\psi_{\mathbb{Z}_2, m}(\widehat{\mathcal{O}}, \mathbf{Sp}(n))$. We already understand $\psi_{\mathbb{Z}_2, 0}(\widehat{\mathcal{O}}, \mathbf{Sp}(n)) = \psi(\widehat{\mathcal{O}}, \mathbf{Sp}(n))$. To enumerate $\psi_{\mathbb{Z}_2, 1}(\widehat{\mathcal{O}}, \mathbf{Sp}(n))$, we need to enumerate twisted homomorphisms $\tilde{\rho}$ given by pairs satisfying $\tilde{\rho}(a)^4 = \tilde{\rho}(b)^3 = -\tilde{\rho}(ab)^2$.

To study them, note that $\tilde{\rho}(a), \tilde{\rho}(b)$ act on $\mathbb{H}^n$. Regard $\mathbb{H}^n$ as $\mathbb{C}^{2n}$, i.e., we partially forget the quaternionic structure and keep only the complex structure. We then define $\rho(a) := -i\tilde{\rho}(a)$ and $\rho(b) := \tilde{\rho}(b)$. They satisfy $\rho(a)^4 = \rho(b)^3 = \rho(ab)^2$, and ρ is a genuine complex representation of Γ . This means that twisted representations $\tilde{\rho}$, in turn, allow an irreducible decomposition in terms of $(i\rho(a), \rho(b))$, where ρ runs over ordinary irreducible representations of Γ . This operation preserves the (complex) dimensions and the action by x , but it changes the reality properties of representations.

Call the resulting irreducible twisted representations

$$\begin{array}{c} \tilde{2}'' \\ | \\ \tilde{1} - \tilde{2} - \tilde{3} - \tilde{4} - \tilde{3}' - \tilde{2}' - \tilde{1}'. \end{array}$$

Due to the multiplication by i for a , their reality conditions do change. It is easy to see that $(\tilde{1}, \tilde{1}'), (\tilde{2}, \tilde{2}'), (\tilde{3}, \tilde{3}')$ form complex conjugate pairs. For $\tilde{2}''$, using explicit matrices given in equation (4.3), we can directly see that both $\tilde{\rho}_{2''}(a)$ and $\tilde{\rho}_{2''}(b)$ are in $\mathrm{SU}(2)$, and so it is pseudoreal. As $\tilde{4} \otimes 2 = \tilde{3} + \tilde{3}' + \tilde{2}''$, it means that $\tilde{4}$ is strictly real.

The multiplication by x still exchanges $\tilde{1}$ and $\tilde{1}'$, $\tilde{2}$ and $\tilde{2}'$, and $\tilde{3}$ and $\tilde{3}'$, while fixing $\tilde{2}''$.

5.1.2. $\psi_{\mathbb{Z}_2, m}(\widehat{\mathcal{O}}, \mathbf{Spin}(2n+1))$. In this case, what we can easily study are homomorphisms to $\mathrm{SO}(2n+1)$ rather than those to $\mathrm{Spin}(2n+1)$. So we need to study homomorphisms to $\mathrm{SO}(2n+1)$, and then discuss when and how they lift to (genuine and twisted) homomorphisms to $\mathrm{Spin}(2n+1)$.

Consider $\tilde{\rho}: \Gamma \rightarrow O(2n+1)$ up to conjugation, which can be easily counted via irreducible decomposition. Say $\tilde{\rho}$ contains n_i copies of the irreducible representation ρ_i , so that $2n+1 = \sum n_i \dim \rho_i$. We need to do three things:

- (1) Restrict to those which are actually in $\mathrm{SO}(2n+1)$.
- (2) Decide whether it lifts to a genuine ($m=0$) or a twisted ($m=1$) homomorphism into $\mathrm{Spin}(2n+1)$.

- (3) In each case, decide whether the lifts are fixed by x or form a pair exchanged by x .

Issues (1) and (2) can be solved using a bit of basic algebraic topology. Each real representation ρ has an associated total Stiefel–Whitney class $w(\rho)$. The degree- k term of $w(\rho)$ is denoted by $w_k(\rho)$, the k -th Stiefel–Whitney class. We only need w_1 and w_2 , so we regard $w(\rho) \in \mathbb{Z}_2[y]/(y^3)$, where y is the generator of $H^1(B\widehat{O}, \mathbb{Z}_2) = \mathbb{Z}_2$, by a slight abuse of notation. As $w(\rho_1 \oplus \rho_2) = w(\rho_1)w(\rho_2)$, we have

$$w(\rho) = \prod_i w(\rho_i)^{n_i}.$$

So to compute $w(\rho)$, we only need to know $w(\rho_i)$. Then

- a real representation ρ (i.e., a homomorphism to O) lifts to an SO representation if and only if $w_1(\rho) = 0$;
- an SO representation lifts to a genuine ($m = 0$) or a twisted ($m = 1$) homomorphism into $\text{Spin}(2n+1)$ depending on whether $w(\rho) = 1$ or $w(\rho) = 1 + y^2$. Equivalently, $w_2(\rho) = my^2$.

Any real representation of $\widehat{O}$ is a direct sum of copies of the strictly real representations 1, 3, $2''$, $3'$, $1'$ or the underlying real representations of pseudoreal representations 2, 4 and $2'$.

The Stiefel–Whitney classes of the underlying real representations of pseudo-real representations are all trivial, for the following reasons. Given a pseudoreal representation $\Gamma \subset \text{Sp}(k) \curvearrowright \mathbb{H}^k$, we are interested in the behavior of $G \subset \text{SO}(4k) \curvearrowright \mathbb{R}^{4k}$, where $G \subset [\text{Sp}(k) \times \text{Sp}(1)]/\{\pm 1\} \subset \text{SO}(4k)$. This inclusion fits into the following diagram:

$$\begin{array}{ccc} \text{Sp}(k) \times \text{Sp}(1) & \hookrightarrow & \text{Spin}(4k) \\ \downarrow & & \downarrow \\ (\text{Sp}(k) \times \text{Sp}(1))/\{\pm 1\} & \hookrightarrow & \text{SO}(4k), \end{array}$$

meaning that $G \subset \text{Sp}(k) \subset \mathbb{H}^k$ regarded as $G \subset \text{SO}(4k) \subset \mathbb{R}^{4k}$ automatically lifts to $\text{Spin}(4k)$.

The Stiefel–Whitney classes of irreducible strictly real representations of $\widehat{O}$ are not hard to determine, either. By definition, $w(1) = 1$ and $w(1') = 1 + y$. Next, $w(3) = 1$, because the action of $\widehat{O}$ to 3 is by definition obtained by reducing the action of $\widehat{O} \subset \text{SU}(2)$ to $\widehat{O} \subset \text{SO}(3)$. To determine $w(3')$, we use $w(1')w(3') = w(2' \otimes 2)$. But a real representation obtained by tensoring a quaternionic representation by the defining representation is automatically a spin representation,

because of the same diagram above; the only difference is that now Γ is embedded diagonally to both $\mathrm{Sp}(k)$ and $\mathrm{Sp}(1)$. Therefore, $w(2' \otimes 2) = 1$, and therefore $w(3') = (1+y)^{-1} = 1+y+y^2$. Applying the same argument to $w(3)w(3')w(2'') = w(4 \otimes 2)$, we get $w(2'') = 1+y$.

The discussions up to this point take care of issues (1) and (2) raised above. We still need to discuss issue (3).

Suppose we are given a homomorphism $\rho: \widehat{\mathcal{O}} \rightarrow \mathrm{SO}(2n+1)$ which lifts to a (genuine or twisted) homomorphism $\tilde{\rho}$ from $\widehat{\mathcal{O}}$ to $\mathrm{Spin}(2n+1)$. The action of x fixes $\tilde{\rho}$ up to conjugation by $\mathrm{Spin}(2n+1)$ if and only if there is a $\tilde{g} \in \mathrm{Spin}(2n+1)$ such that

$$\tilde{g}\tilde{\rho}(a)\tilde{g}^{-1} = -\tilde{\rho}(a), \quad \tilde{g}\tilde{\rho}(b)\tilde{g}^{-1} = \tilde{\rho}(b).$$

Such a $\tilde{g} \in \mathrm{Spin}(k)$ determines a corresponding $g \in \mathrm{SO}(k)$ such that

$$g\rho(a)g^{-1} = \rho(a), \quad g\rho(b)g^{-1} = \rho(b).$$

Now, a representation of a finite group Γ on $\mathbb{R}^k$ has a decomposition

$$\mathbb{R}^k \otimes \mathbb{C} = \bigoplus \rho_{\mathbb{R},i}^{\oplus s_i} \oplus \bigoplus (\rho_{\mathbb{H},i} \oplus \rho_{\mathbb{H},i})^{\oplus t_i} \oplus \bigoplus (\rho_{\mathbb{C},i} \oplus \overline{\rho_{\mathbb{C},i}})^{\oplus u_i},$$

where $\rho_{\mathbb{R},\mathbb{H},\mathbb{C},i}$ list the irreducible representations of Γ of the indicated types. Any $O(k)$ matrix commuting with the Γ action is in the subgroup

$$X := \prod_i O(s_i) \times \prod_i \mathrm{Sp}(t_i) \times \prod_i U(u_i)$$

which acts by permuting the copies of the same irreducible representation; this is a generalization of Schur's lemma from complex representations to real representations. Our g is in $\mathrm{SO}(k)$, so we further need a requirement that $g = (g_{\mathbb{R},i}; g_{\mathbb{H},i}; g_{\mathbb{C},i})$ satisfies $\prod_i (\det g_{\mathbb{R},i})^{\dim \rho_{\mathbb{R},i}} = +1$.

Now, given a $g \in X \cap \mathrm{SO}(k)$, we have

$$(5.6) \quad \tilde{g}\tilde{\rho}(a)\tilde{g}^{-1} = \pm \tilde{\rho}(a)$$

with a $+$ or $-$ sign. This sign is a representation $X \cap \mathrm{SO}(k) \rightarrow \mathbb{Z}_2$. As such it is locally a constant, and therefore it only depends on the connected component of $X \cap \mathrm{SO}(k)$, which is just a product of a number of $\mathbb{Z}_2$'s. To explicitly describe these $\mathbb{Z}_2$'s, let h_i be an element from the component of $O(s_i)$ disconnected from the identity. Then an over-complete basis of these $\mathbb{Z}_2$'s is given by (1) h_i for $\dim \rho_{\mathbb{R},i}$ is even and (2) $h_j h_k$ for $\dim \rho_{\mathbb{R},j}$, $\dim \rho_{\mathbb{R},k}$ are both odd. The sign appearing in (5.6) can be found by a direct computation, and gives

$$\det \rho_i(a)$$

for $g = h_i$ and

$$\det \rho_j(a) \det \rho_k(a)$$

for $g = h_j h_k$.

Applying this consideration for $\Gamma = \widehat{\mathcal{O}}$ and using our knowledge of types of ρ_i and also of $\det \rho_i$, we obtain the following proposition.

Proposition 5.7. *Given a homomorphism $\rho: \Gamma \rightarrow \mathrm{SO}(2n+1)$, consider its complexification $\rho_{\mathbb{C}}: \Gamma \rightarrow U(2n+1)$ and let n_i be the number of copies of the irreducible representation ρ_i appearing in the direct sum decomposition of $\rho_{\mathbb{C}}$. Let $\tilde{\rho}$ be a (genuine or twisted) homomorphism of Γ to $\mathrm{Spin}(2n+1)$ obtained by lifting ρ . Then $\tilde{\rho}$ is fixed by the action of x if and only if*

$$n_{2''} > 0 \text{ or } (n_1 + n_3 > 0 \text{ and } n_{1'} + n_{3'} > 0).$$

§5.2. The proof of the refined conjecture

With the preparation done, we can finally proceed to the proof of Theorem 5.3.

Proposition 5.8. *The set $\psi_{\mathbb{Z}_2,0}(\widehat{\mathcal{O}}, \mathrm{Sp}(n))$ can be identified with the set of integer solutions to*

$$2k_1 + 2n_2 + 6k_3 + 4n_4 + 6k_{3'} + 4k_{2'} + 2k_{1'} + 4k_{2''} = 2n.$$

The action of x is given by

$$(k_1, n_2, k_3, n_4, k_{3'}, k_{2'}, k_{1'}; k_{2''}) \mapsto (k_{1'}, k_{2'}, k_{3'}, n_4, k_3, n_2, k_1; k_{2''}).$$

Similarly, the set $\psi_{\mathbb{Z}_2,1}(\widehat{\mathcal{O}}, \mathrm{Sp}(n))$ can be identified with the set of integer solutions to

$$2n_1 + 4n_2 + 6n_3 + 8k_4 + 2n_{2''} = 2n,$$

and the action of x is trivial.

Proof. For a homomorphism $\rho: \widehat{\mathcal{O}} \rightarrow \mathrm{Sp}(n)$, let $\rho_{\mathbb{C}}: \widehat{\mathcal{O}} \rightarrow U(2n)$ be the homomorphism obtained by composing with $\mathrm{Sp}(n) \rightarrow U(2n)$. Let n_i be the number of copies of irreducible representation ρ_i in the irreducible decomposition of $\rho_{\mathbb{C}}$. As ρ is pseudoreal, n_i for $i = 1, 3, 1', 2''$ are even. We let $n_i = 2k_i$ for these cases. The action of x can be inferred from the data already given above.

For a twisted homomorphism ρ from $\widehat{\mathcal{O}}$ to $\mathrm{Sp}(n)$, let n_i be the number of copies of the irreducible twisted representation $\tilde{\rho}_i$. As ρ is pseudoreal, $n_1 = n_{1'}$, $n_2 = n_{2'}$, $n_3 = n_{3'}$ and n_4 is even. Writing $n_4 = 2k_4$, we obtain the integer equation given above. The action of x can also be inferred from the data already given. $\square$

Proposition 5.9. *The set $\psi_{\mathbb{Z}_2, m}^{\text{fixed by } x}(\widehat{\mathcal{O}}, \text{Spin}(2n+1))$ can be identified with the set of integer solutions to*

$$n_1 + 4k_2 + 3n_3 + 8k_4 + 3n_{3'} + 4k_{2'} + n_{1'} + 2n_{2''} = 2n + 1,$$

with the condition

$$n_{1'} - n_{3'} + n_{2''} = 2m \bmod 4,$$

and

$$n_{2''} > 0 \text{ or } (n_1 + n_3 > 0 \text{ and } n_{1'} + n_{3'} > 0).$$

The set $\psi_{\mathbb{Z}_2, m}^{\text{not fixed by } x}(\widehat{\mathcal{O}}, \text{Spin}(2n+1))$, modulo the action of x , can be identified with the set of integer solutions to

$$n_1 + 4k_2 + 3n_3 + 8k_4 + 3n_{3'} + 4k_{2'} + n_{1'} + 2n_{2''} = 2n + 1,$$

with the condition

$$n_{1'} - n_{3'} + n_{2''} = 2m \bmod 4,$$

and

$$n_{2''} = 0 \text{ and } (n_1 + n_3 = 0 \text{ or } n_{1'} + n_{3'} = 0).$$

Proof. For a homomorphism $\rho: \widehat{\mathcal{O}} \rightarrow \text{SO}(2n+1)$, let $\rho_{\mathbb{C}}: \widehat{\mathcal{O}} \rightarrow U(2n+1)$ be its complexification. Let n_i be the number of copies of ρ_i in $\rho_{\mathbb{C}}$. Then n_i needs to be even when ρ_i is pseudoreal, and therefore $n_2 = 2k_2$, $n_4 = 2k_4$, $n_{2'} = 2k_{2'}$, from which we find $n_1 + 4k_2 + 3n_3 + 8k_4 + 3n_{3'} + 4k_{2'} + n_{1'} + 2n_{2''} = 2n + 1$. From the condition that $\det \rho = 1$, we find that $n_{1'} + n_{3'} + n_{2''}$ is even. Under this condition, the action of x can be studied using Proposition 5.7, resulting in the statements of this proposition. $\square$

The following is a slight rephrasing of our main theorem, Theorem 5.3.

Proposition 5.10. *We have*

$$\dim V_{\mathbb{Z}_2, m}^e(\widehat{\mathcal{O}}, \text{Sp}(n)) = \dim V_{\mathbb{Z}_2, e}^m(\widehat{\mathcal{O}}, \text{Spin}(2n+1))$$

for all $e = 0, 1$ and $m = 0, 1$.

Proof. Using Proposition 5.5, the statement of this proposition is seen to be equivalent to the following four statements:

- For $(e, m) = (0, 0)$, we have

$$\begin{aligned} & \# \psi_{\mathbb{Z}_2, 0}^{\text{fixed}}(\widehat{\mathcal{O}}, \text{Sp}(n)) + \frac{1}{2} \# \psi_{\mathbb{Z}_2, 0}^{\text{not fixed}}(\widehat{\mathcal{O}}, \text{Sp}(n)) \\ (5.11) \quad & = \# \psi_{\mathbb{Z}_2, 0}^{\text{fixed}}(\widehat{\mathcal{O}}, \text{Spin}(2n+1)) + \frac{1}{2} \# \psi_{\mathbb{Z}_2, 0}^{\text{not fixed}}(\widehat{\mathcal{O}}, \text{Spin}(2n+1)). \end{aligned}$$

- For $(e, m) = (1, 0)$, we have

$$(5.12) \quad \begin{aligned} & \frac{1}{2} \# \psi_{\mathbb{Z}_2, 0}^{\text{not fixed}}(\widehat{\mathcal{O}}, \text{Sp}(n)) \\ &= \# \psi_{\mathbb{Z}_2, 1}^{\text{fixed}}(\widehat{\mathcal{O}}, \text{Spin}(2n+1)) + \frac{1}{2} \# \psi_{\mathbb{Z}_2, 1}^{\text{not fixed}}(\widehat{\mathcal{O}}, \text{Spin}(2n+1)). \end{aligned}$$

- For $(e, m) = (0, 1)$, we have

$$(5.13) \quad \begin{aligned} & \# \psi_{\mathbb{Z}_2, 1}^{\text{fixed}}(\widehat{\mathcal{O}}, \text{Sp}(n)) + \frac{1}{2} \# \psi_{\mathbb{Z}_2, 1}^{\text{not fixed}}(\widehat{\mathcal{O}}, \text{Sp}(n)) \\ &= \frac{1}{2} \# \psi_{\mathbb{Z}_2, 0}^{\text{not fixed}}(\widehat{\mathcal{O}}, \text{Spin}(2n+1)). \end{aligned}$$

- For $(e, m) = (1, 1)$, we have

$$(5.14) \quad \frac{1}{2} \# \psi_{\mathbb{Z}_2, 1}^{\text{not fixed}}(\widehat{\mathcal{O}}, \text{Sp}(n)) = \frac{1}{2} \# \psi_{\mathbb{Z}_2, 1}^{\text{not fixed}}(\widehat{\mathcal{O}}, \text{Spin}(2n+1)).$$

Now, note that

$$\text{the left-hand side of (5.11) + the left-hand side of (5.12) = } N(\widehat{\mathcal{O}}, \text{Sp}(n))$$

and that

$$\text{the right-hand side of (5.11) + the right-hand side of (5.12) = } N(\widehat{\mathcal{O}}, \text{SO}(2n+1)),$$

and that the equality $N(\widehat{\mathcal{O}}, \text{Sp}(n)) = N(\widehat{\mathcal{O}}, \text{SO}(2n+1))$ was proved in Theorem 4.1. Therefore, we only have to prove either (5.11) or (5.12).

Note also that, from Proposition 5.8, $\frac{1}{2} \# \psi_{\mathbb{Z}_2, 1}^{\text{not fixed}}(\widehat{\mathcal{O}}, \text{Sp}(n)) = 0$. Therefore, equation (5.13) is equivalent to

$$(5.15) \quad \# \psi_{\mathbb{Z}_2, 1}^{\text{fixed}}(\widehat{\mathcal{O}}, \text{Sp}(n)) = \frac{1}{2} \# \psi_{\mathbb{Z}_2, 0}^{\text{not fixed}}(\widehat{\mathcal{O}}, \text{Spin}(2n+1)),$$

and equation (5.14) is equivalent to

$$(5.16) \quad \# \psi_{\mathbb{Z}_2, 1}^{\text{not fixed}}(\widehat{\mathcal{O}}, \text{Spin}(2n+1)) = 0.$$

We can conclude the proof of this proposition, and therefore the main theorem Theorem 5.3 then, by proving (5.11), (5.15) and (5.16), which are Propositions 5.19, 5.20, 5.21 given below, respectively. $\square$

To prove Propositions 5.19, 5.20, and 5.21, we use generating functions as we did in Section 4.

Proposition 5.17. *We have the following generating functions for $(e, m) = (0, 0)$, $(0, 1)$, $(1, 1)$, which will be used in the proofs of Propositions 5.19, 5.20, 5.21, respectively:*

- For $(e, m) = (0, 0)$,

$$\begin{aligned}
 & \sum_{n=0}^{\infty} q^{2n} \left(\# \psi_{\mathbb{Z}_2, 0}^{\text{fixed}}(\widehat{\mathcal{O}}, \text{Sp}(n)) + \frac{1}{2} \# \psi_{\mathbb{Z}_2, 0}^{\text{not fixed}}(\widehat{\mathcal{O}}, \text{Sp}(n)) \right) \\
 &= \frac{1}{2} \left(\frac{1}{(1-q^2)^4} \frac{1}{(1-q^4)^2} \frac{1}{(1-q^6)^2} + \frac{1}{(1-q^4)^4} \frac{1}{1-q^{12}} \right), \\
 & \sum_{n=0}^{\infty} q^{2n+1} \left(\# \psi_{\mathbb{Z}_2, 0}^{\text{fixed}}(\widehat{\mathcal{O}}, \text{Spin}(2n+1)) + \frac{1}{2} \# \psi_{\mathbb{Z}_2, 0}^{\text{not fixed}}(\widehat{\mathcal{O}}, \text{Spin}(2n+1)) \right) \\
 &= \frac{1}{2} \sum_{a=0}^1 \frac{1}{4} \sum_{b=0}^3 (-1)^a \frac{1}{1-(-1)^a q} \frac{1}{1-(-1)^a i^b q} \frac{1}{1-(-1)^{2a} i^b q^2} \frac{1}{1-(-1)^{3a} q^3} \\
 & \quad \times \frac{1}{1-(-1)^{3a} i^{-b} q^3} \frac{1}{(1-(-1)^{4a} q^4)^2} \frac{1}{1-(-1)^{8a} q^8}.
 \end{aligned}$$

- For $(e, m) = (0, 1)$,

$$\begin{aligned}
 & \sum_{n=0}^{\infty} q^{2n} (\# \psi_{\mathbb{Z}_2, 1}^{\text{fixed}}(\widehat{\mathcal{O}}, \text{Sp}(n))) = \frac{1}{(1-q^2)^2} \frac{1}{1-q^4} \frac{1}{1-q^6} \frac{1}{1-q^8}, \\
 & \sum_{n=0}^{\infty} q^{2n+1} \left(\frac{1}{2} \# \psi_{\mathbb{Z}_2, 0}^{\text{not fixed}}(\widehat{\mathcal{O}}, \text{Spin}(2n+1)) \right) \\
 &= \frac{1}{2} \sum_{a=0}^1 \frac{1}{4} \sum_{b=0}^3 (-1)^a \left(\frac{1}{1-i^b(-1)^a q} \frac{1}{1-i^{-b}(-1)^{3a} q^3} \right. \\
 & \quad \left. + \frac{1}{1-(-1)^a q} \frac{1}{1-(-1)^{3a} q^3} - 1 \right) \\
 & \quad \times \frac{1}{(1-(-1)^{4a} q^4)^2} \frac{1}{1-(-1)^{8a} q^8}.
 \end{aligned}$$

- For $(e, m) = (1, 1)$,

$$\begin{aligned}
 & \sum_{n=0}^{\infty} q^{2n+1} \left(\frac{1}{2} \# \psi_{\mathbb{Z}_2, 1}^{\text{not fixed}}(\widehat{\mathcal{O}}, \text{Spin}(2n+1)) \right) \\
 &= \frac{1}{2} \sum_{a=0}^1 \frac{1}{4} \sum_{b=0}^3 (-1)^a i^{-2b} \left(\frac{1}{1-i^b(-1)^a q} \frac{1}{1-i^{-b}(-1)^{3a} q^3} \right. \\
 & \quad \left. + \frac{1}{1-(-1)^a q} \frac{1}{1-(-1)^{3a} q^3} - 1 \right) \\
 & \quad \times \frac{1}{(1-(-1)^{4a} q^4)^2} \frac{1}{1-(-1)^{8a} q^8}.
 \end{aligned}$$

Proof. The basic idea is the same as in Proposition 4.4. We give only the proof of the case when $(e, m) = (0, 0)$. The others can be proved in a similar way.

First, we consider the Sp side. From Proposition 4.4, we know

$$\sum_{n=0}^{\infty} q^{2n} (\#\psi_{\mathbb{Z}_2,0}(\widehat{\mathcal{O}}, \mathrm{Sp}(n))) = \frac{1}{(1-q^2)^4} \frac{1}{(1-q^4)^2} \frac{1}{(1-q^6)^2}.$$

In the same way, we can find

$$\sum_{n=0}^{\infty} q^{2n} (\#\psi_{\mathbb{Z}_2,0}^{\mathrm{fixed}}(\widehat{\mathcal{O}}, \mathrm{Sp}(n))) = \frac{1}{(1-q^4)^4} \frac{1}{1-q^{12}}.$$

Putting these together, we have

$$\begin{aligned} & \sum_{n=0}^{\infty} q^{2n} \left(\#\psi_{\mathbb{Z}_2,0}^{\mathrm{fixed}}(\widehat{\mathcal{O}}, \mathrm{Sp}(n)) + \frac{1}{2} \#\psi_{\mathbb{Z}_2,0}^{\mathrm{not fixed}}(\widehat{\mathcal{O}}, \mathrm{Sp}(n)) \right) \\ &= \frac{1}{(1-q^4)^4} \frac{1}{1-q^{12}} + \frac{1}{2} \left(\frac{1}{(1-q^2)^4} \frac{1}{(1-q^4)^2} \frac{1}{(1-q^6)^2} - \frac{1}{(1-q^4)^4} \frac{1}{1-q^{12}} \right) \\ &= \frac{1}{2} \left(\frac{1}{(1-q^2)^4} \frac{1}{(1-q^4)^2} \frac{1}{(1-q^6)^2} + \frac{1}{(1-q^4)^4} \frac{1}{1-q^{12}} \right). \end{aligned}$$

Next, we prove the Spin side. From Proposition 5.9, we know that

$$\#\psi_{\mathbb{Z}_2,0}^{\mathrm{fixed}}(\widehat{\mathcal{O}}, \mathrm{Spin}(2n+1)) + \frac{1}{2} \#\psi_{\mathbb{Z}_2,0}^{\mathrm{not fixed}}(\widehat{\mathcal{O}}, \mathrm{Spin}(2n+1))$$

equals the number of integer solutions to

$$n_1 + 4k_2 + 3n_3 + 8k_4 + 3n_{3'} + 4k_{2'} + n_{1'} + 2n_{2''} = 2n + 1,$$

with the condition

$$n_{1'} - n_{3'} + n_{2''} = 0 \pmod{4}.$$

Therefore, the generating function is calculated as follows:

$$\begin{aligned} & \sum_{n=0}^{\infty} q^{2n+1} \left(\#\psi_{\mathbb{Z}_2,0}^{\mathrm{fixed}}(\widehat{\mathcal{O}}, \mathrm{Spin}(2n+1)) + \frac{1}{2} \#\psi_{\mathbb{Z}_2,0}^{\mathrm{not fixed}}(\widehat{\mathcal{O}}, \mathrm{Spin}(2n+1)) \right) \\ &= \frac{1}{2} \sum_{a=0}^1 (-1)^a \sum_{n=0}^{\infty} (-1)^{an} q^n \left(\#\psi_{\mathbb{Z}_2,0}^{\mathrm{fixed}}(\widehat{\mathcal{O}}, \mathrm{Spin}(n)) + \frac{1}{2} \#\psi_{\mathbb{Z}_2,0}^{\mathrm{not fixed}}(\widehat{\mathcal{O}}, \mathrm{Spin}(n)) \right) \\ &= \frac{1}{2} \sum_{a=0}^1 (-1)^a \sum_{n=0}^{\infty} \sum_{\substack{0 \leq l_1, \dots, l_8 \\ l_1 + 4l_2 + 3l_3 + 8l_4 + 3l_5 + 4l_6 + l_7 + 2l_8 = n \\ (l_7 - l_5 + l_8) \equiv 0 \pmod{4}}} (-1)^{an} q^n \\ &= \frac{1}{2} \sum_{a=0}^1 (-1)^a \sum_{n=0}^{\infty} \sum_{\substack{0 \leq l_1, \dots, l_8 \\ l_1 + 4l_2 + 3l_3 + 8l_4 + 3l_5 + 4l_6 + l_7 + 2l_8 = n}} \frac{1}{4} \sum_{b=0}^3 (-1)^{an} i^{b(l_7 - l_5 + l_8)} q^n \end{aligned}$$

$$= \frac{1}{2} \sum_{a=0}^1 \frac{1}{4} \sum_{b=0}^3 (-1)^a \frac{1}{1 - (-1)^a q} \frac{1}{1 - (-1)^a i b q} \frac{1}{1 - (-1)^{2a} i b q^2} \frac{1}{1 - (-1)^{3a} q^3} \\ \times \frac{1}{1 - (-1)^{3a} i^{-b} q^3} \frac{1}{(1 - (-1)^{4a} q^4)^2} \frac{1}{1 - (-1)^{8a} q^8},$$

which matches the desired expression. $\square$

Now let us prove the main propositions. We consider a general case to prove the first one, as in the previous section.

Proposition 5.18. *Let $l \in \mathbb{Z}_{\geq 0}$, $k_1, k_2, v_1, \dots, v_l \in \mathbb{N}$. For $k_1 \neq k_2$, set*

$$\left\{ \begin{array}{l} F(q) = \frac{1}{1 - q^{2k_1+2k_2-2}} \frac{1}{(1 - q^{2k_2-2k_1})^2} \prod_{r=1}^2 \frac{1}{(1 - q^{4k_r-2})^2} \prod_{i=1}^l \frac{1}{1 - q^{2v_i}}, \\ F_0(q) = \frac{1}{1 - q^{2k_1+2k_2-2}} \frac{1}{1 - q^{4k_2-4k_1}} \prod_{r=1}^2 \frac{1}{1 - q^{8k_r-4}} \prod_{i=1}^l \frac{1}{1 - q^{2v_i}}, \\ \tilde{F}(q, t) = \frac{1}{1 - q^{4k_1+4k_2-4}} \frac{1}{1 - q^{4k_2-4k_1}} \frac{1}{1 - tq^{2k_2-2k_1}} \frac{1}{(1 - q^{2k_1-1})(1 - tq^{2k_1-1})} \\ \quad \times \frac{1}{(1 - q^{2k_2-1})(1 - t^{-1}q^{2k_2-1})} \prod_{i=1}^l \frac{1}{1 - q^{2v_i}}. \end{array} \right.$$

Then we have

$$q^{2k_1-1} \frac{F(q) + F_0(q)}{2} = \frac{1}{2} \sum_{a=0}^1 \frac{1}{4} \sum_{b=0}^3 (-1)^a \tilde{F}((-1)^a q, i^b).$$

Proof. It suffices to prove the case when $l = 0$. Then we have

$$\frac{1}{2} \sum_{a=0}^1 \frac{1}{4} \sum_{b=0}^3 (-1)^a \tilde{F}((-1)^a q, i^b) \\ = \frac{1}{2} \frac{1}{1 - q^{4k_1+4k_2-4}} \frac{1}{1 - q^{4k_2-4k_1}} \left\{ \frac{1}{1 - q^{2k_2-2k_1}} \right. \\ \times \frac{q^{2k_1-1}(1 + q^{2k_2-2k_1})(1 + q^{2k_1+2k_2-2})}{(1 - q^{4k_1-2})^2(1 - q^{4k_2-2})^2} \\ \left. + \frac{q^{2k_1-1}(1 + q^{2k_1+2k_2-2})}{(1 - q^{8k_1-4})(1 - q^{8k_2-4})} \right\}$$

$$\begin{aligned}
&= \frac{q^{2k_1-1}}{2} \frac{1}{1-q^{2k_1+2k_2-2}} \left\{ \frac{1}{(1-q^{2k_2-2k_1})^2} \prod_{r=1}^2 \frac{1}{(1-q^{4k_r-2})^2} \right. \\
&\quad \left. + \frac{1}{1-q^{4k_2-4k_1}} \prod_{r=1}^2 \frac{1}{1-q^{8k_r-4}} \right\} \\
&= q^{2k_1-1} \frac{F(q) + F_0(q)}{2}.
\end{aligned}$$

This completes the proof. $\square$

Proposition 5.19. Equation (5.11) holds, i.e.,

$$\begin{aligned}
&q \sum_{n=0}^{\infty} q^{2n} \left(\# \psi_{\mathbb{Z}_2,0}^{\text{fixed}}(\widehat{\mathcal{O}}, \text{Sp}(n)) + \frac{1}{2} \# \psi_{\mathbb{Z}_2,0}^{\text{not fixed}}(\widehat{\mathcal{O}}, \text{Sp}(n)) \right) \\
&= \sum_{n=0}^{\infty} q^{2n+1} \left(\# \psi_{\mathbb{Z}_2,0}^{\text{fixed}}(\widehat{\mathcal{O}}, \text{Spin}(2n+1)) + \frac{1}{2} \# \psi_{\mathbb{Z}_2,0}^{\text{not fixed}}(\widehat{\mathcal{O}}, \text{Spin}(2n+1)) \right).
\end{aligned}$$

Proof. Both sides are explicitly computed in Proposition 5.17. From these expressions, we see that setting $(k_1, k_2; l; v_1) = (1, 2; 1; 2)$ in Proposition 5.18 yields the desired result. $\square$

Proposition 5.20. Equation (5.15) holds, i.e.,

$$q \sum_{n=0}^{\infty} q^{2n} (\# \psi_{\mathbb{Z}_2,1}^{\text{fixed}}(\widehat{\mathcal{O}}, \text{Sp}(n))) = \sum_{n=0}^{\infty} q^{2n+1} \left(\frac{1}{2} \# \psi_{\mathbb{Z}_2,0}^{\text{not fixed}}(\widehat{\mathcal{O}}, \text{Spin}(2n+1)) \right).$$

Proof. We start from

$$\begin{aligned}
&\sum_{n=0}^{\infty} q^{2n+1} \left(\frac{1}{2} \# \psi_{\mathbb{Z}_2,0}^{\text{not fixed}}(\widehat{\mathcal{O}}, \text{Spin}(2n+1)) \right) \\
&= \frac{1}{2} \sum_{a=0}^1 \frac{1}{4} \sum_{b=0}^3 (-1)^a \left(\frac{1}{1-i^b(-1)^a q} \frac{1}{1-i^{-b}(-1)^{3a} q^3} \right. \\
&\quad \left. + \frac{1}{1-(-1)^a q} \frac{1}{1-(-1)^{3a} q^3} - 1 \right) \\
&\quad \times \frac{1}{(1-(-1)^{4a} q^4)^2} \frac{1}{1-(-1)^{8a} q^8}.
\end{aligned}$$

As only the second term in the parentheses survives the sum over a and b , we can continue the computation above as

$$= \frac{1}{8} \frac{1}{(1-q^4)^2(1-q^8)} \sum_{a=0}^1 \sum_{b=0}^3 (-1)^a \frac{1}{1-(-1)^a q} \frac{1}{1-(-1)^{3a} q^3}$$

$$\begin{aligned}
&= \frac{1}{2} \frac{1}{(1-q^4)^2(1-q^8)} \frac{2q(1+q^2)}{(1-q^2)(1-q^6)} \\
&= q \sum_{n=0}^{\infty} q^{2n} (\#\psi_{\mathbb{Z}_2,1}^{\text{fixed}}(\widehat{\mathcal{O}}, \text{Sp}(n))),
\end{aligned}$$

which is what we wanted to prove. $\square$

Note that the proof of Proposition 5.20 essentially came down to Proposition 4.5. The following proposition can be proved in a similar manner.

Proposition 5.21. *Equation (5.16) holds, i.e.,*

$$\sum_{n=0}^{\infty} q^{2n+1} (\#\psi_{\mathbb{Z}_2,1}^{\text{not fixed}}(\widehat{\mathcal{O}}, \text{Spin}(2n+1))) = 0.$$

Proof. Indeed, we have

$$\begin{aligned}
&\sum_{n=0}^{\infty} q^{2n+1} \left(\frac{1}{2} \#\psi_{\mathbb{Z}_2,1}^{\text{not fixed}}(\widehat{\mathcal{O}}, \text{Spin}(2n+1)) \right) \\
&= \frac{1}{2} \sum_{a=0}^1 \frac{1}{4} \sum_{b=0}^3 (-1)^a i^{-2b} \frac{1}{(1 - (-1)^{4a} q^4)^2} \frac{1}{1 - (-1)^{8a} q^8} \\
&\quad \times \left(\frac{1}{1 - i^b (-1)^a q} \frac{1}{1 - i^{-b} (-1)^{3a} q^3} \right. \\
&\quad \left. + \frac{1}{1 - (-1)^a q} \frac{1}{1 - (-1)^{3a} q^3} - 1 \right) \\
&= 0,
\end{aligned}$$

as each of the three terms in the parentheses cancels out upon the summation over a and b . $\square$

This concludes the proofs of the propositions used in the proof of Proposition 5.10, which in turn establishes our main claim in this section, Theorem 5.3.

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