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Optimal Regularity for One-Dimensional Porous Medium Flow

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Abstract

We give a new proof of the Lipschitz continuity with respect to t of the pressure in a one dimensional porous medium flow. As is shown by the Barenblatt solution, this is the optimal t-regularity for the pressure. Our proof is based on the existence and properties of a certain selfsimilar solution.

In recent years there has been considerable interest in the regularity of nonnegative solutions $u = u(x, t)$ to the porous medium equation

$$\frac{\partial u}{\partial t} = \Delta(u^m)$$

in $\mathbb{R}^d \times \mathbb{R}^+$, where $m > 1$ is a constant. For $d > 1$ the theory is still in flux and the optimal global regularity results are as yet unknown. Partial results can be found in [CVW] and [A2]. For $d = 1$ it is known [A1] that

$$v \equiv \frac{m}{m-1} u^{m-1}.$$

is Lipschitz continuous as a function of x, and this is the optimal regularity with respect to x. The Lipschitz continuity of v implies that u is Hölder continuous with exponent $\alpha = \min\{1, 1/(m-1)\}$. Kruzhkov [Kr] proved that for a class of parabolic equations which includes the porous medium equation, Hölder continuity in x with exponent α implies Hölder continuity in t

with exponent $\alpha/(\alpha + 2)$. Gilding [G] refined Kruzhkov's result to obtain the t-exponent $\alpha/2$. On the other hand, by assuming certain monotonicity for v_{xx}, Di Benedetto [DiB] proved that v is Lipschitz in t.

Actually, v is Lipschitz continuous in t without any assumptions on v_{xx}. This result was first proved by Bénilan [B] by means of a clever comparison argument. In this note we give an alternate proof which also uses comparison methods, but which is completely different from Bénilan's. In particular, our proof is based on a selfsimilar solution of the porous medium equation which has some independent interest.

We consider the initial value problem

$$\begin{aligned} u_t &= (u^m)_{xx} \quad \text{in} \quad \mathbb{R} \times \mathbb{R}^+, \\ u(\cdot, 0) &= u_0 \quad \text{in} \quad \mathbb{R}, \end{aligned} \tag{1}$$

where $m > 1$ is constant and $u_0 \geqslant 0$. For simplicity we assume that $u_0 \in L^\infty(\mathbb{R}) \cap$ $\cap L^1(\mathbb{R})$. It is known that problem (1) possesses a unique generalized solution $u = u(x, t)$ in $\mathbb{R} \times \mathbb{R}^+$ with

$$0 \leqslant u \leqslant \|u_0\|_{L^\infty(\mathbb{R})}.$$

For isentropic flow of a perfect gas in a homegeneous porus medium u represents an appropriately scaled density. The corresponding pressure, given by

$$v \equiv \frac{m}{m-1} u^{m-1},$$

satisfies the equation

$$v_t = (m-1) v v_{xx} + v_x^2 \tag{2}$$

on the set where u is positive. For v we have the estimates

$$0 \leqslant v(x, t) \leqslant \|v_0\|_{L^\infty(\mathbb{R})} \quad \text{in} \quad \mathbb{R} \times \mathbb{R}^+, \tag{3}$$

$$|v(x, t)|^2 \leqslant \frac{2}{(m+1)t} \|v_0\|_{L^\infty(\mathbb{R})} \quad \text{a.e.} \quad \text{in} \quad \mathbb{R} \times \mathbb{R}^+, \tag{4}$$

and

$$v_t(x, t) \geqslant -\frac{m-1}{m+1} \frac{\|v_0\|_{L^\infty(\mathbb{R})}}{t} \quad \text{in} \quad \mathfrak{D}'(\mathbb{R} \times \mathbb{R}^+). \tag{5}$$

Here $v_0 = m u_0^{m-1}/(m-1)$. For definitions, proofs and references the reader can consult [A2].

Our main result is the following

Theorem. *Let v be the pressure corresponding to the solution u of problem* (1). *For every $\delta > 0$ there exists a constant $C = C(\delta, m, \|v_0\|_{L^\infty(\mathbb{R})}) \in \mathbb{R}^+$ such that*

$$|v(x, t') - v(x, t)| \leqslant C|t' - t|$$

for all (x, t') and (x, t) in $\mathbb{R} \times [\delta, \infty)$.

The proof of this theorem is based on two propositions. The first describes a selfsimilar solution of the pressure equation (2) which is then used in the second proposition to estimate the growth of v.

Proposition 1. *The initial value problem*

$$\begin{aligned} &v_t = (m-1)vv_{xx} + v_x^2 \quad \text{in} \quad \mathbb{R} \times \mathbb{R}^+ \\ &v(x, 0) = |x| \qquad\qquad\quad\;\; \text{in} \quad \mathbb{R} \end{aligned} \tag{6}$$

possess a unique solution $v = p(x, t)$, where p has the form

$$p(x, t) = rf(\theta) \tag{7}$$

with $r = \{x^2 + t^2\}^{1/2}$ and $\theta = \arctan(x/t)$. Here $f \in C^1[-\frac{\pi}{2}, \frac{\pi}{2}]$ with $f'(0) = 0$, $f(\pm\frac{\pi}{2}) = 1$, $f'(\pm\frac{\pi}{2}) = \mp 1$, and

$$f(\theta) > \cos\theta + |\sin\theta|.$$

Remark. According to the results of [AV], as $m \downarrow 1$ the solution of (6) tends to the solution $v = q(x, t)$ of the initial value problem

$$\begin{aligned} &v_t = v_x^2 \qquad\quad \text{in} \quad \mathbb{R} \times \mathbb{R}^+ \\ &v(x, 0) = |x| \quad \text{in} \quad \mathbb{R}. \end{aligned}$$

In particular,

$$q(x, t) = r(\cos\theta + |\sin\theta|).$$

Thus $f(\theta) \to \cos\theta + |sin \theta|$ as $m \downarrow 1$. The (computed) graphs of $f(\theta)$ are shown in the next page in figure 1 for various values of m.

PROOF. The global existence and uniqueness of the solution $v = p(x, t)$ of (6) follows from the results of Kalashnikov [K]. Moreover, $p > 0$ in $\mathbb{R} \times \mathbb{R}^+$ so that $p \in C^\infty(\mathbb{R} \times \mathbb{R}^+)$. For any $\lambda \in \mathbb{R}^+$ define

$$p_\lambda(x, t) \equiv \frac{1}{\lambda} p(\lambda x, \lambda t).$$

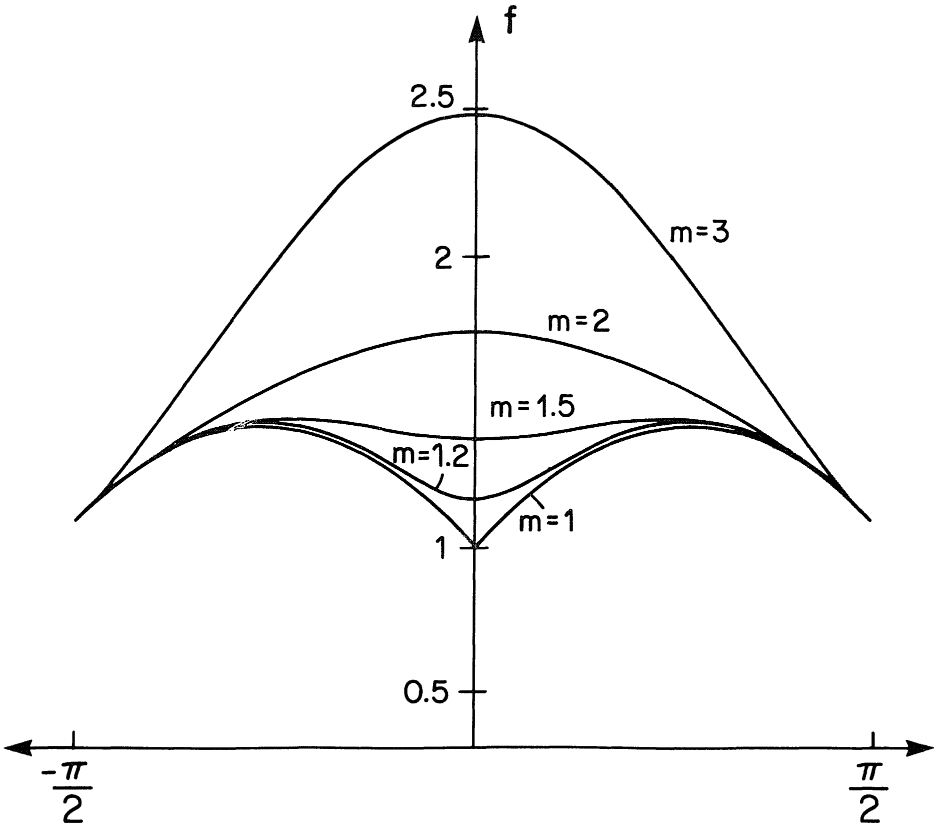

Fig. 1.

It is easy to verify that p_λ is a solution to the pressure equation (2) in $\mathbb{R} \times \mathbb{R}^+$ regardless of the value of $\lambda \in \mathbb{R}^+$. Moreover

$$p_\lambda(x, 0) = \frac{1}{\lambda}|\lambda x| = |x|.$$

Therefore, for every $\lambda \in \mathbb{R}^+$, $p_\lambda(x, t)$ is a solution to problem (6). By uniqueness [K]

$$p(x, t) \equiv p_\lambda(x, t) = \frac{1}{\lambda} p(\lambda x, \lambda t) \tag{8}$$

in $\mathbb{R} \times \mathbb{R}^+$ for every $\lambda \in \mathbb{R}^+$. In particular, for $\lambda = 1/r$ we have

$$p(x, t) = rp(\sin\theta, \cos\theta)$$

so that (7) holds with $f(\theta) = p(\sin\theta, \cos\theta)$.

Since p is an even function of x which is smooth for $t > 0$, it follows that f is even and $f'(0) = 0$. For $x \neq 0$

$$|x| = p(x, 0) = |x| f\left(\pm\frac{\pi}{2}\right)$$

implies that $f\left(\pm\frac{\pi}{2}\right) = 1$. Moreover, $p > 0$ for $t > 0$ implies that $f > 0$ on $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

To derive further properties of f it is convenient to look at another form of the solution of (6). If we take $\lambda = 1/t$ in (8) we find

$$p(x, t) = tp\left(\frac{x}{t}, 1\right) = rp(\tan\theta, 1)\cos\theta.$$

Thus

$$f(\theta) = g(\tan\theta)\cos\theta$$

where $g(s) \equiv p(s, 1)$. By a calculation which is elementary but tedious, one can verify that g satisfies the ordinary differential equation

$$(m-1)gg'' + g'^2 = g - sg', \tag{9}$$

where $' = d/ds$ and $s = \tan\theta$. Note that

$$f'(\theta) = -g(\tan\theta)\sin\theta + \frac{g'(\tan\theta)}{\cos\theta}$$

so that $f'(0) = 0$ implies that

$$g'(0) = 0.$$

On the other hand,

$$1 = \lim_{\theta\to\pi/2} f(\theta) = \lim_{\theta\to\pi/2} \sin\theta\frac{g(\tan\theta)}{\tan\theta} = \lim_{s\to\infty}\frac{g(s)}{s}.$$

Thus

$$g(s) \sim s \quad \text{as} \quad s \to \infty.$$

Moreover, it follows from l'Hôpital's rule that if g' has a limit as $s \to \infty$ then

$$g'(s) \sim 1 \quad \text{as} \quad s \to \infty.$$

Next, we observe that

$$g'' > 0 \quad \text{on} \quad [0, \infty). \tag{10}$$

Since $g(0) = f(0) \neq 0$ and $g'(0) = 0$ it follows from (9) that

$$g''(0) = 1/(m-1).$$

Suppose that for some $\bar{s} \in \mathbb{R}^+$ we have $g''(\bar{s}) = 0$. Then, in view of (9), $g(\bar{s})$ and $g'(\bar{s})$ satisfy

$$g'^2(\bar{s}) + \bar{s}g'(\bar{s}) - g(\bar{s}) = 0$$

so that

$$g'(\bar{s}) = b \equiv \frac{1}{2}(-\bar{s} \pm \{\bar{s}^2 + 4g(\bar{s})\}^{1/2}).$$

The function

$$G(s) \equiv b^2 + bs$$

is a solution to (9) with $G(\bar{s}) = g(\bar{s})$ and $G'(\bar{s}) = g'(\bar{s})$. By standard uniqueness theory we conclude that $g(s) \equiv G(s)$ and this contradicts $g''(0) > 0$.

Set $a = g(0)$. We claim that

$$g'(s) < \sqrt{a} \quad \text{and} \quad g(s) < a + \sqrt{a}\,s$$

on $\mathbb{R}^+$. Suppose there exists an $\tilde{s} \in \mathbb{R}^+$ for which $g'(\tilde{s}) \geqslant \sqrt{a}$. Since $g'(0) = 0$ and g' is increasing, there exists an $\bar{s} \in (0, \tilde{s}]$ such that $g'(\bar{s}) = \sqrt{a}$, $g' < \sqrt{a}$ on $[0, \bar{s})$, and $g(\bar{s}) < a + \sqrt{a}\,\bar{s}$. Then

$$0 = 1 - \frac{\sqrt{a}(\sqrt{a} + \bar{s})}{a + \sqrt{a}\,\bar{s}} > 1 - \frac{g'(\bar{s})(g'(\bar{s}) + \bar{s})}{g(\bar{s})} = (m-1)g''(\bar{s})$$

which contradicts (10).

Since $g'(s) < \sqrt{g(0)}$ and g' is increasing, it follows that $g'(s) \uparrow 1$ as $s \to \infty$. Moreover, $g''(s) \to 0$ as $s \to \infty$. Thus it follows from (9) that

$$g(\bar{s}) \sim 1 + s \quad \text{as} \quad s \to \infty.$$

In view of (10), we also have

$$g(s) > 1 + s \quad \text{on} \quad \mathbb{R}^+.$$

Finally,

$$F'(\theta) \sim -\sin\theta\left(1 + \frac{\sin\theta}{\cos\theta}\right) + \frac{1}{\cos\theta} = -\sin\theta + \cos\theta$$

implies that $f'(\theta) \to 1$ as $\theta \to \pi/2$. □

By the usual approximation procedures (cf. [B]) we can assume that u and v are positive in $\mathbb{R} \times \mathbb{R}^+$. Then, in particular, v_t exists and is continuous in $\mathbb{R} \times \mathbb{R}^+$. It therefore suffices to derive a bound for $|v_t|$ which is independent of the lower bound for v.

Proposition 2. *Fix an arbitrary $\delta > 0$. For each $(x_0, t_0) \in \mathbb{R}^+ \times [2\delta, \infty)$ set $\alpha \equiv v(x_0, t_0)$. There exists constants A and B depending only on δ, m, and $N \equiv \|v_0\|_{L^\infty(\mathbb{R})}$ such that*

$$\frac{\alpha}{4f(0)} \leqslant v(x, t) \leqslant 2\alpha$$

for all (x, t) which satisfy

$$|x - x_0| \leqslant A\gamma \quad \textit{and} \quad 0 \leqslant t_0 - t \leqslant B\gamma,$$

where $\gamma \equiv \min(\alpha, \delta)$.

PROOF. In view of (4)

$$|v(x_0, t_0) - v(x, t_0)| \leqslant L|x - x_0|$$

where L depends on δ, m and N. Thus

$$|x - x_0| \leqslant \delta/2L$$

implies that

$$\frac{\alpha}{2} \leqslant v(x, t_0) \leqslant \frac{3\alpha}{2}.$$

According to (5), for $t \geqslant \delta$ we have

$$v(x, t_0) - v(x, t) \geqslant -K(t_0 - t),$$

where K depends only on δ, m and N. Therefore

$$v(x, t) \leqslant v(x, t_0) + K(t_0 - t) \leqslant 2\alpha$$

if

$$|x - x_0| \leqslant \gamma/2L \quad \text{and} \quad 0 \leqslant t_0 - t \leqslant \gamma \min\left(\frac{1}{2K}, 1\right).$$

We assert that

$$v(x_0, t) \geqslant \frac{\alpha}{2f(0)} \quad \text{for} \quad t \in [t_0 - \gamma E, t_0], \tag{11}$$

where $E = \min(1/8L^2f(0), 1)$. Suppose that (11) is false. Then there is a $\theta \in (0, E)$ such that

$$v(x_0, t_0 - \theta\gamma) < \frac{\alpha}{2f(0)}.$$

Without loss of generality, we can assume that $x_0 = t_0 = 0$. By Taylor's theorem and (4) we have

$$v(x, -\delta\theta) < \frac{\alpha}{2f(0)} + L|x|.$$

Set

$$p^*(x,t) \equiv \sqrt{2}Lp(x, \sqrt{2}L(t + \gamma\eta))$$

for $t > -\gamma\eta$, where p is the solution of problem (6) and η is to be chosen. Note that p^* is a solution of the pressure equation (2). Since $\{a^2 + b^2\}^{1/2} \geqslant (|a| + |b|)/\sqrt{2}$ and $f(0) > 1$ we have

$$p^*(x,t) \geqslant L\{|x| + \sqrt{2}L(t + \gamma\eta)\}.$$

Thus

$$v(x, -\gamma\theta) < \frac{\alpha}{2f(0)} + L|x| = L\{|x| + \sqrt{2}L(\eta - \theta)\gamma\} \leqslant p^*(x, -\gamma\theta)$$

provided that

$$\eta = \frac{\alpha}{2\sqrt{2}\gamma L^2 f(0)} + \theta \leqslant \frac{\alpha}{2\sqrt{2}\gamma L^2 f(0)} + E. \tag{12}$$

By the comparison principle,

$$\alpha = v(0,0) \leqslant p^*(0,0) = 2L^2\gamma\eta f(0).$$

It follows from (12) and the definition of E that

$$\alpha \leqslant 2L^2 f(0)\left\{\frac{\alpha}{2\sqrt{2}L^2 f(0)} + \gamma \min\left(\frac{1}{\delta L^2 f(0)}, 1\right)\right\} \leqslant \alpha\left(\frac{1}{\sqrt{2}} + \frac{1}{4}\right) < \alpha.$$

Thus we have a contradition and conclude that (11) holds.

For any $t \in [t_0 - \gamma E, t_0]$ it follows from (4) and (11) that

$$v(x,t) \geqslant v(x_0, t) - L|x - x_0| \geqslant \frac{\alpha}{2f(0)} - L|x - x_0| \geqslant \frac{\alpha}{4f(0)}$$

provided that $|x - x_0| \leqslant \gamma/4Lf(0)$. Thus the assertion of the proposition holds if we take

$$A = 1/4Lf(0)$$

and

$$B = \min(1/\delta L^2 f(0), 1/2K, 1).$$

PROOF OF THEOREM. Define

$$w(x, t) \equiv \frac{1}{\gamma} v(x_0 + \gamma x, t_0 + \gamma t).$$

Then w is a solution of the pressure equation (2) which satisfies

$$\frac{\alpha}{4f(0)\gamma} \leqslant w(x, t) \leqslant \frac{2\alpha}{\gamma}$$

in the rectangle $|x| \leqslant A$, $-B \leqslant t \leqslant 0$. If $\alpha \leqslant \delta$ then $\gamma = \alpha$ and we have

$$\frac{1}{4f(0)} \leqslant w(x, t) \leqslant 2 \quad \text{for} \quad |x| \leqslant A, -B \leqslant t \leqslant 0.$$

If $\alpha > \delta$ then $\gamma = \delta$ and $\alpha/\gamma > 1$. Then since $\alpha \leqslant N$ we have

$$\frac{1}{4f(0)} \leqslant w(x, t) \leqslant \frac{2N}{\delta} \quad \text{for} \quad |x| \leqslant A, -B \leqslant t \leqslant 0.$$

In both cases we conclude from the standard theory of parabolic equations [LSU] that there is a positive constant C depending only on δ, m and N such that

$$|w_t(0, 0)| \leqslant C.$$

The theorem now follows since $w_t(0, 0) = v_t(x_0, t_0)$ and (x_0, t_0) is arbitrary. □

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