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Square donuts and twistable holes

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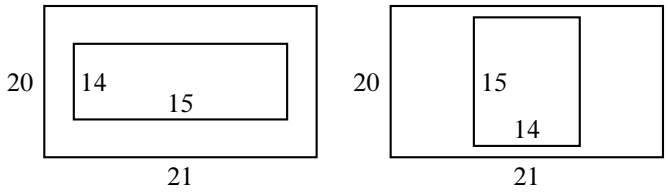
1 Introduction

In their paper, Nirode and Krumpe [3] describe stumbling across a routine problem in a high school geometry textbook [2]. It describes a trend in middle age landscape design, where rectangular courtyards regularly contained a rectangular garden, whose sides were parallel to those of the courtyard itself. For aesthetic reasons, it was prescribed that the garden's area would be exactly half of the courtyard's area.

Without further restrictions, this is simple to accomplish: place a divider down the middle of the courtyard and assign either half of the courtyard to be the garden. However, the geometry text insisted that a path of fixed width should surround the garden so that the courtyard path would be of the same area as the garden.

Nirode and Krumpe set out to investigate such nested rectangles mathematically, and while they did not insist on a fixed width border around the garden, they did require that both rectangles have integral side lengths, which gives rise to the following definition.

Zeichnen Sie ein Rechteck mit ganzzahligen Seiten. Zeichnen Sie nun im Inneren dieses Rechtecks ein zweites Rechteck, dessen ebenfalls ganzzahlige Seiten parallel zu denen des äusseren Rechtecks sind und zwar so, dass die Fläche des kleineren Rechtecks genau halb so gross ist wie die des grösseren. Dies ist das Konzept eines «mathematischen Donuts». Beispielsweise kann ein 2×3-Rechteck vollständig innerhalb eines 3×4-Rechtecks platziert werden, während es kein Rechteck mit ganzzahligen Seitenlängen mit einer Fläche von 5 gibt, welches innerhalb eines 2×5-Rechtecks platziert werden kann. In diesem Artikel werden mathematische Donuts, bei denen entweder das äussere oder das innere Rechteck ein Quadrat ist, vollständig beschrieben. Es stellt sich heraus, dass die Klassifizierung in engem Zusammenhang mit den pythagoreischen Tripeln steht.



(a) (21, 20, 15, 14) (b) (21, 20, 14, 15)

Figure 1. Two configurations of the donut 441

Define a *rectangular donut* D to be a positive integer $D = ab$, where a and b are positive integers strictly less than D, with $b \leq a$, and $D = 2xy$, for some positive integers x and y with $1 \leq x < a$ and $1 \leq y < b$. A *configuration* of D is the ordered 4-tuple (a, b, x, y). Visually, a configuration of D is an $a \times b$ rectangle, which is called the *exterior* of D, with the smaller $x \times y$ rectangle, referred to as the *hole* of D, contained strictly inside it, and the corresponding sides of the two rectangles are parallel. While a donut is a single positive integer, it may have numerous configurations; Figure 1 illustrates two configurations of the donut 441.

Nirode and Krumpe classify mathematical donuts, finding a connection with *Pythagorean triples*, a set of three positive integers a, b, and c such that $a^2 + b^2 = c^2$. This classification, given in Theorem 1.1, is not a one-to-one correspondence, since a donut may have multiple configurations. However, the theorem does introduce a further refinement of the previous definitions. A configuration (a, b, x, y) is called *primitive* if $\gcd(a, x) = \gcd(b, y) = 1$ (that is, corresponding side lengths of the exterior and the hole have no common factors), and we define a donut to be *primitive* if it has only a single configuration that is itself primitive. Consequently, a *nonprimitive* donut is a donut with more than one configuration, or, with a single configuration that is not primitive. Those nonprimitive donuts that have only a single primitive configuration (that is, a donut with two or more configurations, exactly one of which is primitive) are called *quasi-primitive*. While donuts cannot have more than one primitive configuration (the proof of which, involving prime factorization, is left to the reader), a donut does not necessarily have a primitive configuration (such as 24, which has only two configurations: $(8, 3, 6, 2)$ and $(6, 4, 4, 3)$).

Note that this terminology is borrowed from basic number theory, because *primitive Pythagorean triples* are those Pythagorean triples consisting of three coprime positive integers. Likewise, *quasi-primitive Pythagorean triples* are those for which the three positive integers share a common factor greater than 1.

Observing that the primitive or quasi-primitive donuts less than 100 are 12, 30, 40, 56, 70, 84, and 90, and that these are precisely the first seven perimeters of primitive right triangles [4, A024364] (where a right triangle is *primitive* if its three side lengths form a primitive Pythagorean triple), Nirode and Krumpe completed the classification problem with the following theorem.

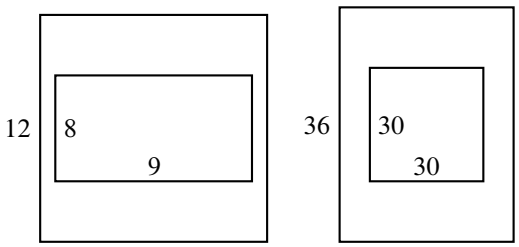
Theorem 1.1 ([3]). *A positive integer is a primitive or quasi-primitive donut if and only if it is the sum of a primitive Pythagorean triple.*

The proof of this relies on a well-known tool for generating primitive Pythagorean triples: Euclid's formula, Proposition XXIX in Book X of his *Elements* [1]. Given arbitrary relatively prime positive integers m and n ($m > n$) that are not odd, the formula states that $a = m^2 - n^2$, $b = 2mn$, and $c = m^2 + n^2$ form a primitive Pythagorean triple.

Because a donut is by definition even, any donut can be written in the form $D = 2k$. From this value k and a given configuration (a, b, x, y) of D, it is shown that $k < b < 2k$, so that $b = k + n$ for some positive integer $n < k$. These values of k and n turn out to be relatively prime and not both odd. The authors then show that the primitive Pythagorean triple generated via Euclid's formula sums to D, providing one direction of the desired correspondence.

The converse relies also on Euclid's formula. It is known that any primitive Pythagorean triple (a, b, c) arises from a unique coprime pair of integers m and n ($m > n$), where $a = m^2 - n^2$, $b = 2mn$, and $c = m^2 + n^2$. Since m and n are coprime with $m > n$, it follows that $(2m, m + n, m + n, m)$ is indeed a primitive donut configuration of the donut $2m(m + n) = a + b + c$.

The focus of our paper is geometric. We aim to completely classify both *square* donut configurations (those of the form (a, a, x, y)) and *square-holed* configurations (those of the form (a, b, x, x)). Figure 2 illustrates two such configurations. While square-holed square donut configurations do not exist (since $a^2 = 2x^2$ has no integer solutions), square configurations are a subcollection of configurations where the rectangular holes, visually, can be rotated 90°. This inspires the following definition, where we say that the configuration (a, b, x, y) has a *twistable hole* if (a, b, y, x) is also a configuration. Figure 1 illustrates a configuration in which the hole is twistable. Not all configurations have twistable holes, however. For example, $(4, 3, 3, 2)$ is a configuration of the donut 12, while $(4, 3, 2, 3)$ is not. To this end, then, in addition to classifying square and square-holed configurations, a necessary and sufficient condition for a hole to be twistable is proven.



(a) $(12, 12, 9, 8)$ (b) $(50, 36, 30, 30)$

Figure 2. (a) Square and (b) square-holed donut configurations

2 Twistable holes and squares

When drawing donuts from given configurations, the notation prescribes an orientation for the hole. For $(22, 20, 20, 11)$, the exterior side of length 22 is parallel to the hole side of length 20. In this case, the hole of the donut cannot be rotated 90°, as $(22, 20, 11, 20)$ is not a donut configuration. To determine if (a, b, x, y) has a twistable hole, one need only consider two inequalities relative to the largest dimension of the configuration's exterior.

Theorem 2.1. *The donut configuration (a, b, x, y) has a twistable hole if and only if both $2x > a$ and $2y > a$.*

Proof. Given that (a, b, x, y) is a donut configuration, we know $ab = 2xy$, as well as

$$1 \leq y < b. \tag{1}$$

From (1), we have that $2xy < 2xb$, or equivalently, $ab < 2xb$, showing $a < 2x$. Assuming its hole is twistable, we have that $1 \leq x < b$. From this, we see $2xy < 2by$, or equivalently, $ab < 2yb$, showing that $a < 2y$.

To show that the donut configuration (a, b, x, y) has a twistable hole, given both $2x > a$ and $2y > a$, we must show $y < a$ and $x < b$. The first of these follows from (1) and since $b \leq a$. The assumption that $2y > a$ yields $ab < 2yb$, or equivalently, $2xy < 2yb$, so that $x < b$, as required. ∎

Two types of configurations clearly have twistable holes: square and square-holed donut configurations. For some positive integers n, there are square or square-holed donuts where the exterior or hole, respectively, has side length n, and for others, none exist. It is natural, then, to ask how to classify these two types of configurations. Theorems 2.2 and 2.3 do precisely this.

Theorem 2.2. *Given positive integers a, b, and n such that $n < b \leq a$, then (a, b, n, n) is a donut configuration if and only if there exist relatively prime factors p and q of n such that $p < q < 2p$.*

Proof. Let n be a positive integer that has relatively prime factors p and q satisfying $p < q < 2p$. Then, defining $a = 2pn/q$ and $b = qn/p$ yields the donut configuration (a, b, n, n), since both

$$1 < \frac{2p}{q} \quad \text{and} \quad 1 < \frac{q}{p},$$

so that $a > n$ and $b > n$, as well as $ab = 2n^2$. Hence (a, b, n, n) is indeed a donut configuration.

Conversely, suppose that (a, b, n, n) is a donut configuration, for positive integers a, b, and n, with $a \geq b > n$ and $ab = 2n^2$. Let n have prime factorization given by

$$n = p_1^{e_1} \dots p_k^{e_k},$$

where the p_i are distinct primes and e_j are positive integers. Note that it is impossible to have $k = 1$, because in this scenario

$$ab = 2p_1^{2e_1},$$

and the product $2p_1^{2e_1}$ cannot be partitioned into two terms so that both

$$a > 2p_1^{\alpha_1} \quad \text{and} \quad b > p_1^{\alpha_2},$$

where both α_1 and α_2 are nonnegative integers. Thus $k \geq 2$. In order to show the existence of p and q, the desired relatively prime factors of n satisfying $p < q < 2p$, because we have

$$\frac{ab}{n^2} = 2,$$

it must be the case, without loss of generality, that

$$\frac{a}{n} = \frac{2p}{q} \quad \text{and} \quad \frac{b}{n} = \frac{q}{p},$$

where p and q are a product of some of the prime factors of n. Since $k \geq 2$ and we can simply reduce each fraction to lowest terms, we have that $\gcd(p, q) = 1$. Lastly, because both $a > n$ and $b > n$, and

$$\frac{2p}{q} > 1 \quad \text{and} \quad \frac{q}{p} > 1,$$

the result holds. ∎

Having classified all square-holed donut configurations, we move now to classifying all square donut configurations. It is worth reminding ourselves of the result in [3]: a positive integer, which is the product of the dimensions of the outer rectangle of a donut configuration, is a donut if and only if it is the sum of a Pythagorean triple. In Theorem 2.3, the classification of donut configurations of the form (a, a, x, y) depends not on a^2 being the sum of a Pythagorean triple (as it does in Theorem 1.1), but rather, it depends on a itself being such a sum.

Theorem 2.3. *For a positive integer a, there exists a square donut configuration (a, a, x, y) for some positive integers x and y if and only if a is the sum of a Pythagorean triple.*

Proof. Begin by assuming $a = x' + y' + z'$, where (x', y', z') is a Pythagorean triple. Then $x' = k(p^2 - q^2)$, $y' = k(2pq)$, and $z' = k(p^2 + q^2)$, via Euclid's formula for Pythagorean triples (where $q < p$) [1]. Then

$$a = k(2p^2 + 2pq), \tag{2}$$

so that

$$a^2 = 2[2k^2p^2(p + q)^2]. \tag{3}$$

If we set $x = 2kp^2$ and $y = k(p + q)^2$, we claim that (a, a, x, y) is a donut configuration. As (3) demonstrates one of the requirements to be a donut, we need only show that both $x < a$ and $y < a$. The first of these follows from (2). Since $q < p$, we have that

$$p^2 + 2pq + q^2 < p^2 + 2pq + p^2,$$

or equivalently,

$$(p + q)^2 < 2p^2 + 2pq,$$

yielding $y < a$.

Next, assume that (a, a, x, y) is a donut configuration. Because $a^2 = 2xy$, it is not primitive (since both $\gcd(a, x) \neq 1$ and $\gcd(a, y) \neq 1$). Let $h = \gcd(a, x)$, $k = \gcd(a, y)$, where $a = h \cdot a_h = k \cdot a_k$, $x = h \cdot x_h$, and $y = k \cdot y_k$. Note that both $\gcd(a_h, x_h) = 1$ and $\gcd(a_k, y_k) = 1$.

By this construction, the configuration (a_h, a_k, x_h, y_k) is primitive. Additionally, by Theorem 1.1,

$$a_h a_k = x' + y' + z', \tag{4}$$

for some primitive Pythagorean triple (x', y', z'). We claim that

$$a = \gcd(h, k)[x' + y' + z'].$$

Note that proving this claim consequently proves the result, as any constant multiple of a Pythagorean triple is itself a Pythagorean triple. Moreover, if we are able to show that $a = \operatorname{lcm}(h, k)$, this claim will be proven, since (4) yields

$$a^2 = hk[x' + y' + z'] = \gcd(h, k)\operatorname{lcm}(h, k)[x' + y' + z'].$$

To that end, suppose a has prime factorization given by

$$a = 2^{e_0} p_1^{e_1} \dots p_m^{e_m},$$

written in this form since a is necessarily even (where the p_i are odd primes and e_j are positive integers). Because $a^2 = 2xy$, we know the prime factorizations of x and y take similar forms,

$$x = 2^{x_0} p_1^{x_1} \dots p_m^{x_m} \quad \text{and} \quad y = 2^{y_0} p_1^{y_1} \dots p_m^{y_m},$$

with the x_i and y_i nonnegative integers (that is, they may possibly be 0). Then take $h = \gcd(a, x)$ and $k = \gcd(a, y)$. In terms of prime factorizations, we have that

$$\begin{aligned} h &= 2^{\min(e_0, x_0)} p_1^{\min(e_1, x_1)} \dots p_m^{\min(e_m, x_m)}, \\ k &= 2^{\min(e_0, y_0)} p_1^{\min(e_1, y_1)} \dots p_m^{\min(e_m, y_m)}. \end{aligned}$$

It follows that

$$\operatorname{lcm}(h, k) = 2^{\max(\min(e_0, x_0), \min(e_0, y_0))} p_1^{\max(\min(e_1, x_1), \min(e_1, y_1))} \dots p_m^{\max(\min(e_m, x_m), \min(e_m, y_m))}.$$

Because $a^2 = 2xy$, we have that

$$x_0 + y_0 + 1 = 2e_0 \tag{5}$$

and

$$x_i + y_i = 2e_i$$

for $i \geq 1$. In either case, it follows that $x_i \leq e_i \leq y_i$ or $y_i \leq e_i \leq x_i$ (with the inequalities following from (5) since $x_0 \neq y_0$). Hence, for all i,

$$e_i = \max(\min(e_i, x_i), \min(e_i, y_i)),$$

showing that $\operatorname{lcm}(h, k) = a$, as desired. ∎

3 Conclusion

The results in this paper are just the tip of the iceberg when it comes to investigating mathematical donuts. A multitude of directions for potential research appear in [3], but it is worth noting some of the geometrically aligned questions here.

A very natural generalization of a mathematical donut (a two-dimensional object) would be to consider three- or higher-dimensional donuts. Is there a classification of these donuts similar to that in [3]? How about classifying cube or cube-holed three-dimensional donut configurations, a generalization of square or square-holed configurations?

Back in two-dimensions, there is no shortage of potential avenues to explore. Can two-holed (or for that matter, arbitrary n-holed) donuts be classified? Are there similar results for waffle-shaped figures: a rectangle encompassing an $n \times m$ array of equally spaced holes (where all lengths are integral)? Both of these questions can be specified further by requiring the respective figures to be *perfect*, where the border between the hole(s) and/or the exterior is a constant width.

Defining a cost function for donuts (for example, should the donut configurations $(90, 90, 81, 50)$ and $(90, 90, 75, 54)$ "cost" the same, however cost is defined, even though the holes have different dimensions?) or colorings on donuts (perhaps playing the role of icing on a donut) are two examples of connecting donuts to other mathematical areas.

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