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Short note

A note on a result of Makowski

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Abstract. In this note, we fix a gap in a proof of the first author that 28 is the only even perfect number which is the sum of two perfect cubes. We also discuss the situation for higher powers.

In this note, we fix a gap in a proof of the first author. Makowksi [3] proved that 28 is the only even perfect number of the form $x^3 + 1$. Motivated by this, the first author [1] stated that if n is an even perfect number, and $n = x^3 + a^3$ for some positive integers x and a, then we must have $n = 28$.[1] However, there is a step in that proof that has a gap. In particular, if n is an even perfect number, then we may factor n as

$$n = x^3 + a^3 = (x + a)(x^2 - ax + a^2) \tag{1}$$

and, with some work, reach a contradiction. However, the contradiction reached required assuming that $(x + a, x^2 - ax + a^2) = 1$, which is not necessarily the case. The main purpose of this note to is to address this situation. We then discuss representing even perfect numbers as the sum of two higher powers.

We recall the Euclid–Euler theorem, which states that n is an even perfect number if and only if there is a prime $2^p - 1$ such that

$$n = (2^p - 1)2^{p-1}.$$

Note that if $2^p - 1$ is prime, then p is prime, but the converse does not follow.

We will, following the first author's prior work [1], factor n as in equation (1). We will also assume that there is a non-trivial common factor between $x + a$ and $x^2 - ax + a^2$. Note that, since the only prime factor repeated in the factorization of n is 2, it must be the case that our common factor is a power of 2.

We first want to show that $2^p - 1$ is not a factor of $x + a$. Assume $2(2^p - 1) \mid x + a$. We consider then two cases. In Case I, $2(2^p - 1) = x + a$. In Case II, $4(2^p - 1) \mid x + a$.

Case I: Assume that

$$2(2^p - 1) = x + a. \tag{2}$$

Then we have

$$2^{p-2} = x^2 - ax + a^2. \tag{3}$$

[1] That the cubes are positive is needed here. A referee of this paper pointed out that the prior paper also implicitly used that the cubes were positive without stating it explicitly. If one does not insist on the cubes being positive, one has other examples such as $8128 = 28^3 - 24^3$.

Then we may combine equation (2) and equation (3) to get that

$$\frac{x+a+2}{8} = x^2 - ax + a^2$$

which has a negative discriminant if $a \geq 1$, and thus has no solution in positive integers. We may thus assume that we are in Case II, where $4(2^p - 1) \mid x + a$. Thus we have that $2^p - 1 \leq \frac{x+a}{4}$, and so

$$x^3 + a^3 = (2^p - 1)2^{p-1} \leq \Big(\frac{x+a}{4}\Big)\Big(\frac{x+a}{8} + \frac{1}{2}\Big),$$

which is a contradiction when $x \geq 1$ and $a \geq 1$.

We thus may assume that $2^p - 1 \mid x^2 - ax + x^2$. For the remainder of the proof, we will assume that

$$x + a = 2^h \tag{4}$$

and

$$x^2 - ax + a^2 = (2^p - 1)2^{p-1-h}. \tag{5}$$

We have then

$$2^{2h} = (x+a)^2 = x^2 + 2ax + a^2 > x^2 - ax + a^2.$$

Thus we get $2^{2h} > (2^p - 1)2^{p-1-h}$, which implies that $2^{3h} > 2^{2p-1} - 2^{p-1}$ and hence that $3h \geq 2p - 1$.

We also have that

$$2^{2h} = (x+a)^2 \leq 4(x^2 - ax + a^2) = 4(2^p - 1)2^{p-1-h} < 2^{2p+1}2^{-h}.$$

We have from the above chain of inequalities that $3h \leq 2p$. Thus we have either

$$3h = 2p - 1 \quad \text{or} \quad 3h = 2p.$$

If $3h = 2p$, then $p = 3$ and thus we have $n = 28 = 3^3 + 1^3$. We thus need only to consider the situation where $3h = 2p - 1$.

Note that we must have $p \equiv 2 \pmod 3$. We set $p = 3q + 2$. So

$$h = \frac{6q+3}{3} = 2q + 1.$$

We then may apply equations (4) and (5) to get that

$$x + a = 2^{2q+1} = 2(2^{2q}) \tag{6}$$

and

$$x^2 - ax + a^2 = (2^{3q+2} - 1)(2^{3q+1-(2q+1)}) = 2^{4q+2} - 2^q. \tag{7}$$

If we set $N = 2^q$, we get from equation (6) and equation (7) that

$$x + a = 2N^2 \quad \text{and} \quad x^2 - ax + a^2 = 4N^4 - N.$$

Thus we have

$$x^2 - ax + a^2 = (x + a)^2 - \sqrt{\frac{x + a}{2}},$$

but this equation does not have any integer solutions. The proof is now complete.

We strongly suspect the following holds.

Conjecture 1. *If n is an even perfect number, and $n = x^m + a^m$, where x and a are positive integers and m is a positive integer which is at least 2, $m \geq 2$, then we must have $m = 3$, and $n = 28$.*

This conjecture is motivated by three things. First, we know it is true for $m = 3$ by the earlier part of this note. Second, it is true for any even m, since every Mersenne prime is 3 (mod 4), and thus no even perfect number is the sum of two perfect squares. Third, for any $m \geq 5$, the set of numbers of the form $x^m + a^m$ has a very low density.

However, trying to apply the method above to prove this seems insufficient even for the case of $m = 5$. To see where this breaks down, let us assume that we have an even perfect number where $m = 5$ and try to follow through the same line of logic we had above.

Let us assume that

$$n = x^5 + a^5 \tag{8}$$

for positive integers, x and a. As before, we may write $n = 2^{p-1}(2^p - 1)$. We have, from equation (8),

$$n = (x + a)(x^4 - x^3a + x^2a^2 - xa^3 + a^4).$$

We will write $A = x + a$ and $B = (x^4 - x^3a + x^2a^2 - xa^3 + a^4)$. Since x and a are not both equal to 1, we get $2(x + a)^3 \leq x^5 + a^5$ and so

$$2A^2 < B. \tag{9}$$

We thus have that $2^p - 1 \mid B$, since $2^p - 1$ is prime, $2^p - 1 \mid AB$, and if we had $2^p - 1 \mid A$, we would be forced to violate inequality (9). We thus have a situation similar to what we had before, where

$$A = 2^h, \tag{10}$$

$$B = 2^{p-1-h}(2^p - 1). \tag{11}$$

We have then from inequalities (9), (10), (11) that $3h \leq 2p - 2$. We are thus in a situation very similar to our $m = 3$ situation. But we need to get a lower bound on h as well, and now we run into a problem. The best we seem to be able to do is $A^4 \geq B$, and this only gives us after simplifying $3h \geq p - 1$. And this is too wide a range of possible h values for our earlier strategy to work without some additional insight.

We can show, subject to an explicit version of the ABC conjecture, that we can rule out even perfect numbers which are the sum of two non-trivial powers of high degree with certain properties.

Conjecture 2. *Let A, B, C be positive integers such that $A + B = C$ and $(A, B) = 1$. Then $\max(A, B, C) < \mathrm{rad}(ABC)^{7/4}$, where the radical $\mathrm{rad}(n)$ of a positive integer n is defined as the product of the distinct prime numbers dividing n.*

Note that Conjecture 2 follows from Baker's explicit version of the ABC conjecture [2]. Then we have the following.

Proposition 3. *Assume that Conjecture 2 holds. Assume that $2^p - 1$ is prime. Then there is no simultaneous solution in positive integers satisfying the following list of conditions:*

(a) $m > 29$,

(b) $(x + a)^{m-2} \leq x^m + a^m$,

(c) *both x and a are odd,*

(d) $x + a = 2^h$,

(e) $\frac{x^m + a^m}{x+a} = (2^p - 1) \cdot 2^{p-1-h}$.

Proof. Assume Conjecture 2 (ABC with exponent $7/4$) holds. Let p be a prime number such that $2^p - 1$ is also prime. Moreover, assume that all conditions (a), (b), (c), (d) and (e) hold. Note that we may assume that $p > 3$. Put $A = x^m$, $B = a^m$ and $C = (2^p - 1) \cdot 2^{p-1}$. Note that (d) and (e) imply that $2^{p-1} \cdot (2^p - 1) = x^m + a^m$.

By conditions (c) and (d), we have $\gcd(A, B) = 1$. Thus Conjecture 2 implies that

$$(2^p - 1) \cdot 2^{p-1} < (2 \cdot \mathrm{rad}(ax) \cdot (2^p - 1))^{7/4}. \tag{12}$$

From (12) raised to the 4-th power, we get

$$(2^p - 1)^4 \cdot 2^{4p-4} < 2^7 \cdot \mathrm{rad}(ax)^7 \cdot (2^p - 1)^7. \tag{13}$$

Thus (13) implies

$$2^{4p-11} < \mathrm{rad}(ax)^7 \cdot (2^p - 1)^3. \tag{14}$$

But since $4a\ x \leq (x + a)^2$ and $\mathrm{rad}(ax) \leq a\ x$, it follows from (d) that $\mathrm{rad}(ax) \leq 2^{2h-2}$. Thus (14) implies

$$2^{4p-11} < 2^{14h-14} \cdot (2^p - 1)^3 < 2^{14h-14} \cdot 2^{3p}. \tag{15}$$

We deduce from (15) that $2^{p-11} < 2^{14h-14}$.

Therefore, we have

$$14h - 14 \geq p - 9 \tag{16}$$

since p is odd. Inequality (16) implies that

$$h \geq \frac{p + 5}{14}. \tag{17}$$

Now, we will find an upper bound for h. Using (b), (d) and (e), we deduce that

$$2^{h\cdot(m-2)} \leq 2^h \cdot (2^p - 1) \cdot 2^{p-1-h} \leq 2^h \cdot 2^p \cdot 2^{p-1-h}. \tag{18}$$

Inequality (18) divided by 2^h becomes

$$2^{h\cdot(m-3)} \leq 2^{2p-1-h}. \tag{19}$$

From (19), we get

$$2p - 1 - h \geq h \cdot (m - 2) - h. \tag{20}$$

In other words, (20) says that

$$2p - 1 \geq h \cdot (m - 2). \tag{21}$$

And (21) can be also written as

$$h \leq \frac{2p - 1}{m - 2}. \tag{22}$$

Putting together the lower bound (17) of h with the upper bound (22) of h, we get

$$\frac{p + 5}{14} \leq \frac{2p - 1}{m - 2},$$

which is equivalent to

$$m - 2 \leq f(p), \tag{23}$$

where

$$f(p) := \frac{14 \cdot (2p - 1)}{p + 5} = \frac{-154}{p + 5} + 28. \tag{24}$$

It follows from (23) and (24) that

$$m \leq f(p) + 2 = 30 - \frac{154}{p + 5}. \tag{25}$$

Then inequality (25) contradicts condition (a) and the result follows. ∎

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