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Confluent Vandermonde matrix and related topics

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1 Introduction

Let $\Lambda = \{\lambda_1, \dots, \lambda_d\}$ be a set of d distinct real numbers. The corresponding *Vandermonde matrix* is the $d \times d$ matrix

$$V_\Lambda = \begin{bmatrix} 1 & \lambda_1 & \cdots & \lambda_1^{d-1} \\ 1 & \lambda_2 & \cdots & \lambda_2^{d-1} \\ \vdots & \vdots & & \vdots \\ 1 & \lambda_d & \cdots & \lambda_d^{d-1} \end{bmatrix}. \tag{1}$$

The Vandermonde matrix has been used in polynomial interpolation, coding theory, signal processing, differential equations, and recursive relations; see [3, 5, 14, 17, 24] and the references therein. It was also used to give alternative proofs for some classical results. For example, it was used to prove a theorem of Abel [11], whose classical proof is based on the residue theorem. There are many elegant results on the Vandermonde matrix, and

Die Vandermonde-Matrix spielt eine zentrale Rolle bei der Polynominterpolation, in der Codierungstheorie, in der Signalverarbeitung, bei der Lösung von Differentialgleichungen und bei Rekursionen. Wenn ihre definierenden Werte verschieden sind, ist die Matrix invertierbar, wobei die Inverse mit der Lagrange-Interpolation zusammenhängt. Aber was passiert, wenn sich Werte wiederholen? Der vorliegende Artikel befasst sich mit der sogenannten konfluenten Vandermonde-Matrix und stellt einen einheitlichen Ansatz vor, der für jeden Körper funktioniert und der es erlaubt, die LR-Zerlegung, die Determinantenformel und Verbindungen zur Hermite-Interpolation und zu Begleitmatrizen abzuleiten.

one of them is the following formula for the determinant of V_Λ which appeared in many textbooks:

$$\det(V_\Lambda) = \prod_{i<j}(\lambda_j - \lambda_i). \tag{2}$$

Rushanan [24] gave the LU factorization of V_Λ. Many formulas for V_Λ^{-1} were studied and rediscovered by researchers in pure and applied subjects; see, e.g., [15, 18]. In particular, Parker [21] pointed out that the columns of V_Λ^{-1} are exactly the coefficients of the Lagrange interpolation polynomials. The generalization of V_Λ whose columns are assigned with nonconsecutive powers was considered and revealed a strong connection to symmetric functions through Schur functions; see, e.g., [1, 6, 7, 10, 12, 16, 20].

Using an example, Aitken and Turnbull [26, Chapter VI] demonstrated an interesting fact that $V_\Lambda C_\Lambda V_\Lambda^{-1}$ is a diagonal matrix, where C_Λ is the companion matrix of $(x-\lambda_1)\cdots(x-\lambda_d)$. In the same chapter, the authors mentioned that if $\lambda_1, \ldots, \lambda_d$ are not distinct, then one can extend the definition of the Vandermonde matrix to define the confluent Vandermonde matrix V_Λ using the higher derivatives of the polynomial $(x-\lambda_1)\cdots(x-\lambda_d)$ at the points λ_j, which appear more than once; see the definition in Section 2. Then $V_\Lambda C_\Lambda V_\Lambda^{-1}$ will be the Jordan canonical form of C_Λ. The same points were later emphasized by Brand [2].

Researchers have obtained results on the confluent Vandermonde matrix because of its connections and applications to many different subjects. However, these results are scattered across different areas of the mathematical literature. In this paper, we give a systematic introduction to the confluent Vandermonde matrix. In particular, using a unified approach based on polynomial evaluations and derivative computations at selected points, we provide accessible proofs that not only clarify key results but also offer insights for both experienced researchers and those new to the subject.

2 Evaluations of a polynomial at specific points

Let

$$p(x) = c_0 + c_1 x + \cdots + c_{d-1}x^{d-1}$$

be a polynomial over $\mathbb{R}$. Let $\Lambda = \{\lambda_1, \ldots, \lambda_d\}$ be a set of d distinct real numbers and let V_Λ be as in equation (1). Then

$$V_\Lambda \begin{bmatrix} c_0 \\ \vdots \\ c_{d-1} \end{bmatrix} = \begin{bmatrix} p(\lambda_1) \\ \vdots \\ p(\lambda_d) \end{bmatrix}. \tag{3}$$

Therefore, V_Λ can be viewed as a transformation from a polynomial of degree at most $d-1$ to its evaluations at the points in Λ. In fact, V_Λ is a linear transformation from the linear space of polynomials with degrees bounded by $d-1$ to $\mathbb{R}^d$. Since $\det(V_\Lambda) \neq 0$, V_Λ is invertible and any assignment of $p(\lambda_1), \ldots, p(\lambda_d)$ defines a unique polynomial $p(x)$ of degree bounded by $d-1$.

When Λ contains repeated values, the repeated λ_i should provide not just the information $p(\lambda_i)$. Instead, more information should be revealed at the point of λ_i, and the

evaluations at its derivatives $p(\lambda_i), p'(\lambda_i), \ldots$, are natural choices. In fact, one may view the higher derivatives at λ_i as the information obtained by the limit process when several distinct points converge to λ_i.

Note that if $p(x) = c_0 + c_1 x + \cdots + c_{d-1}x^{d-1}$, then

$$\begin{aligned}
\frac{1}{0!}\cdot p(x) &= c_0 + c_1 x + c_2 x^2 + \cdots + c_{d-1}x^{d-1},\\
\frac{1}{1!}\cdot \frac{d}{dx} p(x) &= \quad c_1 + c_2 2x + \cdots + c_{d-1}(d-1)x^{d-2},\\
\frac{1}{2!}\cdot \frac{d^2}{dx^2} p(x) &= \quad\quad c_2 + \cdots + c_{d-1}\Big(\frac{(d-1)(d-2)}{2}\Big)x^{d-3},\\
&\ \ \vdots
\end{aligned}$$

and in general,

$$\frac{1}{r!}\cdot \frac{d^r}{dx^r} p(x) = \sum_{i=r}^{d-1} c_i \binom{i}{r} x^{i-r}.$$

Let $\mathcal{D}^{(r)} = \frac{1}{r!}\cdot\frac{d^r}{dx^r}$. Then, for any $m \le d$,

$$\begin{bmatrix}
1 & \lambda & \lambda^2 & & \cdots & \lambda^{d-1}\\
0 & \binom{1}{1} & \binom{2}{1}\lambda & & \cdots & \binom{d}{1}\lambda^{d-2}\\
0 & 0 & \binom{2}{2} & & \cdots & \binom{d}{2}\lambda^{d-3}\\
 & & & & \vdots & \\
0 & \cdots & 0 & \binom{m-1}{m-1} & \cdots & \binom{d}{m-1}\lambda^{d-m}
\end{bmatrix}
\begin{bmatrix} c_0 \\ \vdots \\ c_{d-1}\end{bmatrix}
=
\begin{bmatrix} \mathcal{D}^{(0)}p(\lambda) \\ \vdots \\ \mathcal{D}^{(m-1)}p(\lambda)\end{bmatrix}.$$

Kalman [14] gave a detailed discussion on this approach, while the confluent Vandermonde matrix to be defined below can be traced back to Aitken and Turnbull [26] and Aitken [1].

Definition 2.1. Let λ be a real number and m, d integers with $m \le d$. Define the $m \times d$ matrix

$$V_{\lambda^{(m)}}^{(d)} = \begin{bmatrix}
1 & \lambda & \lambda^2 & & \cdots & \lambda^{d-1}\\
0 & \binom{1}{1} & \binom{2}{1}\lambda & & \cdots & \binom{d}{1}\lambda^{d-2}\\
0 & 0 & \binom{2}{2} & & \cdots & \binom{d}{2}\lambda^{d-3}\\
 & & & & \vdots & \\
0 & \cdots & 0 & \binom{m-1}{m-1} & \cdots & \binom{d}{m-1}\lambda^{d-m}
\end{bmatrix}.$$

Let $\Lambda = \{\lambda_1^{(m_1)}, \ldots, \lambda_q^{(m_q)}\}$ be a multiset consisting of distinct numbers $\lambda_1, \ldots, \lambda_q$ with multiplicities $m_1, \ldots, m_q$ such that $m_1 + \cdots + m_q = d$. Define

$$V_\Lambda = \begin{bmatrix} - & V_{\lambda_1^{(m_1)}}^{(d)} & - \\ & \vdots & \\ - & V_{\lambda_q^{(m_q)}}^{(d)} & - \end{bmatrix}.$$

The square matrix V_Λ is called the *confluent Vandermonde matrix* (a.k.a. *the generalized Vandermonde matrix*) of Λ.

Example 2.2. Suppose $p(x) = c_0 + c_1 x + \cdots + c_4 x^4$ and $\Lambda = \{\lambda_1^{(1)}, \lambda_2^{(2)}, \lambda_3^{(2)}\}$. Then

$$V_\Lambda = \begin{bmatrix} 1 & \lambda_1 & \lambda_1^2 & \lambda_1^3 & \lambda_1^4 \\ 1 & \lambda_2 & \lambda_2^2 & \lambda_2^3 & \lambda_2^4 \\ 0 & 1 & 2\lambda_2 & 3\lambda_2^2 & 4\lambda_2^3 \\ 1 & \lambda_3 & \lambda_3^2 & \lambda_3^3 & \lambda_3^4 \\ 0 & 1 & 2\lambda_3 & 3\lambda_3^2 & 4\lambda_3^3 \end{bmatrix}.$$

The confluent Vandermonde matrix V_Λ of

$$\Lambda = \{\lambda_1^{(m_1)}, \ldots, \lambda_q^{(m_q)}\} \quad \text{with } m_1 + \cdots + m_q = d$$

has the property that

$$V_\Lambda \begin{bmatrix} c_0 \\ \vdots \\ c_{d-1} \end{bmatrix} = \begin{bmatrix} \mathcal{D}^{(0)} p(\lambda_1) \\ \vdots \\ \mathcal{D}^{(m_1-1)} p(\lambda_1) \\ \vdots \\ \mathcal{D}^{(0)} p(\lambda_q) \\ \vdots \\ \mathcal{D}^{(m_q-1)} p(\lambda_q) \end{bmatrix}. \tag{4}$$

Let $[p(x)] = [c_0, \ldots, c_{d-1}]^\top$ be the vector of coefficients of

$$p(x) = c_0 + c_1 x + \cdots + c_{d-1} x^{d-1}$$

and $\mathrm{eval}_\Lambda(p(x))$ the evaluations of $p(x)$ as shown in the right-hand side of equation (4). Hence we have $V_\Lambda[p(x)] = \mathrm{eval}_\Lambda(p(x))$, extending the classical result (3).

In summary, V_Λ is a matrix that transforms a polynomial into its evaluations at designated points and their derivatives according to the multiplicities.

3 Hasse derivative and confluent Vandermonde matrices over arbitrary fields

Let $\mathbb{F}[x]$ be the set of polynomials over a general field $\mathbb{F}$, and let

$$p(x) = c_0 + c_1 x + \cdots + c_{d-1} x^d \in \mathbb{F}[x].$$

For any nonnegative integer r, define the *Hasse derivative* of $p(x)$ by

$$\mathcal{D}^{(r)} p(x) = \sum_{i=r}^{d-1} c_i \binom{i}{r} x^{i-r}.$$

When $\mathbb{F} = \mathbb{R}$, it is the same as the definition $\mathcal{D}^{(r)} = \frac{1}{r!} \cdot \frac{d^r}{dx^r}$ mentioned in the last section.

Note that the binomial coefficient $\binom{n}{k}$ is the coefficient of x^k in the expansion of $(1+x)^n$, which is well defined in any field and is the same as $\frac{n!}{k!(n-k)!}$ when $\mathbb{F}$ is of characteristic 0. For this sake, an alternative definition of the Hasse derivative is

$$\mathcal{D}^{(r)} p(x) = [y^r] p(x+y),$$

which is the coefficient of y^r in $p(x+y)$ when treated as a polynomial of y. This can be verified by direct computation as follows:

$$\begin{aligned}[y^r]p(x+y) &= [y^r]\sum_{i=0}^{d-1} c_i (x+y)^i = \sum_{i=0}^{d-1} c_i \cdot [y^r](x+y)^i \\ &= \sum_{i=r}^{d-1} c_i \cdot [y^r](x+y)^i = \sum_{i=r}^{d-1} c_i \cdot \binom{i}{r} x^{i-r} = \mathcal{D}^{(r)} p(x).\end{aligned}$$

More intuitively, the evaluation $\mathcal{D}^{(r)} p(\lambda)$ is exactly the coefficient of $(x-\lambda)^r$ in the "Taylor expansion" of $p(x)$ at λ as shown in the following.

Proposition 3.1. *Let $\lambda \in \mathbb{F}$. When $p(x) = c_0 + c_1 x + \cdots + c_{d-1} x^{d-1}$ is written as*

$$p(x) = a_0 + a_1(x-\lambda) + \cdots + a_{d-1}(x-\lambda)^{d-1},$$

then

$$\mathcal{D}^{(r)} p(x) = \sum_{i=r}^{d-1} a_i \binom{i}{r} (x-\lambda)^{i-r}.$$

Therefore, $\mathcal{D}^{(r)} p(\lambda) = a_r$ can be viewed as the r-th order term of $p(x)$ in its Taylor expansion at λ.

Proof. With the alternative definition, we may compute

$$\begin{aligned}\mathcal{D}^{(r)} p(x) = [y^r]p(x+y) &= [y^r]\sum_{i=0}^{d-1} a_i (x+y-\lambda)^i = \sum_{i=0}^{d-1} a_i [y^r](x-\lambda+y)^i \\ &= \sum_{i=r}^{d-1} a_i [y^r](x-\lambda+y)^i = \sum_{i=r}^{d-1} a_i \binom{i}{r} (x-\lambda)^{i-r}.\end{aligned}$$

When λ is plugged into $\mathcal{D}^{(r)} p(x)$, each summand in the formula is zero except for

$$a_r \binom{r}{r} (x-\lambda)^{r-r} = a_r.$$

∎

This view of the Taylor expansion leads to the following identities.

Corollary 3.2. *Let $p(x)$ be a polynomial in $\mathbb{F}[x]$. Then*

$$\mathcal{D}^{(r)} (x-\lambda)^m p(x)|_{x=\lambda} = \begin{cases} \mathcal{D}^{(r-m)} p(\lambda) & \text{if } r \geq m, \\ 0 & \text{otherwise.} \end{cases}$$

In the subsequent discussion, $\mathcal{D}^{(r)}$ will always be treated as the Hasse derivative so that results obtained are valid for arbitrary fields. The simple observation in the corollary turns out to be a very useful tool for proving results on the confluent Vandermonde matrix as we shall see.

One may see [9] for more information about the Hasse derivative.

4 LU factorization

In the subsequent discussion, we shall always let $\Lambda = \{\lambda_1^{(m_1)}, \ldots, \lambda_q^{(m_q)}\}$ for q distinct elements in $\lambda_1, \ldots, \lambda_q$ in a field $\mathbb{F}$ with multiplicities $m_1, \ldots, m_q$ such that $d = m_1 + \cdots + m_q$. We will find an LU factorization of the confluent Vandermonde matrix V_Λ in the following. To achieve our goal, consider the $d + 1$ polynomials

$$\begin{aligned} &p_0(x) = 1, \quad p_1(x) = (x - \lambda_1), \quad \ldots, \quad p_{m_1}(x) = (x - \lambda_1)^{m_1}, \\ &p_{m_1+1} = (x - \lambda_1)^{m_1}(x - \lambda_2), \quad \ldots, \quad p_d(x) = (x - \lambda_1)^{m_1} \cdots (x - \lambda_q)^{m_q}. \end{aligned}$$

Thus $p_i(x)$ is a degree i polynomial with leading coefficient equal to 1. Here we only use $p_0(x), \ldots, p_{d-1}(x)$, and consider

$$V_\Lambda \begin{bmatrix} | & & | \\ [p_0(x)] & \cdots & [p_{d-1}(x)] \\ | & & | \end{bmatrix} = \begin{bmatrix} | & & | \\ \mathrm{eval}_\Lambda(p_0(x)) & \cdots & \mathrm{eval}_\Lambda(p_{d-1}(x)) \\ | & & | \end{bmatrix}. \quad (5)$$

Let $U^{-1} = [[p_0(x)] \ \cdots \ [p_{d-1}(x)]]$ and $L = [\mathrm{eval}_\Lambda(p_0(x)) \ \cdots \ \mathrm{eval}_\Lambda(p_{d-1}(x))]$. Note that U^{-1} is an upper triangular matrix with 1's on the diagonal. Thus we can write it as the inverse of some matrix U, where $U = (U^{-1})^{-1}$ is an upper triangular matrix. Moreover, by Corollary 3.2, L is a lower triangular matrix with the diagonal entries 1 for m_1 times, $(\lambda_2 - \lambda_1)^{m_1}$ for m_2 times, $(\lambda_3 - \lambda_1)^{m_1}(\lambda_3 - \lambda_2)^{m_2}$ for m_3 times, and so on.

This approach was used by Rushanan [24] to develop the LU factorization of V_Λ when all elements in Λ are distinct and V_Λ is the classical Vandermonde matrix. As we have seen, the same approach leads to similar consequences for the confluent Vandermonde matrix. Therefore, we have the following.

Theorem 4.1. *Equation (5), when written as $V_\Lambda U^{-1} = L$, leads to the LU factorization $V_\Lambda = LU$.*

This leads to a generalization of equation (2). The next theorem was mentioned in [26] and proved in [1]. The proof in [1] requires arithmetic on homogeneous symmetric functions and taking limits when $\mathbb{F} = \mathbb{R}$, while using the LU factorization allows us to observe the formula directly for V_A over arbitrary $\mathbb{F}$.

Theorem 4.2. *For any $\Lambda = \{\lambda_1^{(m_1)}, \ldots, \lambda_q^{(m_q)}\}$, $\det(V_\Lambda) = \prod_{i,j\,(i<j)}(\lambda_j - \lambda_i)^{m_i m_j}$. Consequently, V_Λ is always invertible.*

Proof. This is immediate from the fact that $V_\Lambda U^{-1} = L$ and the determinant of an upper triangular matrix (or a lower triangular matrix) is the product of its diagonal entries. ∎

Using Example 2.2 with $(\lambda_1, \lambda_2, \lambda_3) = (1, 3, 6)$, we have the following numerical example.

Example 4.3. Let $\Lambda = \{1^{(1)}, 3^{(2)}, 6^{(2)}\}$ and $\mathbb{F}$ a field, where $1, 3, 6$ are distinct, or equivalently, $\mathrm{char}(\mathbb{F}) \neq 2, 3, 5$. Then

$$p_0(x) = 1, \quad p_1(x) = (x-1), \quad p_2(x) = (x-1)(x-3),$$
$$p_3(x) = (x-1)(x-3)^2, \quad p_4(x) = (x-1)(x-3)^2(x-6).$$

Also,

$$V_\Lambda = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 3 & 9 & 27 & 81 \\ 0 & 1 & 6 & 27 & 108 \\ 1 & 6 & 36 & 216 & 1296 \\ 0 & 1 & 12 & 108 & 864 \end{bmatrix} \quad \text{and} \quad U^{-1} = \begin{bmatrix} 1 & -1 & 3 & -9 & 54 \\ 0 & 1 & -4 & 15 & -99 \\ 0 & 0 & 1 & -7 & 57 \\ 0 & 0 & 0 & 1 & -13 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Thus

$$V_\Lambda U^{-1} = \begin{bmatrix} p_0(1) & \cdots & p_4(1) \\ p_0(3) & \cdots & p_4(3) \\ \mathcal{D}^{(1)} p_0(3) & \cdots & \mathcal{D}^{(1)} p_4(3) \\ p_0(6) & \cdots & p_4(6) \\ \mathcal{D}^{(1)} p_0(6) & \cdots & \mathcal{D}^{(1)} p_4(6) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 \\ 1 & 5 & 15 & 45 & 0 \\ 0 & 1 & 8 & 39 & 45 \end{bmatrix} = L.$$

Since $\det(U^{-1}) = 1$, it follows that $\det(V_\Lambda) = \det(L) = (3-1)^2(6-1)^2(6-3)^4$. Since $1, 3, 6$ are distinct in $\mathbb{F}$, we have $\det(L) \neq 0$.

Remark 4.4. The Chinese remainder theorem states that, given the residues of a polynomial modulo $p_i(x)$ for some coprime family of polynomials $\{p_1(x), \ldots, p_q(x)\}$, the polynomial always exists and is uniquely determined, modulo $\prod_{i=1}^{q} p_i(x)$; see, e.g., [25]. When

$$p_i(x) = (x - \lambda_i)^{m_i} \quad \text{for some multiset } \Lambda = \{\lambda_1^{(m_1)}, \ldots, \lambda_q^{(m_q)}\},$$

the Chinese remainder theorem is equivalent to the invertibility of V_Λ.

5 Partial fraction decomposition

Partial fraction decomposition is an important tool for many areas, e.g., calculus, complex analysis, and more. However, when the denominator has repeated roots, the computation of the partial fraction decomposition is not so straightforward; see, e.g., [19].

For example, let $\Lambda = \{1^{(1)}, 3^{(2)}, 6^{(2)}\}$. Given a polynomial $p(x)$ of degree at most 4, how to find the coefficients a, b, c, d, e such that

$$\frac{p(x)}{(x-1)(x-3)^2(x-6)^2} = \frac{a}{(x-1)} + \frac{b + c(x-3)}{(x-3)^2} + \frac{d + e(x-6)}{(x-6)^2} \tag{6}$$

holds?

For the real and complex field, when a root λ is simple, one may multiply on both sides by $(x-\lambda)$ and take the limit $x \to \lambda$ to find the corresponding coefficients. For example,

$$a = p(1) \lim_{x\to 1} \frac{(x-1)}{(x-1)(x-3)^2(x-6)^2}.$$

However, the same idea does not extend to the case for repeated roots and polynomials over arbitrary fields. Here we demonstrate an efficient approach using the evaluation $\mathrm{eval}(p(x))$ and the Vandermonde matrix V_Λ, which is similar to the methods in [19].

We first formulate the problem. Let $\Lambda = \{\lambda_1^{(m_1)}, \ldots, \lambda_q^{(m_q)}\}$ with $d = m_1 + \cdots + m_q$. Define

$$\hat{h}_{j,m}(x) = (x-\lambda_j)^m \prod_{\substack{k=1,\ldots,q \\ k\neq j}} (x-\lambda_k)^{m_k}$$

for $j = 1, \ldots, q$ and $m = 0, \ldots, m_j - 1$. Note that there are d polynomials. When we write $\hat{h}_{1,0}(x), \ldots, \hat{h}_{q,m_q-1}(x)$, we mean $\hat{h}_{1,0}(x), \ldots, \hat{h}_{1,m_1-1}(x), \ldots, \hat{h}_{q,0}(x), \ldots, \hat{h}_{q,m_q-1}(x)$ in order. By writing equation (6) in a common denominator, we observe that finding the partial fraction decomposition is the same as writing $p(x)$ as a linear combination of $\hat{\mathscr{H}}_\Lambda = \{\hat{h}_{1,0}(x), \ldots, \hat{h}_{q,m_q-1}(x)\}$. Let

$$\hat{H}_\Lambda = \begin{bmatrix} | & & | \\ [\hat{h}_{1,0}] & \cdots & [\hat{h}_{q,m_q-1}] \\ | & & | \end{bmatrix}.$$

Then coefficients of the partial fraction decomposition can be found by $\hat{H}_\Lambda^{-1}[p(x)]$.

Example 5.1. Let $\Lambda = \{1^{(1)}, 3^{(2)}, 6^{(2)}\}$ and $\mathbb{F}$ a field, where $1, 3, 6$ are distinct. Then

$$\begin{aligned} \hat{h}_{1,0}(x) &= (x-1)^0(x-3)^2(x-6)^2, \\ \hat{h}_{2,0}(x) &= (x-1)(x-3)^0(x-6)^2, \quad \hat{h}_{2,1}(x) = (x-1)(x-3)^1(x-6)^2, \\ \hat{h}_{3,0}(x) &= (x-1)(x-3)^2(x-6)^0, \quad \hat{h}_{3,1}(x) = (x-1)(x-3)^2(x-6)^1. \end{aligned}$$

By expanding all polynomials, we have

$$\hat{H}_\Lambda = \begin{bmatrix} 324 & -36 & 108 & -9 & 54 \\ -324 & 48 & -180 & 15 & -99 \\ 117 & -13 & 87 & -7 & 57 \\ -18 & 1 & -16 & 1 & -13 \\ 1 & 0 & 1 & 0 & 1 \end{bmatrix}.$$

Thus, solving the coefficients in equation (6) is equivalent to solving

$$\hat{H}_\Lambda \begin{bmatrix} a \\ b \\ c \\ d \\ e \end{bmatrix} = [p(x)].$$

However, it is time consuming to find the inverse of $\hat{H}_\Lambda$.

The confluent Vandermonde matrix V_Λ makes the computation easier. Suppose that we want to solve $\hat{H}_\Lambda \mathbf{x} = [p(x)]$ for $\mathbf{x}$. By multiplying V_Λ to the both sides, we have $(V_\Lambda \hat{H}_\Lambda)\mathbf{x} = \text{eval}(p(x))$. We will see that $T_\Lambda = V_\Lambda \hat{H}_\Lambda$ is invertible and has nice structure, so that the answer can be readily determined by $\mathbf{x} = T_\Lambda^{-1} \text{eval}(p(x))$.

Since V_Λ transforms a polynomial into its evaluations, we examine the structure of $V_\Lambda \hat{H}_\Lambda$ by taking the evaluations of $\hat{h}_{j,m}(x)$. We observe that

$$\mathcal{D}^{(r)}\hat{h}_{j,m}(\lambda_i) = \begin{cases} 0 & \text{if } i \neq j \text{ and } r < m_i, \\ 0 & \text{if } i = j \text{ and } r < m, \\ \star & \text{if } i = j \text{ and } r = m, \\ ? & \text{if } i = j \text{ and } r > m. \end{cases}$$

Here $\star$ stands for a nonzero number and ? stands for any value in the field. To be more precise, by Corollary 3.2, we have

$$\mathcal{D}^{(m)}\hat{h}_{j,m}(\lambda_j) = \mathcal{D}^{(0)} \prod_{\substack{k=1,\dots,q \\ k\neq j}} (x-\lambda_k)^{m_k}|_{x=\lambda_j} = \prod_{\substack{k=1,\dots,q \\ k\neq j}} (\lambda_j - \lambda_k)^{m_k} \neq 0$$

if $\lambda_i - \lambda_j \neq 0$ whenever $i \neq j$. Moreover, Corollary 3.2 also guarantees

$$\mathcal{D}^{(r)}\hat{h}_{j,m}(\lambda_j) = \mathcal{D}^{(r-1)}\hat{h}_{j,m-1}(\lambda_j)$$

if $r \geq 1$ and $m \geq 1$. Remark 5.2 summarizes all these properties.

Remark 5.2. For any multiset Λ, the matrix $T_\Lambda = V_\Lambda \hat{H}_\Lambda$ is a lower triangular matrix with nonzero diagonal entries, so T_Λ, V_Λ, and $\hat{H}_\Lambda$ are all invertible, which shows the existence and the uniqueness of the partial fraction decomposition. Moreover, T_Λ is the direct sum of q Toeplitz matrices, where q is the number of distinct values in Λ.

Continuing Example 5.1, we solve the partial fraction decomposition through $T_\Lambda = V_\Lambda \hat{H}_\Lambda$.

Example 5.3. We first compute $T = V_\Lambda \hat{H}_\Lambda$ through evaluations. By direct computation, $\mathcal{D}^{(0)}\hat{h}_{1,0}(1) = 100$. Also, we may compute that

$$\begin{aligned}(x-1)(x-6)^2 &= (x-3+2)(x-3-3)^2 \\ &= 18 - 3(x-3) - 4(x-3)^2 + (x-3)^3,\end{aligned}$$

so $\mathcal{D}^{(0)}\hat{h}_{2,0}(3) = 18$, $\mathcal{D}^{(1)}\hat{h}_{2,0}(3) = -3$, and $\mathcal{D}^{(0)}\hat{h}_{2,1}(3) = 0$, $\mathcal{D}^{(1)}\hat{h}_{2,1}(3) = 18$. Other evaluations can be done in similar manners. Therefore,

$$T = V_\Lambda \hat{H}_\Lambda = \begin{bmatrix} \hat{h}_{1,0}(1) & \cdots & \hat{h}_{3,1}(1) \\ \hat{h}_{1,0}(3) & \cdots & \hat{h}_{3,1}(3) \\ \mathcal{D}^{(1)}\hat{h}_{1,0}(3) & \cdots & \mathcal{D}^{(1)}\hat{h}_{3,1}(3) \\ \hat{h}_{1,0}(6) & \cdots & \hat{h}_{3,1}(6) \\ \mathcal{D}^{(1)}\hat{h}_{1,0}(6) & \cdots & \mathcal{D}^{(1)}\hat{h}_{3,1}(6) \end{bmatrix} = \begin{bmatrix} 100 & 0 & 0 & 0 & 0 \\ 0 & 18 & 0 & 0 & 0 \\ 0 & -3 & 18 & 0 & 0 \\ 0 & 0 & 0 & 45 & 0 \\ 0 & 0 & 0 & 39 & 45 \end{bmatrix}.$$

By the structure of T_Λ, its inverse can be easily found as

$$T_\Lambda^{-1} = \begin{bmatrix} \frac{1}{100} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{18} & 0 & 0 & 0 \\ 0 & \frac{1}{108} & \frac{1}{18} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{45} & 0 \\ 0 & 0 & 0 & -\frac{13}{675} & \frac{1}{45} \end{bmatrix}.$$

Note that $\mathrm{char}(\mathbb{F}) \neq 2, 3, 5$; the entries in T_Λ^{-1} are well defined. With $\mathrm{eval}(p(x))$ computed, the coefficients of partial fraction decomposition of $p(x)$ are recorded in $T_\Lambda^{-1}\,\mathrm{eval}(p(x))$.

Theorem 5.4. *Let $\Lambda = \{\lambda_1^{(m_1)}, \ldots, \lambda_q^{(m_q)}\}$ with $m_1 + \cdots + m_q = d$. For any given polynomial $p(x)$ of degree at most $d-1$,*

$$\hat{H}_\Lambda^{-1}[p(x)] = T_\Lambda^{-1}\,\mathrm{eval}(p(x))$$

records the coefficients of $f(x)$ as a linear combination of $\hat{\mathcal{H}}_\Lambda$, which gives the partial fraction decomposition of $f(x)/\prod_{i=1}^{q}(x-\lambda_i)^{m_i}$.

In conclusion, Figure 1 illustrates the relations between $[p(x)]$, $\mathrm{eval}(p(x))$, and the partial fraction decomposition. With Λ given, each of V_Λ, $\hat{H}_\Lambda$, and T_Λ can be computed directly. By Remark 5.2, it is not too hard to find T_Λ^{-1}. Thus Theorem 5.4 states that $\hat{H}_\Lambda^{-1} = T_\Lambda^{-1} V_\Lambda$ provides a more efficient way to compute the partial fraction decomposition.

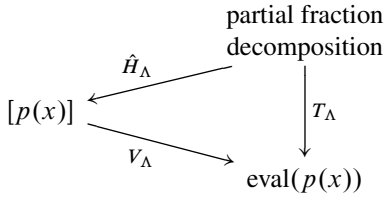


Figure 1. Relations between $[p(x)]$, $\mathrm{eval}(p(x))$, and the partial fraction decomposition.

6 Hermite interpolation polynomials

Given the evaluations of a polynomial at different points, the Lagrange interpolation polynomials provide a powerful tool in reconstructing the polynomial. What happens if the evaluations have derivatives involved?

Let $\Lambda = \{\lambda_1^{(m_1)}, \ldots, \lambda_q^{(m_q)}\}$ with $d = m_1 + \cdots + m_q$. Let $H_\Lambda = V_\Lambda^{-1}$. Then we may find polynomials $\mathcal{H}_\Lambda = \{h_{1,0}(x), \ldots, h_{q,m_q-1}(x)\}$ such that

$$H_\Lambda = \begin{bmatrix} | & & | \\ [h_{1,0}(x)] & \cdots & [h_{q,m_q-1}(x)] \\ | & & | \end{bmatrix}.$$

The set of polynomials $\mathcal{H}_\Lambda$ is called the *Hermite interpolation polynomials* of Λ. Given any polynomial $p(x)$ of degree at most $d-1$, one may recover the polynomial from the

information $\operatorname{eval}_\Lambda(p(x))$ by

$$[p(x)] = V_\Lambda^{-1} \operatorname{eval}_\Lambda(p(x)) = H_\Lambda \operatorname{eval}_\Lambda(p(x))$$

and write $p(x)$ as a linear combination of polynomials in $\mathcal{H}_\Lambda$.

In general, computing V_Λ^{-1} and formulating the Hermite interpolation polynomials are not straightforward; see, e.g., Respondek [23] for a survey. With Figure 1 in mind, finding the Hermite interpolation polynomials becomes easier with $V_\Lambda^{-1} = \hat{H}_\Lambda^{-1} T_\Lambda^{-1}$.

Example 6.1. Let $\Lambda = \{1^{(1)}, 3^{(2)}, 6^{(2)}\}$ and $\mathbb{F}$ a field, where $1, 3, 6$ are distinct. Suppose that $p(x) \in \mathbb{F}[x]$ is a polynomial of degree at most 4, and

$$p(1) = c_1, \quad p(3) = c_2, \quad \mathcal{D}^{(1)}(p(3)) = c_3, \quad p(6) = c_4, \quad \mathcal{D}^{(1)}(p(6)) = c_5$$

are given. Then

$$V_\Lambda[p(x)] = \operatorname{eval}_\Lambda(p(x)) = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \end{bmatrix}.$$

With $H_\Lambda = V_\Lambda^{-1}$, we have

$$H_\Lambda \operatorname{eval}_\Lambda(p(x)) = [p(x)].$$

Let T_Λ^{-1} and $\hat{h}_{1,0}, \ldots, \hat{h}_{3,1}$ be as in Example 5.3. Since $H_\Lambda = V_\Lambda^{-1} = \hat{H}_\Lambda T_\Lambda^{-1}$, the Hermite interpolation polynomials are

$$\begin{aligned}
h_{1,0}(x) &= \frac{1}{100}\hat{h}_{1,0}(x) = \Big(\frac{1}{100}\Big)(x-3)^2(x-6)^2, \\
h_{2,0}(x) &= \frac{1}{18}\hat{h}_{2,0}(x) + \frac{1}{108}h_{2,1}(x) = (x-1)\Big(\frac{1}{18} + \frac{1}{108}(x-3)\Big)(x-6)^2, \\
h_{2,1}(x) &= \frac{1}{18}\hat{h}_{2,1}(x) = (x-1)\Big(\frac{1}{18}(x-3)\Big)(x-6)^2, \\
h_{3,0}(x) &= \frac{1}{45}\hat{h}_{3,0}(x) - \frac{13}{675}\hat{h}_{3,1}(x) = (x-1)(x-3)^2\Big(\frac{1}{45} - \frac{13}{675}(x-6)\Big), \\
h_{3,1}(x) &= \frac{1}{45}\hat{h}_{3,1}(x) = (x-1)(x-3)^2\Big(\frac{1}{45}(x-6)\Big).
\end{aligned}$$

For each of these polynomials, its evaluations given by Λ are all 0 except for a unique 1. Therefore, for any given evaluations $c_1, \ldots, c_5$, we have the Hermite interpolation

$$p(x) = c_1 h_{1,0}(x) + c_2 h_{2,0}(x) + c_3 h_{2,1}(x) + c_4 h_{3,0}(x) + c_5 h_{3,1}(x).$$

Note that computing T_Λ^{-1} is much easier than computing V_Λ^{-1}. In fact, the author of [13] used similar ideas to compute the inverse of a confluent Vandermonde matrix, and a faster algorithm was later proposed by Respondek [22]. The Hermite interpolation is also used to calculate the functions of companion matrices [5, 17].

7 Companion matrix

Let

$$p(x) = c_0 + c_1 x + \cdots + c_d x^d \quad \text{with } c_d = 1$$

be a polynomial of degree d. The *companion matrix* of $p(x)$ is defined as

$$C = \begin{bmatrix} 0 & \cdots & 0 & -c_0 \\ 1 & \ddots & \vdots & -c_1 \\ & \ddots & 0 & \vdots \\ 0 & & 1 & -c_{d-1} \end{bmatrix}.$$

When

$$p(x) = \prod_{\lambda \in \Lambda} (x - \lambda)$$

is the monic polynomial with roots in Λ, we write C_Λ as the companion matrix of $p(x)$. Here, we always assume that the roots lie in the field. In particular, the discussion always works for an algebraically closed field.

Suppose $\Lambda = \{\lambda_1^{(m_1)}, \ldots, \lambda_q^{(m_q)}\}$. Let $J_{\lambda,m}$ be the $m \times m$ lower-triangular Jordan block of λ and define

$$J_\Lambda = \begin{bmatrix} J_{\lambda_1,m_1} & & \\ & \ddots & \\ & & J_{\lambda_q,m_q} \end{bmatrix}.$$

Aitken and Turnbull [26] stated that

$$V_\Lambda C_\Lambda V_\Lambda^{-1} = J_\Lambda$$

is the Jordan canonical form of C_Λ with exactly 1 Jordan block for each distinct eigenvalue. In particular, when all elements in Λ are distinct, $V_\Lambda C_\Lambda V_\lambda^{-1}$ is a diagonal matrix.

Example 7.1. Let $\Lambda = \{\lambda_1, \lambda_2^{(2)}, \lambda_3^{(2)}\}$ and $p(x) = (x-\lambda_1)(x-\lambda_2)^2(x-\lambda_3)^2$ such that $\lambda_i - \lambda_j \neq 0$ whenever $i \neq j$. Then $V_\Lambda C_f V_\lambda^{-1}$ will be a direct sum of

$$[\lambda_1], \quad \begin{bmatrix} \lambda_2 & 0 \\ 1 & \lambda_2 \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} \lambda_3 & 0 \\ 1 & \lambda_3 \end{bmatrix}.$$

This Jordan form result is elegant, but no formal proof was given in [26]. With the discussions in the previous sections, we can prove the Jordan form result above and gain better insight into it. Recall that $\text{eval}_\Lambda(f(x))$ is a vector evaluating the polynomial $f(x)$. By definition, we have $\text{eval}_\Lambda(p(x)) = \mathbf{0}$. Moreover, each evaluation has the following properties.

Proposition 7.2. *Let Λ be a multiset and $f(x) \in \mathbb{F}[x]$. Then*

(1) $\mathcal{D}^{(0)}(xf(x))|_{x=\lambda} = \lambda \mathcal{D}^{(0)}(f(x))|_{x=\lambda}$,

(2) $\mathcal{D}^{(r)}(xf(x))|_{x=\lambda} = \mathcal{D}^{(r-1)}(f(x))|_{x=\lambda} + \lambda \mathcal{D}^{(r)}(f(x))|_{x=\lambda}$ *if* $r \geq 1$, *and*

(3) $\text{eval}_\Lambda(xf(x)) = J_\Lambda \,\text{eval}_\Lambda(f(x))$.

Proof. The first identity holds since both sides are equal to $\lambda f(\lambda)$.

For the field of real numbers, the second identity could be written as

$$\frac{1}{r!}\cdot\frac{d^r}{dx^r}(xf(x))|_{x=\lambda} = \frac{1}{r!}\left(\binom{r}{1}\frac{d^{r-1}}{dx^{r-1}}(f(x)) + x\cdot\frac{d^r}{dx^r}(xf(x))\right)\Bigg|_{x=\lambda}$$

and is the product rule applied to $xf(x)$. However, we emphasize that the identity remains valid from the Hasse derivative point of view, and the proof is even more intuitive. By direct computation and Corollary 3.2, we have

$$\begin{aligned}\mathcal{D}^{(r)}(xf(x))|_{x=\lambda} &= \mathcal{D}^{(r)}((x-\lambda)f(x)+\lambda f(x))|_{x=\lambda}\\ &= \mathcal{D}^{(r)}((x-\lambda)f(x))|_{x=\lambda} + \mathcal{D}^{(r)}(\lambda f(x))|_{x=\lambda}\\ &= \mathcal{D}^{(r-1)}(f(x))|_{x=\lambda} + \lambda\mathcal{D}^{(r)}(f(x))|_{x=\lambda}.\end{aligned}$$

Finally, the third identity is simply a matrix-vector form of the first two identities. ∎

Let $p_j(x) = x^j$. Then the columns of the $d\times d$ identity matrix I are

$$[p_0(x)],\ldots,[p_{d-1}(x)]$$

and

$$C_\Lambda = \begin{bmatrix} | & & | & | \\ [xp_0(x)] & \cdots & [xp_{d-2}(x)] & [xp_{d-1}(x)-p(x)] \\ | & & | & | \end{bmatrix}.$$

With Proposition 7.2 and the fact $\mathrm{eval}_\Lambda(p(x)) = \mathbf{0}$, we have

$$V_\Lambda C_\Lambda = J_\Lambda\begin{bmatrix} | & & | \\ \mathrm{eval}_\Lambda(p_0(x)) & & \mathrm{eval}_\Lambda(p_{d-1}(x)) \\ | & & | \end{bmatrix} = J_\Lambda V_\Lambda I = J_\Lambda V_\Lambda.$$

Therefore, $V_\Lambda C_\Lambda V_\Lambda^{-1} = J_\Lambda$ follows.

Moreover, with our discussion in earlier sections, V_Λ^{-1} comes from the Lagrange interpolation polynomials when all elements in Λ^{-1} are distinct and from the Hermite interpolation polynomials in general. Let $\mathcal{E}_d = \{p_0(x),\ldots,p_{d-1}(x)\}$ and $\mathcal{H}_\Lambda$ the Hermite interpolation polynomials. Then both $\mathcal{E}_d$ and $\mathcal{H}_\Lambda$ are bases of the space $\mathcal{P}_{d-1}$ of polynomials of degree at most $d-1$. Consider the linear transformation $\phi\colon \mathcal{P}_{d-1}\to\mathcal{P}_{d-1}$ defined by $\phi(f(x)) = xf(x) \pmod{p(x)}$. Then C_Λ can be viewed as the matrix representation of ϕ with respect to $\mathcal{E}_d$, while J_Λ is the matrix representation of ϕ with respect to $\mathcal{H}_\Lambda$. Thus V_Λ and $H_\Lambda = V_\Lambda^{-1}$ are the change of basis matrices between $\mathcal{E}_d$ and $\mathcal{H}_\Lambda$. This provides a comprehensive understanding of the Jordan decomposition of C_Λ.

The decomposition $V_\Lambda C_\Lambda V_\Lambda^{-1} = J_\Lambda$ was then used for solving differential equations in [3] and yet more applications in [4, 17, 27].

Moreover, Gilbert [8] and the references therein considered integer matrices with integer eigenvalues with integer (left and right) generalized eigenvectors. The paper pointed out that C_Λ is one of such matrices when Λ contains only two distinct values that are integers with difference 1. This can again be seen from our discussion. When Λ contains only

one integer value or two distinct integer values with difference 1, we have $\det(V_\Lambda) = 1$ and V_Λ^{-1} is an integer matrix. In these cases, V_Λ, C_Λ, V_Λ^{-1}, and J_Λ are all integer matrices.

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