Doc. Math. (Online first)
DOI 10.4171/DM/1100

© 2026 Deutsche Mathematiker-Vereinigung
Published by EMS Press
This work is licensed under a CC BY 4.0 license

Fields of Toeplitz algebras form the principal symbols of regular 2-step nilpotent groups

Clément Cren

Abstract. We show that the C^*-algebra of a regular 2-step nilpotent Lie group can be recovered using continuous fields of Toeplitz algebras and a crossed product. We generalize this result to polycontact manifolds in the sense of van Erp which are endowed with fields of such groups. We also investigate those manifolds with a more rigid structure, namely those modeled on H-type groups. In all those cases, there is a certain pseudodifferential calculus named filtered calculus, we show that the algebra of principal symbols can also be recovered from the field of Toeplitz algebras.

1. Introduction

The study of certain hypoelliptic differential operators led to the introduction of a new pseudodifferential calculus, taking into account a certain anisotropy in the ambient space in the definition of its principal symbols. Here we will be interested in manifolds M endowed with a distribution $H \subset TM$ such that:

$$\Gamma(H) + \big[\Gamma(H), \Gamma(H)\big] = \Gamma(TM).$$

Here Γ denotes the global sections of a vector bundle (the same statement works with compactly supported sections instead, the manifold M will be assumed compact anyway). The idea of this new calculus, following e.g. [2, 10], is to consider vector fields in $\Gamma(H)$ still as differential operators of order 1, but the other vector fields as having order 2. The principal symbol of an operator is then understood as a convolution operator on a family of graded nilpotent groups, here of step 2, the osculating groups. This family arises from the vector bundle $\mathfrak{t}_H M = H \oplus {}^{TM}\!/_{H}$, endowed with the Lie bracket on sections given by the curvature form:

$$\begin{aligned} B\colon \Gamma(H) \times \Gamma(H) &\to \Gamma\big({}^{TM}\!/_{H}\big) \\ (X, Y) &\mapsto [X, Y]_{\Gamma(TM)} \mod \Gamma(H). \end{aligned}$$

We have $B \in \Gamma(\Lambda^2 H^* \otimes {}^{TM}\!/_{H})$ so the Lie brackets can be localized on each fiber of $\mathfrak{t}_H M$ which is thus a family of Lie algebras, the osculating Lie algebras. These algebras being nilpotent, they can be integrated into a family of connected, simply connected

Mathematics Subject Classification 2020: 47B35 (primary); 47L80, 35R03, 43A80, 53C17 (secondary).
Keywords: Toeplitz operators, representation theory of nilpotent groups, Harmonic analysis, operator algebras.

Lie groups $T_H M$ using the Baker–Campbell–Hausdorff formula, these groups are called the osculating Lie groups. This family of groups is in general not locally trivial as the group structure might change abruptly at a given point. The analysis of the operators in the calculus requires a good understanding of the algebra of principal symbols. The ellipticity condition for instance (called Rockland condition in this context, after [25]) requires invertibility of the principal symbols in all the non-trivial irreducible unitary representations of the osculating groups. Understanding these representations can be done through Kirillov theory [18]. This theory provides a homeomorphism between the unitary dual of a nilpotent Lie group with the Fell topology and the set of co-adjoint orbits with the quotient topology. This calculus has gained a lot of attraction lately.

In this paper we assume that our groups have a specific structure, namely regular 2-step nilpotent groups to allow a good understanding of their unitary dual. Manifolds having all their osculating groups 2-step regular are called polycontact after [26]. We show that the C^*-algebra of principal symbols (of order 0) on these manifolds can be constructed from a field of Toeplitz algebras. We give two different proofs of this identification. The first reduces the result to the case of Heisenberg group which was treated in [5]. The second relies on a finer analysis of symbols on regular step-2 groups, where we identify them with certain families of Weyl operators.

A particular example of regular 2-step nilpotent groups are the groups of Heisenberg type introduced in [13]. These groups can be constructed from representations of Clifford algebras, in particular, the moduli space of their isomorphism classes is discrete. Because of that, we show that if a polycontact manifold has all its osculating groups of Heisenberg type then they are all isomorphic and the osculating groups form a locally trivial group bundle. In particular, the symbol algebra also becomes a locally trivial bundle of C^*-algebras over M. These manifolds include contact manifolds, for which the result can be found in the author's previous work [5], as well as quaternionic contact or octonionic contact manifolds [3], and the cocompact quotients of any group of Heisenberg type.

2. Toeplitz algebra of a symplectic vector space

Given a complex hermitian vector space W, we can construct a Toeplitz C^*-algebra $\mathcal{T}(W)$ in natural way. Now, given a symplectic vector space (V, ω), we can choose a complex structure $J: V \to V$ compatible with the symplectic form. We have then two complex hermitian vector spaces (of the same real dimension as V), $V^{1,0}, V^{0,1}$. The two algebras $\mathcal{T}(V^{1,0}), \mathcal{T}(V^{0,1})$ are opposed to one another and depend naturally on the triple (V, ω, J) with respect to unitary symplectomorphisms. Replacing J by $-J$ corresponds to changing the orientation of the symplectic vector space (i.e., replacing ω by $-\omega$ to keep compatibility) so it swaps the two algebras.

We describe this C^*-algebra as a subalgebra of bounded operators on a specific Hilbert space, namely the symmetric Fock space. For more details see [5]. With the same notations as above, consider a complex hermitian vector space W. We then take the positive

symmetric tensor powers $\mathrm{Sym}^k(W)$ with the induced hermitian metric. The symmetric Fock space is then given by the ℓ^2-completion:

$$\mathcal{F}^+(W) = \ell^2 - \bigoplus_{k\geq 0}^{\perp} \mathrm{Sym}^k(W).$$

If $w \in W$, it induces a shift operator on $\mathcal{F}^+(W)$ by:

$$S_w = \mathrm{Sym}(w \otimes \cdot).$$

The Toeplitz algebra is then generated by these operators and their adjoints:

$$\mathcal{T}(W) = C^*(S_w | w \in W) \subset \mathcal{B}\big(\mathcal{F}^+(W)\big).$$

This means that we consider all (non-commutative) polynomials between S_w's and their adjoints, as well as the limits of such operators. If we fix an orthonormal basis $e_1, \ldots, e_n$, then the Toeplitz algebra is generated by $S_{e_1}, \ldots, S_{e_n}$ and their adjoints. The commutation relations between these operators are then easier to express. In particular, they commute modulo the compact operators. Considering their joint spectrum, we obtain the exact sequence:

$$0 \longrightarrow \mathcal{K}\big(\mathcal{F}^+(W)\big) \longrightarrow \mathcal{T}(W) \longrightarrow \mathcal{C}(\mathbb{S}^*W) \longrightarrow 0. \tag{1}$$

The quotient map is realized as follows. A vector $w \in W$, seen as an element of $(W^*)^*$ defines a linear form on W^*. By homogeneity, we can see this map as defined on the sphere $\mathbb{S}^*W$. We can also take the adjoints of such operators (we get the complex conjugate function), products of such operators (so homogeneous polynomial functions on W^*) and sums (arbitrary polynomials). This corresponds to the quotient map in the above exact sequence. This can be summarized as

$$\forall w \in W, \quad S_w \equiv w \mod \mathcal{K}\big(\mathcal{F}^+(W)\big). \tag{2}$$

Remark 2.1. This symbolic description of the Toeplitz algebra is due to Arveson [1]. Toeplitz operators can also be defined as multiplication operators on some spaces of (weighted) holomorphic L^2-functions (composed with the orthogonal projection onto the subspace of holomorphic functions). The resulting C^*-algebras are isomorphic.

This algebra is compatible with unitary morphisms between the underlying vector spaces. Let W_1, W_2 be two hermitian vector spaces, and $T: W_1 \to W_2$ a unitary map. It induces a map between the symmetric tensor powers and hence between the symmetric Fock spaces:

$$\widetilde{T}: \mathcal{F}^+(W_1) \to \mathcal{F}^+(W_2)$$

which is also unitary. We then have, for any $w \in W$, the relation

$$\widetilde{T} \circ S_w = S_{T(w)} \circ \widetilde{T}.$$

Therefore the isomorphism $\widetilde{T}$ intertwines the two Toeplitz algebras.

Consequently, if $E \to M$ is a hermitian vector bundle, one can define a bundle of symmetric Fock spaces $\mathcal{F}^+(E) \to M$. This is a locally trivial bundle of Hilbert spaces thanks to the unitary equivariance of the construction. We get corresponding bundle of Toeplitz algebras $(\mathcal{T}(E_m))_{m\in M}$ which is also locally trivial by the same remark. Denote by $\mathcal{T}(E)$ the algebra of continuous sections of this bundle, vanishing at infinity if M is non-compact. If $X \in \Gamma_c(M, E)$ is a compactly supported section, we can build a section of $\mathcal{T}(E)$ by S_X, which on every fiber over $m \in M$ takes the value $S_{X(m)}$. The C^*-algebra $\mathcal{T}(E)$ is generated by these sections.

Given a symplectic vector space (V, ω), we can take a compatible complex structure $J \in \mathrm{End}(V)$. We get a hermitian vector space $V^{1,0}$. We denote by

$$\mathcal{T}(V, J) := \mathcal{T}(V^{1,0}),$$

the corresponding symmetric Fock space. The identification $V \cong V^{1,0}$ identifies the respective (co)-spheres. This way we rewrite the exact sequence (1) into:

$$0 \longrightarrow \mathcal{K}\left(\mathcal{F}^+(V^{1,0})\right) \longrightarrow \mathcal{T}(V, J) \longrightarrow \mathcal{C}(\mathbb{S}^*V) \longrightarrow 0. \tag{3}$$

We obtain a similar construction for a symplectic vector bundle $H \to M$ (i.e., there is a section $\omega \in \Gamma(M, \Lambda^2 H^*)$ which restricts to a symplectic form on each fiber). The space of compatible complex structures on a symplectic vector space is contractible. Therefore we may also choose a complex structure $J \in \mathrm{End}(H)$ compatible with the symplectic form on each fiber. We then have a hermitian vector bundle $H^{1,0} \to M$ and take the corresponding Toeplitz algebra (so the one given by sections of Toeplitz algebras over each fiber). Similarly as for symplectic vector spaces, we denote by

$$\mathcal{T}(H, J) := \mathcal{T}(H^{1,0}),$$

this Toeplitz algebra. The bundle of Hilbert spaces $\mathcal{F}^+(H^{1,0})$ is a bundle of infinite dimensional, separable Hilbert spaces and is therefore trivializable, by a result of Dixmier and Douady [7]. Consequently, the corresponding field of compact operators on each fiber is trivializable as well. The algebra of continuous sections vanishing at infinity on this field $\Gamma_0(M, \mathcal{K}(\mathcal{F}^+(H^{1,0})))$ is thus isomorphic to $\mathcal{C}_0(M, \mathcal{K})$. Therefore, this Toeplitz algebra fits into the exact sequence:

$$0 \longrightarrow \mathcal{C}_0(M, \mathcal{K}) \longrightarrow \mathcal{T}(H, J) \longrightarrow \mathcal{C}(\mathbb{S}^*H) \longrightarrow 0. \tag{4}$$

3. Regular 2-step nilpotent groups and H-type groups

In this paper we will work with the following kind of groups.

Definition 3.1. A regular 2-step nilpotent Lie algebra is a graded Lie algebra $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ such that $\mathfrak{g}_2 = \mathfrak{z}(\mathfrak{g}) = [\mathfrak{g}_1, \mathfrak{g}_1]$ and moreover, for any $x \in \mathfrak{g}_1$, the map $\mathrm{ad}_x \colon \mathfrak{g}_1 \to \mathfrak{g}_2$ is surjective. A regular 2-step nilpotent group is a connected, simply connected group, whose Lie algebra is regular 2-step nilpotent Lie algebra.

These groups are also called polycontact groups in [26] or H-groups in the sense of Métivier (and not to be confused with the H-type groups later on). This class of groups was introduced in [21]. It was shown than on this group, hypoellipticity and analytic hypoellipticity were equivalent for homogeneous differential operators (so both equivalent to the Rockland condition). Here we show that the regularity condition gives a nice structure to the space of coadjoint orbits.

Proposition 3.2. *Let $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ be a 2-step nilpotent Lie algebra. The following are equivalent:*

(1) *$\mathfrak{g}$ is regular.*

(2) *For every $\theta \in \mathfrak{g}_2^* \setminus 0$, the form $\omega_\theta \in \Lambda^2 \mathfrak{g}_1^*$ defined by $\omega_\theta \colon (u, v) \mapsto \theta([u, v])$ is non-degenerate (i.e., $(\mathfrak{g}_1, \omega_\theta)$ is a symplectic vector space).*

Proof. If $\mathfrak{g}$ is not regular consider $x \in \mathfrak{g}_1$ for which $\mathrm{ad}_x \colon \mathfrak{g}_1 \to \mathfrak{g}_2$ is not surjective. If $\theta \in \mathfrak{g}_2^* \setminus 0$ vanishes on $\mathrm{im}(\mathrm{ad}_x)$, then $\omega_\theta(x, \cdot) = 0$ and the form ω_θ is degenerate.

If $\mathfrak{g}$ is regular, assume there is a $\theta \in \mathfrak{g}_2^*$ for which ω_θ has a kernel, and take $x \in \ker(\omega_\theta) \setminus 0$. Let $z \in \mathfrak{g}_2$, by regularity, we can take $y \in \mathfrak{g}_1$ with $[x, y] = z$. Then $\theta(z) = \omega_\theta(x, y) = 0$ so $\theta = 0$ and $\mathfrak{g}$ satisfies (2). ∎

We now compute the coadjoint orbits of a regular 2-step graded group. It will be used in the next section to describe its irreducible unitary representations using Kirillov theory.

Proposition 3.3. *Let G be a regular 2-step graded group. Its coadjoint orbits consist of:*

- *Affine subspaces of the form $\mathfrak{g}_1^* \oplus \{\theta\}$, $\theta \in \mathfrak{g}_2^* \setminus 0$.*
- *Points $\{\eta\}$, $\eta \in \mathfrak{g}_1^*$.*

Consequently, we get a homeomorphism $\mathfrak{g}^*/G \cong (\mathfrak{g}_2^* \setminus 0) \sqcup \mathfrak{g}_1^*$. The topology on the right-hand side being the euclidean one on each component, the first being open and the second closed. A net in $\mathfrak{g}_2^* \setminus 0$ converging to 0 converges to all the points in $\mathfrak{g}_1^*$.

Proof. Let $\eta \in \mathfrak{g}_1^*$, $x \in \mathfrak{g}$. The image of ad_x is included in $\mathfrak{g}_2$, thus $\mathrm{ad}_x^*(\eta) = 0$. Consequently, $\forall g \in G$, $\mathrm{Ad}^*(g)(\eta) = \eta$ and $\{\eta\}$ is a coadjoint orbit.

Now consider $(\eta, \theta) \in \mathfrak{g}^*$, $x \in \mathfrak{g}_1$. We have $\mathrm{ad}_x^*(\eta, \theta) = (\mathrm{ad}_x^*(\theta), 0)$. The mapping $x \mapsto \mathrm{ad}_x^* \theta$ from $\mathfrak{g}_1$ to its dual is the musical isomorphism induced by the form ω_θ and is in particular surjective. Therefore

$$T_{(\eta,\theta)}\big(\mathrm{Ad}^*(G) \cdot (\eta, \theta)\big) = \mathfrak{g}_1^*.$$

Consequently, the coadjoint orbit going through (η, θ) is the affine space $\mathfrak{g}_1^* \oplus \{\theta\}$. ∎

Remark 3.4. We see in the proof the importance of the regularity assumption. Without it we would still have the first kind of coadjoint orbits. However, the other kind of orbits would just be included in the affine spaces and not necessarily equal to them. These particular kind of orbits (called flat orbits) have been studied extensively since the work of Moore and Wolf [23], as they were proved to correspond to square-integrable representations of the group (for any kind of nilpotent group).

Now, using Kirillov theory, we can make these coadjoint orbits correspond to representations of the group. Given a general nilpotent group G, Kirillov [18] defines a one-to-one correspondence between the irreducible unitary representations of the group, and the co-adjoint orbits in the dual of its Lie algebra. This correspondence

$$\widehat{G} \xrightarrow{\sim} {\mathfrak{g}^*}\!\!\diagup_{\mathrm{Ad}^*(G)}$$

was also shown to be an homeomorphism (the spaces being endowed with the hull kernel topology and the quotient of the euclidean topology respectively). Kirillov also links certain properties and constructions of the representations with those of the co-adjoint orbits. The one we will use is the fact that the representation corresponding to an orbit of dimension $2d$, $d \in \mathbb{N}$, can be realized on $L^2(\mathbb{R}^d)$ (in a very specific way). This implies that the representations corresponding to 0-dimensional co-adjoint orbits are characters and all the others are infinite dimensional.

Recall that for a locally compact group G, we may form its C^*-algebra $C^*(G)$ (for more details and background on this construction, see e.g., [6]). First consider the algebra $\mathcal{C}_c(G)$ with convolution

$$(f_1, f_2) \mapsto f_1 * f_2 = \left(g \mapsto \int_{\gamma \in G} f_1(\gamma) f_2(\gamma^{-1} g) \mathrm{d}\gamma\right).$$

The group C^*-algebra is then obtained by taking the closure of $\mathcal{C}_c(G)$ acting by convolution on $L^2(G)$. This is the reduced C^*-algebra, there is also a maximal C^* algebra but both coincide for amenable groups, in particular for nilpotent groups. We will only consider groups for which this is true, so that we may use properties of the maximal C^*-algebra as well. Given a unitary representation $\pi\colon G \to \mathcal{U}(H)$, we can extend it to a $*$-representation of $C^*(G)$ by the formula:

$$\pi(f) = \int_{g \in G} f(g)\pi(g)\mathrm{d}g, \quad f \in \mathcal{C}_c(G).$$

This formula extends to a bounded C^*-algebra homomorphism $\pi\colon C^*(G) \to \mathcal{B}(H)$. Conversely, any non-zero algebra homomorphism of this sort comes from a unitary representation of G. The irreducibility of such representations is also preserved and we get a one-to-one correspondence between unitary irreducible representations of the group G and irreducible representations of its C^*-algebra $C^*(G)$.

Remark 3.5. To define the C^*-algebra, we could also have started with $\mathcal{C}_c^\infty(G)$ instead of all compactly supported continuous functions, the norm closure would stay the same. All the integrals considered here are with respect to a fixed Haar measure on the group but the resulting algebra does not depend on it up to isomorphism. Finally, this construction applies verbatim to smooth families of groups (more generally for Lie groupoids). That is, a pair of spaces with a surjective submersion $G \to M$, and a smooth map $m\colon G \times_M G \to G$ which restricts to a group law on each fiber (the inverse also needs to be smooth). We get a similar construction of C^*-algebra if we consider the group convolution fiberwise (the completion is also similar). This will be used in Section 9.

We now go back to a regular 2-step graded group G. Elements of $\mathfrak{g}_1^*$ are fixed points for the co-adjoint action so they represent characters of the group. In particular, $0 \in \mathfrak{g}_1^*$ corresponds to the trivial representation. The other representations, corresponding to some $\theta \in \mathfrak{g}_2^*$ are infinite dimensional.

Corollary 3.6. *Let G be a regular 2-step nilpotent group with Lie algebra $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$. Then $C^*(G)$ is a continuous field of C^*-algebras over $\mathfrak{g}_2^*$. The fiber over a non-zero point is the algebra of compact operators over an infinite-dimensional separable Hilbert space, $\mathcal{K}$. The fiber over $0 \in \mathfrak{g}_2^*$ is the commutative algebra $\mathcal{C}_0(\mathfrak{g}_1^*)$. The continuous field is trivial on $\mathfrak{g}_2^* \setminus 0$.*

Proof. The map $\widehat{G} \to \mathfrak{g}_2^*$ which is the identity on $\mathfrak{g}_2^* \setminus 0$ and sends the rest to 0 is continuous and open. The result is then a direct consequence of a theorem of Lee [19, Theorem 4]. For the triviality away from 0, we need to show that if $f \in \mathcal{C}_c^\infty(G)$, the function $\theta \in \mathfrak{g}_2^* \setminus 0 \mapsto \pi_\theta(f) \in \mathcal{K}$ is continuous. This will be clear in the next section where we give a concrete description of these representations. ∎

Corollary 3.7. *Let G be a regular 2-step nilpotent group with Lie algebra $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$. Then we have the extension of the group C^*-algebra:*

$$0 \longrightarrow \mathcal{C}_0(\mathfrak{g}_2^* \setminus 0, \mathcal{K}) \longrightarrow C^*(G) \longrightarrow \mathcal{C}_0(\mathfrak{g}_1^*) \longrightarrow 0.$$

Proof. The quotient map is the evaluation at zero in the previous continuous field description of $C^*(G)$. The kernel of this map is exactly the sections that vanish at $0 \in \mathfrak{g}_2^*$. The field of compact operators being trivial away from 0 we obtain the corresponding ideal. ∎

4. Representation theory for regular 2-step nilpotent groups

The representation theory of regular 2-step nilpotent groups was already described in the last section through Kirillov theory. In this section we give a more concrete description of the representations, particularly the infinite-dimensional ones. This description will allow a precise description for the symbol algebra through Weyl operators in the next section, giving the link with Toeplitz operators.

The representations corresponding to $\mathfrak{g}_1^*$ are characters so we focus on the other ones, which are infinite dimensional. Let $\theta \in \mathfrak{g}_2^* \setminus 0$. We obtain a symplectic form ω_θ on $\mathfrak{g}_1$. If we choose a compatible complex structure J, we obtain a complex hermitian vector space $(\mathfrak{g}_1, J)^{1,0}$ and consider the corresponding symmetric Fock space $\mathcal{F}^+_{\theta,J} := \mathcal{F}^+((\mathfrak{g}_1, J)^{1,0})$.

We construct the representation π_θ of the Lie algebra $\mathfrak{g}$ by anti-self-adjoint, unbounded operators on $\mathcal{F}^+_{\theta,J}$. This representation lifts to a representation of the group G, as it satisfies the condition of Nelson [24] (we will write π_θ for both the representation of the group and the Lie algebra). By Schur's Lemma, an irreducible representation has to map the center into scalar operators. In particular, it induces a linear form on $\mathfrak{z}(\mathfrak{g}) = \mathfrak{g}_2$. This map is, following the construction of Kirillov, given by $\theta \in \mathfrak{g}_2$, i.e.,

$$\pi_{\theta|\mathfrak{g}_2} = \mathrm{i}\theta.$$

We then need to construct an operator for each element of $\mathfrak{g}_1$ such that their Lie bracket, which is an operator corresponding to an element of $\mathfrak{g}_2$, is one of the previous scalar ones. Let $X_1, \dots, X_n, Y_1, \dots, Y_n$ be a Darboux basis for $(\mathfrak{g}_1, \omega_\theta)$, compatible with J, that is $JX_j = -Y_j$, $JY_j = X_j$, $j \geq 1$. Consider then $W_j = \frac{1}{\sqrt{2}}(X_j + \mathrm{i}Y_j)$. The vectors $W_1, \dots, W_n$ for an orthonormal basis for $(\mathfrak{g}_1, J)^{1,0}$. This induces a basis on the symmetric Fock space given by the family of vectors $\mathrm{Sym}(W_1^{\otimes\alpha_1} \otimes \cdots \otimes W_n^{\otimes\alpha_n})$, $\alpha \in \mathbb{N}^n$. We denote by $|\alpha\rangle$ the vectors obtained after normalization. The family $(|\alpha\rangle)_{\alpha\in\mathbb{N}^n}$ is a Hilbert basis of the symmetric Fock space $\mathcal{F}^+_{\theta,J}$.

Now the operators $\pi_\theta(W_j), \pi_\theta(\overline{W_j})$ that we want to obtain have to satisfy the canonical commutation relations:

$$\left[\pi_\theta(W_j), \pi_\theta(\overline{W_j})\right] = \pi_\theta\left([W_j, \overline{W_j}]\right) = -\mathrm{i}^2\omega_\theta(X_j, Y_j) = 1.$$

Similarly, if $j \neq k$ the operators $\pi_\theta(W_j)$ commutes with $\pi_\theta(W_k)$ and $\pi_\theta(\overline{W_k})$. Consider the creation and annihilation operators on the Fock space A_j^*, A_j, $1 \leq j \leq n$. They are unbounded self-adjoint operators defined by:

$$\begin{aligned} A_j|\alpha\rangle &= \sqrt{\alpha_j}|\alpha_1, \dots, \alpha_j - 1, \dots, \alpha_n\rangle, \\ A_j^*|\alpha\rangle &= \sqrt{\alpha_j + 1}|\alpha_1, \dots, \alpha_j + 1, \dots alpha_n\rangle. \end{aligned}$$

We can also define these operators using the shifts of the Toeplitz algebra and the so-called number operator N, which is densely defined, self-adjoint and defined by:

$$N|\alpha\rangle = |\alpha||\alpha\rangle, \quad |\alpha| = \sum_{j=1}^{n} \alpha_j,\ \alpha \in \mathbb{N}^n.$$

We then have the following relations, which can be deduced from direct computation (see [1,5]):

$$\forall 1 \leq j \leq n, \quad A_j^* = N^{1/2}S_{W_j} = S_{W_j}(N+1)^{1/2}. \tag{5}$$

In this way, the creation operators can be seen as an unbounded version of the shifts defining the Toeplitz algebra (in the shift, some constant appear from the symmetrization making it a bounded operator). The main feature of these operators is that they satisfy the canonical commutation relations:

$$\forall 1 \leq j, k \leq n, \quad [A_j, A_k^*] = \delta_{j,k}.$$

We can therefore define

$$\pi_\theta(W_j) = \mathrm{i}A_j^*, \quad \pi_\theta(\overline{W_j}) = \mathrm{i}A_j. \tag{6}$$

This defines an irreducible representation of the Lie algebra $\mathfrak{g}$. To integrate this representation into a representation of the Lie group G, Nelson's theorem [24, Corollary 9.1] requires the operator $\sum_j \pi_\theta(V_j)^2$ (the V_j form a basis of $\mathfrak{g}$) to be essentially self-adjoint

on a dense linear subspace. This subspace has to be contained in all the domains of $\pi_\theta(V)$ and $\pi_\theta(V)\pi_\theta(W)$ where V, W are elements of $\mathfrak{g}$. Here this operator is $-2N + C\,\mathrm{Id}$ where $C \in \mathbb{R}$ is a constant. It is easily seen to be essentially self-adjoint on the dense subspace of finite linear combinations of vectors $|\alpha\rangle$, $\alpha \in \mathbb{N}^n$. The representation π_θ can thus be integrated to a unitary representation of the group G. As a representation of $\mathfrak{g}$, π_θ is irreducible, thus it is still the case as a representation of the group G.

Finally, we are interested in the corresponding representation of $C^*(G)$, which we still denote π_θ. The image of $C^*(G)$ by this representation is known to be the compact operators on $\mathcal{F}^+_{\theta,J}$. It actually suffices to show that $\pi_\theta(f)$ is compact for every $f \in C^*(G)$. This follows from the following lemma by density.

Lemma 4.1. *If $f \in \mathcal{C}^\infty_c(G)$, then $\pi_\theta(f)$ compact.*

Proof. Let $D \in \mathcal{U}(\mathfrak{g})$ be an element of the universal enveloping algebra (basically, non-commutative polynomials in elements of $\mathfrak{g}$) such that $\pi_\theta(D) = N + 1$, where N is the number operator as above. Let $Z_\theta \in \mathfrak{g}_2$ be an element with $\theta(Z_\theta) = 1$. Then we can set:

$$D = -\frac{1}{2}\left(\sum_{j=1}^{n} X_j^2 + Y_j^2\right) - \mathrm{i}\left(1 - \frac{n}{2}\right) Z_\theta.$$

For $f \in \mathcal{C}^\infty_c(G)$, we obtain

$$\pi_\theta(f) = (N+1)^{-1}\pi_\theta(Df).$$

Now the operator $\pi_\theta(Df)$ is bounded since Df still has Schwarz decay. Moreover, the operator $(N+1)^{-1}$ is compact thus $\pi_\theta(f)$ is compact. ∎

Remark 4.2. Notice that the same proof show that $\pi_\theta(f)$ is trace class. It suffices to replace D by D^n because $(N+1)^{-n}$ is trace class (this follows from a quick computation since the operator N is diagonal in the basis $(|\alpha\rangle)_{\alpha\in\mathbb{N}^n}$). This argument of trace class even extends to elements $f \in C^\infty(G)$ with rapid decay at infinity.

Recall that $\mathfrak{g}$ is endowed with a family of inhomogeneous dilations δ_λ, $\lambda > 0$. They act by Lie algebra automorphisms on $\mathfrak{g}$ so they also act on G and $\widehat{G}$.

On the characters it becomes the usual dilation on $\mathfrak{g}_1^*$. On the infinite-dimensional representations, we have:

$$\delta_\lambda^* \pi_\theta = \pi_{\iota\delta_\lambda\theta} = \pi_{\lambda^2\theta}.$$

This can be seen the following way. We have $\omega_{\lambda^2\theta} = \lambda^2\omega_\theta$ by construction. Therefore we can take the same complex structure for both symplectic forms and then if $X_1, \ldots, X_n$, $Y_1, \ldots, Y_n$ is a Darboux basis for ω_θ then $\lambda^{-1}X_1, \ldots, \lambda^{-1}X_n, \lambda^{-1}Y_1, \ldots, \lambda^{-1}Y_n$ is a Darboux basis for $\omega_{\lambda^2\theta}$.

Remark 4.3 (Polar coordinates parameterization). Using this homogeneity, we can parameterize the representations in polar coordinates by $\mathbb{R}^*_+ \times \mathbb{S}^*\mathfrak{g}_2$. This requires a choice of

metric on $\mathfrak{g}_2^*$ (or equivalently, directly on $\mathfrak{g}_2$) to embed the sphere into $\mathfrak{g}_2^*$. This choice is a priori arbitrary. In the case of H-type groups however, such a metric comes naturally in the group's definition.

If we do not make such a choice the bundle $\mathfrak{g}_1 \times \mathbb{S}^*\mathfrak{g}_2$ only has a conformal symplectic structure (over a point θ we have the conformal class $(t\omega_\theta)_{t>0}$). The choice of an embedding $\mathbb{S}^*\mathfrak{g}_2 \to \mathfrak{g}_2^* \setminus 0$ allows to (continuously) fix a representative for each of these classes.

Remark 4.4 (Changing orientation). We can use a similar reasoning to compare π_θ and $\pi_{-\theta}$. For the symplectic forms we have $\omega_{-\theta} = -\omega_\theta$. Therefore we have to take the opposite complex structure. The corresponding hermitian vector space is then the same but with conjugate complex structure. For the symmetric Fock spaces we get $\mathcal{F}_{-\theta,-J} = \overline{\mathcal{F}_{\theta,J}}$ and the representations switches the roles of X_j's and Y_j's.

5. Toeplitz algebra of a regular 2-step nilpotent group

Let G be a 2-step regular nilpotent group. We fix a metric on $\mathfrak{g}_2$ and $(J_\theta)_{\theta\in\mathbb{S}^*\mathfrak{g}_2}$ a smooth family of complex structures on $\mathfrak{g}_1$ compatible with the forms ω_θ, $\theta \in \mathbb{S}^*\mathfrak{g}_2$. As explained before, the compatibility condition does not depend on the choice of θ up to multiplication by a positive scalar. We may and will assume, that all the corresponding inner products $\omega_\theta(\cdot, J_\theta\cdot)$ do not depend on θ. This is possible because a choice of inner product induces a compatible complex structure. Such a choice gives an unambiguous meaning to the spheres on $\mathfrak{g}_1$ and $\mathfrak{g}_1^*$.

For each $\theta \in \mathbb{S}^*\mathfrak{g}_2$ we can then form the Toeplitz algebra $\mathcal{T}(\mathfrak{g}_1, J_\theta)$. Choosing local trivializations of the complex bundle then allows to glue these algebras to a locally trivial bundle of C^*-algebras. Indeed the choice of J reduces the structural group of the symplectic bundle $\mathbb{S}^*\mathfrak{g}_2 \times \mathfrak{g}_1$ from $Sp(2n, \mathbb{R})$ to $U(n)$ (here $2n = \dim_{\mathbb{R}}(\mathfrak{g}_1)$) and the Toeplitz algebra of a hermitian vector space is invariant under unitary transformations of that vector space. We denote by $\mathcal{T}(\mathfrak{g}, J)$ the C^*-algebra of continuous sections of that bundle.

Remark 5.1. If we replace J by another complex structure J', we get two complex structures on the bundle $\mathbb{S}^*\mathfrak{g}_2 \times \mathfrak{g}_1$. The two complex structures are compatible with the same symplectic structure so the corresponding complex vector bundles are homeomorphic. However, this homeomorphism might not preserve the symplectic structure. Indeed if it did, it would preserve the hermitian metrics induced by the symplectic form and the complex structures. Already at the level of a single symplectic vector space, we see that not all complex vector space isomorphisms are unitary.

For each $\theta \in \mathbb{S}^*\mathfrak{g}_2$, the symplectic space on the fiber gives a Toeplitz short exact sequence with ideal the compact operators and quotient continuous functions on the cosphere of $\mathfrak{g}_1$.

$$0 \longrightarrow \mathcal{K}\left(\mathcal{F}^+\left((\mathfrak{g}_1, J_\theta)^{1,0}\right)\right) \longrightarrow \mathcal{T}(\mathfrak{g}_1, J_\theta) \longrightarrow \mathcal{C}(\mathbb{S}^*\mathfrak{g}_1) \longrightarrow 0.$$

Although the hermitian bundle structure induced by J on $\mathbb{S}^*\mathfrak{g}_2 \times \mathfrak{g}_1$ is not trivial, the corresponding co-sphere bundle only depends on the real structure as we can obtain it as $\mathbb{S}^*\mathfrak{g}_2 \times (\mathfrak{g}_1^* \setminus 0 /\!\!/_{\mathbb{R}_+^*}) = \mathbb{S}^*\mathfrak{g}_2 \times \mathbb{S}^*\mathfrak{g}_1$. We can therefore glue the quotient algebras and obtain $\mathcal{C}(\mathbb{S}^*\mathfrak{g}_2 \times \mathbb{S}^*\mathfrak{g}_1)$ seen as a continuous field of C^*-algebras over $\mathbb{S}^*\mathfrak{g}_2$. The corresponding ideal is a locally trivial bundle of compact operators on the bundle of symmetric Fock spaces of $(\mathcal{F}^+((\mathfrak{g}_1, J_\theta)^{1,0}))_{\theta \in \mathbb{S}^*\mathfrak{g}_2}$. This bundle of Hilbert spaces is infinite dimensional, hence trivializable by a result of Dixmier and Douady [7], providing an isomorphism between the ideal and $\mathcal{C}(\mathbb{S}^*\mathfrak{g}_2) \otimes \mathcal{K}$. This discussion gives the following result.

Proposition 5.2. *We have the exact sequence:*

$$0 \longrightarrow \mathcal{C}(\mathbb{S}^*\mathfrak{g}_2) \otimes \mathcal{K} \longrightarrow \mathcal{T}(\mathfrak{g}, J) \longrightarrow \mathcal{C}(\mathbb{S}^*\mathfrak{g}_2 \times \mathbb{S}^*\mathfrak{g}_1) \longrightarrow 0.$$

Our goal is to relate this bundle of Toeplitz algebras to the C^*-algebra of the group. Ultimately this should yield a Morita equivalence between our algebra and $C_0^*(G) \rtimes \mathbb{R}_+^*$ where $C_0^*(G) \subset C^*(G)$ is the kernel of the trivial representation. This algebra has a similar short exact sequence but the quotient is $\mathcal{C}(\mathbb{S}^*\mathfrak{g}_1)$ so the algebra $\mathcal{T}(\mathfrak{g}, J)$ is too big.

Definition 5.3. Let J be a compatible complex structure on the symplectic vector bundle $\mathbb{S}^*\mathfrak{g}_2 \times \mathfrak{g}_1$. We denote by $\mathcal{T}_0(\mathfrak{g}, J) \subset \mathcal{T}(\mathfrak{g}, J)$ the subalgebra of elements for which the image in the quotient $\mathcal{C}(\mathbb{S}^*\mathfrak{g}_2 \times \mathbb{S}^*\mathfrak{g}_1)$ is constant on the $\mathbb{S}^*\mathfrak{g}_2$ direction.

Let us give a more concrete description of this subalgebra. Each element of $X \in \mathfrak{g}_1$ provides a continuous section of the complex vector bundle $(\mathbb{S}^*\mathfrak{g}_2 \times (\mathfrak{g}_1, J)^{1,0})$ by

$$X_c : \theta \mapsto \frac{1}{\sqrt{2}}(X - \mathrm{i}J_\theta X).$$

From this, we create a family of shifts $S_X : \theta \mapsto S_{X_c(\theta)} \in \mathcal{T}(\mathfrak{g}_1, \theta)$. By construction, their image on the quotient is the constant map $\theta \mapsto X \in (\mathfrak{g}_1^*)^* \cong \mathfrak{g}_1$ using equation (2) and the identification of real vector spaces $\mathfrak{g}_1 \cong (\mathfrak{g}_1, J_\theta)^{1,0}$ (this identification is the same one we used to go from X to X_c above). If we only consider these shifts however, we will only get constant sections of the bundle of compact operators. Therefore we obtain the following result.

Proposition 5.4. *With the same notations as before $\mathcal{T}_0(\mathfrak{g}, J)$ is the subalgebra of sections of $\mathcal{T}(\mathfrak{g}, J)$ generated by the sections $S_X, X \in \mathfrak{g}_1$ and sections of the bundle of C^*-algebras $(\mathcal{K}(\mathcal{F}^+((\mathfrak{g}_1, J_\theta)^{1,0})))_{\theta \in \mathbb{S}^*\mathfrak{g}_2}$.*

Proof. The second subalgebra is a subalgebra of $\mathcal{T}_0(\mathfrak{g}, J)$. For elements of the form S_X, $X \in \mathfrak{g}_1$ this results from the discussion above. Sections of the bundle of compact operators have zero image in the quotient, in particular it is constant. Now this second subalgebra have the same ideal and quotient so it is equal to $\mathcal{T}_0(\mathfrak{g}, J)$. ∎

Example 5.5. If (V, ω) is a symplectic vector space, let $\mathfrak{heis}(V, \omega)$ be the corresponding Heisenberg Lie algebra obtained by seeing ω as a 2-cocycle on the abelian group V. We

have $\mathfrak{g}_2 = \mathbb{R}$, thus $\mathbb{S}^*\mathfrak{g}_2 = \{\pm 1\}$. The symplectic vector space corresponding to $+1$ is (V, ω) and the one to -1 is $(V, -\omega)$. Then, if $J : V \to V$ is a complex structure compatible with ω, $-J$ is compatible with $-\omega$ and we have $(V, -J)^{1,0} = (V, J)^{0,1}$. We will then write $V^{1,0} = (V, J)^{1,0}$ and $V^{0,1}$ accordingly. The Toeplitz algebra of the group is then a fibered product:

$$\mathcal{T}_0\big(\mathfrak{heis}(V, \omega), J\big) = \mathcal{T}(V^{1,0}) \bigoplus_{\mathcal{C}(\mathbb{S}^*V)} \mathcal{T}(V^{0,1}).$$

This means that an element in $\mathcal{T}_0(\mathfrak{heis}(V, \omega), J)$ corresponds to a pair of elements

$$(T_+, T_-) \in \mathcal{T}(V^{1,0}) \oplus \mathcal{T}(V^{0,1}),$$

such that the images of $T_\pm$ modulo compact operators are the same elements in $\mathcal{C}(\mathbb{S}^*V)$. This Toeplitz algebra is the one considered in [5].

Remember that the algebra $\mathcal{T}_0(\mathfrak{g}, J)$ acts on the bundle $\mathcal{F}^+((\mathfrak{g}_1, J)^{1,0})$. On this bundle, we can define a number operator N. This is an unbounded operator which, on the subbundle $\mathrm{Sym}^k((\mathfrak{g}_1, J)^{1,0})$, $k \in \mathbb{N}$, is given by multiplication by k. The operator $N + 1$ is then positive. We can consider the complex powers $(N + 1)^{\mathrm{i}t/2}$, $t \in \mathbb{R}$, which then form a continuous family of unitary operators. We can consider the conjugation $\mathrm{Ad}((N + 1)^{\mathrm{i}t/2})$ as a group of automorphisms of $\mathcal{B}(\mathcal{F}^+((\mathfrak{g}_1, J)^{1,0}))$. This action of $\mathbb{R}$, that we denote by α, restricts fiberwise. It is proved in [5, Corollary 4.2.1] that it preserves the Toeplitz algebra, and is trivial on the quotient. Therefore we have an action $\mathbb{R} \curvearrowright \mathcal{T}(\mathfrak{g}, J)$. Since the action is trivial on the quotient, then sections that have constant image in the quotients are also preserved so we also get an action $\mathbb{R} \curvearrowright \mathcal{T}_0(\mathfrak{g}, J)$. The key relation is that for $X \in \Gamma((\mathfrak{g}_1, J)^{1,0})$, then:

$$\alpha_t(S_X) \equiv S_X \mod \mathcal{C}(\mathbb{S}^*\mathfrak{g}_2, \mathcal{K}).$$

Proposition 5.6. *We have the exact sequence:*

$$0 \longrightarrow \mathcal{C}_0\big(\mathfrak{g}_2^* \setminus \{0\}\big) \otimes \mathcal{K} \longrightarrow \mathcal{T}_0(\mathfrak{g}, J) \rtimes_\alpha \mathbb{R} \longrightarrow \mathcal{C}_0\big(\mathfrak{g}_1^* \setminus \{0\}\big) \longrightarrow 0.$$

Proof. As said before the action is trivial on the quotient so:

$$\begin{aligned}\mathcal{C}(\mathbb{S}^*\mathfrak{g}_1) \rtimes_\alpha \mathbb{R} &\cong \mathcal{C}(\mathbb{S}^*\mathfrak{g}_1) \otimes C^*(\mathbb{R}) \\ &\cong \mathcal{C}(\mathbb{S}^*\mathfrak{g}_1) \otimes \mathcal{C}_0(\mathbb{R}_+^*) \\ &\cong \mathcal{C}_0\big(\mathfrak{g}_1^* \setminus \{0\}\big).\end{aligned}$$

For the ideal, notice that $(N + 1)^{\mathrm{i}t/2}$ is a multiplier of $\mathcal{C}(\mathbb{S}^*\mathfrak{g}_2, \mathcal{K})$. The crossed product is then isomorphic to a trivial one and we get the result by similar computations. ∎

The exact sequence of that proposition looks like the one of Corollary 3.7 when the trivial representation (which corresponds to $0 \in \mathfrak{g}_1^*$) is removed. If we denote by $C_0^*(G) \triangleleft C^*(G)$ the kernel of the trivial representation, we should expect an isomorphism $\mathcal{T}_0(\mathfrak{g}, J) \rtimes_\alpha \mathbb{R} \cong C_0^*(G)$. This is proved in [5] when G is a Heisenberg group. We generalize this result to all the 2-step regular nilpotent groups. We do this by reducing to the case of Heisenberg groups (but for families).

For a symplectic vector space (V,ω), recall from Corollary 3.6 that the group C^*-algebra $C^*(\mathrm{Heis}(V,\omega))$ is a continuous field of C^*-algebras over $\mathbb{R}$. If we fix a compatible complex structure J, the fiber at $t\in\mathbb{R}$ is identified with the compact operators $\mathcal{K}(\mathcal{F}^+((V,\mathrm{sign}(t)J)^{1,0}))$ if $t\neq 0$ and $\mathcal{C}_0(V^*)$ if $t=0$. Denote by $I_+(V,\omega)$ the quotient of $C^*(\mathrm{Heis}(V,\omega))$ by the sections vanishing for negative values (this corresponds to restricting the sections from $\mathbb{R}$ to $\mathbb{R}_+$).

Because G is regular, we have a symplectic bundle $\mathfrak{g}_1\times\mathbb{S}^*\mathfrak{g}_2\to\mathbb{S}^*\mathfrak{g}_2$ with the forms ω_θ, $\theta\in\mathbb{S}^*\mathfrak{g}_2$. Consider the bundle of C^*-algebras $(I_+(\mathfrak{g}_1,\omega_\theta))_{\theta\in\mathbb{S}^*\mathfrak{g}_2}$ and denote by $\widetilde{C}^*(G)$ the algebra of continuous sections of this bundle. We have the exact sequence:

$$0\longrightarrow\mathcal{C}_0(\mathfrak{g}_2^*\setminus 0,\mathcal{K})\longrightarrow\widetilde{C}^*(G)\longrightarrow\mathcal{C}_0(\mathfrak{g}_1^*\times\mathbb{S}^*\mathfrak{g}_2)\longrightarrow 0.$$

Here the quotient should be seen as $\mathcal{C}(\mathbb{S}^*\mathfrak{g}_2,\mathcal{C}_0(\mathfrak{g}_1^*))$. It corresponds to the field of quotients corresponding to the characters for each group.

Consider $\widetilde{C}_0^*(G)\subset\widetilde{C}^*(G)$ to be the subalgebra of sections that vanish at the points $(0_{\mathbb{R}},0_{\mathfrak{g}_1^*})$, i.e., whose image in the quotient in the above exact sequence, belongs to the subspace $\mathcal{C}(\mathbb{S}^*\mathfrak{g}_2,\mathcal{C}_0(\mathfrak{g}_1^*\setminus 0))$.

Theorem 5.7. *The algebra $C^*(G)$ is isomorphic to the subalgebra of $\widetilde{C}^*(G)$ corresponding to sections $F\in C^*(G)$ for which the images in the commutative quotients $F(\theta)(0)\in\mathcal{C}_0(\mathfrak{g}_1^*)$ do not depend on $\theta\in\mathbb{S}^*\mathfrak{g}_2$.*

Under this identification, the algebra $C_0^*(G)$ corresponds to the elements in the image of $C^*(G)$ that are also in $\widetilde{C}_0^*(G)$.

Proof. Consider $f\in C^*(G)$. We see it as a section of a continuous field of C^*-algebras over $\mathfrak{g}_2^*$ as in Corollary 3.6. Then to $\theta\in\mathbb{S}^*\mathfrak{g}_2$ we map the restriction of f to $\mathbb{R}_+\theta$:

$$F(\theta)(t)=f(t\theta),\quad t\geq 0.$$

This provides an element $F(\theta)\in I_+(\mathfrak{g}_1,\omega_\theta)$ with a continuous dependence in θ. The image in the commutative quotients for each θ is then given by $f(0)$ which does not depend on θ.

This mapping is a continuous $*$-algebra morphism $C^*(G)\to\widetilde{C}^*(G)$. Conversely, every element of F which has constant commutative quotients is of this form. ∎

Theorem 5.8. *Let G be a regular 2-step nilpotent group. Let J be a compatible complex structure on the symplectic vector bundle $\mathbb{S}^*\mathfrak{g}_2\times\mathfrak{g}_1$. We have an isomorphism:*

$$\mathcal{T}(\mathfrak{g},J)\rtimes_\alpha\mathbb{R}\xrightarrow{\sim}\widetilde{C}_0^*(G).$$

This identification induces an isomorphism:

$$\mathcal{T}_0(\mathfrak{g},J)\rtimes_\alpha\mathbb{R}\cong C_0^*(G).$$

The last isomorphism is $\mathbb{R}_+^*$-equivariant for the dual action on the left and δ^* on the right.

Proof. We will use [5, Theorem 5.9], which says that if (V, ω) is a symplectic vector space, then $\mathcal{T}(V^{1,0}) \rtimes \mathbb{R}$ is isomorphic to $I_{+,0}(V, \omega)$, where $I_{+,0}(V, \omega)$ is the kernel of the map $f \mapsto f(0_{\mathbb{R}})(0_{V^*})$. Moreover, this isomorphism is $\mathbb{R}_+^*$-equivariant and also $U(n)$-invariant. Therefore it generalizes to symplectic vector bundles (with fixed compatible complex structure).

We now apply this result to $\widetilde{C}_0^*(G)$ which is the subalgebra of sections of the bundle $(I_{+,0}(\mathfrak{g}_1, \omega_\theta))_{\theta \in \mathfrak{g}_2^* \setminus 0}$. This gives the first part of the theorem. This isomorphism induces the following diagram:

$$\begin{array}{ccc} \mathcal{T}(\mathfrak{g}, J) \rtimes_\alpha \mathbb{R} & \longrightarrow & \mathcal{C}_0\big(\mathbb{S}^*\mathfrak{g}_2 \times \big(\mathfrak{g}_1^* \setminus \{0\}\big)\big) \\ \downarrow & & \| \\ \widetilde{C}_0^*(G) & \longrightarrow & \mathcal{C}_0\big(\mathbb{S}^*\mathfrak{g}_2 \times \big(\mathfrak{g}_1^* \setminus \{0\}\big)\big). \end{array}$$

If we take on both sides the preimage of $\mathcal{C}_0(\mathfrak{g}_1^* \setminus 0)$ (i.e., the sections that have constant commutative quotient) then we get the second part of the theorem. ∎

6. Fields of Weyl algebras

In the previous section, we have reduced the proof to the case of the Heisenberg group to use the result from [5]. Another possibility is to adapt directly the proof for the Heisenberg group to the case of an arbitrary 2-step regular nilpotent group. This is the purpose of this section. Given an element $T \in \mathcal{T}_0(\mathfrak{g}_1, J)$, we want to construct a family of multipliers of $C^*(G)$ seen as a continuous field over $\mathfrak{g}_2^*$ by taking $F_T(r\theta) = T_\theta, r > 0, \theta \in \mathbb{S}^*\mathfrak{g}_2$. The previous section also gives that $F_T(0)$ is the multiplication by $T_\theta \mod \mathcal{K} \in \mathcal{C}(\mathbb{S}^*\mathfrak{g}_1)$ which does not depend on θ by construction of $\mathcal{T}_0(\mathfrak{g}_1, J)$. The only problem with that approach is to understand the continuity at $0 \in \mathfrak{g}_2^*$ of this field of multipliers. To understand this, we will see the Toeplitz operators as a particular case of Weyl operators. The families of Weyl operators then arise from a pseudodifferential calculus, naturally associated to the group G and its filtration, called the filtered calculus. We do not need a full understanding of this calculus as what appears here are the principal symbols of such operators (which, as in the case of classical pseudodifferential calculus, can be understood as "local models" for the operators themselves). In particular, we will only define principal symbols and their product, leaving out the question of how to quantize them into actual operators, even though the symbols can already be seen as operators themselves. The interested reader can check e.g. [4, 9, 27] and their references.

Principal symbols of order $k \in \mathbb{C}$ in the filtered calculus for the group G are smooth functions on $\mathfrak{g}^* \setminus 0$, homogeneous of degree k for the dilations $\delta_\lambda, \lambda > 0$. These functions multiply in a way that takes the group structure into account. By homogeneity, such a function σ is characterized by:

- a function on $\mathfrak{g}_1^* \setminus 0$, homogeneous of degree k (now for the usual dilations), hence a function σ_0 on the sphere $\mathbb{S}^*\mathfrak{g}_1$.
- a smooth family of functions σ_θ on $\mathfrak{g}_1^*$, parameterized by $\theta \in \mathbb{S}^*\mathfrak{g}_2$.

If we get two symbols $a \in \Sigma^k(G), b \in \Sigma^\ell(G)$, they multiply the following way: $(ab)_0 = a_0 b_0$ is the usual (commutative) product of functions, and $(ab)_\theta = a_\theta \#_\theta b_\theta$. The product $\#_\theta$ is constructed using the symplectic form ω_θ^* on $\mathfrak{g}_1^*$ and is non-commutative. This product is given by the integral formula, where $2n = \dim(\mathfrak{g}_1)$,

$$\forall u \in \mathfrak{g}_1^*, \quad a_\theta \#_\theta b_\theta(u) = \frac{1}{\pi^{2n}} \int_{v,w \in \mathfrak{g}_1^*} e^{2i\omega_\theta(v,w)} a_\theta(u+v) b_\theta(u+w) \mathrm{d}v \, \mathrm{d}w. \tag{7}$$

As is, this product is only defines for a_θ and b_θ having enough decay at infinity (e.g. if they are in $\mathcal{S}(\mathfrak{g}_1^*)$). We can see it as a composition formula of symbols for the Weyl quantization. This will give its sense to the integral formula above. To this end, notice like in [8,11], that the homogeneity of a on the subspace $\mathfrak{g}_1^* \oplus \mathbb{R} \cdot \theta$ gives that a_θ is an element of the class $S^m_{\mathrm{iso}}(\mathfrak{g}_1^*)$ and admits the polyhomogeneous expansion at infinity:

$$a_\theta(v) \sim \sum_{k \geq 0} a_\theta^{(k)}(v) \|v\|^{m-2k}$$

with $a_\theta^{(k)}$ homogeneous of degree $m-2k$, and $\|\cdot\|$ any norm on V (we can take the one obtained from a choice of a complex structure J_θ compatible with ω_θ^*). The class of functions $S^m_{\mathrm{iso}}(\mathfrak{g}_1^*)$ is similar to the Hörmander classes but here, no separation of variables into $\mathbb{R}^n \times \mathbb{R}^n$ is considered. For a_θ this means that for every multi-index α:

$$\exists C_\alpha \geq 0, \ \forall u \in \mathfrak{g}_1^*, \quad \left|\partial_u^\alpha a_\theta(u)\right| \leq C_\alpha \big(1 + \|u\|\big)^{m-|\alpha|}. \tag{8}$$

The polyhomogeneous expansion comes from the fact that the sphere at infinity of $\mathfrak{g}_1^* \times \{\theta\} \subset \mathfrak{g}_1^* \oplus \mathbb{R} \cdot \theta$ is $\mathbb{S}^*\mathfrak{g}_1 \times \{0\}$. The expansion is then just a Taylor expansion for the function $a_{|\mathbb{S}(\mathfrak{g}_1^* \oplus \mathbb{R} \cdot \theta)}$ near the equator. In particular, the first term of this extension is $a_{|\mathbb{S}^*\mathfrak{g}_1 \times \{0\}} = a_0$, independent of θ. This is reinterpreted below in terms of Weyl principal symbol.

The norm estimates (8) give sense to the composition formula (7), as we even have:

$$\forall k, \ell \in \mathbb{C}, \quad S^k_{\mathrm{iso}}(\mathfrak{g}_1^*) \#_\theta S^\ell_{\mathrm{iso}}(\mathfrak{g}_1^*) \subset S^{k+\ell}_{\mathrm{iso}}(\mathfrak{g}_1^*).$$

Remark 6.1. The references [8, 11] only treat the case of the Heisenberg group. The situation here is however very similar and can actually be reduced to the Heisenberg group using the following trick. If we fix $\theta \in \mathbb{S}^*\mathfrak{g}_2$, we restricted function on $\mathfrak{g}^*$ to $\mathfrak{g}_1^* \oplus \mathbb{R} \cdot \theta$. This subspace is equal to $\mathfrak{h}_\theta^\perp \subset \mathfrak{g}^*$, where $\mathfrak{h}_\theta := \ker(\theta) \subset \mathfrak{g}$ is an ideal. The space $\mathfrak{h}_\theta^\perp$ is stable under the co-adjoint action, and elements from the subgroup H_θ corresponding to $\mathfrak{h}_\theta$ act trivially. Under Kirillov correspondence, the coadjoint orbits of $\mathfrak{h}_\theta^\perp$ are the co-adjoint orbits of the quotient group G/H_θ. This group is isomorphic to $\mathrm{Heis}(\mathfrak{g}_1, \omega_\theta)$.

Finally, on the symbol side, we can push-forward symbols by submersions between Lie groups. Here this gives:

$$\Sigma^k(G) \to \Sigma^k\big(\mathrm{Heis}(\mathfrak{g}_1, \omega_\theta)\big).$$

This push-forward corresponds, by Kirillov's correspondence, to the restriction of functions on $\mathfrak{g}^*$ to the subspace $\mathfrak{h}_\theta^\perp$.

In particular, the function a_θ above, can be computed by first restricting a to $\mathfrak{h}_\theta^\perp$, where the results of [8, 11] apply.

We now describe the product in terms of composition of operators. Let $L_\theta \oplus L'_\theta = \mathfrak{g}_1^*$ be a decomposition into a pair of complementary lagrangians for ω_θ^*. If we have a compatible complex structure J, we can take any lagrangian subspace L_θ and set $L'_\theta = JL_\theta$. We can then quantize the symbol a_θ into a Weyl operator $\mathrm{Op}^W(a_\theta)\colon \mathcal{S}(L_\theta) \to \mathcal{S}(L_\theta)$, by the formula:

$$\forall f \in \mathcal{S}(L_\theta),\ \forall x \in L_\theta, \quad \mathrm{Op}^W(a_\theta)(f)(x) = \frac{1}{(2\pi)^n} \int_{\substack{y \in L_\theta \\ \xi \in L'_\theta}} a_\theta\left(\frac{x+y}{2}, \xi\right) f(y)\mathrm{d}y\,\mathrm{d}\xi.$$

A straightforward computation then gives:

$$\mathrm{Op}^W(a_\theta) \circ \mathrm{Op}^W(b_\theta) = \mathrm{Op}^W(a_\theta \#_\theta b_\theta).$$

The formula for the product $\#_\theta$ shows that this algebra of operators does not depend on the choice of lagrangians. We thus denote by $\mathcal{W}^k(\mathfrak{g}_1^*, \omega_\theta^*)$ the vector space $S^k_{\mathrm{iso}}(\mathfrak{g}_1^*)$ which we now see with the product $\#_\theta$ satisfying:

$$\#_\theta\colon \mathcal{W}^k(\mathfrak{g}_1^*, \omega_\theta^*) \times \mathcal{W}^\ell(\mathfrak{g}_1^*, \omega_\theta^*) \to \mathcal{W}^{k+\ell}(\mathfrak{g}_1^*, \omega_\theta^*), \quad k, \ell \in \mathbb{C}.$$

Given $a_\theta \in \mathcal{W}^k(\mathfrak{g}_1^*, \omega_\theta^*)$, taking the principal part in the polyhomogeneous expansion $a_\theta^{(0)}$ gives a continuous map which we will call the Weyl principal symbol:

$$\sigma_W^k\colon \mathcal{W}^k(\mathfrak{g}_1^*, \omega_\theta^*) \to \mathcal{C}^\infty(\mathbb{S}^*\mathfrak{g}_1).$$

This map is compatible with the product of Weyl operators. If $a_\theta \in \mathcal{W}^k(\mathfrak{g}_1^*, \omega_\theta^*)$ and $b_\theta \in \mathcal{W}^\ell(\mathfrak{g}_1^*, \omega_\theta^*)$ then

$$\sigma_W^{k+\ell}(a_\theta \#_\theta b_\theta) = \sigma_W^k(a_\theta)\sigma_W^\ell(b_\theta).$$

As mentioned above with the polyhomogeneous expansion, in the case where the operator a_θ comes from an element $\Sigma^k(G)$, the Weyl principal symbol is given by $\sigma_W^k(a_\theta) = a_0$. Therefore the Weyl principal symbol of the elements a_θ does not depend on θ.

Example 6.2. Consider $X \in \mathfrak{g}_1$ or $X \in \mathfrak{g}_2$, seen as the principal symbol of a differential operator of order 1 or 2 respectively. They define (linear) functions ℓ_X on $\mathfrak{g}^*$ by duality. Since X is of degree 1 or 2, this function is homogeneous for the dilations and is thus an element of $\Sigma^1(G)$ or $\Sigma^2(G)$ respectively.

Notice that in the second case $X \in \mathfrak{g}_2$, the function ℓ_X vanishes on the characters. This means that $(\ell_X)_0 = 0$. This can also be seen from the polyhomogeneous expansion in the other representations. Let $\theta \in \mathbb{S}^*\mathfrak{g}_2$. The operator induced on the representation is $(\ell_X)_\theta = \mathrm{i}\theta(X)\mathrm{Id}$. Since the symbol ℓ_X has order 2, we have $(\ell_X)_\theta \in S^2_{\mathrm{iso}}(\mathfrak{g}_1^*)$. The first term in the homogeneous expansion is thus quadratic. Since $(\ell_X)_\theta$ is bounded this means that the principal part of its polyhomogeneous expansion, hence $(\ell_X)_0$, vanishes.

Combining what is said above we get the following. The spaces of Weyl operators $\mathcal{W}^k(\mathfrak{g}_1^*, \omega_\theta^*)$ form a smooth bundle of Fréchet spaces over $\mathbb{S}^*\mathfrak{g}_2$. We denote its sections by $\mathcal{W}^k(\mathfrak{g}^*)$. We can thus see principal symbols $a \in \Sigma^k(G)$ as smooth sections of the field of algebras of Weyl operators $(\mathcal{W}^k(\mathfrak{g}_1^*, \omega_\theta^*))_{\theta\in\mathbb{S}^*\mathfrak{g}_2}$, such that the principal symbol is constant. Of course not all such fields arise from elements in $\Sigma^k(G)$ because of the gluing condition on these operators on $\mathbb{S}^*\mathfrak{g}_1$ only gives the continuity and not smoothness (see [11] for such a discussion in the case of the Heisenberg group). At the level of C^*-completion for operators of order 0 it is however enough.

The estimate (8) shows that Weyl operators of order 0 extend continuously to operators on $L^2(L_\theta)$ (for a lagrangian splitting of $(\mathfrak{g}_1^*, \omega_\theta^*)$ as previously). We denote by $W(\mathfrak{g}_1^*, \omega_\theta^*)$ the C^*-algebra obtained by closure of $\mathcal{W}^0(\mathfrak{g}_1^*, \omega_\theta^*)$ for the operator norm in $L^2(L_\theta)$. This C^*-algebra does not depend on the choice of lagrangian splitting. The principal symbol extends continuously to a C^*-algebra homomorphism to $\mathcal{C}(\mathbb{S}^*\mathfrak{g}_1)$. We get the exact sequence:

$$0 \longrightarrow \mathcal{K}\big(L^2(L_\theta)\big) \longrightarrow W(\mathfrak{g}_1^*, \omega_\theta^*) \xrightarrow{\sigma_W^0} \mathcal{C}(\mathbb{S}^*\mathfrak{g}_1) \longrightarrow 0. \tag{9}$$

The ideal $\mathcal{K}(L^2(L_\theta))$ here appears as the completion of $\mathcal{W}^{-1}(\mathfrak{g}_1^*, \omega_\theta^*) \subset \mathcal{W}^0(\mathfrak{g}_1^*, \omega_\theta^*)$.

We define $W_0(\mathfrak{g})$ the C^*-algebra of sections of the field $(W(\mathfrak{g}_1^*, \omega_\theta^*))_{\theta\in\mathbb{S}^*\mathfrak{g}_2}$ obtained by completion of $\mathcal{W}$.

Theorem 6.3. *Let $\Sigma(G)$ be the completion of $\Sigma^0(G)$ in the multipliers of $\ker(\pi_0) \subset C^*(G)$. The map $\Sigma^0(G) \to \mathcal{W}^0(\mathfrak{g}^*)$ extends to an embedding $\Sigma(G) \to W(\mathfrak{g}^*)$ whose image is exactly $W_0(\mathfrak{g}^*)$. Consequently, we have an isomorphism $\Sigma(G) \cong W_0(\mathfrak{g}^*)$.*

Proof. Let $\sigma \in \Sigma^0(G)$. We have

$$\|\sigma\| = \sup_{\pi\in\widehat{G}\setminus 1} \big\|\pi(\sigma)\big\| = \sup_{\theta\in\mathbb{S}^*\mathfrak{g}_2} \big\|\pi_\theta(\sigma)\big\|.$$

The reduction from $\widehat{G} \setminus 1$ to $\mathbb{S}^*\mathfrak{g}_2$ is done using homogeneity and density of the representations $\pi_\eta, \eta \in \mathfrak{g}_2^* \setminus 0$ in $\widehat{G}$. We can realize the representation π_θ the following way. Take an element $f \in \mathcal{C}_c^\infty(G)$ and consider its euclidean Fourier transform $\hat{f} \in \mathcal{C}^\infty(\mathfrak{g}^*)$. We can then restrict it to the co-adjoint orbit $\mathfrak{g}_1^* \times \{\theta\}$ and see it as a Weyl symbol (of order $-\infty$ because it is compactly supported). To get actual operators on a Hilbert space, we can take a pair of transverse lagrangian subspaces for ω_θ^*, $\mathfrak{g}_1^* = L_\theta \oplus L'_\theta$. We have

$$\pi_\theta(f) = \mathrm{Op}^W(\hat{f}_{|\mathfrak{g}_1^*\times\{\theta\}}) \in \mathcal{K}\big(L^2(L_\theta)\big).$$

The operator $\pi_\theta(\sigma)$ is then represented by $\mathrm{Op}^W(\sigma_\theta)$. This shows that the map $\Sigma^0(G) \to \mathcal{W}^0(\mathfrak{g}^*)$ is isometric, hence continuous, so we can extend it to the respective closures.

We now have an isometric embedding

$$\iota\colon \Sigma(G) \to W(\mathfrak{g}^*).$$

By construction, its image is included in $W_0(\mathfrak{g}^*)$. The algebra $W_0(\mathfrak{g}^*)$ sits in the exact sequence:

$$0 \longrightarrow \mathcal{C}(\mathbb{S}^*\mathfrak{g}_2, \mathcal{K}) \longrightarrow W_0(\mathfrak{g}^*) \xrightarrow{\sigma_W^0} \mathcal{C}(\mathbb{S}^*\mathfrak{g}_1) \longrightarrow 0.$$

The map ι is compatible with the quotient maps in the sense that $\sigma_W^0 \circ \iota : \sigma \mapsto \sigma_0$. Therefore we have to show that the induced map from the kernel of $\sigma \mapsto \sigma_0$ to $\mathcal{C}(\mathbb{S}^*\mathfrak{g}_2, \mathcal{K})$ is onto (hence an isomorphism). We will denote by $I(G) \subset \Sigma(G)$ the kernel of $\sigma \mapsto \sigma_0$ (these symbols are the symbols of the so-called Hermite-type operators, hence the notation). Both $I(G)$ and $\mathcal{C}(\mathbb{S}^*\mathfrak{g}_2, \mathcal{K})$ are $\mathcal{C}(\mathbb{S}^*\mathfrak{g}_2)$-algebras and the map induced by ι preserves that structure. We can thus check that the induced map between the fibers over θ is an isomorphism for each $\theta \in \mathbb{S}^*\mathfrak{g}_2$. For a fixed θ this map corresponds to the same map for the group $\mathrm{Heis}(\mathfrak{g}_1, \omega_\theta)$ where it is known to be an isomorphism. ∎

On the other hand, given a symplectic vector space (V, ω) with compatible complex structure J, we can identify $\mathcal{T}(V, J)$ with $W(V^*, \omega^*)$. To this end, remember that $\mathcal{T}(V, J)$ is generated by shift operators $S_1, \ldots, S_d$ and their adjoints. These shifts are obtained by taking an orthonormal basis $W_1, \ldots, W_d$ of $V^{1,0}$ and setting $S_j = S_{W_j}$. Now we can also see $W_1, \ldots, W_d$ as homogeneous functions on V^*, hence elements in $\mathcal{W}^1(V^*, \omega^*)$. Similarly for their conjugates $\overline{W_j}$.

These elements act as i times the annihilation operators, and the $\overline{W_j}$ as i times the creation operators by (6). This can also be seen by direct computation. Let us write $W_j = \frac{1}{\sqrt{2}}(X_j + \mathrm{i}Y_j)$, $JX_j = -Y_j$, $1 \le j \le d$ and take the lagrangian subspaces $L = \mathrm{Span}(Y_j^*, 1 \le j \le d) \subset V^*$ and $L' = J^*L = \mathrm{Span}(X_j^*, 1 \le j \le d)$. For a function $f \in \mathcal{S}(L)$ we obtain:

$$\begin{aligned}
\mathrm{Op}^W(Y_j) f(y) &= \frac{1}{(2\pi)^d} \iint_{x,\xi \in \mathbb{R}^d} e^{\mathrm{i}\omega^*(y-x,\xi)} X_j\left(\frac{x+y}{2}, \xi\right) f(y) \mathrm{d}x \, \mathrm{d}\xi \\
&= \frac{1}{(2\pi)^d} \iint_{x,\xi \in \mathbb{R}^d} e^{\mathrm{i}\langle \xi, y-x\rangle} \frac{x_j + y_j}{2} f(y) \mathrm{d}x \, \mathrm{d}\xi \\
&= \int_{x \in \mathbb{R}^d} \delta_{x=y} \frac{x_j + y_j}{2} f(y) \mathrm{d}x \\
&= y_j f(y).
\end{aligned}$$

Similarly, $\mathrm{Op}^W(X_j) = -\mathrm{i}\frac{\partial}{\partial y_j}$. This gives:

$$\mathrm{Op}^W(W_j) = \frac{i}{\sqrt{2}}\left(x_j - \frac{\partial}{\partial x_j}\right).$$

The operators $\frac{1}{\sqrt{2}}(x_j - \frac{\partial}{\partial x_j})$ and $\frac{1}{\sqrt{2}}(x_j + \frac{\partial}{\partial x_j})$ are then well-known representatives of the creation and annihilation operators on $L^2(\mathbb{R}^d)$.

If we now want to represent the shifts of the Toeplitz algebra, we can use the relation (5) rewritten as:

$$\forall 1 \leq j \leq d, \quad S_j = A_j^*(N+1)^{-1/2}.$$

Recall that we can also recover N from the A_j^* so here we have:

$$\mathrm{Op}^W\left(-\sum_{j=1}^{d} W_j\overline{W_j} - (d-1)\right) = N+1. \tag{10}$$

The operator $N+1$ is positive and invertible. Since $-\sum_{j=1}^{d} W_j\overline{W_j} - (d-1) \in \mathcal{W}^2(V^*,\omega^*)$, we may find $\eta \in \mathcal{W}^{-1}(V^*,\omega^*)$ such that $\mathrm{Op}^W(\eta) = (N+1)^{-1/2}$.

Lemma 6.4. *There is an embedding $\mathcal{T}(V,J) \to W(V^*,\omega^*)$ sending S_j to $-\mathrm{i}W_j\eta$ and S_j^* to $-\mathrm{i}\eta\overline{W_j}$. Moreover, the composition with the Weyl principal symbol σ_W^0 is the quotient map for Toeplitz operators their compact ideal.*

Proof. We have for every $1 \leq j \leq d$, $-\mathrm{i}W_j\eta, -\mathrm{i}\eta\overline{W_j} \in \mathcal{W}^0(V^*,\omega^*)$. Moreover, their image by Op^W are S_j and S_j^* respectively by construction. This shows that the C^*-subalgebra W' generated by these operators is isomorphic to $\mathcal{T}(V,J)$. All that remains to show is $W' = W(V^*,\omega^*)$. Consider the Weyl principal symbol restricted to W'. If $z_1,\ldots,z_d$ denote the coordinates in the basis $W_1,\ldots,W_d$ we have

$$\sigma_W^1(W_j)(z) = z_j, \quad \sigma_W^1(\overline{W_j})(z) = \overline{z_j},$$

as well as

$$\sigma_W^2\left(-\sum_{j=1}^{d} W_j\overline{W_j} - (d-1)\right)(z) = |z|^2.$$

By multiplicativity of the principal symbol we obtain:

$$\sigma_W^0(-\mathrm{i}W_j\eta)(z) = -\mathrm{i}\frac{z_j}{|z|}, \quad \sigma_W^0(-\mathrm{i}\eta\overline{W_j})(z) = -\mathrm{i}\frac{\overline{z_j}}{|z|}.$$

Under the identification $\mathcal{C}^\infty(\mathbb{S}^*V) \cong \mathcal{C}^\infty(\mathbb{S}^*V^{1,0})$, these symbols are the images of S_j and S_j^* in the by quotient of $\mathcal{T}(V,J)$ by compact operators as in (2). ∎

We want this embedding to be an isomorphism. We will use a 2 out of 3 lemma thanks to the exact sequences of $\mathcal{T}(V,J)$ and $W(V^*,\omega^*)$ respectively. The quotient was treated in the previous lemma. We now need to show that the algebra W' contains all compact operators on $L^2(\mathbb{R}^d)$. It suffices to show that the representation of the Canonical Commutation Relations constructed above is irreducible, i.e., that the intersection of the kernels of the annihilation operators is 1-dimensional. If this is the case, fix a non-zero vector in this intersection, then the vectors

$$\mathrm{Op}^W(\mathrm{i}\overline{W_{j_1}})\cdots\mathrm{Op}^W(\mathrm{i}\overline{W_{j_k}})v_0, \quad 1 \leq j_1,\ldots,j_k \leq d,\ k \geq 0,$$

span a dense subspace of $L^2(\mathbb{R}^d)$.

Lemma 6.5. *The realization of the Canonical Commutation Relations on $L^2(\mathbb{R}^d)$ given by $\mathrm{i}\,\mathrm{Op}^W(W_j)$ and $\mathrm{i}\,\mathrm{Op}^W(\overline{W_j})$ for $1 \leq j \leq d$ is irreducible. That is*

$$\dim\left(\bigcap_{j=1}^{d} \ker\left(\mathrm{Op}^W(\mathrm{i}\overline{W_j})\right)\right) = 1.$$

Proof. Since

$$\mathrm{i}\,\mathrm{Op}^W(\overline{W_j}) = \frac{1}{\sqrt{2}}\left(x_j + \frac{\partial}{\partial x_j}\right)$$

the intersection of the kernels is the line directed by

$$x \mapsto \exp\left(-\frac{|x|^2}{2}\right).$$

∎

Remember that the choice of complex structure gives us an action of $U(n)$ on V, preserving ω and J. Likewise, we get an action on V^* preserving the dual objects. Elements in $\mathcal{W}^k(V^*, \omega^*)$ functions on V^*, and the estimates (8) are invariant by the $U(n)$-action. Therefore, $U(n)$ acts on $\mathcal{W}^k(V^*, \omega^*)$. Moreover, this action is compatible with the product of Weyl operators, which can be seen in equation (7). In particular, the C^*-algebra $W(V^*, \omega^*)$ is endowed with an action of $U(n)$.

Corollary 6.6. *The embedding $\mathcal{T}(V, J) \to W(V^*, \omega^*)$ sending S_j to $-\mathrm{i}W_j\eta$ and S_j^* to $-\mathrm{i}\eta\overline{W_j}$ is onto, i.e., it is an isomorphism. More precisely we have the commutative diagram:*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{K}\left(L^2(\mathbb{R}^d)\right) & \longrightarrow & W(V^*, \omega^*) & \xrightarrow{\sigma_W^0} & \mathcal{C}(\mathbb{S}^*V) & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \| & & \\
0 & \longrightarrow & \mathcal{K}\left(\mathcal{F}^+(V^{1,0})\right) & \longrightarrow & \mathcal{T}(V, J) & \longrightarrow & \mathcal{C}(\mathbb{S}^*V) & \longrightarrow & 0.
\end{array}$$

Finally, this isomorphism does not depend on the choice of orthonormal basis for V, i.e., is equivariant under the respective $U(n)$-actions on both spaces.

Proof. By irreducibility of the representation of the Canonical Commutation Relations, we can construct a unitary isomorphism $U\colon \mathcal{F}^+(\mathbb{C}^d) \to L^2(\mathbb{R}^d)$ intertwining both representations of the Canonical Commutation Relations. The map $\mathcal{T}(V, J) \to W(V^*, \omega^*)$ then becomes the conjugation by U. This preserves the ideal of compact operators in both spaces. We thus get the commutative diagram as claimed. The respective quotient maps have been identified in Lemma 6.4. Since the map on ideals and quotients are isomorphisms, then the map $\mathcal{T}(V, J) \to W(V^*, \omega^*)$ is an isomorphism as well.

To check the $U(n)$-invariance, we just have to check that η is invariant by the $U(n)$-action. This follows from the fact that $N + 1$ is $U(n)$-invariant, which is immediate from equation (10). ∎

We can then extend this idea to fields of Weyl operators to get a new description of $\Sigma(G)$ for a 2-step regular nilpotent group G.

Theorem 6.7. *Let G is a 2-step regular nilpotent group, $\Sigma(G)$ the C^*-algebra of principal symbols of order 0 in the filtered calculus on G. Let J be a family of almost complex structures on $\mathfrak{g}_1$ compatible with the symplectic forms. We then have an isomorphism:*

$$\Sigma(G) \cong \mathcal{T}_0(\mathfrak{g}, J).$$

Proof. We fix a compatible complex structure on the symplectic bundle $\mathfrak{g}_1 \times \mathbb{S}^*\mathfrak{g}_2 \to \mathbb{S}^*\mathfrak{g}_2$. We have two continuous fields of C^*-algebras. On the one hand we have $(W(\mathfrak{g}_1^*, \omega_\theta^*))_{\theta \in \mathbb{S}^*\mathfrak{g}_2}$. On the other hand we have $(\mathcal{T}(\mathfrak{g}_1, J_\theta))_{\theta \in \mathbb{S}^*\mathfrak{g}_2}$. By the corollary above, the fibers of the two fields are isomorphic. Moreover, these fields are actually bundles of C^*-algebras. This is because we can locally trivialize $\mathfrak{g}_1 \times \mathbb{S}^*\mathfrak{g}_2 \to \mathbb{S}^*\mathfrak{g}_2$ as a $U(n)$-bundle. In this local trivialization, the isomorphism between the fibers induces a continuous bundle map. This implies that the notion of continuity for sections is the same for both fields. Hence the respective algebras of sections are isomorphic. Now we identified $\Sigma(G)$ with the sections of $(W(\mathfrak{g}_1^*, \omega_\theta^*))_{\theta \in \mathbb{S}^*\mathfrak{g}_2}$ for which the principal Weyl symbol is constant in Theorem 6.3. This subalgebra of section is then isomorphic to the subalgebra of sections $\mathcal{T}_0(\mathfrak{g}, J)$ by the identification of quotient maps in Lemma 6.4. Therefore $\Sigma(G) \cong \mathcal{T}_0(\mathfrak{g}, J)$. ∎

This identification gives another proof of Theorem 5.8 by then applying [4, Theorem 5.10]. This theorem shows that if we transfer our $\mathbb{R}$-action on $\mathcal{T}_0(\mathfrak{g}, J)$ to $\Sigma(G)$ through the isomorphism of the previous theorem, then the corresponding map

$$\Sigma(G) \rtimes \mathbb{R} \to C_0^*(G)$$

is an isomorphism. Proposition 5.6 of the same paper also gives another explanation on why elements of $\mathcal{T}_0(\mathfrak{g}, J)$ act as bounded multipliers of $C_0^*(G)$. Indeed it is a general fact for graded groups that elements in $\Sigma(G)$ act as bounded multipliers of $C_0^*(G)$.

Remark 6.8. Take $\mathfrak{g}$ as previously, with family of complex structures J corresponding to an inner product $|\cdot|_{\mathfrak{g}_1}$. We write the same norm for the one induced on the dual space.

As a Weyl operator on $(\mathfrak{g}_1^*, \omega_\theta^*)$, we have

$$N + \frac{n}{2} = \mathrm{Op}^W\left(-\frac{1}{2}|\cdot|_{\mathfrak{g}_1}^2\right).$$

This does not depend on θ. In particular, we can extend this to every θ by setting:

$$\sigma = -\frac{1}{2}|\cdot|_{\mathfrak{g}_1}^2 \in \mathcal{C}^\infty(\mathfrak{g}^* \setminus 0).$$

This function is homogeneous of order 2 and satisfies $\pi_\theta(\sigma) = N + \frac{n}{2}$ for every $\theta \in \mathbb{S}^*\mathfrak{g}_2$. In particular, $\sigma \in \Sigma^2(G)$.

In order to get $N + 1$ precisely, one could subtract a multiple of the identify for each θ. This globalizes as a multiple of $|\cdot|_{\mathfrak{g}_2^*}$ where $|\cdot|_{\mathfrak{g}_2^*}$ is the norm corresponding

to the sphere $\mathbb{S}^*\mathfrak{g}_2$. This function is however not smooth on $\mathfrak{g}_2^*$ but only on $\mathfrak{g}_2^* \setminus 0$. This is why the function:

$$\sigma' \colon (\xi, \eta) \mapsto -\frac{1}{2}|\xi|^2_{\mathfrak{g}_1} - \left(1 - \frac{n}{2}\right)|\eta|_{\mathfrak{g}_2^*}$$

is not smooth and thus does not define an element of $\mathfrak{g}_2^*$.

This is what Theorem 6.7 circumvents. The workaround is that we can work with θ fixed, use the functional calculus to get a Weyl operator acting as $(N+1)^{-1/2}$, and construct shifts by left multiplying with vectors from $\mathfrak{g}_1^{1,0}$. Families of such operators of order 0 do not define elements of $\Sigma^0(G)$, even locally on $\mathbb{S}^*\mathfrak{g}_2$ as we lack some smoothness in the directions of $\mathfrak{g}_2^*$. However they are still continuous so they define elements of the completion $\Sigma(G)$.

7. H-type groups

In this section we investigate the special features of our construction for H-type groups. This is a special class of regular step-2 groups introduced by Kaplan [13]. The motivation was that on these groups there is an explicit fundamental solution for the sublaplacian $\sum_{i=1}^{2n} X_i^2$ where $X_1, \dots, X_{2n}$ is a basis of $\mathfrak{g}_1$. These groups are endowed with a metric that is compatible with the symplectic structure in a way we will make explicit below. As such, they will be related to representations of Clifford algebras and thus become rather rigid. On the one hand, if we start with a H-type Lie algebras and consider a deformation within the class of H-type Lie algebras, the algebras stay in the same isomorphism class along the deformation, see Lemma 10.4 below. On the other hand, if we allow the deformation to take place within the class of regular step-2 Lie algebras, it is very easy to get a non-H-type Lie algebra, see [20].

We approach these groups with the following problem. Our construction of Toeplitz algebra for a regular step-2 group G relied on the symplectic vector bundle $\mathbb{S}^*\mathfrak{g}_2 \times \mathfrak{g}_1 \to \mathbb{S}^*\mathfrak{g}_2$ (we have fixed a metric on $\mathfrak{g}_2$ and then see $\mathbb{S}^*\mathfrak{g}_2 \hookrightarrow \mathfrak{g}_2^*$ as the unit sphere). Despite the bundle $\mathbb{S}^*\mathfrak{g}_2 \times \mathfrak{g}_1$ being trivial as a real vector bundle, it is not necessarily trivial as a symplectic vector bundle. The question is then, for which regular step-2 group can the symplectic vector bundle $\mathbb{S}^*\mathfrak{g}_2 \times \mathfrak{g}_1$ be trivialized?

Remark 7.1. All the reasonings below will implicitly use connectedness of the sphere $\mathbb{S}^*\mathfrak{g}_2$. For that we need to assume that $\dim(\mathfrak{g}_2) \geq 2$. The case where $\dim(\mathfrak{g}_2) = 1$ corresponds to the Heisenberg groups. Indeed if $\mathfrak{g}$ is a regular step-2 Lie algebra with $\mathfrak{g}_2$ one-dimensional, take any non-zero element $\theta \in \mathfrak{g}_2^*$. Then $G \cong \mathrm{Heis}(\mathfrak{g}_1, \omega_\theta)$.

Let $N = \dim(\mathfrak{g}_2) - 1 \geq 1$ be the dimension of the sphere, $2n = \dim(\mathfrak{g}_1)$. Real (oriented) vector bundles of rank $2n$ over $\mathbb{S}^*\mathfrak{g}_2$ are classified by $\pi_{N-1}(\mathrm{SO}(2n))$. If we choose a compatible complex structure on the symplectic bundle, the triviality of the symplectic bundle (as a symplectic vector bundle) is equivalent to the triviality of the complex bundle

(as a complex vector bundle). The latter does not depend on the choice of compatible complex structure because all the resulting complex vector bundles are isomorphic. Complex vector bundles over $\mathbb{S}^*\mathfrak{g}_2$ are classified by $\pi_{N-1}(\mathrm{U}(n))$. The exact sequence of homotopy groups of the fibration $\mathrm{U}(n) \to \mathrm{SO}(2n) \to {}^{\mathrm{SO}(2n)}\!/_{\mathrm{U}(n)}$ gives:

$$\begin{array}{ccc} \pi_N\big(\mathrm{SO}(2n)\big) & \longrightarrow & \pi_N\big({}^{\mathrm{SO}(2n)}\!/_{\mathrm{U}(n)}\big) \\ & \overset{\delta}{\swarrow} & \\ \pi_{N-1}\big(\mathrm{U}(n)\big) & \longrightarrow & \pi_{N-1}\big(\mathrm{SO}(2n)\big). \end{array}$$

We have chosen a lift $\xi \in \pi_{N-1}(\mathrm{U}(n))$ of our real vector bundle. By exactness we can write $\xi = \delta(x)$ for some $x \in \pi_N({}^{\mathrm{SO}(2n)}\!/_{\mathrm{U}(n)})$. We have an explicit representative for this homotopy class. Indeed, remember that the set of symplectic forms on $\mathfrak{g}_1$ is homeomorphic to ${}^{\mathrm{SO}(2n)}\!/_{\mathrm{U}(n)}$. Then the map $\omega\colon \theta \mapsto \omega_\theta$ defines a homotopy class $[\omega] \in \pi_N({}^{\mathrm{SO}(2n)}\!/_{\mathrm{U}(n)})$ with $\delta([\omega]) = \xi$ (this follows from the compatibility between the complex structure and the symplectic structure). Finally, we want $\xi = 1$. By exactness this means that $[\omega]$ is the image of an element in $\pi_N(\mathrm{SO}(2n))$. Consider the following diagram:

$$\begin{array}{ccc} \mathbb{S}^*\mathfrak{g}_2 & \xrightarrow{\omega} & {}^{\mathrm{SO}(2n)}\!/_{\mathrm{U}(n)} \\ & \overset{\exists ?\Omega}{\searrow} & \uparrow \\ & & \mathrm{O}(2n) \longleftarrow \mathrm{U}(n). \end{array}$$

The right-hand side of the diagram is a fibration. We want a lift of ω to $\mathrm{SO}(2n)$. The sphere $\mathbb{S}^*\mathfrak{g}_2$ only has a cell in dimension 0 and N. The obstruction to this lift lives in

$$H^N\big(\mathbb{S}^*\mathfrak{g}_2, \pi_{N-1}\big(\mathrm{U}(n)\big)\big) \cong \pi_{N-1}\big(\mathrm{U}(n)\big).$$

Indeed since ${}^{\mathrm{SO}(2n)}\!/_{\mathrm{U}(n)}$ is simply connected, the bundle of local coefficients is trivial, therefore the cohomology group is a genuine (singular) cohomology group. The identification then comes from the universal coefficients theorem. The problem is now that this class is precisely the class ξ above. Therefore we find that ξ is trivial if and only if it is trivial... so we do not get any new information. We can however give a geometric meaning to the lift Ω.

Remark 7.2. If $N - 1 < 2n$ and N is odd (so $N - 1$ even), we have $\pi_{N-1}(U(n)) = 0$. Therefore there is no obstruction to the lift above and the symplectic bundle is trivial. However, we will see below that the groups satisfying this triviality condition are quite rigid, so not all values of n and N can be taken (compare with Remark 7.9 below).

The map Ω takes values in $\mathrm{SO}(2n)$ so we may fix a metric on $\mathfrak{g}_1$ so that all the occurrences of $\mathrm{SO}(2n)$ above correspond to $\mathrm{SO}(\mathfrak{g}_1, \langle , \rangle)$. For every θ we have an orthogonal

map $\Omega_\theta \in \mathrm{SO}(\mathfrak{g}_1, \langle , \rangle)$. It defines a lift of ω if and only if for every $X, Y \in \mathfrak{g}_1$, the map:

$$\theta \mapsto \omega_\theta(\Omega_\theta X, \Omega_\theta Y)$$

is constant. Let $J_\theta \in \mathrm{End}(\mathfrak{g}_1)$ be the complex structure compatible with ω_θ such that the corresponding metric is $\langle , \rangle$. Then the endomorphism $\Omega_\theta^{-1} J_\theta \Omega_\theta$ does not depend on $\theta \in \mathbb{S}^*\mathfrak{g}_2$. Moreover, this complex structure, and hence all the J_θ, is orthogonal for the inner product on $\mathfrak{g}_1$. This is precisely the definition of an H-type group.

Definition 7.3 (Kaplan). A step-2 Lie algebra is of H-type if it can be endowed with a metric such that $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are orthogonal and such that the map $J_z \in \mathrm{End}(\mathfrak{g}_1)$, $z \in \mathfrak{g}_2^*$ defined by:

$$\forall X, Y \in \mathfrak{g}_1, \quad \big\langle z, [X, Y] \big\rangle = \langle J_z X, Y \rangle,$$

is orthogonal whenever z has norm 1.

In this definition, we do not consider the metric to be part of the data. Indeed, it is proved in [16] that two H-type groups with a compatible metric, that are isomorphic as groups, are also isomorphic in an isometric way.

Theorem 7.4. *A regular step-2 Lie algebra is of H-type if and only if the symplectic bundle $\mathbb{S}^*\mathfrak{g}_2 \times \mathfrak{g}_1 \to \mathbb{S}^*\mathfrak{g}_2$ is trivial.*

Proof. Thus far we have proved that if the bundle was trivial then the Lie algebra had to be of H-type. Conversely if $\mathfrak{g}$ is of H-type then the map $\Omega\colon \theta \mapsto J_\theta$ lifts the map ω. Consequently, the symplectic bundle is trivial. ∎

Remark 7.5. We can include the Heisenberg groups in the previous theorem by saying that the bundle is trivial on each connected component of $\mathbb{S}^*\mathfrak{g}_2$.

Corollary 7.6. *If $\mathfrak{g}$ is an H-type Lie algebra, then trivializing the symplectic bundle gives a complex structure $J_{\mathrm{cst}} \in \mathrm{End}(\mathfrak{g}_1)$. Then we have:*

$$\mathcal{T}(\mathfrak{g}, J_{\mathrm{cst}}) \cong \mathcal{C}(\mathbb{S}^*\mathfrak{g}_2) \otimes \mathcal{T}(\mathfrak{g}_1^{1,0}).$$

Moreover, $\mathcal{T}_0(\mathfrak{g}, J_{\mathrm{cst}})$ is then the algebra of $\mathcal{T}(\mathfrak{g}_1^{1,0})$-valued functions on $\mathbb{S}^*\mathfrak{g}_2$ such that the image in the quotient $\mathcal{C}(\mathbb{S}^*\mathfrak{g}_1)$ is constant.

Remark 7.7. If we take the Heisenberg group $\mathrm{Heis}(V, \omega)$, then there is no complex structure compatible with both ω and $-\omega$. The algebras $\mathcal{T}(V^{1,0})$ and $\mathcal{T}(V^{0,1})$ are still isomorphic (since they are both isomorphic to $\mathcal{T}(\mathbb{C}^n)$ where $\dim(V) = 2n$). Under this non-natural isomorphism, we can also see $\mathcal{T}(\mathfrak{heis}(V, \omega))$ in the form as in the last corollary.

Example 7.8. Consider (V, ω) a complex symplectic space and let $\mathfrak{heis}_{\mathbb{C}}(V, \omega)$ be the complex Lie algebra constructed from the abelian V with the 2-cocycle ω. If we see (V, ω) as the complexification of a real symplectic space (V_0, ω_0), then $\mathfrak{heis}_{\mathbb{C}}(V, \omega)$ is

the complexification of $\mathfrak{heis}(V_0, \omega_0)$. In particular, we have now a 2-dimensional center. Trivializing the symplectic bundle, we get:

$$\mathcal{T}\big(\mathfrak{heis}_{\mathbb{C}}(V, \omega), J_{\mathrm{cst}}\big) = \mathcal{C}(\mathbb{S}^1) \otimes \mathcal{T}(V_0^{1,0} \oplus V_0^{0,1}).$$

The H-type groups correspond to Clifford modules. Indeed, consider for $z \in \mathfrak{g}_2 \cong \mathfrak{g}_2^*$, the endomorphism $J_z \in \mathrm{End}(\mathfrak{g}_1)$ characterized by the identity: $\langle x, y\rangle = \omega_z(J_z x, y)$. Then for $x \in \mathfrak{g}_1$, $J_z(x)$ is the unique other element of $\mathfrak{g}_1$ satisfying $[J_z(x), x] = |x|^2 z$. Some quick computations yield:

$$\forall x, y \in \mathfrak{g}_1, \forall z, w \in \mathfrak{g}_2, \quad \langle J_z x, J_w y\rangle + \langle J_w x, J_z y\rangle = 2\langle x, y\rangle\langle z, w\rangle.$$

Now using anti-self adjointness, we obtain the Clifford relations:

$$J_z J_w + J_w J_z = -2\langle z, w\rangle \mathrm{Id}.$$

Consequently, the data of a (metric) H-group is the data of a (unitary) $\mathrm{Cl}(\mathfrak{g}_2)$-module structure on $\mathfrak{g}_1$. There is then a one to one correspondence between isomorphisms of Clifford modules and isomorphisms of Lie algebras being the identity on the center $\mathfrak{g}_2$. In general an isomorphism of Lie algebras does not have to be the identity on the center. Because of this, non-isomorphic Clifford modules could yield isomorphic H-type Lie algebras. It is shown in [14] that, if we fix the dimension of the center, there is only one irreducible H-type Lie algebra with a center of that dimension up to isomorphism. Now if $\mathfrak{v}_1, \mathfrak{v}_2$ are two complementary Clifford submodules of $\mathfrak{g}_1$ then we have $[\mathfrak{v}_1, \mathfrak{v}_2] = 0$. Indeed, assume $[x, y] = z \neq 0$ with $x \in \mathfrak{v}_1$, $y \in \mathfrak{v}_2$. Then $J_z(y) = |y|^2 x$, contradiction. Therefore an H-type Lie algebra is completely classified, up to isomorphism, by the pair $(\dim(\mathfrak{g}_2), \dim(\mathfrak{g}_1))$.

Remark 7.9. From the theory of Clifford modules, the dimension of $\mathfrak{g}_1$ has to be a non-zero multiple of the dimension of the irreducible Clifford modules of $\mathrm{Cl}(\mathfrak{g}_2)$. The latter only depends on the dimension of $\mathfrak{g}_2$. If we write d_m for the dimension of that module, where $m = \dim(\mathfrak{g}_2)$, we have $d_{m+8} = 16 d_m$. Moreover, the dimensions for small values of m is given by:

$$d_1 = 2, \quad d_2 = d_3 = 4, \quad d_4 = d_5 = d_6 = d_7 = 8, \quad d_8 = 16.$$

Therefore H-type groups can only have dimensions:

$$\big(\dim(\mathfrak{g}_1), \dim(\mathfrak{g}_2)\big) = (k d_m, m), \quad m, k \in \mathbb{N}.$$

8. Polycontact manifolds

Definition 8.1 (van Erp). A polycontact structure on a smooth manifold M is a smooth distribution $H \subset TM$ such that, if $\theta \in \Gamma_{\mathrm{loc}}(H^{\perp})$ then $\mathrm{d}\theta_{|H \times H}$ is non-degenerate. The pair (M, H) is called a polycontact manifold.

Given a smooth manifold M and a subbundle $H \subset TM$, we can create a bundle of nilpotent groups the following way. Consider the vector bundle $H \oplus {}^{TM}\!/_{H}$. The curvature form of H is the bundle map

$$H \times H \to {}^{TM}\!/_{H}$$

induced by the Lie bracket of vector fields:

$$(X_m, Y_m) \mapsto [X, Y]_m \mod H_m,\ m \in M.$$

Such a map is indeed well defined and does not depend on the choice of vector fields representing the elements $X_m, Y_m \in H_m$. This form endows $H \oplus {}^{TM}\!/_{H}$ with a Lie algebra structure on each fiber by taking the Lie bracket with any element of ${}^{TM}\!/_{H}$ to be zero. This Lie algebra structure depends smoothly on the base-point, but is not necessarily locally trivial.

The Lie algebras being nilpotent, we can integrate them all at once using the Baker–Campbell–Hausdorff formula. This gives a smooth family $T_H M$ of nilpotent groups on M called the osculating groups.

Proposition 8.2. *Let M be a manifold endowed with a smooth distribution $H \subset TM$. The following are equivalent:*

(1) *(M, H) is a polycontact manifold.*

(2) *For every $m \in M, \theta \in H_m^\perp \setminus 0$, the form $\omega_\theta \in \Lambda^2 H_m^*$ defined by composing the curvature 2-form with θ is non-degenerate.*

(3) *All osculating groups are regular 2-step nilpotent groups.*

Proof. Let $X, Y \in \Gamma_{\mathrm{loc}}(H)$ and $\theta \in \Gamma_{\mathrm{loc}}(H^\perp)$. We have:

$$\mathrm{d}\theta(X, Y) = X \cdot \theta(Y) - Y \cdot \theta(X) - \theta\big([X, Y]\big) = -\theta\big([X, Y]\big),$$

thus the two first conditions are equivalent. The equivalence between the last two conditions is Proposition 3.2. ∎

Example 8.3. Contact manifolds, quaternionic and octonionic contact manifolds [3] are all examples of polycontact manifold.

A regular 2-step nilpotent group, with the left invariant distribution induced by the filtration of its Lie algebra is a polycontact manifold. If $\Gamma \subset G$ is a discrete subgroup, the manifold ${}^{G}\!/_{\Gamma}$ is a polycontact manifold. If Γ is cocompact we then have a compact polycontact manifold.

Remark 8.4. In all the previous examples, the osculating groups are the same no matter the point on a manifold. This is because these examples are either constructed from the group, or are particular cases of H-type manifolds, as defined below, where this phenomenon is to be expected.

It is however not the general case. It is shown in [20] how a H-type group can be deformed into a family of non-isomorphic regular 2-step nilpotent groups.

9. Fields of Toeplitz algebras

Let (M, H) be a polycontact manifold. From the family of osculating groups $T_H M$ we obtain a C^*-algebra $C^*(T_H M)$. This C^*-algebra is a continuous field of C^*-algebras over M, with fiber at $x \in M$ equal to the group C^*-algebra $C^*(T_{H,x} M)$. The spectrum of $C^*(T_H M)$ is homeomorphic to ${}^{\mathfrak{t}_H M^*}\!/_{\mathrm{Ad}^*(T_H M)}$. We again have a dichotomy between the flat orbits and points. The flat orbits are parameterized by $H^\perp \setminus M$ and correspond to infinite-dimensional representations. They provide an ideal of $C^*(T_H M)$ isomorphic to $\mathcal{C}_0(H^\perp \setminus M, \mathcal{K})$. The quotient corresponds to the characters, i.e., fixed points for the coadjoint action, parameterized by H^*.

Proposition 9.1. *Let (M, H) be a polycontact manifold, we have the exact sequence:*

$$0 \longrightarrow \mathcal{C}_0(H^\perp \setminus M, \mathcal{K}) \longrightarrow C^*(T_H M) \longrightarrow \mathcal{C}_0(H^*) \longrightarrow 0.$$

Proof. The maps are well defined globally. The sequence is then exact on each fiber by Corollary 3.7, so it is itself exact. Real codimension 1 submanifolds in hypercomplex spaces (with strong pseudoconvexity assumptions) are an example of polycontact manifold where the osculating groups are not necessarily all the same [26]. ∎

Using this proposition, we define $C_0^*(T_H M)$ to be the preimage by the quotient map of $\mathcal{C}_0(H^* \setminus M)$. It is the algebra of sections of the continuous field of C^*-algebras formed by the $(C_0^*(T_{H,x} M))_{x \in M}$ (seen as a subfield of the one underlying $C^*(T_H M)$). We want to express $C_0^*(T_H M)$ as a crossed product, gluing our previous construction with all the osculating groups at the same time.

Consider for each $(x, \theta) \in H^\perp \setminus M$, the form $\omega_\theta \in \Lambda^2 H_x$ obtained by composition with the curvature form (i.e., the Lie bracket of $\mathfrak{t}_H M$). Since we are on a polycontact manifold, ω_θ is a symplectic form. We therefore get a symplectic bundle $H \times_M (H^\perp \setminus M) \to H^\perp \setminus M$. We may fix a metric on ${}^{TM}\!/_H$, hence on $H^\perp$, and restrict the bundle to $H \times_M \mathbb{S}H^\perp \to \mathbb{S}H^\perp$. Denote by $\pi : \mathbb{S}H^\perp \to M$ the bundle map, so that the previous bundle becomes $\pi^* H$. For any given $x \in M$, we can restrict the bundle to $H_x \times \mathbb{S}H_x^\perp \to \mathbb{S}H_x^\perp$, which is exactly the symplectic bundle for the osculating group $T_{H,x} M$.

Now that we have a symplectic bundle, we may choose a compatible complex structure $J \in \mathrm{End}(\pi^* H)$ and the corresponding C^*-algebra of sections of the Toeplitz bundle $\mathcal{T}(\mathfrak{t}_H M, J)$. The symplectic bundle restricts over each $\mathbb{S}H_x^\perp$, $x \in M$ to the one of the osculating group. We can thus see $\mathcal{T}(\mathfrak{t}_H M, J)$ as the algebra of sections of a continuous field of C^*-algebras over M, whose fiber at $x \in M$ is:

$$\mathcal{T}(\mathfrak{t}_H M, J)_x = \mathcal{T}(\mathfrak{t}_{H,x} M, J_x),$$

where $J_x = J_{|H_x \times \mathbb{S}H_x^\perp}$. We denote by $\mathcal{T}_0(\mathfrak{t}_H M, J)$ the subalgebra of $\mathcal{T}(\mathfrak{t}_H M, J)$ consisting of sections whose value at $x \in M$ is in $\mathcal{T}_0(\mathfrak{t}_{H,x} M, J_x)$. We have the exact sequence:

$$0 \longrightarrow \mathcal{C}_0(M, \mathcal{K}) \longrightarrow \mathcal{T}_0(\mathfrak{t}_H M, J) \longrightarrow \mathcal{C}_0(\mathbb{S}H^*) \longrightarrow 0.$$

Theorem 9.2. *There is an isomorphism of C^*-algebras:*

$$\mathcal{T}_0(\mathfrak{t}_H M, J) \cong \Sigma(T_H M).$$

Proof. The proof is similar as for a single group. The algebra $\Sigma(T_H M)$ is the subalgebra of sections of Weyl operators $\mathcal{W}^0(H_x^*, \omega_\theta^*)$, $(x,\theta) \in \mathbb{S}H^\perp$ such that the corresponding field of principal symbols is constant on each $\mathbb{S}H_x^\perp$. This is then the same as Theorem 6.7. ∎

Like on a group, we can define the number operator. The algebra $\mathcal{T}(\mathfrak{t}_H M, J)$ acts on the Hilbert bundle of symmetric Fock spaces $\mathcal{F}^+((\pi^* H, J)^{1,0})$. The number operator N acts on $\mathrm{Sym}^k((\pi^* H, J)^{1,0})$ by $k\mathrm{Id}$ and defines this way an unbounded operator. Its complex powers define a strongly continuous group of bounded operators. As for the group, conjugation by $N^{it/2}$, $t \in \mathbb{R}$ defines a group of automorphisms of $\mathcal{T}(\mathfrak{t}_H M, J)$ and $\mathcal{T}_0(\mathfrak{t}_H M, J)$.

Theorem 9.3. *Let (M, H) be a compact polycontact manifold. Let $J \in \mathrm{End}(\pi^* H)$ be a compatible complex structure on the symplectic bundle. Then we have an $\mathbb{R}_+^*$-equivariant isomorphism:*

$$\mathcal{T}_0(\mathfrak{t}_H M, J) \rtimes \mathbb{R} \cong C_0^*(T_H M).$$

Proof. As for the group case, we identify $C_0^*(T_H M)$ with a subalgebra of sections of the bundle $(I_{+,0}(H_x, \omega_\theta))_{(x,\theta)\in\mathbb{S}H_x^\perp}$. These sections must satisfy that, for $x \in M$ fixed, their restriction at 0 in $I_{+,0}(H_x, \omega_\theta)$, $\theta \in \mathbb{S}H_x^\perp$ is constant as a family of elements of $\mathcal{C}_0(H_x^* \setminus 0)$.

Let $\widetilde{C}_0^*(T_H M)$ be the algebra of (all the) sections of the bundle of C^*-algebras given by $(I_{+,0}(H_x, \omega_\theta))_{(x,\theta)\in\mathbb{S}H_x^\perp}$. We want to use the same arguments as in 5.8 to show an isomorphism

$$\mathcal{T}(\mathfrak{t}_H M, J) \rtimes \mathbb{R} \xrightarrow{\sim} \widetilde{C}_0^*(T_H M).$$

This is similar to [5, Theorem 7.7]. ∎

Remark 9.4. The construction of the map $\mathcal{T}(V^{1,0}) \rtimes \mathbb{R} \to I_{+,0}(V, \omega)$ amounts to the existence of an operator $\Delta \in \Sigma^2(\mathrm{Heis}(V, \omega))$ such that $\pi_\lambda(\Delta) = \lambda^2(N+1)$ for every positive λ and then send S_X to $-\mathrm{i}X\Delta^{-1/2}$ (compare with Corollary 6.6). In our situation, we would need such a Δ for each point (x, θ). In case M is not compact, we also need to control the norm of these type of elements at infinity and it is not clear it should decay at infinity. This is the reason we only state the previous theorem when M is compact (similarly as in the previous [5]).

10. The case of H-type manifolds

Definition 10.1. A polycontact manifold (M, H) is called a H-type manifold if all its osculating groups are H-type groups.

A geometric characterization of these manifolds is given by the following.

Proposition 10.2 ([15, Proposition 5.1]). *A distribution $H \subset TM$ makes M a H-type manifold if and only if it admits a compatible (sub)conformal structure.*

Example 10.3. Contact, quaternionic contact and octonionic contact manifolds are all examples of H-type manifolds.

Lemma 10.4. *If (M, H) is a H-type manifold then all its osculating groups are mutually isomorphic.*

Proof. The isomorphism class of an H-type Lie algebra is completely characterized by the dimension and codimension of its center. The distribution $H \subset TM$ having constant rank (and co-rank), if the osculating groups are all H-type, they have to be mutually isomorphic. ∎

Theorem 10.5. *If (M, H) is a H-type manifold then $T_H M$ is a locally trivial bundle of Lie groups.*

Proof. This is a consequence of a more general fact, that a (smooth) family of Lie algebras that form a (locally trivial) vector bundle and where all the fibers are isomorphic has to be a locally trivial bundle of Lie algebras [17]. ∎

Remark 10.6. In that context where the osculating groups form a locally trivial group bundle, we should probably even have local diffeomorphisms to the typical fiber, in a way that preserves the respective filtrations. This would be in contrast with some rigidity results known in quaternionic contact geometry. It is indeed proved in [12] that for those manifolds, the curvature of the Biquard connection [3] prevents a quaternionic contact manifold to be isomorphic to the local model $\mathbb{H}^n \times \Im(\mathbb{H})$. This notion of isomorphism however requires to preserve the conformal class on the metric on the distribution, which is stronger than the local isomorphism we want. Compare also this claim with [22] where it is proved that for generic distributions, finding a local frame that generates a finite-dimensional Lie algebra (which is equivalent to our claim) is only possible in very specific cases. However here we are not asking for generic distributions since requiring to have the same osculating group everywhere (at least locally) is quite rigid.

Corollary 10.7. *If (M, H) is a H-type manifold with typical osculating group G, then the algebras $\Sigma(T_H M), C^*(T_H M), C_0^*(T_H M)$ are algebras of sections of locally trivial bundles of C^*-algebras with respective fiber $\Sigma(G), C^*(G), C_0^*(G)$.*

Corollary 10.8. *Let (M, H) be a H-type manifold with typical osculating group G. Choose a complex structure $J \in \mathrm{End}(\pi^*(H))$ compatible with the symplectic form and that restricts to J_{cst} on each fiber, where $\pi : \mathbb{S}H^\perp \to M$ is the projection. then the bundle of Toeplitz algebras $\mathcal{T}(\pi^* H, J) \to \mathbb{S}H^\perp$ can be trivialized over subsets of the form $\pi^{-1}(U)$. More precisely, locally over $U \subset M$ where we trivialize the bundle $\mathbb{S}H^\perp$, we get*

$$\mathcal{T}(\pi^* H, J)_{|\pi^{-1}(U)} \cong \mathcal{C}_0(U \times \mathbb{S}^* \mathfrak{g}_2) \otimes \mathcal{T}_0(\mathfrak{g}, J_{\mathrm{cst}}).$$

References

[1] W. Arveson, Subalgebras of C^*-algebras. III. Multivariable operator theory. *Acta Math.* **181** (1998), no. 2, 159–228 Zbl 0952.46035 MR 1668582

[2] R. Beals and P. Greiner, *Calculus on Heisenberg manifolds*. Ann. of Math. Stud. 119, Princeton University Press, Princeton, NJ, 1988 Zbl 0654.58033 MR 0953082

[3] O. Biquard, Métriques d'Einstein asymptotiquement symétriques. *Astérisque* **265** (2000), vi+109 Zbl 0967.53030 MR 1760319

[4] C. Cren, Filtered calculus and crossed products by $\mathbb{R}$-actions. *J. Operator Theory* **94** (2025), no. 2, 473–511 Zbl 08143118 MR 5001197

[5] C. Cren, Toeplitz algebras and the Heisenberg group. *J. Geom. Phys.* **225** (2026), article no. 105829 Zbl 08196999 MR 5048917

[6] J. Dixmier, *C^*-algebras*. North-Holland Math. Library 15, North-Holland Publishing, Amsterdam, 1977 Zbl 0372.46058 MR 0458185

[7] J. Dixmier and A. Douady, Champs continus d'espaces hilbertiens et de C^*-algèbres. *Bull. Soc. Math. France* **91** (1963), 227–284 Zbl 0127.33102 MR 0163182

[8] C. L. Epstein, Lectures on indices and relative indices on contact and CR-manifolds. In *Woods Hole mathematics*, pp. 27–93, Ser. Knots Everything 34, World Sci. Publ., Hackensack, NJ, 2004 MR 2123367

[9] C. Fermanian-Kammerer and V. Fischer, Defect measures on graded Lie groups. *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)* **21** (2020), 207–291 Zbl 1469.35272 MR 4288603

[10] G. B. Folland and E. M. Stein, Estimates for the $\bar{\partial}_b$ complex and analysis on the Heisenberg group. *Comm. Pure Appl. Math.* **27** (1974), 429–522 Zbl 0293.35012 MR 0367477

[11] A. Gorokhovsky and E. van Erp, The Heisenberg calculus, index theory and cyclic cohomology. *Adv. Math.* **399** (2022), article no. 108229 Zbl 1489.58008 MR 4383012

[12] S. Ivanov and D. Vassilev, Conformal quaternionic contact curvature and the local sphere theorem. *J. Math. Pures Appl. (9)* **93** (2010), no. 3, 277–307 Zbl 1194.53044 MR 2601333

[13] A. Kaplan, Fundamental solutions for a class of hypoelliptic PDE generated by composition of quadratic forms. *Trans. Amer. Math. Soc.* **258** (1980), no. 1, 147–153 Zbl 0393.35015 MR 0554324

[14] A. Kaplan and F. Ricci, Harmonic analysis on groups of Heisenberg type. In *Harmonic analysis (Cortona, 1982)*, pp. 416–435, Lecture Notes in Math. 992, Springer, Berlin, 1983 Zbl 0521.43008 MR 0729367

[15] A. Kaplan and M. Subils, On the equivalence problems for bracket-generating distributions. In *Hodge theory, complex geometry, and representation theory*, pp. 157–171, Contemp. Math. 608, American Mathematical Society, Providence, RI, 2014 Zbl 1303.53036 MR 3220089

[16] A. Kaplan and M. Subils, Heisenberg algebras from division algebras and parabolic subalgebras of simple Lie algebras. *J. Lie Theory* **28** (2018), no. 2, 561–575 Zbl 1395.22002 MR 3744014

[17] B. S. Kiranagi, Lie algebra bundles. *Bull. Sci. Math. (2)* **102** (1978), no. 1, 57–62 Zbl 0375.55016 MR 0500955

[18] A. A. Kirillov, *Lectures on the orbit method*. Grad. Stud. Math. 64, American Mathematical Society, Providence, RI, 2004 Zbl 1229.22003 MR 2069175

[19] R. Y. Lee, On the C^*-algebras of operator fields. *Indiana Univ. Math. J.* **25** (1976), no. 4, 303–314 Zbl 0306.46073 MR 0410400

[20] F. Levstein and A. Tiraboschi, Classes of 2-step nilpotent Lie algebras. *Comm. Algebra* **27** (1999), no. 5, 2425–2440 Zbl 0921.17002 MR 1683874

[21] G. Métivier, Hypoellipticité analytique sur des groupes nilpotents de rang 2. *Duke Math. J.* **47** (1980), no. 1, 195–221 Zbl 0433.35015 MR 0563376

[22] R. Montgomery, Generic distributions and Lie algebras of vector fields. *J. Differential Equations* **103** (1993), no. 2, 387–393 Zbl 0782.58008 MR 1221912

[23] C. C. Moore and J. A. Wolf, Square integrable representations of nilpotent groups. *Trans. Amer. Math. Soc.* **185** (1973), 445–462 (1974) Zbl 0274.22016 MR 0338267

[24] E. Nelson, Analytic vectors. *Ann. of Math. (2)* **70** (1959), 572–615 Zbl 0091.10704 MR 0107176

[25] C. Rockland, Hypoellipticity on the Heisenberg group-representation-theoretic criteria. *Trans. Amer. Math. Soc.* **240** (1978), 1–52 Zbl 0326.22007 MR 0486314

[26] E. van Erp, Contact structures of arbitrary codimension and idempotents in the Heisenberg algebra. [v1] 2010, [v2] 2011, arXiv:1001.5426v2

[27] E. van Erp and R. Yuncken, A groupoid approach to pseudodifferential calculi. *J. Reine Angew. Math.* **756** (2019), 151–182 Zbl 1433.58025 MR 4026451

Communicated by Clotilde Fermanian Kammerer

Received 16 January 2026; revised 15 July 2026.

Clément Cren
Mathematisches Institut, Georg-August Universität Göttingen, Bunsenstraße 3-5, 37073 Göttingen, Germany; clement.cren@mathematik.uni-goettingen.de
Author IDs: zbMATH cren.clement MR 1627974 ORCID 0000-0001-9217-4953

Licensed under the Creative Commons Attribution 4.0 International License CC BY 4.0 (https://creativecommons.org/licenses/by/4.0/), which permits use, sharing, adaptation, and redistribution with appropriate credit, a link to the license, and indication of changes.