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Singular Integrals on Product H^P Spaces

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1. Introduction

We shall begin describing some terminology and notation. By «Calderón-Zygmund space» we shall mean the class of all bounded operators, T, on $L^2(R^1)$ given by a kernel $k(x,y)$ so that $Tf(x) = \int_{R^1} k(x,y)f(y)\,dy$ and so that for each fixed $x \in R^1$, $k \in C^\infty(R^1/\{x\})$ as a function of y and satisfies

$$\left|\left(\frac{\partial}{\partial y}\right)^\alpha k(x,y)\right| \leqslant C_\alpha |x-y|^{-1-\alpha} \quad \text{for} \quad \alpha > 0. \tag{*}$$

We shall often identify the operator T with its kernel k. For a particular choice of a positive integer N, we define the norm of T in Calderón-Zygmund space $\|T\|_{CZ}$ by $\|T\|_{CZ} = \|T\|_{L^2,L^2} + \sum_{\alpha=1}^{N} C_\alpha$ where here C_α denotes the smallest constant for which (*) is valid.

Suppose $k(x)$ now stands for a kernel on R^1 with values in Calderón-Zygmund space satisfying:

(1) $|k(x)|_{CZ} \leqslant \dfrac{C}{|x|}$

(2) $\left|\left(\dfrac{d}{dx}\right)^j k(x)\right|_{CZ} \leqslant \dfrac{C}{|x|^{j+1}}, \qquad j \leqslant N$

and

(3) $\int_{\alpha<|x|<\beta} k(x)\,dx = 0 \qquad \forall 0<\alpha<\beta.$

Then k defines an integral operator taking functions $f(x_1, x_2)$ on R^2 to functions $Hf(x_1, x_2)$ on R^2 as follows:

$$Hf(x_1, x_2) = \iint k_0(x_1, y_1, x_2, y_2) f(y_1, y_2)\,dy_1 dy_2 \qquad (\neq)$$

where

$$k_0(x_1, y_1, x_2, y_2) = k(x_1 - y_1)\cdot(x_2, y_2).$$

Another way of understanding how k gives rise to H is as follows: Suppose we associate to each function $f(x_1, x_2)$ on R^2, the function $\tilde{f}$ on R^1 taking its values in the space of functions (on R^1) of x_2 given by $\tilde{f}(x_1)(x_2) = f(x_1, x_2)$. Then

$$\widetilde{Hf}(x_1) = \int k(x_1 - y_1)\cdot \tilde{f}(y_1)\,dy_1.$$

Our theorem is then:

Theorem. Let $0<p<1$. Then there exists N so that if $k(x_1)$ is a kernel satisfying (1), (2) and (3) above and H is as in $(\neq)$, then H maps $H^P(R^2_+ \times R^2_+)$ boundedly to $L^P(R^2)$.

Recall that $f \in H^P(R^2_+ \times R^2_+)$, product H^P, means that f is a distribution on R^2 with the property that

$$\sup_{\delta_1, \delta_2 > 0} \left| f * \phi_{\delta_1, \delta_2}(x_1, x_2)\right| \in L^P(R^2)$$

for

$$\phi \in C_0^\infty(R^2),\ \phi_{\delta_1,\delta_2}(x_1, x_2) = \delta_1^{-1}\delta_2^{-1}\phi\left(\frac{x_1}{\delta_1}, \frac{x_2}{\delta_2}\right).$$

Before proving this theorem let us put it into some perspective. First of all, in case the values of $k(x)$ are convolution operators, then this theorem is already known. (See the article of E. *M.* Stein and the author [1]). The spirit of the proof we give here is along the lines of C. Fefferman's theorem on the maximal double Hilbert transform [2]. There, the product structures of the kernel plays a role, where here, we assume no such structure. The other main ingredient of the argument below is the atomic decomposition of H^P spaces on product domains ([3], [4], [5]). We shall assume that the reader is familiar with the properties of product H^P atoms.

Finally, we should mention the interesting work of J. L. Journé [6], where non convolution operators in the product setting are treated, and proven bounded on the L^P spaces and from L^∞ to product *BMO* (for the properties of product *BMO*, see [3]). This paper, we feel, should have simple generalizations to cover operators like Journé's bi-commutators, and the proof is probably very similar to the one given here.

PROOF OF THE THEOREM. In order to simplify things a little, we shall assume $p = 1$. The case $p < 1$ requires no major changes. We shall let $a(x_1, x_2)$ be an $H^1(R^2_+ \times R^2_+)$ atom supported in an open set Ω, which by dilation invariance, we may assume to have measure 1. For this atom a, we then show that if $\phi \in C^\infty(R^1)$, Ψ supported in $[-1, -1]$ is even and has its first N moments vanishing, then the corresponding square function in the first variable is in $L^1(R^2)$:

$$\left(\sum_{k=-\infty}^{+\infty} |\Psi_{2^k} *_1 (k * a)(x_1, x_2)|^2\right)^{1/2} \quad \in L^1(R^2).$$

This will prove the theorem.
(Here $\Psi_{2^k}(x_1) = 2^{-k}\Psi(x_1/2^k)$, $*_1$ refers to a convolution taken only in the first variable for each fixed x_2, and $k * a$ is the function such that $\widetilde{k * a}(x_1) = = \int_{R^1} k(x_1 - t) \cdot \tilde{a}(t)\,dt$; see the introduction for an explanation of the $\sim$ notation).

We shall sometimes find it convenient to assume that Ψ has the form $\Psi^{(1)} * \Psi^{(1)}$ where $\Psi^{(1)}$ has properties similar to those listed for Ψ. Also it will suit our needs to define a cutoff function $\phi(x_1) \in C^\infty(R^1)$, which is even, is supported in $\frac{1}{4} < |x_1| < 4$ and so that $\sum_j \phi(x/2^j) \equiv 1$. Define $k_j(x) = k(x)\phi(x/2^j)$ and $k_{k,j}(x) = \Psi_{2k} * k_j(x)$.

Our proof will show that the norms $\|(\sum_k |k_{k,k+j} * a(x_1, x_2)|^2)^{1/2}\|_1$ decrease geometrically as $|j| \to \infty$. Then summing over j finishes the proof. To see what is going on let us first consider the case $j = 0$, and write $k_k = k_{k,k}$. To estimate $(\sum_k |k_k * a|^2)^{1/2}$ we next decompose a as follows: Using the notation of [3], [4] and [5], since a is an atom, it can be written as $a = \sum_{R \subset \Omega} e_R$ where the sum is taken over the dyadic subrectangles of Ω, and we are going to use this representation of a to do a splitting of a which depends upon the point (x_1, x_2). For an integer $r \geqslant 0$ and an integer l consider the following definitions: R_l is the collection of all dyadic subrectangles of Ω, $R = I \times J$ where $|I| = 2^l$ and where J is a maximal dyadic interval such that $I \times J \subset \Omega$. Split the rectangles in R_l into disjoint subclasses $R_{(x_1,x_2),l,r}$ by setting $R_{x,l,r}$ to be those rectangles of R_l, $R = I \times J$ where $2^{-r-1} < M(X_J)(x_2) \leqslant 2^{-r}$. Then for each fixed $(x_1, x_2) = x$,

$$\bigcup_{\substack{l \in Z \\ r \in Z, r \geq 0}} R_{x,l,r}$$

is precisely the collection of all dyadic subrectangles of Ω, $R \cong I \times J$ so that J is maximal.

Finally, we let

$$a_{l,r}^{(x_1,x_2)} = \sum_{R \in R_{x,l,r}} \Bigl(\sum_{S \subset R;\, x_1 \text{ length of } S = 2^l} e_S \Bigr),$$

so that

$$a = \sum_{l,r} a_{l,r}^x.$$

Now, we claim that $(\sum_k |k_k * a_{k+j,r}^x|^2)^{1/2}$ has an L^1 norm which is $0(c_1^{|j|} c_2^r)$ as $|j|, r \to \infty$ where the $c_i < 1$ for $i = 1, 2$. Summing these estimates on j and r finishes the argument that

$$\Bigl(\sum_k |k_k * a|^2 \Bigr)^{1/2} \in L^1.$$

Again, we consider the special case $j = 0$ and show that

$$\Bigl\| \Bigl(\sum_k |k_k * a_{k,r}^x|^2 \Bigr)^{1/2} \Bigr\|_{L^1} = 0(c^r), \qquad c < 1, \quad \text{as} \quad r \to \infty. \tag{0}$$

If $r \geqslant 1$, by the way we constructed $a_{k,r}^x$ it is clear that $k_k * a_{k,r}^x$ has support in $\{(x_1, x_2) \mid M_s(X_\Omega)(x_1, x_2) > \frac{1}{10 \cdot 2^r}\} = \tilde{\Omega}_r$. Since by the strong maximal theorem, $|\tilde{\Omega}_r| = 0(r2^r)$, by the Cauchy-Schwartz inqueality, to show (0) it will be sufficient to show

$$\Bigl\| \Bigl(\sum_k |k_k * a_{k,r}^x|^2 \Bigr)^{1/2} \Bigr\|_{L^2} = 0(2^{-Nr})$$

We shall estimate $\|k_k * a_k^x\|_{L^2}$ by using the following trivial lemmas:

Lemma 1. Suppose $b(x)$ is a function on R^1 whose support lies in the union of the disjoint intervals I_k, and which has its first N moments vanishing over each I_k. Suppose a point $x \in R^1$ lies outside the union of the doubles of the I_k. Then for any Calderon-Zygmund operator T on R^1,

$$|Tb(x)| \leqslant C|T|_{CZ} \sup_k M(X_{I_k})^N(x) \cdot \mathcal{G}(b)(x),$$

where

$$\mathcal{G}(b)(x) = \bigl(\sum M^2(X_{I_k}) \bigr)^{1/2} \cdot \bigl(\sum M^2(bX_{I_k})(x) \bigr)^{1/2}.$$

Lemma 2. Let $b(x_1, x_2)$ be a function on R^2 supported in an open set θ of finite measure. Denote by b_{x_1} the function given by $b_{x_1}(x_2) = b(x_1, x_2)$, and by

$L_k(x_1)$ the component intervals of $\theta_{x_1} = \{x_1 \mid (x_1, x_2) \in \theta\}$. Let $\mathcal{G}$ denote the operator which acts in the x_2 variable for each x_1, defined by

$$\mathcal{G}(b)(x_1, x_2) = \Big(\sum_k M^2(X_{I_k(x_1)}(x_2)\Big)^{1/2}\Big(\sum M^2(X_{I_k(x_1)}b_{x_1})(x_2)\Big)^{1/2}$$

Then $\|\mathcal{G}(b)\|_{L^2} \leqslant C|\theta|^{1/2}\|b\|_{L^2}$.

The point of these lemmas is that, as a function of x_2 for fixed x_1, $a^x_{k,r}(x_1, \cdot)$ has the properties of the function in lemma 1, so that $k_k(x_1 - t)$ as a singular integral in the x_2 variable applied to $a^x_{k,r}(t, \cdot)$ is dominated by

$$|k_k(x_1 - t)|_{CZ} \cdot \mathcal{G}[a^x_{k,r}(t, \cdot)] \cdot 2^{-Nr} \leqslant |k_k(x_1 - t)|_{CZ} \cdot \mathcal{G}[a_k(t, \cdot][a_k(t, \cdot)] \cdot 2^{-Nr}$$

where

$$a_k = \sum_{\substack{R \subset \Omega \\ R = I \times J \\ |I| = 2^k}} e_R.$$

We also have $\int_{R^1} |k_k(t)|_{CZ}\, dt \leqslant C$.

Therefore $|k_k * a^x_{k,r}(x_1, x_2)| \leqslant |k_k|_{CZ} *_1 \mathcal{G}(a_k)(x_1)(x_1, x_2) \cdot 2^{-Nr}$ and by lemma 2 we see that $\|\mathcal{G}(a_k)\|_2 \leqslant C\|a_k\|_2$. It follows that

$$\Big\|\Big(\sum_k |k_k * a^k_{k,r}|^2\Big)^{1/2}\Big\|_1 \leqslant 2^{-Nr}C\Big(\sum_k \|a_k\|_2^2\Big)^{1/2} \leqslant C'2^{-Nr}.$$

If $r = 0$ we estimate $\big(\sum_k |k_k * a^x_{k,0}|^2\big)^{1/2}$ by observing that this function is supported in the set $\{M_S(X_\Omega) > \frac{1}{10}\}$ which has measure $\leqslant C$. To estimate its L^1 norm we estimate its L^2 norm by observing that

$$\|\big(\sum |k_k * a^x_{k,0}|^2\big)^{1/2}\|_{L^2} \leqslant$$
$$\leqslant \sum_{r \geq 1} \Big\|\Big(\sum_k |k_k * a^x_{k,r}|^2\Big)^{1/2}\Big\|_{L^2} + \Big\|\Big(\sum_k |k_k * a_k|^2\Big)^{1/2}\Big\|_{L^2},$$

so we have only to estimate $\|(\sum |k_k * a_k|^2)^{1/2}\|_{L^2}$. But this is easy since $\int_{R^1} |k_k(t)|_{CZ}\, dt \leqslant C$, so $\|k_k * a_k\|_2 \leqslant C\|a_k\|_2$ and so $\sum_k \|k_k * a_k\|_2^2 \leqslant$ $\leqslant C\sum_k \|a_k\|_2^2 \leqslant C'$. Now, let us pass to the next case where $j > 0$ and $r > 0$. That is, we require an estimate of

$$\Big\|\Big(\sum_k |k_k * a^x_{k+j,r}|^2\Big)^{1/2}\Big\|_1 = 0(c^{j+r}) \text{ for some } c < 1.$$

Again, the support of $k_k * a^x_{k+j,r}$ is contained in $\tilde{\Omega}_r$ of measure $\leqslant Cr2^r$. So as above we need to estimate $\|k_k * a^x_{k+j,r}\|_2$. By using the fact that $\psi = \psi^{(1)} * \psi^{(1)}$ we may write $k_k * a^x_{k+j,r} = k'_k * (\psi^{(1)}_{2^k} *_1 a^x_{k+j,r})$ where k'_k has similar properties to k_k. Now we use the special form of $a^x_{k+j,r}$ to estimate $k'_k * (\psi^{(1)}_{2^k} *_1 a^x_{k+j,r})$. It turns out that essentially $a^x_{k+j,r}(x_1, x_2)$ over a dyadic interval in the x_1 variable of length 2^{k+j}, say the interval $[0, 2^{k+j}]$, is of the form

$\psi(x_1/2^{k+j}) \cdot \alpha_{x,r}(x_2)$. Then, for $x_1 \in [0, 2^{k+j}]$ we have $|\psi_{2^k}^{(1)} * [\psi(x_1/2^{k+j})]| \leqslant \leqslant C2^{-jN}$ because of the vanishing moments of $\psi^{(1)}$. It follows that

$$|k_k' * (\psi_{2^k}^{(1)} *_1 a_{k+j,r}^x)(x_1, x_2)| \leqslant 2^{-jN-rN} \frac{1}{2^k} \int_{x_1-2^k}^{x_1+2^k} (\alpha_{k+j})(t, x_2)\, dt$$

where $\alpha_{k+j}(x_2)$ is the function such that $a_{k+j}(x_1, x_2) = \psi(x_1/2^{k+j}) \cdot \alpha_{k+j}(x_2)$. Again, from the lemmas, it follows that

$$\| k_k' * (\psi_{2^k}^{(1)} *_1 a_{k+j,r}^x)\|_2 \leqslant C2^{-jN-rN} \| a_{k+j}\|_2,$$

and this is the estimate we needed.
(Actually $a_{j+r,r}^x$ and a_{k+j} and a_{k+j} over dyadic intervals of length 2^{k+j} (say $[0, 2^{k+j}]$ are averages over $\tau \in [0, 2^{k+j}]$ of functions of the form $\psi(x_1 - \tau/2^{k+j})\alpha_\tau(x_2)$; this average does not, however, interfere with any of the estimates done above; see [5]). The case $r = 0$ and $j > 0$ follows from the cases $r > 0$, $j < 0$ as was carried out previously. If $r > 0$ and $j < 0$, then we estimate $\|(\sum_k |k_k * a_{k-|j|,r}^x|^2)^{1/2}\|_1$ by observing that the support of $k_k * a_{k-|j|,r}^x(x_1, x_2)$ is contained in $\tilde{\Omega}_{r+|j|}$ of measure $\leqslant C(r + |j|)2^{r+|j|}$. We estimate, as before, the L^2 norm of $k_k * a_{k-|j|,r}^x$ by using the fact that for each fixed x_2, $a_{k+|j|,r}^x(\cdot, x_2)$ has the following property: There exist disjoint intervals of length $2^{k-|j|}$ over which the first N moments of $a_{k-|j|,r}^x(\cdot, x_2)$ vanish. Just as for the familiar case of scalar valued kernels, we may take advantage of the smoothness of $k_k(x)$ and subtract off the correct Taylor approximation to k_k to produce $\widehat{k_k}$ such that $k_k * a_{k-|j|,r}^x = \widehat{k_k} * a_{k-|j|,r}^x$, but $|\widehat{k_k}(z)|_{CZ} \leqslant 2^{-k_2-|j|N}$. Now we use the fact that for fixed x_1, $a_{k-|j|,r}^x(x_1, \cdot)$ has N vanishing moments over the component intervals of $\{x_2 \mid (x_1, x_2) \in \in \bigcup_{R \in R_{x,k-|j|,r}} R\}$ to dominate $\widehat{k_k}(t)$ acting on $a_{k-|j|,r}^x(x_1 - t, \cdot)$ by $2^{-Nr}\mathcal{G}(a_{k-|j|,r}^x(x_1 - t, \cdot))\, |\widehat{k_k}(t)|_{CZ}$ so that

$$|k_k * a_{k-|j|,r}^x(x_1, x_2)| \leqslant C2^{-Nr}|\widehat{k_k}| *_1 \mathcal{G}(a_{k-|j|})$$

and obviously

$$\| \widehat{k_k} *_1 \mathcal{G}(a_{k-|j|})\|_1 \leqslant C2^{-|j|N} \| a_{k-|j|}\|_2,$$

which proves that

$$\left\| \left(\sum_k |k_k * a_{k-|j|,r}^x|^2 \right)^{1/2} \right\| \leqslant C2^{-N(r+|j|)}.$$

Passing to the estimate of $\|(\sum_k |k_k * a_{k-|j|,0}^x|^2)^{1/2}\|_1$ is routine and left to the reader.

Now we are almost finished. We have shown that $\|(\sum_k |k_k * a|^2)^{1/2}\|_1 \leqslant C$ and all that is left is to show that $\|(\sum_k |k_{k,k+j} * a|^2)^{1/2}\|_1$ tend to 0 geometrically. But this is immediate, since for say $j > 0$, $k_{k,k+j}(x)$ satisfies all the same estimates as $2^{-jN}k_{k+j}(x)$ and so by the estimates above,

$$\left\| \left(\sum_k |k_{k,k+j} * a|^2 \right)^{1/2} \right\|_1 = 0(2^{-jN}).$$

This proves our theorem.

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