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On Discrete Subgroups of Lie Groups and Elliptic Geometric Structures

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In this paper we continue the investigation of [7]-[10] concerning the actions of discrete subgroups of Lie groups on compact manifolds.

Let H be a connected semisimple Lie group with finite center and suppose that the $\mathbb{R}$-rank of every simple factor of H is at least 2. Let $\Gamma \subset H$ be a lattice subgroup and M^n a compact n-manifold with a volume density. Let $P \to M$ be a G-structure on M where G is a real algebraic group. More precisely, let $GL(n, \mathbb{R})^{(k)}$ be the k-jets of diffeomorphisms of $\mathbb{R}^n$ fixing 0, and $P^{(k)} \to M$ the principal $GL(n, \mathbb{R})^{(k)}$-bundle of k-frames on M([1], [13]). Then $GL(n, \mathbb{R})^{(k)}$ is a real algebraic group and the (k-th order) G-structure $P \to M$ is a reduction of $P^{(k)}$ to the real algebraic subgroup $G \subset GL(n, \mathbb{R})^{(k)}$. We shall let $\mathrm{Aut}(P) \subset \mathrm{Diff}(M)$ denote the subgroup of diffeomorphisms of M that preserve the G-structure.

In [6] we showed that under the above hypotheses any volume preserving action of H on M which preserves the G-structure is either trivial or implies the existence of a non-trivial Lie algebra homomorphism $L(H) \to L(G)$, or equivalently, a Lie algebra embedding $L(H') \to L(G)$ for some simple factor H' of H. In [7], [8] we put forward the following conjecture.

Conjecture. *With hypotheses as above, assume there is a smooth volume preserving action of* Γ *on* M *defining a homomorphism* $\Gamma \to \mathrm{Aut}(P)$. *Then either:*

a) There is Lie algebra embedding $L(H') \to L(G)$ *for some simple factor* H' *of* H; *or*
b) *there is a* Γ*-invariant Riemannian metric on* M.

We remark that, as explained in [7], the conjecture would imply in particular that actions of Γ on low dimensional manifolds are trivial on a subgroup of finite index.

In [8] this conjecture was proven under the additional assumptions that the G-structure is of finite type (in the sense of E. Cartan [3], or more generally in the sense of Tanaka [5]), and that the Γ-action is ergodic. (In [8] the existence of a Γ-invariant C^0-Riemannian metric is deduced. However the arguments of [10] show that in fact an invariant C^∞ metric exists.) In this paper we weaken the assumption of finite type to that of ellipticity at the expense of assuming that $\mathrm{Aut}(P)$ acts transitively on M. (However the ergodicity assumption is no longer needed.) We recall that the G-structure is elliptic if the infinitesimal automorphisms of P (i.e. the vector fields defined by 1-parameter subgroups of $\mathrm{Aut}(P)$) are characterized as those vector fields satisfying an elliptic partial differential equation. For first order structures, this is equivalent to the simple condition on G that the linear Lie algebra $L(G) \subset \mathfrak{Gl}(n, \mathbf{R})$ contains no matrices of rank 1 [3, Prop I.1.4]. (For higher order structures see [1, p. 71].) One of the salient features of an elliptic G-structure is that $\mathrm{Aut}(P)$ is a (finite dimensional) Lie group. The main result of this paper is the following.

Theorem 1. *With* H, Γ, M, G, P *as above, suppose that* $\Gamma \to \mathrm{Aut}(P)$ *is a volume preserving, G-structure preserving action of* Γ *on* M. *Assume that*

a) *Aut(P) is a Lie group (e.g., G elliptic), and*
b) *Aut(P) acts transitively on M.*

Then either

1. *there is a Lie algebra embedding* $L(H') \to L(G)$ *for some simple factor* H' *of* H; *or*
2. *there is a* Γ-in*variant Riemannian metric on* M.

We remark that if $\mathrm{Aut}(P)$ is almost connected (or more generally if a subgroup of Γ of finite index is mapped into the connected component of the identity, $\mathrm{Aut}(P)^0$) then Theorem 1 follows from the work of Margulis [4] combined with the result of [6] described above and Kazhdan's property for Γ [2], [12]. In general, if course, $\mathrm{Aut}(P)/\mathrm{Aut}(P)^0$ may be infinite.

There are two basic known results we need for the proof of Theorem 1. The first is that the above conjecture is true if conclusion (2) is weakened to asserting the existence of a Γ-invariant measurable Riemannian metric on M. (By a measurable Riemannian metric on a vector bundle we of course mean a measurable assignment of an inner product to each fiber of the bundle.) This is a consequence of the superrigidity theorem for cocycles [11], [12, Thm. 5.2.5]. More precisely, we have:

Lemma 2. (Cf. [7, sections 2, 3]). *Let $P \to M$ be a principal G-bundle where G is a real algebraic group. Let H be a connected semisimple Lie group with finite center such that the* $\mathbf{R}$*-rank of every simple factor of H is at least 2. Let $\Gamma \subset H$ be a lattice. Assume that every Lie algebra homomorphism $L(H) \to L(G)$ is trivial. Let V be a vector space on which G acts linearly (and smoothly), and $E \to M$ the associated vector bundle. If Γ acts by principal bundle automorphisms of P covering a finite volume preserving action on M, then there is a measurable Γ-invariant Riemannian metric on the vector bundle E.*

The second result we need, proved in [7] enables us to give an estimate for the integrability properties of the measurable invariant metric in lemma 2.

Lemma 3. [7, Theorem 4.1]. *Let Γ be a discrete Kazhdan group (i.e., group with Kazhdan's property T* [2], [12]), *and $\Gamma_0 \subset \Gamma$ a fixed finite generating set. Then there exists $K > 1$ with the following property. If (S, μ) is a standard Borel ergodic Γ-space with Γ-invariant probability measure, and $f\colon S \to \mathbb{R}$ is a measurable function satisfying $|f(s\gamma)| \leqslant K|f(s)|$ for almost all s and all $\gamma \in \Gamma_0$, then $f \in L^1(S)$.*

Now let V be a finite dimensional real vector space. If η, ξ are inner products on V, we set (as in [7, section 3])

$$M(\eta/\xi) = \max\{\, \|v\|_\eta / \|v\|_\xi \,|\, v \neq 0, v \in V\},$$

and if η_m, $\xi_m (m \in M)$ are measurable Riemannian metrics on a vector bundle $E \to M$ we let $M(\eta/\xi)\colon M \to \mathbf{R}$ be $M(\eta/\xi)(m) = M(\eta_m/\xi_m)$. Suppose Γ acts on E by vector bundle automorphisms, that M is compact, that η is a measurable Γ-invariant metric, and ξ is a smooth metric. Then for $\gamma \in \Gamma$, and $m \in M$,

$$\begin{aligned} M(\eta/\xi)(m\gamma) &= M(\gamma^*\eta/\gamma^*\xi)(m) = M(\eta/\gamma^*\xi)(m) \\ &\leqslant M(\eta/\xi)(m) M(\xi/\gamma^*\xi)(m). \end{aligned}$$

(Cf. [7, Cor. 4.2]). We thus deduce that there exists $B > 0$ such that $m \in M$ and $\gamma \in \Gamma_0$ implies $M(\eta/\xi)(m\gamma) \leqslant BM(\eta/\xi)(m)$. From these remarks and lemma 3, we obtain:

Lemma 4. *Let Γ be a Kazhdan group acting smoothly on a compact manifold M. Suppose Γ preserves a smooth probability measure μ on M. We* *let $\mu = \int_E^{\oplus} \mu_t d\nu(t)$ be an ergodic decomposition of μ under the Γ-action. (Thus (E, ν) is the space of ergodic components.) Suppose η is a measurable Γ-invariant Riemannian metric and that ξ is a smooth metric. Then for q sufficiently large, we have $M(\eta/\xi) \in L^{2/q}(M, \mu_t)$ for almost all t.*

We now assume the hypotheses of Theorem 1, and suppose that every Lie algebra homomorphism $L(H) \to L(G)$ is trivial. By lemma 2, there is a measurable Γ-invariant metric η on TM. Choose q as in lemma 4. If f: $M \to (0, \infty)$ is a measurable Γ-invariant function, then $f\eta$ is also a measurable Γ-invariant metric. There is clearly a measurable h: $E \to (0, \infty)$ such that $\int_E h(t)^{2/q}\left(\int M(\eta/\xi)^{2/q} d\mu_t\right) d\nu(t) < \infty$ and thus if we let $f = h \circ p$ where p: $M \to E$ is the map defining the decomposition into ergodic components we have that $f\eta$ is a measurable Γ-invariant Riemannian metric satisfying $M(f\eta/\xi) \in L^{2/q}(M, \mu)$. Thus, replacing η by $f\eta$, we shall assume $M(\eta/\xi) \in L^{2/q}(M, \mu)$. Let Y be the set of (globally defined) infinitesimal automorphisms of P, so that (by hypothesis (*a*) of Theorem 1) Y is a finite dimensional vector space of smooth sections of TM and (by hypothesis (*b*)), for each $m \in M$ the evaluation map e_m: $Y \to TM_m$ is surjective. For $F \in Y$, let $\Phi(F) = \int_M \|F(m)\|_{\eta m}^{1/q} d\mu$. Since $M(\eta/\xi) \in L^{2/q}(M)$, $0 \leqslant \Phi(F) < \infty$, and it is clear that $\Phi(F) = 0$ if and only if $F = 0$. Furthermore Φ is continuous. (To see this simply observe that

$$\begin{aligned}|\Phi(F)| &\leqslant \int |M(\eta/\xi)|^{1/q} \|F(m)\|_{\xi m}^{1/q} \\ &\leqslant \|M(\eta/\xi)^{1/q}\|_2 \left(\int \|F(m)\|_{\xi m}^{2/q}\right)^{1/2}.\end{aligned}$$

Thus, if $\max_{m \in M} \|F(m)\|_{\xi m} \to 0$, we have $\Phi(F) \to 0$.) We also observe that Φ: $Y \to [0, \infty)$ is homogeneous of degree $\frac{1}{q}$. It follows from these properties of Φ (and the fact that $\dim Y < \infty$) that $\{F \mid |\Phi(F)| < 1\}$ is a (non-empty) open set with compact closure. Since η is Γ-invariant, it is clear that Φ is also Γ-invariant, and the preceding sentence implies that the representation of Γ on Y is uniformly bounded. Since $\dim Y < \infty$, there is a Γ-invariant inner product on Y. Via the maps $\{e_m\}$ this defines a smooth metric on TM, and it is clear that Γ-invariance of the inner product on Y implies that this metric on TM is Γ-invariant. This proves Theorem 1.

Remarks (*a*). If k is a local field of characteristic 0, H an almost k-simple algebraic k-group, with k-rank$(H) > 2$, and $\Gamma \subset H_k$ is a lattice, then super-rigidity and Kazhdan's property hold for Γ [2], [12]. Thus, the above argument shows:

Theorem 10.5. *Let M be a compact manifold, P a G-structure on M such that Aut(P) is a Lie group acting transitively on M. Let $\Gamma \subset H_k$ be as above,* and assume that Γ *acts on M so as to preserve a volume density and the G-structure P. Then there is a* Γ*-invariant Riemannian metric on M.*

(b) If Aut(P) is a Lie group which is not transitive on M but the globally defined infinitesimal automorphisms of P define a foliation of M (which is then of necessity Γ-invariant), then the above argument shows that there is a Γ-invariant smooth metric on the tangent bundle to the foliation.

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