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Mathematical Analysis. – *Semigroups on generalized Sobolev spaces associated with Laplacians with applications to stochastic PDEs with singular boundary conditions*, by Sergio Albeverio, Zdzisław Brzeźniak and Szymon Peszat, accepted on 1 October 2026.

We dedicate this work to the dear memory of Giuseppe Da Prato. His fundamental work in the broad area of analysis, stochastic analysis, and infinite-dimensional analysis has opened the way to the theory of singular partial differential equations with important applications in mathematical physics and many other areas. Besides being a great scientist, he has also been a true mentor to us and a wonderful person. We would like to stress our deep gratitude to Giuseppe, both as a scientist and as a close friend. The departure of Giuseppe is an immense loss for all of us, but he will remain with us through his beautiful scientific production and his great humanity

Abstract. – A systematic functional analytic study of the Laplace operator on generalized Sobolev spaces over n-dimensional Euclidean space without a point is presented, as a prototype for the study of second-order differential operators with perturbations on singular domains, and of the semigroups that they generate. In particular, extensions of the Laplacian acting on C_0 spaces and on L^p spaces are characterized, depending on the underlying space dimensions and on p. Also their relations with traces and boundary operators are clarified. These new results are then applied to the study of mild solutions of stochastic heat equations with boundary noise at the origin. The well-posedness of the corresponding stochastic boundary value problems in space dimension less than or equal to 3 for several boundary operators and on L^p spaces for suitable p is proven and precise conditions for the existence of invariant measures are identified. The results of this paper are intended to open a way for investigating more complex SPDEs with singular boundary conditions and more complicated domains.

Keywords. – Laplace operator on $\mathbb{R}^n \setminus \{0\}$, point interactions, singular boundary conditions, generalized Sobolev spaces, extensions of operators, C_0-semigroups on L^p spaces, Green formula, stochastic heat equation, boundary noise, Gaussian invariant measures, polar sets.

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1. Introduction

The description of many dynamical systems in the material world, both by deterministic and stochastic methods, often has to take into account the natural presence of boundary conditions.

Boundary conditions may depend on the geometry of the underlying spaces and are often characterized, in the deterministic case, by specifying values for the solutions and their derivatives on the boundary, such as Dirichlet-type, Neumann-type, or mixed conditions.

Sturm–Liouville problems provide simple examples of boundary value problems for ordinary deterministic second-order differential equations. Probabilistic methods offer new insights into these problems, see, e.g., [58, 98, 127, 128, 130–132].

Vast extensions of these insights and problems to PDEs and stochastic PDEs (SPDEs) have been discussed, see, e.g., [54, 67, 90, 123, 127] for elliptic and parabolic systems, and [39, 60, 75, 78, 82] for hyperbolic systems.

In many areas of the applications of mathematics, it has turned out to be useful to consider other types of boundary conditions, namely, singular ones, in the sense that they are defined either by generalized functions supported on the boundary or, more generally, by suitable limits of regularized systems and boundary conditions. A simple example was encountered first in the modeling of very short-range strong interactions in (quantum) physical systems, see [33, 59] and references in [12, 18, 19, 50, 62, 70, 79].

The first prototype is described by the Laplacian (possibly perturbed by regular and potential terms) on $\mathbb{R}^n$, with $n = 1, 2, 3$, defined first on smooth functions with compact support in $\mathbb{R}^n \setminus \{0\}$, i.e. on $\mathcal{C}_0^\infty(\mathbb{R}^n \setminus \{0\})$. Self-adjoint extensions of this operator acting in $L^2(\mathbb{R}^n)$ have been investigated mathematically by Berezin, Faddeev, and Minlos in [30, 104]. They are related to the theory of extensions of symmetric operators on Hilbert spaces and correspond to a limit problem starting from a regularized weighted potential term $\lambda_\varepsilon \delta_\varepsilon(\cdot)$ in the limit as $\varepsilon \to 0$, where $\delta_\varepsilon(\cdot)$ is a regularized scaled Dirac delta distribution at the origin and λ_ε is a suitable real constant depending on the parameter ε, see [12, 15, 109]. These extensions generate self-adjoint semigroups in $L^2(\mathbb{R}^n)$. Some of them are Markovian and constitute transition semigroups of Markov diffusion processes with boundary conditions at the origin, which have been discussed in detail, see, e.g., [9, 12, 16, 17, 69]. A probabilistic description of some of these can be obtained through the theory of quadratic forms and associated diffusion processes,

see [16] and, e.g., [4, 12]. Another direct description, using the Feynman–Kac formula, is given in [64, 65], see also [108] for an excellent very recent work.[1]

Let us also mention deterministic and stochastic (Wentzell-type) boundary conditions in connection with SPDEs, which often arise also in other areas of science, see, e.g., [16, 24–27, 29, 32, 42, 43, 47, 49, 55, 66, 71, 83, 92, 96, 97, 99, 100, 113, 115, 118–122, 125] and references therein.

Semigroups associated with singular boundary conditions, in particular those associated with point interaction models, have also been studied in the L^p spaces, see [38] and references, e.g., in [6, 21, 42].

In the present paper, we provide a novel view on boundary value problems, both in the deterministic and stochastic case, taking into account a large class of "singular boundary conditions" (SBCs). These SBCs are defined either by generalized functions supported on boundaries of regions in $\mathbb{R}^n$ or, more generally, by suitable limits of regularized systems, the regularizations depending only on "classical boundary conditions". The solutions of such singular boundary problems belong to general spaces like Sobolev and Lebesgue ones, e.g. L^p for suitable p, having in mind also applications to SPDEs.

The framework for discussing such problems is first presented through a detailed discussion of the operator describing a perturbation of the Laplacian by a suitable singular interaction term located at a point in $\mathbb{R}^n$, a particular case of the "point-interaction models" mentioned above, in particular in Footnote 1. Whereas in the quantum physics literature the space is usually $L^2(\mathbb{R}^n)$, we consider here the cases of $L^p(\mathbb{R}^n)$ spaces or suitable Sobolev spaces.

(1) In general, operators and boundary conditions given by such constructions, starting with approximations with potentials concentrated on some points, have been called "point interactions". The use of such singular boundary conditions has been vastly extended to many models and situations, e.g., moving point interactions on infinitely many discrete manifolds, or on submanifolds, or on random sets, see, e.g., [40, 41, 44, 56, 68, 74, 101, 106, 116] and further references in [12, 19]. The use of singular boundary conditions in probabilistic problems also comes from other sources, e.g., from the study of SPDEs (elliptic, parabolic, hyperbolic) in connection with problems in quantum field theory, hydrodynamics, or electromagnetism, see, e.g., [14, 18, 23, 84, 85, 87, 109] for early works and [20–22, 45, 46, 48, 53, 57, 61, 63, 75–78, 80, 86–89, 91, 116, 117, 119] for recent developments. Here, the SPDEs exhibit interesting singularities and non-linearities due to physical constraints, and boundary conditions naturally arise when considering limits with space-time regularization, local and global, which are later removed by a procedure defined by the generic expression "renormalization". In the construction of models in such areas problems arise that are similar to those we mentioned above, concerning point interaction models. In fact, the latter models were partly developed to understand better mathematical problems arising in the study of simplified models in quantum field theory, see, e.g., [8, 50, 52, 62, 81, 95, 103, 104, 112, 113, 124].

We also investigate SPDEs associated with the operators and the semigroups that they generate and study the SPDEs with SBC in this general setting, also providing invariant measures for them.

We stress that the results presented in detail for our "case study" of a point interaction model can be extended to much more general SPDEs with singular coefficients and boundary data.

The content of the paper is roughly as follows. Section 2 presents some notation and definitions. In Section 3, we study Sobolev spaces $W^{\Delta,p}(\mathbb{R}^n \setminus \{0\})$, with integrability exponent $p \in (1, +\infty)$, associated with the Laplacian Δ on the open set $\mathbb{R}^n \setminus \{0\}$.

Let $\Delta_{\mathbb{R}^n\setminus\{0\}}$ be the distributional Laplace operator on $\mathcal{D}'(\mathbb{R}^n \setminus \{0\})$. Given a function[2] $u \in L^p(\mathbb{R}^n \setminus \{0\}) = L^p(\mathbb{R}^n \setminus \{0\}, \mathbb{C})$ such that $\Delta_{\mathbb{R}^n\setminus\{0\}}u$ belongs to $L^p(\mathbb{R}^n \setminus \{0\})$, we consider

$$R(u) := \widetilde{\Delta_{\mathbb{R}^n\setminus\{0\}}u} - \Delta\tilde{u},$$

where for a function $v\colon \mathbb{R}^n \setminus \{0\} \to \mathbb{C}$, we put

$$\tilde{v}(x) = \begin{cases} v(x) & \text{if } x \in \mathbb{R}^n \setminus \{0\}, \\ 0 & \text{if } x = 0. \end{cases}$$

Our first result, see Lemma 3.5, characterizes the difference $R(u)$. Next, see Theorem 3.13, we describe how the space $W^{\Delta,p}(\mathbb{R}^n \setminus \{0\})$ differs from the space $W^{2,p}(\mathbb{R}^n)$. Then we are concerned with the operators $(A, D(A))$ on $L^p(\mathbb{R}^n \setminus \{0\})$ such that

$$\begin{aligned} &D(A) \cap D(A^*) \supset \mathcal{C}_0^\infty\big(\mathbb{R}^n \setminus \{0\}\big), \\ &Au = A^*u = -\Delta_{\mathbb{R}^n\setminus\{0\}}u, \quad u \in \mathcal{C}_0^\infty\big(\mathbb{R}^n \setminus \{0\}\big), \end{aligned}$$

with A^* denoting the adjoint to A.

To this end, crucial roles are played by Theorem 3.13 and a Green formula on $\mathbb{R}^n \setminus \{0\}$, see Theorem 4.2.

The main result in Section 3 is given in Theorem 3.13, which provides a characterization of the elements of these spaces. The independence of the representation of an element with respect to the exponent p is specified in Theorem 3.14 and its Corollaries 3.15, 3.16, and 3.17.

In particular, the relations between the spaces $W^{\Delta,p}(\mathbb{R}^n \setminus \{0\})$ and the classical spaces $W^{2,p}(\mathbb{R}^n)$ are discussed in Proposition 3.20. In Corollary 3.10, it is stressed that something exceptional happens when $n = 2$: then $W^{\Delta,p}(\mathbb{R}^2 \setminus \{0\}) \neq W^{2,p}(\mathbb{R}^2 \setminus \{0\})$ for every $p \in (1, +\infty)$. Examples 3.21 and 3.22 further illustrate the case $n = 1$. In Remark 3.24, we show that $W^{\Delta,p}(\mathbb{R}^n \setminus \{0\}) = W^{2,p}(\mathbb{R}^n \setminus \{0\}) = W^{2,p}(\mathbb{R}^n)$ if $n \geq 3$ and $p \geq \frac{n}{n-2}$, and that $W^{\Delta,p}(\mathbb{R} \setminus \{0\}) = W^{2,p}(\mathbb{R} \setminus \{0\})$ if $n = 1$.

[2] In the whole paper we consider complex valued functions and distributions. We will not introduce any specific notation for these objects.

In Section 4, the Green, or Stokes-type, formula on $\mathbb{R}^n \setminus \{0\}$ is established, for elements in the corresponding space $W^{\Delta,p}$.

In Section 5, we prove a slightly more general result than the one in [107] for closed extensions of operators and related C_0-semigroups on complex Banach spaces, see Theorem 5.1. We then discuss the special case of the linear unbounded operator $A_{\min}$, coinciding with minus the Laplacian $-\Delta$ on the space $C_0^\infty(\mathbb{R}^n \setminus \{0\})$. We consider the operator $A_{0,p}$, which coincides with $-\Delta$ on the space $W^{2,p}(\mathbb{R}^n)$, and show, see Theorem 5.4, that it is the unique extension of $A_{\min}$ in the class of negative generators of C_0-semigroups on $L^p(\mathbb{R}^n)$ if $n \geq 3$ and $p \leq \frac{n}{2}$. If $p > \frac{n}{2}$, then there are infinitely many such extensions. Some open problems are pointed out in Remarks 5.7 and 5.8. This discussion is further extended in Section 6, where the closure of $\mathcal{C}_0^\infty(\mathbb{R}^n \setminus \{0\})$ in $W^{2,p}(\mathbb{R}^n)$ is characterized, see Proposition 6.3.

In Section 7, symmetric extensions of the Laplace operator on $\mathbb{R}^n \setminus \{0\}$ are discussed in detail. In particular, an extension of the theory of defect indices and spaces to the L^p-setting is provided (for the L^2-theory or Hilbert space theory see, e.g., [3,5,10,11,13]).

In Section 8, we consider the cases $n = 2, 3$ in all detail, depending on p. In particular, we provide boundary operators at the origin and extensions of $-\Delta_{\mathbb{R}^n\setminus\{0\}} = A_{\min}$ in $L^p(\mathbb{R}^n)$. We finally present corresponding spectral results.

Previous results are recovered in a systematic way and discussed in Section 9, which is devoted to the heat equation and the corresponding semigroups with singular boundary conditions, in the cases $n = 2, 3$. The relations with Gaussian Markov families in the $L^p(\mathbb{R}^n \setminus \{0\})$ spaces are also pointed out.

In more detail, in Section 9, we study for $n = 2, 3$ boundary value problems involving the Laplacian and the corresponding heat equation

$$\frac{\partial u}{\partial t} = \Delta u,$$

with a time-dependent boundary condition given by $\tau_\beta u = \dot{W}$, $\dot{W}$ being Gaussian white noise in time and τ_β the boundary operator, depending on a parameter $\beta \in \mathbb{R}$. In the mild form, it involves the kernel of the semigroup generated by a Laplacian with boundary condition given by τ_β and the Bessel potential G_n, see (3.6), and, e.g., [31].

In Corollary 9.5, we exhibit the mild solution for $\beta \neq 0$, under the assumptions of Proposition 9.2.

For $\beta = 0$ we have a solution involving the standard heat semigroup. The boundary problem in this case with initial condition in $L^p(\mathbb{R}^n)$ is well posed and defines a Gaussian Markov family for $p \in (1, 2)$ if $n = 2$ and for $p \in (1, \frac{3}{2})$ if $n = 3$, see Proposition 9.6.

For $\beta = 0$ and $\ell > n/2$, the problem is well posed in $H^{-\ell}(\mathbb{R}^n)$, and a Gaussian invariant measure on $H^{-\ell}(\mathbb{R}^n)$ exists if and only if $n = 3$, see Proposition 9.9.

For $\beta > 0$ and $\ell > 1$, the well-posedness in $H^{-\ell}_{\beta}(\mathbb{R}^n \setminus \{0\})$ is proved in Proposition 9.10.

Various further estimates are provided in Section 9.3.

For $n = 3$, $0 < \beta \leq 4\pi$ and $p \in (\frac{3}{2}, 3)$, the well-posedness in $L^p(\mathbb{R}^3)$ is proved in Proposition 9.14.

Appendix A provides a detailed discussion of relations with previous work and also stresses that our treatment goes well beyond previous ones that were limited to the symmetric case in L^2, whereas in the present work we handle operators in L^p spaces and Sobolev spaces.

Our detailed results involve the use of a modified Laplace operator (of the "point interaction type") in the particular singular domain $\mathbb{R}^n \setminus \{0\}$ but are expected to be extendable to more complicated domains and singular differential operators.

Appendix B is dedicated to the derivation of density results in Sobolev spaces, adapting and extending to our setting, involving singular operators and domains, classical results, see, e.g., [1, 37].

1.1. Notations

The set of all natural numbers, including zero, will be denoted by $\mathbb{N}$. By $\mathbb{N}^*$ we will denote the subset of $\mathbb{N}$ consisting of all non-zero natural numbers. By D_i, $i = 1, \ldots, n$, we will denote the weak, that is, the distributional, partial derivatives in open subsets of $\mathbb{R}^n$ (some authors prefer the notation $\frac{\partial}{\partial x_i}$). When $n = 1$, the weak derivative D_1 will be simply denoted by D. Next, for a multi-index $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ we set $|\alpha| = \alpha_1 + \cdots + \alpha_n$ and

$$D^{\alpha} = D_1^{\alpha_1} \cdots D_n^{\alpha_n}.$$

As usual we put $D^0 f := f$. If $\mathcal{O}$ is a subset of some topological space, then we will denote the closure of $\mathcal{O}$ by $\overline{\mathcal{O}}$.

1.2. Sign convention

Throughout the paper the unbounded operators denoted by A, with or without a decoration, are *minus* the Laplacian; i.e. they act as

$$Au = -\Delta u,$$

the Laplacian being understood in the sense specified in each case. Thus, every such A is bounded from below; it is $-A$, and not A, that generates the corresponding C_0-semigroup, and the heat equation reads $\frac{\partial u}{\partial t} = -Au = \Delta u$. Accordingly, spectra are contained in $[0, +\infty)$ up to at most one negative eigenvalue, resolvent sets contain a left half-line, and a resolvent is written $(\lambda I + A)^{-1}$ with $\lambda > 0$ sufficiently large. In particular, the Friedrichs Laplacian on $L^p(\mathbb{R}^n)$ is denoted by $\Delta_{n,p}$, and the associated operator in the sense above is $A_{0,p} = -\Delta_{n,p}$.

2. Preliminaries

If not explicitly stated otherwise, all functional spaces treated in this paper, cf. Footnote 2, are assumed to consist of complex-valued functions. By δ_0 we denote the Dirac delta distribution at the point $0 \in \mathbb{R}^n$. By $\mathcal{L}(X)$ we denote the space of all bounded and linear operators from a Banach space X into itself.

Assume that $\mathcal{O} \subset \mathbb{R}^n$ is an open set. By $\mathcal{D}(\mathcal{O})$ we denote the space of test functions on $\mathcal{O}$, i.e. $\mathcal{D}(\mathcal{O}) = \mathcal{C}_0^\infty(\mathcal{O})$ endowed with the locally convex topology as defined in [114, Definition 6.3]. By $\mathcal{D}'(\mathcal{O})$ we denote the space of distributions on $\mathcal{O}$ in the sense of [114, Chapter 6].

If $k \in \mathbb{N}$ and $p \in (1, +\infty)$, $W^{k,p}(\mathcal{O})$ is defined, see [1, Section 3.2], to be the space of all functions $f \in L^p(\mathcal{O})$ such that the weak derivatives $D^\alpha f$, for $|\alpha| \leq k$, belong to $L^p(\mathcal{O})$ as well. It is known that $W^{k,p}(\mathcal{O})$ is a Banach space with the norm

$$|f|_{k,p} := \left\{ \sum_{|\alpha| \leq k} \int_{\mathcal{O}} \left| D^\alpha f(x) \right|^p \mathrm{d}x \right\}^{\frac{1}{p}}. \tag{2.1}$$

Let us also recall, see [1, Section 3.2], that $W^{k,p}(\mathcal{O})$ can be defined equivalently as the completion of the space $\mathcal{C}^{k,p}(\mathcal{O}) := \{f \in \mathcal{C}^k(\mathcal{O}) \colon |f|_{k,p} < +\infty\}$, see [102] and/or [1, Theorem 3.17]. Thus, in particular, $\mathcal{C}^{k,p}(\mathcal{O})$ is dense in $W^{k,p}(\mathcal{O})$.

Remark 2.1. Assume that the boundary $\partial\mathcal{O}$ is of class $\mathcal{C}^k$ and $|\alpha| < k - \frac{1}{p}$ with $\frac{1}{p} < k$. Let $B_q^{r,p}(\partial\mathcal{O})$ be the Besov space on the boundary $\partial\mathcal{O}$, see [129, Definition 3.6.1]. Let us point out that the Besov spaces $B_q^{r,p}(\partial\mathcal{O})$ and the classical spaces $\mathcal{C}^k(\partial\mathcal{O})$ on the boundary are defined via localization, and this explains the need for the $\mathcal{C}^k$ condition for the boundary regularity.

It is well known, see, e.g., [129, Theorem 4.7.1] or [2], that there exists a unique linear bounded mapping, called the trace operator,

$$\gamma_\alpha \colon W^{k,p}(\mathcal{O}) \to B_p^{k-\frac{1}{p}-|\alpha|,p}(\partial\mathcal{O}), \tag{2.2}$$

such that

$$\gamma_\alpha(u) = D^\alpha u|_{\partial\mathcal{O}}, \quad u \in \mathcal{C}^k(\overline{\mathcal{O}}).$$

One first defines a map $\widetilde{\gamma}_\alpha \colon \mathcal{C}^k(\overline{\mathcal{O}}) \cap \mathcal{C}^{k,p}(\mathcal{O}) \ni u \mapsto D^\alpha u|_{\partial\mathcal{O}} \in \mathcal{C}^{k-|\alpha|}(\partial\mathcal{O})$ and proves that it is continuous in the norms (2.1). Making use of the density of $\mathcal{C}^k(\overline{\mathcal{O}}) \cap \mathcal{C}^{k,p}(\mathcal{O})$ in the space $W^{k,p}(\mathcal{O})$, see [129, Theorem 3.2.5], one proves that $\widetilde{\gamma}_\alpha$ has exactly one continuous extension which is the map γ_α from (2.2). For this and related results we refer to [129, Sections 2.9.1 and 4.7.1] or [2]. However, if the boundary $\partial\mathcal{O}$ is not regular enough, the situation becomes more complicated. It is our aim to consider here an example of a domain $\mathcal{O}$ with irregular boundary, in a sense made precise below.

Assume that $m \in \mathbb{N}$. Let B be a differential operator of order $\leq m$ with coefficients of $\mathcal{C}^\infty(\mathcal{O})$ class, i.e.

$$Bu = \sum_{|\alpha| \leq m} a_\alpha(\cdot) D^\alpha u \in \mathcal{D}'(\mathcal{O}), \quad u \in \mathcal{D}'(\mathcal{O}). \tag{2.3}$$

In other words, B is a linear operator in the space $\mathcal{D}'(\mathcal{O})$ of distributions on $\mathcal{O}$, defined by

$$(Bu, \varphi) := (u, B^*\varphi), \quad \varphi \in \mathcal{D}(\mathcal{O}),$$

where B^* is the formal adjoint operator of B defined by

$$\begin{aligned} B^*\varphi &= \sum_{|\alpha| \leq m} (-1)^{|\alpha|} D^\alpha \big(a_\alpha(\cdot)\varphi\big) \\ &= \sum_{|\alpha| \leq m} \sum_{\beta \leq \alpha} (-1)^{|\alpha|} \binom{\alpha}{\beta} \big(D^{\alpha-\beta} a_\alpha(\cdot)\big) D^\beta, \quad \varphi \in \mathcal{D}(\mathcal{O}). \end{aligned} \tag{2.4}$$

In what follows, the Laplace operator acting on the space of distributions $\mathcal{D}'(\mathcal{O})$ will be denoted by $\Delta_{\mathcal{O}}$, i.e.

$$\Delta_{\mathcal{O}} \colon \mathcal{D}'(\mathcal{O}) \to \mathcal{D}'(\mathcal{O}).$$

In the special case when $\mathcal{O} = \mathbb{R}^n$, we will use the notation $\Delta_{\mathbb{R}^n}$ or simply Δ:

$$\Delta_{\mathbb{R}^n} = \Delta : \mathcal{D}'(\mathbb{R}^n) \to \mathcal{D}'(\mathbb{R}^n).$$

We will say that $Bu \in L^p(\mathcal{O})$ if and only if there exists $v \in L^p(\mathcal{O})$ such that

$$(v, \varphi) = (u, B^*\varphi), \quad \varphi \in \mathcal{D}(\mathcal{O}).$$

The next definition forms a basis for our whole paper.

DEFINITION 2.2. Assume that $N \in \mathbb{N}^*$ and that $\mathcal{B} = \{B_1, \ldots, B_N\}$ is a finite set of differential operators of the form (2.3). Let us define $W^{\mathcal{B},p}(\mathcal{O})$ to be the space of all $u \in L^p(\mathcal{O})$ such that $B_i u \in L^p(\mathcal{O})$, $i = 1, \ldots, N$, i.e.

$$W^{\mathcal{B},p}(\mathcal{O}) := \big\{u \in L^p(\mathcal{O}) \colon B_i u \in L^p(\mathcal{O}),\ i = 1, \ldots, N\big\}. \tag{2.5}$$

The space $W^{\mathcal{B},p}(\mathcal{O})$ is endowed with the following norm:

$$\|u\|_{\mathcal{B},p} := \left\{ |u|^p_{L^p(\mathcal{O})} + \sum_{i=1}^N |B_i u|^p_{L^p(\mathcal{O})} \right\}^{\frac{1}{p}}.$$

The space $W^{\mathcal{B},p}(\mathcal{O})$ is the domain of a linear map $\mathbb{B}$ defined as follows:

$$D(\mathbb{B}) := W^{\mathcal{B},p} \subset L^p(\mathcal{O}),$$
$$\mathbb{B}u := (B_i u)_{i=1}^N \in L^p(\mathcal{O}, \mathbb{C}^N).$$

When the family $\mathcal{B} = \{B\}$ is a singleton, we will simply use the simpler notation $W^{B,p}(\mathcal{O})$.

EXAMPLE 2.3. The space $W^{k,p}(\mathcal{O})$, where $k \in \mathbb{N}$, is an example of the space $W^{\mathcal{B},p}(\mathcal{O})$, with $\mathcal{B} = \{D^\alpha : \alpha \in \mathbb{N}^n,\ |\alpha| \le k\}$. The space $W^{\Delta,p}(\mathcal{O}) := W^{\{\Delta\},p}(\mathcal{O})$ is another example of the space $W^{\mathcal{B},p}(\mathcal{O})$.

It is standard to prove that $W^{\mathcal{B},p}(\mathcal{O})$ is a Banach space. Moreover, if $\mathcal{B}_1 \subset \mathcal{B}_2$, then

$$W^{\mathcal{B}_2,p}(\mathcal{O}) \subset W^{\mathcal{B}_1,p}(\mathcal{O}) \tag{2.6}$$

and the corresponding embedding

$$W^{\mathcal{B}_2,p}(\mathcal{O}) \hookrightarrow W^{\mathcal{B}_1,p}(\mathcal{O})$$

is continuous. It may happen that $W^{\mathcal{B}_2,p}(\mathcal{O}) = W^{\mathcal{B}_1,p}(\mathcal{O})$ with equivalent norms.

In many cases, for example, if B is a uniformly elliptic operator of even order, i.e. $m = 2l$ for some $l \in \mathbb{N}^*$, and $\mathcal{O}$ is a bounded domain with a $\mathcal{C}^\infty$ boundary (subject to suitable boundary conditions) or if $\mathcal{O} = \mathbb{R}^n$, see [129], the space $W^{B,p}(\mathcal{O}) := W^{\{B\},p}(\mathcal{O})$ coincides with the classical Sobolev space $W^{m,p}(\mathcal{O})$ and the norms are equivalent. For example, we have

$$W^{\Delta,p}(\mathbb{R}^n) = W^{2,p}(\mathbb{R}^n) \quad \text{(with equivalent norms)}.$$

The last assertion fails in the case of $\mathcal{O} = \mathbb{R}^n \setminus \{a\}$ for a fixed point $a \in \mathbb{R}^n$. We will come back to this question further on in Lemma 3.5. However, in view of (2.6), we have

$$W^{2,p}\big(\mathbb{R}^n \setminus \{a\}\big) \subset W^{\Delta,p}\big(\mathbb{R}^n \setminus \{a\}\big).$$

For simplicity of exposition we will always take $a = 0$. Before we state our first result, let us introduce some further notation.

DEFINITION 2.4. Assume that $p \in (1, +\infty)$ and $\mathcal{O}$ is an open subset of $\mathbb{R}^n$. If $u \in L^p(\mathcal{O})$, then by $\tilde{u}$ we denote the equivalence class in $L^p(\mathbb{R}^n)$ of a function that is equal to 0 outside $\mathcal{O}$ and equal to u in $\mathcal{O}$. Thus, with a slight abuse of notation,

$$\tilde{u}(x) = \begin{cases} u(x) & \text{if } x \in \mathcal{O}, \\ 0 & \text{if } x \in \mathbb{R}^n \setminus \mathcal{O}. \end{cases}$$

Let us observe that if the closed set $\mathbb{R}^n \setminus \mathcal{O}$ is of Lebesgue measure 0, for example, when $\mathcal{O} = \mathbb{R}^n \setminus \{0\}$, then the map

$$\sim: L^p(\mathcal{O}) \ni u \mapsto \tilde{u} \in L^p(\mathbb{R}^n)$$

is an isometric isomorphism.

DEFINITION 2.5. Let $\mathcal{O}$ be an open subset of $\mathbb{R}^n$ and let T be a distribution on $\mathbb{R}^n$, i.e., $T \in \mathcal{D}'(\mathbb{R}^n)$. By $T_{\mathcal{O}}$ we denote the distribution on $\mathcal{O}$ which is the restriction of T to $\mathcal{O}$, i.e., $T_{\mathcal{O}} \in \mathcal{D}'(\mathcal{O})$ is defined by

$$(T_{\mathcal{O}}, \varphi) = (T, \widetilde{\varphi}) \quad \text{for } \varphi \in \mathcal{D}(\mathcal{O}).$$

Let us point out that if $i_{\mathcal{O}}: \mathcal{D}(\mathcal{O}) \hookrightarrow \mathcal{D}(\mathbb{R}^n)$ is the natural embedding and

$$i'_{\mathcal{O}}: \mathcal{D}'(\mathbb{R}^n) \to \mathcal{D}'(\mathcal{O})$$

is the adjoint map defined by

$$\big(i'_{\mathcal{O}}(T), \varphi\big) := \big(T, i_{\mathcal{O}}(\varphi)\big), \quad \varphi \in \mathcal{D}(\mathcal{O}),\ T \in \mathcal{D}'(\mathbb{R}^n),$$

then $T_{\mathcal{O}} = i'_{\mathcal{O}}(T)$.

Note that for $T \in \mathcal{D}'(\mathbb{R}^n)$,

$$\Delta_{\mathcal{O}}\big(i'_{\mathcal{O}}(T)\big) = i'_{\mathcal{O}}(\Delta T).$$

Note also the following trivial identity:

$$\Delta\big(i_{\mathcal{O}}(\varphi)\big) = i_{\mathcal{O}}(\Delta_{\mathcal{O}}\varphi), \quad \varphi \in \mathcal{D}(\mathcal{O}).$$

In particular, if $T = \tilde{u}$ with $u \in W^{\Delta,p}(\mathcal{O})$, then

$$\Delta_{\mathcal{O}} u = (\Delta\tilde{u})|_{\mathcal{O}}, \tag{2.7}$$

i.e.

$$(\Delta_{\mathcal{O}} u, \varphi) = \big((\Delta\tilde{u})|_{\mathcal{O}}, \varphi\big), \quad \varphi \in \mathcal{C}_0^\infty(\mathcal{O}).$$

Indeed, $\tilde{u}|_{\mathcal{O}} = u$ in the $\mathcal{D}'(\mathcal{O})$ sense.

3. THE SOBOLEV SPACE $W^{\Delta,p}(\mathbb{R}^n \setminus \{0\})$

From now on we will consider the case of

$$\mathcal{O} = \mathbb{R}^n \setminus \{0\}.$$

Let us recall that the spaces $W^{\Delta,p}(\mathbb{R}^n \setminus \{0\})$ and $W^{2,p}(\mathbb{R}^n \setminus \{0\})$, $p \in (1, +\infty)$, have been defined in Definition 2.2 and Example 2.3. It is obvious that the following restriction map

$$\pi\colon W^{\Delta,p}(\mathbb{R}^n) = W^{2,p}(\mathbb{R}^n) \ni u \mapsto u|_{\mathbb{R}^n\setminus\{0\}} \in W^{2,p}\big(\mathbb{R}^n \setminus \{0\}\big) \hookrightarrow W^{\Delta,p}\big(\mathbb{R}^n \setminus \{0\}\big) \tag{3.1}$$

is well defined, injective, and continuous.

We begin with the following two "local" properties of the space $W^{\Delta,p}(\mathbb{R}^n \setminus \{0\})$.

PROPOSITION 3.1. *Let $\widetilde{\mathcal{O}}$ be an open subset of $\mathbb{R}^n$ such that its closure $\overline{\overline{\widetilde{\mathcal{O}}}} \subset \mathbb{R}^n \setminus \{0\}$, and let $u \in W^{\Delta,p}(\mathbb{R}^n \setminus \{0\})$. Then*

$$u|_{\widetilde{\mathcal{O}}} \in W^{2,p}(\widetilde{\mathcal{O}}).$$

PROOF. Let us choose and fix $u \in W^{\Delta,p}(\mathbb{R}^n \setminus \{0\})$. Set

$$B(0,r) := \big\{x \in \mathbb{R}^n\colon |x| \le r\big\}.$$

Let us denote by $B^{\mathrm{c}}(0,r)$ the complement of $B(0,r)$ in $\mathbb{R}^n$. Then we can find a number $r > 0$ such that $\widetilde{\mathcal{O}} \subset B^{\mathrm{c}}(0,r)$. Since $\partial B^{\mathrm{c}}(0,r)$ is regular, we infer that $u \in W^{\Delta,p}(B^{\mathrm{c}}(0,r)) = W^{2,p}(B^{\mathrm{c}}(0,r))$, and the desired conclusion follows as

$$v \in W^{2,p}\big(B^{\mathrm{c}}(0,r)\big) \Longrightarrow v|_{\widetilde{\mathcal{O}}} \in W^{2,p}(\widetilde{\mathcal{O}}). \qquad \blacksquare$$

In the next result, we use the transformation $\sim$ introduced in Definition 2.4 with respect to the open set $\mathbb{R}^n \setminus \{0\} \subset \mathbb{R}^n$.

PROPOSITION 3.2. *Assume that $\phi \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ is such that $\phi = 1$ in a neighborhood of the origin 0. Assume that $u \in L^p(\mathbb{R}^n \setminus \{0\})$. Then the following two conditions are equivalent:*

(i) $u \in W^{\Delta,p}(\mathbb{R}^n \setminus \{0\})$,

(ii) $\phi u \in W^{\Delta,p}(\mathbb{R}^n \setminus \{0\})$ *and* $(1-\phi)\tilde{u} \in W^{2,p}(\mathbb{R}^n)$.

PROOF. The implication (ii)⇒(i) is obvious since the second condition in (ii) implies that the restriction of the function $(1-\phi)\tilde{u}$ to the open set $\mathbb{R}^n \setminus \{0\}$ is equal to $(1-\phi)u$. Therefore, since $W^{\Delta,p}(\mathbb{R}^n \setminus \{0\})$ is a vector space, u belongs to $W^{\Delta,p}(\mathbb{R}^n \setminus \{0\})$.

To prove the implication (i)⇒(ii) we assume that $u \in W^{\Delta,p}(\mathbb{R}^n \setminus \{0\})$. First, we will show that $(1-\phi)\tilde{u} \in W^{2,p}(\mathbb{R}^n) = W^{\Delta,p}(\mathbb{R}^n)$. Since obviously $(1-\phi)\tilde{u} \in L^p(\mathbb{R}^n)$, in view of [126, Proposition III.1.3], it is sufficient to prove that $\Delta((1-\phi)\tilde{u}) \in L^p(\mathbb{R}^n)$. It is easy to see that for every $w \in W^{2,p}(B^{\mathrm{c}}(0,r))$ and any $R > r$ there exists a function

$\widetilde{w} \in W^{2,p}(\mathbb{R}^n)$ such that $\widetilde{w}|_{B^c(0,R)} = w|_{B^c(0,R)}$. Therefore, we can find a function $v \in W^{2,p}(\mathbb{R}^n)$ such that

$$(1-\phi)\tilde{u} = (1-\phi)v.$$

Then

$$\begin{aligned}(3.2)\qquad \Delta\big((1-\phi)\tilde{u}\big) &= \Delta\big((1-\phi)v\big)\\ &= v\Delta(1-\phi) + (1-\phi)\Delta v + 2\sum_{i=1}^{n}\big(D_i(1-\phi)\big)(D_i v)\end{aligned}$$

in the weak sense, and all terms on the right-hand side of (3.2) belong to $L^p(\mathbb{R}^n)$.

Proposition 3.1 can be used in the proof that $\phi u \in W^{\Delta,p}(\mathbb{R}^n \setminus \{0\})$. Indeed, there exists a number $r > 0$ such that $\phi_{|B(0,r)}$ is constant and $u_{|B^c(0,r)} \in W^{2,p}(B^c(0,r))$. Then by a simple continuity argument

$$\begin{aligned}\Delta_{|\mathbb{R}^n\setminus\{0\}}(\phi u) &= u\Delta_{|\mathbb{R}^n\setminus\{0\}}\phi + \phi\Delta_{|\mathbb{R}^n\setminus\{0\}}u\\ &\quad + 2\times \mathbb{1}_{B^c(0,r)}\Bigg(\sum_{i=1}^{n} D_{i\,|B^c(0,r)}\phi_{|B^c(0,r)}\, D_{i\,|B^c(0,r)}u_{|B^c(0,r)}\Bigg)^{\sim},\end{aligned}$$

where $\mathbb{1}_\Gamma$ denotes the characteristic function of a set Γ and $f^{\sim}$, only in this part of the paper, is an arbitrary extension to $\mathbb{R}^n \setminus \{0\}$ of a function f defined on $B^c(0,r)$. ∎

We conclude this part by proving the following scaling invariance of the Sobolev spaces $W^{\Delta,p}(\mathbb{R}^n \setminus \{0\})$ and $W^{2,p}(\mathbb{R}^n \setminus \{0\})$, $p \in (1,+\infty)$.

PROPOSITION 3.3. *Assume that $p \in (1,+\infty)$ and $\lambda > 0$. Assume that $u \in W^{\Delta,p}(\mathbb{R}^n \setminus \{0\})$, resp. $u \in W^{2,p}(\mathbb{R}^n \setminus \{0\})$, and define u_λ by the following formula:*

$$u_\lambda(x) = u(\sqrt{\lambda}x), \quad x \in \mathbb{R}^n \setminus \{0\}.$$

Then $u_\lambda \in W^{\Delta,p}(\mathbb{R}^n \setminus \{0\})$, resp. $u_\lambda \in W^{2,p}(\mathbb{R}^n \setminus \{0\})$.

PROOF. We only prove the first case. Let us choose and fix $p \in (1,+\infty)$ and $\lambda > 0$. If $u \in W^{\Delta,p}(\mathbb{R}^n \setminus \{0\})$, then u and $\Delta_{\mathbb{R}^n\setminus\{0\}}u$ belong to $L^p(\mathbb{R}^n \setminus \{0\})$. Obviously, u_λ also belongs to this space. Moreover, (in the weak sense)

$$\Delta_{\mathbb{R}^n\setminus\{0\}}u_\lambda(x) = \lambda[\Delta_{\mathbb{R}^n\setminus\{0\}}u](\sqrt{\lambda}x), \quad x \in \mathbb{R}^n \setminus \{0\}.$$

Hence, also $\Delta_{\mathbb{R}^n\setminus\{0\}}u_\lambda \in L^p(\mathbb{R}^n \setminus \{0\})$ as claimed. The proof is complete. ∎

In what follows, by $W^{-2,p}(\mathbb{R}^n)$ we denote the space

$$W^{-2,p}(\mathbb{R}^n) = \big(W^{2,p^*}(\mathbb{R}^n)\big)',$$

where p^* is such that $1/p + 1/p^* = 1$. Note that for every $\psi \in L^p(\mathbb{R}^n)$, $\Delta\psi \in W^{-2,p}(\mathbb{R}^n)$. Finally, we understand that

$$\frac{n}{n-2} = +\infty \quad \text{if } n = 2 \quad \text{and} \quad \frac{n}{n-1} = +\infty \quad \text{if } n = 1.$$

LEMMA 3.4. *Assume that $n \in \mathbb{N}^*$ and $N \in \mathbb{N}$ and $p \in (1, +\infty)$. Assume that $c_\alpha \in \mathbb{C}$, for $|\alpha| \le N$, and*

$$\omega = \sum_{|\alpha| \le N} c_\alpha D^\alpha \delta_0. \tag{3.3}$$

Then the following assertions hold:

(i) *if $n \ge 3$ and $p \ge \frac{n}{n-2}$, then $\omega \in W^{-2,p}(\mathbb{R}^n)$ if and only if $c_\alpha = 0$ for each α,*

(ii) *if $n \ge 2$ and $\frac{n}{n-1} \le p < \frac{n}{n-2}$, then $\omega \in W^{-2,p}(\mathbb{R}^n)$ if and only if $c_\alpha = 0$ for each α such that $|\alpha| \ge 1$,*

(iii) *if $1 < p < \frac{n}{n-1}$, then $\omega \in W^{-2,p}(\mathbb{R}^n)$ if and only if $c_\alpha = 0$ for each α such that $|\alpha| \ge 2$.*

PROOF. In the proof we will use the following consequences of the Sobolev Embedding Theorem. Assume that $q \in (1, +\infty)$. Then

(a) $W^{2,q}(\mathbb{R}^n) \subset \mathcal{C}^1(\mathbb{R}^n)$ if and only if $q > n$;

(b) $W^{2,q}(\mathbb{R}^n) \subset \mathcal{C}(\mathbb{R}^n)$ if and only if $q > n/2$;

(c) $W^{2,q}(\mathbb{R}^n) \not\subset \mathcal{C}^2(\mathbb{R}^n)$.

By the above we deduce that if $p \in (1, +\infty)$, then the following assertions hold:

(d) if $D^\alpha \delta_0 \in W^{-2,p}(\mathbb{R}^n)$, then $|\alpha| \le 1$;

(e) $D^\alpha \delta_0 \in W^{-2,p}(\mathbb{R}^n)$ for some α: $|\alpha| = 1$ if and only if $\frac{1}{p} > \frac{n-1}{n}$, i.e. $p < \frac{n}{n-1}$;

(f) $\delta_0 \in W^{-2,p}(\mathbb{R}^n)$ if and only if $\frac{1}{p} > \frac{n-2}{n}$.

In each of the three cases (i)–(iii), the implication $\Leftarrow$ is a consequence of assertions (d)–(f) above. Therefore, it remains to prove, in every case (i)–(iii), the implication $\Rightarrow$. For this purpose let us choose and fix a tempered distribution ω of the form (3.3) belonging to the space $W^{-2,p}(\mathbb{R}^n)$. Let us set $\mathcal{I} = \{\alpha : c_\alpha \ne 0\}$. It is sufficient to prove that every component $c_\alpha D^\alpha \delta_0$ belongs to $W^{-2,p}(\mathbb{R}^n)$ as well. For this purpose we consider $\phi \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ such that $\phi(x) = 1$ in a neighborhood of 0. Set $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ and $\phi_\alpha(x) = x^\alpha \phi(x)$. Next, let $\alpha^k = (\alpha_1, \ldots, \alpha_k - 1, \ldots, \alpha_n)$ if $\alpha_k > 0$. Then for every $\psi \in \mathcal{C}_0^\infty(\mathbb{R}^n)$,

$$\begin{aligned} D^\alpha\big(\phi_\alpha(x)\psi(x)\big)\big|_{x=0} &= \alpha_1! \cdots \alpha_n! \psi(0), \\ D^\alpha\big(\phi_{\alpha^k}(x)\psi(x)\big)\big|_{x=0} &= \alpha! D_k \psi(0) \quad \text{if } \alpha_k > 0. \end{aligned}$$

Case (i). Assume that $p \geq \frac{n}{n-2}$ if $n \geq 3$. We will show that $\mathcal{I} = \emptyset$. If not, i.e. $\mathcal{I} \neq \emptyset$, we can choose $\alpha \in \mathcal{I}$ such that $|\alpha| \geq |\beta|$ for every $\beta \in \mathcal{I}$. Since the multiplication operator

$$W^{2,p^*}(\mathbb{R}^n) \ni \psi \mapsto \phi_\alpha \psi \in W^{2,p^*}(\mathbb{R}^n)$$

is continuous and by (f), $\delta_0 \notin (W^{2,p^*}(\mathbb{R}^n))' = W^{-2,p}(\mathbb{R}^n)$, and

$$\omega(\phi_\alpha \psi) = c_\alpha \alpha_1! \cdots \alpha_n! \delta_0(\psi),$$

we infer that $\omega \notin W^{-2,p}(\mathbb{R}^n)$, which is the desired contradiction.

Case (ii). Assume that $n \geq 2$ and $p \in [\frac{n}{n-1}, \frac{n}{n-2})$. We will show that $\mathcal{I} \cap \{\alpha : |\alpha| \geq 1\} = \emptyset$. Assume that $\alpha \in \mathcal{I}$ is such that $|\alpha| \geq |\beta|$ for every $\beta \in \mathcal{I}$ and that $|\alpha| \geq 1$. Let $\alpha_k \geq 1$. Then there are constants c and a_j, $j \neq k$, such that

$$\omega(\phi_{\alpha^k} \psi) = (-1)^{|\alpha|} c_\alpha \, \alpha! \, (D_k \psi)(0) + \sum_{j \neq k} a_j (D_j \psi)(0) + c \, \delta_0(\psi).$$

Since $\delta_0 \in W^{-2,p}(\mathbb{R}^n)$ and $c_\alpha \neq 0$, this contradicts the fact that no non-zero linear combination of $D_1 \delta_0, \ldots, D_n \delta_0$ belongs to $W^{-2,p}(\mathbb{R}^n)$.

Case (iii). Assume that $p \in (1, \frac{n}{n-1})$ if $n \geq 2$.

We will show that $\mathcal{I} \cap \{\alpha : |\alpha| \geq 2\} = \emptyset$. Using the above arguments with the function $\phi_{(\alpha^k)^j}$, we can easily show that $\mathcal{I} \cap \{\alpha : |\alpha| \geq 3\} = \emptyset$. Now we will show that $\mathcal{I} \cap \{\alpha : |\alpha| = 2\} = \emptyset$. Let us assume by contradiction that there exists $\alpha \in \mathcal{I} \cap \{\alpha : |\alpha| = 2\}$. Since, by (e) and (f), δ_0 and $D_k \delta_0$, $k = 1, \ldots, n$, belong to $W^{-2,p}(\mathbb{R}^n)$, we can assume that

$$\omega = \sum_{\alpha : |\alpha| = 2} c_\alpha D^\alpha \delta_0.$$

Changing variables, we can assume that

$$\omega = \sum_{j=1}^{n} a_j D_j^2 \delta_0.$$

Without loss of generality, we can assume that $a_1 \neq 0$. Let us choose a function $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^{n-1})$ such that each $D_k^2 \varphi(0) = 0$ for $k = 1, \ldots, n-1$ and $\varphi(x) = 1$ in a neighborhood of 0. Consider the following linear continuous map:

$$\Phi : W^{2,p^*}(\mathbb{R}) \ni \psi \mapsto \psi \otimes \varphi \in W^{2,p^*}(\mathbb{R}^n),$$

where $\psi \otimes \varphi$ is the tensor product of functions ψ and φ, i.e.

$$\psi \otimes \varphi(x_1, x_2, \ldots, x_n) = \psi(x_1) \varphi(x_2, \ldots, x_n).$$

Then we have

$$\omega\big(\Phi(\psi)\big) = a_1 D^2\delta_0(\psi), \quad \psi \in W^{2,p^*}(\mathbb{R}).$$

By assumptions, the functional on the left-hand side of the last identity is continuous on $W^{2,p^*}(\mathbb{R})$, while, by (c), the right-hand side is not. This contradiction completes the proof of case (iii). The proof of Lemma 3.4 is complete. ∎

The following result presents a characterization of the Sobolev spaces $W^{\Delta,p}(\mathbb{R}^n\setminus\{0\})$ depending on p and n. Here, by Δ, we understand the map $\Delta_{\mathbb{R}^n}$.

LEMMA 3.5. *Assume that $n\in\mathbb{N}^*$ and $p\in(1,+\infty)$. If $u\in W^{\Delta,p}(\mathbb{R}^n\setminus\{0\})\subset L^p(\mathbb{R}^n\setminus\{0\})$, $v := \Delta_{\mathbb{R}^n\setminus\{0\}}u \in L^p(\mathbb{R}^n\setminus\{0\})$ and, see Definition 2.4 above, $\tilde{u}\in L^p(\mathbb{R}^n)$ and*

$$\tilde{v} := \widetilde{\Delta_{\mathbb{R}^n\setminus\{0\}}u} \in L^p(\mathbb{R}^n),$$

then there exists a finite set of numbers $c_\alpha\in\mathbb{C}$ such that

$$-\Delta\tilde{u}+\tilde{v} = \begin{cases} c_0\delta_0 + \sum_{j=1}^n c_j D_j\delta_0 & \text{if } 1<p<\frac{n}{n-1},\\ c_0\delta_0 & \text{if } \frac{n}{n-1}\le p<\frac{n}{n-2},\\ 0 & \text{if } p\ge\frac{n}{n-2}. \end{cases} \tag{3.4}$$

PROOF. The proof of this lemma will be preceded by a somewhat more general result. Namely, let us assume that $p\in(1,+\infty)$ and $\mathcal{O}$ is an open subset of $\mathbb{R}^n$. If $u\in W^{\Delta,p}(\mathcal{O})$ and $v := \Delta_{\mathcal{O}}u\in L^p(\mathcal{O})$, then

$$\operatorname{supp}(-\Delta\tilde{u}+\tilde{v})\subset\partial(\mathcal{O}). \tag{3.5}$$

To do this let us point out that since $v\in L^p(\mathcal{O})$, the extension $\tilde{v}$ is a well-defined element of $L^p(\mathbb{R}^n)$. Similarly, $\tilde{u}\in L^p(\mathbb{R}^n)$ and so $\Delta\tilde{u}\in W^{-2,p}(\mathbb{R}^n)$. Hence, also $-\Delta\tilde{u}+\tilde{v}\in W^{-2,p}(\mathbb{R}^n)\subset\mathcal{D}'(\mathbb{R}^n)$. To prove (3.5), let us first consider an arbitrary $\varphi\in\mathcal{C}_0^\infty(\mathbb{R}^n)$ with support in $\mathcal{O}$. Then by (2.7) we have

$$(-\Delta\tilde{u},\varphi) + (\widetilde{\Delta_{\mathcal{O}}u},\varphi) = 0.$$

Next let us consider an arbitrary $\varphi\in\mathcal{C}_0^\infty(\mathbb{R}^n)$ with support in $\mathbb{R}^n\setminus\overline{\mathcal{O}}$. Then

$$(\Delta\tilde{u},\varphi) = (\tilde{u},\Delta\varphi) = 0$$

because $\operatorname{supp}(\Delta\varphi)\subset\mathbb{R}^n\setminus\overline{\mathcal{O}}$ and $\tilde{u}=0$ on $\mathbb{R}^n\setminus\overline{\mathcal{O}}$. For the same reason,

$$(\widetilde{\Delta_{\mathcal{O}}u},\varphi) = (\tilde{v},\varphi) = 0.$$

Thus, the proof of (3.5) is complete.

To see (3.4), let us choose and fix a $u \in W^{\Delta,p}(\mathbb{R}^n \setminus \{0\})$. Then $v := \Delta_{\mathbb{R}^n\setminus\{0\}}u \in L^p(\mathbb{R}^n \setminus \{0\})$, and with our convention concerning the map $\sim$, by (3.5) we have

$$\operatorname{supp}(-\Delta\tilde{u} + \tilde{v}) \subset \{0\}.$$

Then by [114, Theorem 6.25], there exist an $N \in \mathbb{N}$ and a set of complex numbers c_α with $|\alpha| \le N$ such that

$$-\Delta\tilde{u} + \tilde{v} = \sum_{|\alpha|\le N} c_\alpha D^\alpha \delta_0.$$

Since $\tilde{u} \in L^p(\mathbb{R}^n)$,

$$\tilde{v} - \sum_{|\alpha|\le N} c_\alpha D^\alpha \delta_0 \in W^{-2,p}(\mathbb{R}^n),$$

and the desired conclusion follows from Lemma 3.4. ∎

LEMMA 3.6. *In the framework of Lemma 3.5, the representation (3.4) is unique. In other words, for every $u \in W^{\Delta,p}(\mathbb{R}^n \setminus \{0\})$ there exists exactly one $\tilde{v} \in L^p(\mathbb{R}^n)$ and, depending on the parameters n and p, exactly one system of constants c_α such that (3.4) holds.*

PROOF. Our claim is a direct consequence of the following simple observation. If $f \in L^p(\mathbb{R}^n)$, $N \in \mathbb{N}$, and $c_\alpha \in \mathbb{C}$ for $|\alpha| \le N$ are such that

$$f - \sum_{|\alpha|\le N} c_\alpha D^\alpha \delta_0 = 0 \quad \text{in } \mathcal{D}'(\mathbb{R}^n),$$

then $f = 0$ and $c_\alpha = 0$ for all α. ∎

Define the *Bessel potential* G_n on $\mathbb{R}^n$ to be the function $\mathbb{R}^n \to [0, +\infty]$ given by

$$\begin{aligned} G_n(x) &:= \int_0^{+\infty} \mathrm{e}^{-\frac{|x|^2}{4t} - t} (4\pi t)^{-\frac{n}{2}}\, \mathrm{d}t \\ &= (4\pi)^{-\frac{n}{2}} 2^{\frac{n}{2}} |x|^{1-\frac{n}{2}} K_{\frac{n}{2}-1}\big(|x|\big) = (2\pi)^{-\frac{n}{2}} |x|^{1-\frac{n}{2}} K_{\frac{n}{2}-1}\big(|x|\big), \end{aligned} \tag{3.6}$$

where $K_{\frac{n}{2}-1}$ is the Macdonald function, or modified Bessel function of the third kind, see [73, entries 8.432.1 and 8.432.6] and [105, Theorem 7.2.1], defined by

$$\begin{aligned} K_\nu(z) &:= \int_0^{+\infty} \mathrm{e}^{-z\cosh u} \cosh(\nu u)\mathrm{d}u \\ &= \frac{1}{2}\left(\frac{z}{2}\right)^\nu \int_0^{+\infty} t^{-\nu-1}\, \mathrm{e}^{-t-\frac{z^2}{4t}}\, \mathrm{d}t, \quad |\arg z| < \frac{\pi}{2} \text{ or } \operatorname{Re} z = 0 \text{ and } \nu = 0. \end{aligned} \tag{3.7}$$

Note that if $n \ge 2$, then $\lim_{x\to 0} G_n(x) = +\infty$.

The functions K_ν for $\nu = -\frac{1}{2}$ and $\nu = \frac{1}{2}$ can be explicitly calculated.

PROPOSITION 3.7. *If $z \in \mathbb{C}$ satisfies the conditions listed in equality (3.7), then*

$$K_{-\frac{1}{2}}(z) = \left(\frac{\pi}{2z}\right)^{\frac{1}{2}} \mathrm{e}^{-z} = K_{\frac{1}{2}}(z).$$

PROOF. By the change of variables $t = \sigma^2$ in the following identity:

$$\int_0^{+\infty} \mathrm{e}^{-(\sigma^2 + c\sigma^{-2})} \mathrm{d}\sigma = \frac{\sqrt{\pi}}{2} \mathrm{e}^{-2\sqrt{c}}, \quad c > 0, \tag{3.8}$$

see, e.g., [133, pp. 233–234], with $\sqrt{c} = \frac{z}{2}$, we get

$$\begin{aligned} K_{-\frac{1}{2}}(z) &= \frac{1}{2}\left(\frac{z}{2}\right)^{-\frac{1}{2}} \int_0^{+\infty} t^{-\frac{1}{2}} \mathrm{e}^{-t - \frac{z^2}{4t}} \, \mathrm{d}t \\ &= 2 \times \frac{1}{2}\left(\frac{2}{z}\right)^{\frac{1}{2}} \int_0^{+\infty} \mathrm{e}^{-(\sigma^2 + \frac{z^2}{4\sigma^2})} \, \mathrm{d}\sigma = \left(\frac{2}{z}\right)^{\frac{1}{2}} \frac{\sqrt{\pi}}{2} \mathrm{e}^{-2\frac{z}{2}} = \left(\frac{\pi}{2z}\right)^{\frac{1}{2}} \mathrm{e}^{-z}. \end{aligned}$$

Using the change of variables $s = t^{-\frac{1}{2}}$ in (3.7), and then (3.8), we get

$$\begin{aligned} K_{\frac{1}{2}}(z) &= \frac{1}{2}\left(\frac{z}{2}\right)^{\frac{1}{2}} \int_0^{+\infty} t^{-\frac{3}{2}} \mathrm{e}^{-t - \frac{z^2}{4t}} \, \mathrm{d}t = 2 \times \frac{1}{2}\left(\frac{z}{2}\right)^{\frac{1}{2}} \int_0^{+\infty} \mathrm{e}^{-(s^{-2} + s^2 \frac{z^2}{4})} \, \mathrm{d}s \\ &= \left(\frac{z}{2}\right)^{\frac{1}{2}} \frac{2}{z} \int_0^{+\infty} \mathrm{e}^{-(\sigma^2 + \sigma^{-2} \frac{z^2}{4})} \, \mathrm{d}\sigma = \left(\frac{1}{2z}\right)^{\frac{1}{2}} 2\frac{\sqrt{\pi}}{2} \mathrm{e}^{-z} = \left(\frac{\pi}{2z}\right)^{\frac{1}{2}} \mathrm{e}^{-z}. \end{aligned}$$

■

Moreover, see [94, Theorem 6.23], the function G_n induces a unique tempered distribution on $\mathbb{R}^n$, and, as such, it satisfies

$$G_n - \Delta G_n = \delta_0 \quad \text{in } \mathcal{D}'(\mathbb{R}^n). \tag{3.9}$$

One can prove the above equality by using the Fourier transform, see also [28] and/or [126].

It follows from (3.9) that $\Delta G_n = G_n$ in $\mathcal{D}'(\mathbb{R}^n \setminus \{0\})$. Therefore, we have the following direct consequence of (3.9).

COROLLARY 3.8. *Assume that $n \in \mathbb{N}^*$ and $p \in (1, +\infty)$. If G_n denotes the restriction of the function G_n defined above to the open set $\mathbb{R}^n \setminus \{0\}$, then the following implication holds:*

$$G_n \in L^p\big(\mathbb{R}^n \setminus \{0\}\big) \implies G_n \in W^{\Delta,p}\big(\mathbb{R}^n \setminus \{0\}\big).$$

Obviously, by (2.5), the converse implication holds as well.

Recall that by $D_i = \frac{\partial}{\partial x_i}$ we mean the weak partial derivative. Finally, in Lemma 3.9 by G_n we understand the restriction of the previous G_n to the open set $\mathbb{R}^n \setminus \{0\}$.

LEMMA 3.9. *For any* $n \in \mathbb{N}^*$,

$$G_n = \Delta_{\mathbb{R}^n\setminus\{0\}} G_n \quad \text{pointwise on } \mathbb{R}^n \setminus \{0\} \text{ and in } \mathcal{D}'\big(\mathbb{R}^n \setminus \{0\}\big). \tag{3.10}$$

Moreover, if $p \in (1, +\infty)$, $i = 1, \dots, n$ *and* $j, k = 1, \dots, n$, $D_{jk} := D_j D_k$, *then*

$$G_n \in L^p\big(\mathbb{R}^n \setminus \{0\}\big) \quad \text{if and only if } n = 1, 2 \text{ or } n \geq 3 \text{ and } p < \frac{n}{n-2}, \tag{3.11}$$

$$D_i G_n \in W^{\Delta,p}\big(\mathbb{R}^n \setminus \{0\}\big) \quad \text{if and only if } n = 1 \text{ or } n \geq 2 \text{ and } p < \frac{n}{n-1}, \tag{3.12}$$

$$D_{jk} G_n \in W^{\Delta,p}\big(\mathbb{R}^n \setminus \{0\}\big) \quad \text{if and only if } n = 1. \tag{3.13}$$

PROOF. By definition, $(I - \Delta)G_n$ is a distribution with the support $\{0\}$. On the other hand, $(I - \Delta)G_n$ restricted to $\mathbb{R}^n \setminus \{0\}$ is a $\mathcal{C}^\infty$-function. Therefore, $(I - \Delta)G_n(x) = 0$ for $x \neq 0$ and hence (3.10) follows as required.

Claim (3.11) follows easily from the well-known asymptotics of G_n. Indeed, for $n = 1$, G_1 is bounded and continuous on $\mathbb{R}$, and for $n \geq 1$, G_n decays exponentially at ∞. Finally, one has

$$G_n(x) = \begin{cases} \Gamma\big(\frac{n-2}{2}\big) \frac{1}{4\pi^{n/2}|x|^{n-2}} \big(1 + \mathrm{o}(1)\big) & \text{if } n \geq 3, \\ \frac{1}{2\pi} \ln \frac{1}{|x|} \big(1 + \mathrm{o}(1)\big) & \text{if } n = 2, \end{cases} \quad \text{as } |x| \to 0. \tag{3.14}$$

Since the distributional operators Δ and D_i commute and in view of the earlier exact formula for G_n, to prove assertion (3.12), it is enough to show that for $n \geq 2$ and $i = 1, \dots, n$,

$$D_i G_n \in L^p\big(\mathbb{R}^n \setminus \{0\}\big) \quad \text{if and only if } p < \frac{n}{n-1}.$$

From the definition (3.6) of the function G_n it follows that

$$\big|\nabla G_n(x)\big| = \frac{|x|}{2} \int_0^{+\infty} \mathrm{e}^{-\frac{\pi|x|^2}{t} - \frac{t}{4\pi}} \, t^{-\frac{n+2}{2}} \, \mathrm{d}t, \quad x \in \mathbb{R}^n \setminus \{0\},$$

and the result follows by applying the asymptotic formula above once again. ∎

The following result follows from identity (3.9) and Lemma 3.9.

COROLLARY 3.10. *If* $n \in \mathbb{N}^*$ *and* $p \in (1, +\infty)$, *then* $G_n \notin W^{\Delta,p}(\mathbb{R}^n)$. *Moreover, if* $n \geq 2$ *and* $1 < p < \frac{n}{n-2}$, *then the spaces* $W^{\Delta,p}(\mathbb{R}^n \setminus \{0\})$ *and* $W^{2,p}(\mathbb{R}^n \setminus \{0\})$ *do not coincide.*

PROOF. $G_n \in W^{\Delta,p}(\mathbb{R}^n)$ means that $G_n, \Delta G_n \in L^p(\mathbb{R}^n)$. This contradicts equality (3.9). Assume that $n \geq 2$ and $1 < p < \frac{n}{n-2}$. It is enough to observe that by Lemma 3.9, $G_n \in W^{\Delta,p}(\mathbb{R}^n \setminus \{0\})$, whereas $G_n \notin W^{2,p}(\mathbb{R}^n \setminus \{0\})$ by the last part of the lemma. ∎

In the particular cases of $n = 1$ and $n = 3$, the potential G_n is given in an explicit form, see below. Moreover, all the calculations from the proof of Lemma 3.9 can be performed in a direct manner.

LEMMA 3.11. *We have*

$$G_1(x) = \frac{1}{2}e^{-|x|}, \quad x \in \mathbb{R}, \tag{3.15}$$

and

$$G_3(x) = \frac{1}{4\pi|x|}e^{-|x|}, \quad x \in \mathbb{R}^3, \tag{3.16}$$

G_3 *is of* $\mathcal{C}^\infty$*-class on* $\mathbb{R}^3 \setminus \{0\}$, *and*

$$G_3(x) = \Delta_{\mathbb{R}^3\setminus\{0\}} G_3(x), \quad x \in \mathbb{R}^3 \setminus \{0\}.$$

Moreover,

(i) $G_3 \in W^{\Delta,p}(\mathbb{R}^3 \setminus \{0\})$, *or equivalently* $G_3 \in L^p(\mathbb{R}^3)$, *if and only if* $p \in (1, 3)$;

(ii) $D_i G_3 \in W^{\Delta,p}(\mathbb{R}^3 \setminus \{0\})$, *or equivalently* $D_i G_3 \in L^p(\mathbb{R}^3)$, *if and only if* $p \in (1, \frac{3}{2})$.

Consequently, if $p \in [\frac{3}{2}, 3)$, *then* $G_3 \in W^{\Delta,p}(\mathbb{R}^3 \setminus \{0\})$ *but* $G_3 \notin W^{1,p}(\mathbb{R}^3 \setminus \{0\})$.

PROOF. Using the second identity of Proposition 3.7 and (3.6) for $n = 3$ we get (3.16). Using the first identity of Proposition 3.7 and (3.6) for $n = 1$ we get (3.15).

Note that

$$\begin{aligned} D_i G_3(x) &= -\frac{x_i}{|x|^2} G_3(x) - \frac{x_i}{|x|} G_3(x) \\ &= -x_i \left[\frac{1}{|x|^2} + \frac{1}{|x|} \right] G_3(x). \end{aligned} \tag{3.17}$$

Therefore,

$$\Delta_{\mathbb{R}^3\setminus\{0\}} G_3(x) = G_3(x) \left[-3\left(\frac{1}{|x|^2} + \frac{1}{|x|} \right) + \left(\frac{1}{|x|} + 1 \right)^2 + \frac{2}{|x|^2} + \frac{1}{|x|} \right] = G_3(x),$$

which gives the first claim. The remaining parts follow easily from the first one and (3.16) and (3.17). ∎

We finish this part with the following result, which will be used later. It can be proved by direct calculations if $n = 3$, and, for $n = 2$, by the Plancherel formula and direct calculations.

LEMMA 3.12. *We have*

$$\|G_2\|_{L^2}^2 = \frac{1}{4\pi}, \tag{3.18}$$

$$\|G_3\|_{L^2}^2 = \frac{1}{8\pi}. \tag{3.19}$$

3.1. Representation theorem

Let us assume that $u \in W^{\Delta,p}(\mathbb{R}^n \setminus \{0\})$ and let us put

$$v := \Delta_{\mathbb{R}^n\setminus\{0\}} u \in L^p\big(\mathbb{R}^n \setminus \{0\}\big).$$

Clearly, $\tilde{u} - \tilde{v} \in L^p(\mathbb{R}^n)$. Applying the operator $(1-\Delta)^{-1}$, which is an isomorphism on the space $\mathcal{S}'(\mathbb{R}^n)$ of tempered distributions, to the following version of equality (3.4):

$$\tilde{u} - \Delta\tilde{u} = c_0\delta_0 + \sum_{j=1}^n c_j D_j\delta_0 + \tilde{u} - \tilde{v},$$

we obtain

$$\tilde{u} = c_0(1-\Delta)^{-1}\delta_0 + \sum_{j=1}^n c_j D_j(1-\Delta)^{-1}\delta_0 + f$$

with $f := (1-\Delta)^{-1}(\tilde{u} - \tilde{v})$.

Since $\tilde{u} - \tilde{v} \in L^p(\mathbb{R}^n)$, the lift property of the Sobolev spaces, see [129, Theorem 2.3.4], yields that $f \in W^{2,p}(\mathbb{R}^n)$. Moreover, in the sense of tempered distributions,

$$(1-\Delta)^{-1}\delta_0 = G_n.$$

Clearly, $\frac{p}{p-1} > n$ if and only if $p < \frac{n}{n-1}$, and $2\frac{p}{p-1} > n$ if and only if $p < \frac{n}{n-2}$. Therefore, we have the following consequence of Lemmata 3.5, 3.6, and 3.9.

THEOREM 3.13. *Assume that $n \in \mathbb{N}^*$ and $p \in (1, +\infty)$.*

(i) *If $n \geq 1$ and $1 < p < \frac{n}{n-1}$, then every function $u \in W^{\Delta,p}(\mathbb{R}^n \setminus \{0\})$ can be uniquely written in the following form:*

$$u = \sum_{|\alpha|\leq 1} c_\alpha D^\alpha G_n + f, \tag{3.20}$$

where $c_\alpha \in \mathbb{C}$ and $f \in W^{2,p}(\mathbb{R}^n)$. Conversely, if $u \in \mathcal{S}'(\mathbb{R}^n)$ has representation (3.20) with $c_\alpha \in \mathbb{C}$ and $f \in W^{2,p}(\mathbb{R}^n)$, then $u \in W^{\Delta,p}(\mathbb{R}^n \setminus \{0\})$. Moreover, the mapping

$$W^{\Delta,p}\big(\mathbb{R}^n \setminus \{0\}\big) \ni u \mapsto \big((c_\alpha)_{|\alpha|\leq 1}, f\big) \in \mathbb{C}^{1+n} \times W^{2,p}(\mathbb{R}^n)$$

is an isomorphism (and hence a bijection).

(ii) *If* $n \geq 2$ *and* $\frac{n}{n-1} \leq p < \frac{n}{n-2}$, *then every function* $u \in W^{\Delta,p}(\mathbb{R}^n \setminus \{0\})$ *can be uniquely written in the following form:*

$$u = c_0 G_n + f, \tag{3.21}$$

where $c_0 \in \mathbb{C}$ *and* $f \in W^{2,p}(\mathbb{R}^n)$. *Conversely, if* $u \in \mathcal{S}'(\mathbb{R}^n)$ *has representation* (3.21) *with* $c_0 \in \mathbb{C}$ *and* $f \in W^{2,p}(\mathbb{R}^n)$, *then* $u \in W^{\Delta,p}(\mathbb{R}^n \setminus \{0\})$. *Moreover, the mapping*

$$W^{\Delta,p}\big(\mathbb{R}^n \setminus \{0\}\big) \ni u \mapsto (c_0, f) \in \mathbb{C} \times W^{2,p}(\mathbb{R}^n)$$

is an isomorphism.

(iii) *If* $n \geq 3$ *and* $p \geq \frac{n}{n-2}$, *then* $u \in W^{\Delta,p}(\mathbb{R}^n \setminus \{0\})$ *if and only if* $\tilde{u} \in W^{2,p}(\mathbb{R}^n)$, *i.e. the map*

$$W^{\Delta,p}\big(\mathbb{R}^n \setminus \{0\}\big) \ni u \mapsto \tilde{u} \in W^{2,p}(\mathbb{R}^n) \tag{3.22}$$

is an isomorphism. If we replace the space $W^{2,p}(\mathbb{R}^n)$ *on the right-hand side of* (3.22) *by the space* $W^{\Delta,p}(\mathbb{R}^n)$, *the map above will be an isometric isomorphism.*

In some sense, the representation of $u \in W^{\Delta,p}(\mathbb{R}^n \setminus \{0\})$ does not depend on p. The symbol $\tilde{v}$ used here has nothing to do with the symbol $\tilde{v}$ used to denote the extension by zero. In fact, we have the following result.

THEOREM 3.14. *Assume that* $p, \tilde{p} \in (1, +\infty)$. *If* $v \in W^{2,p}(\mathbb{R}^n)$ *and* $\tilde{v} \in W^{2,\tilde{p}}(\mathbb{R}^n)$, $c_0, \tilde{c}_0, c_k, \tilde{c}_k \in \mathbb{C}$ *for* $k = 1, \ldots, n$, *and* $u \in W^{\Delta,p}(\mathbb{R}^n \setminus \{0\}) \cap W^{\Delta,\tilde{p}}(\mathbb{R}^n \setminus \{0\})$ *satisfies*

$$\begin{aligned} u &= c_0 G_n + \sum_{k=1}^{n} c_k D_k G_n + v, \\ u &= \tilde{c}_0 G_n + \sum_{k=1}^{n} \tilde{c}_k D_k G_n + \tilde{v}, \end{aligned} \tag{3.23}$$

then

$$c_0 = \tilde{c}_0, \quad c_k = \tilde{c}_k, \quad v = \tilde{v}.$$

PROOF. Without loss of generality, we can assume that $\tilde{p} \leq p$. Let $B(0, R)$ be the ball in $\mathbb{R}^n$ with center at 0 and radius R, and let $B(0, R)^c$ be its complement. Clearly, $W^{2,p}(B(0, R)) \hookrightarrow W^{2,\tilde{p}}(B(0, R))$. Since G_n decays exponentially fast at infinity, so does $v - \tilde{v}$. We infer that $v|_{B(0,R)^c} \in W^{2,\tilde{p}}(B(0, R)^c)$ as $\tilde{v}|_{B(0,R)^c} \in W^{2,\tilde{p}}(B(0, R)^c)$. Thus, $v \in W^{2,\tilde{p}}(\mathbb{R}^n)$. Since also (by assumptions) $\tilde{v} \in W^{2,\tilde{p}}(\mathbb{R}^n)$, by the uniqueness of the representation of $u \in W^{\Delta,\tilde{p}}(\mathbb{R}^n \setminus \{0\})$ of the form (3.23), see Theorem 3.13, we conclude the proof. ∎

COROLLARY 3.15. *Assume that $n \geq 3$ and $p \geq \frac{n}{n-2}$ and $\tilde{p} > 1$. If*

$$u \in W^{\Delta,p}\big(\mathbb{R}^n \setminus \{0\}\big) \cap W^{\Delta,\tilde{p}}\big(\mathbb{R}^n \setminus \{0\}\big),$$

then $\tilde{u} \in W^{2,p}(\mathbb{R}^n) \cap W^{2,\tilde{p}}(\mathbb{R}^n)$.

PROOF. The claim follows from Theorem 3.13 part (iii) and Theorem 3.14. ∎

COROLLARY 3.16. *Assume that $n \geq 2$ and $1 < p < \frac{n}{n-2}$. If $u \in W^{\Delta,p}(\mathbb{R}^n \setminus \{0\})$, then the following two conditions are equivalent:*

(i) $u \in W^{2,p}(\mathbb{R}^n \setminus \{0\})$,

(ii) $\tilde{u} \in W^{2,p}(\mathbb{R}^n)$.

PROOF. Let us choose and fix $n \geq 2$, $1 < p < \frac{n}{n-2}$ and $u \in W^{\Delta,p}(\mathbb{R}^n \setminus \{0\})$. Then, see Lemma 3.9, neither G_n nor $D_k G_n$, for $k = 1, \ldots, n$, belongs to the space $W^{2,p}(\mathbb{R}^n \setminus \{0\})$. Therefore, if also $u \in W^{2,p}(\mathbb{R}^n \setminus \{0\})$, by Theorem 3.13 parts (i) and (ii), we infer that $u = f$ for some $f \in W^{2,p}(\mathbb{R}^n)$. Hence, $\tilde{u} \in W^{2,p}(\mathbb{R}^n)$ as claimed.

The implication (ii)⇒(i) is trivial. The proof is complete. ∎

Note that $2 < \frac{n}{n-1}$ if and only if $n = 1$, and, for $n > 2$, $2 < \frac{n}{n-2}$ if and only if $n \leq 3$. Therefore, we have the following consequence of Theorem 3.13.

COROLLARY 3.17. (i) *If $p \in (1, +\infty)$, then every function $u \in W^{\Delta,p}(\mathbb{R} \setminus \{0\})$ can be uniquely written in the following form:*

$$u = c_0 G_1 + c_1 DG_1 + f, \tag{3.24}$$

where $c_0, c_1 \in \mathbb{C}$ and $f \in W^{2,p}(\mathbb{R})$. Moreover, the mapping

$$W^{\Delta,p}\big(\mathbb{R} \setminus \{0\}\big) \ni u \mapsto (c_0, c_1, f) \in \mathbb{C}^2 \times W^{2,p}(\mathbb{R})$$

is an isomorphism.

(ii) *Assume that $n = 2, 3$. Then every function $u \in W^{\Delta,2}(\mathbb{R}^n \setminus \{0\})$ can be uniquely written in the following form:*

$$u = c_0 G_n + f,$$

where $c_0 \in \mathbb{C}$ and $f \in W^{2,2}(\mathbb{R}^n)$. Moreover, the mapping

$$W^{\Delta,2}\big(\mathbb{R}^n \setminus \{0\}\big) \ni u \mapsto (c_0, f) \in \mathbb{C} \times W^{2,2}(\mathbb{R}^n)$$

is an isomorphism.

(iii) *Assume that $n \geq 4$. Then $W^{\Delta,2}(\mathbb{R}^n \setminus \{0\}) \equiv W^{2,2}(\mathbb{R}^n)$.*

Case	n	p	Relevant formula
(i)	$n \geq 1$	$1 < p < \frac{n}{n-1}$	(3.20)
(ii)	$n \geq 2$	$\frac{n}{n-1} \leq p < \frac{n}{n-2}$	(3.21)
(iii)	$n \geq 3$	$p \geq \frac{n}{n-2}$	(3.22)

TABLE 1. The three cases of the pair (n, p) used in Definitions 3.18 and 3.19.

In the definitions below we shall use Table 1.

The above results, especially Corollary 3.17, lead us to the following definitions in which the case number refers to Table 1.

DEFINITION 3.18. Assume that $n \in \mathbb{N}^*$, $p \in (1, +\infty)$. Let $u \in W^{\Delta,p}(\mathbb{R}^n \setminus \{0\})$. In cases (i) and (ii) of Table 1, we denote by $\Pi_p u$ the unique function $f \in W^{2,p}(\mathbb{R}^n)$ in the appropriate identity from Theorem 3.13: either (3.20) or (3.21). In case (iii), by $\Pi_p u$ we denote the unique function $\tilde{u} \in W^{2,p}(\mathbb{R}^n)$ from (3.22). We will call Π_p the *natural projection from* $W^{\Delta,p}(\mathbb{R}^n \setminus \{0\})$ *onto* $W^{2,p}(\mathbb{R}^n)$.

DEFINITION 3.19. Assume that $n \in \mathbb{N}^*$, $p \in (1, +\infty)$. In case (ii) of Table 1 we define the following map:

$$\xi_p \colon W^{\Delta,p}\big(\mathbb{R}^n \setminus \{0\}\big) \to \mathbb{C}, \quad \xi_p(u) = c_0 \quad \text{if } c_0 G_n = u - \Pi_p(u)|_{\mathbb{R}^n \setminus \{0\}}.$$

In case (i) we define the following map:

$$\xi_p \colon W^{\Delta,p}\big(\mathbb{R}^n \setminus \{0\}\big) \to \mathbb{C}^{1+n}, \quad \xi_p(u) = (c_0, c_1, \ldots, c_n)$$
$$\text{if } c_0 G_n + \sum_{j=1}^{n} c_j D_j G_n = u - \Pi_p(u)|_{\mathbb{R}^n \setminus \{0\}}.$$

Next let us put

$$\xi_p(u) = \big(\xi_{0,p}(u), \xi_{1,p}(u), \ldots, \xi_{n,p}(u)\big).$$

In case (iii), ξ_p is the map into the zero-dimensional space $\mathbb{C}^0 := \{0\}$, i.e.

$$\xi_p \colon W^{\Delta,p}\big(\mathbb{R}^n \setminus \{0\}\big) \ni u \mapsto 0 \in \mathbb{C}^0.$$

The following straightforward result is a consequence of Theorem 3.13.

PROPOSITION 3.20. *The map Π_p is linear and bounded from $W^{\Delta,p}(\mathbb{R}^n \setminus \{0\})$ to $W^{2,p}(\mathbb{R}^n)$. The map ξ_p is linear and bounded on $W^{\Delta,p}(\mathbb{R}^n \setminus \{0\})$. Moreover, for every $f \in \mathrm{Range}(\Pi_p)$,*

$$\Delta_{\mathbb{R}^n \setminus \{0\}} f = \Delta f. \tag{3.25}$$

Finally, for every $u \in W^{\Delta,p}(\mathbb{R}^n \setminus \{0\})$,

$$u - \Pi_p u \in \mathcal{Y}_{n,p} := \begin{cases} \operatorname{linspan}\left\{D_1 G_n, \dots, D_n G_n\right\} & \text{if } n \geq 1 \text{ and } 1 < p < \frac{n}{n-1}, \\ \operatorname{linspan}\{G_n\} & \text{if } n \geq 2 \text{ and } \frac{n}{n-1} \leq p < \frac{n}{n-2}, \\ \{0\} & \text{if } n \geq 3 \text{ and } p \geq \frac{n}{n-2}, \end{cases}$$

where the difference $u - \Pi_p u$ *is understood in the space* $L^p(\mathbb{R}^n)$ *to which both* u *and* $\Pi_p u$ *belong.*

Proof. We only need to prove (3.25) which follows from the inclusion

$$\operatorname{Range}(\Pi_p) \subset W^{2,p}(\mathbb{R}^n). \qquad \blacksquare$$

Example 3.21. Assume that $n = 1$ and $p \in (1, +\infty)$. Then every function $u \in W^{\Delta,p}(\mathbb{R} \setminus \{0\})$ can be identified with an element of $W^{2,p}(-\infty, 0) \oplus W^{2,p}(0, +\infty)$ and therefore determines four numbers $u(0^{\pm})$ and $u'(0^{\pm})$. The numbers c_0, c_1 and the values $f_0 := f(0)$, $f_1 := f'(0)$ in the representation (3.24) of u can be found by solving the following non-degenerate 4×4 system of linear equations:

$$\begin{cases} c_0 - c_1 + 2f_0 = 2u(0^+), \\ c_0 + c_1 + 2f_0 = 2u(0^-), \\ -c_0 + c_1 + 2f_1 = 2u'(0^+), \\ c_0 + c_1 + 2f_1 = 2u'(0^-). \end{cases}$$

This system can be solved giving

$$\begin{cases} c_0 = \left[u'(0^-) - u'(0^+)\right], \\ c_1 = \left[u(0^-) - u(0^+)\right], \\ f_0 = \frac{1}{2}\left[u(0^+) + u(0^-) + u'(0^+) - u'(0^-)\right], \\ f_1 = \frac{1}{2}\left[u(0^+) - u(0^-) + u'(0^+) + u'(0^-)\right]. \end{cases}$$

Example 3.22. Assume again that $n = 1$. Assume that $\beta > 0$ and let us consider the function u defined by $u(x) = G_1(\beta x)$, where G_1 has been defined in (3.15). Then

$$u(0^+) = \frac{1}{2} = u(0^-), \quad u'(0^+) = -\frac{\beta}{2}, \quad u'(0^-) = \frac{\beta}{2}.$$

Thus, the constants c_0, c_1 and the values $f_0 := f(0)$, $f_1 := f'(0)$ in the representation (3.24) of u, see Example 3.21, are

$$\begin{cases} c_0 = u'(0^-) - u'(0^+) = \beta & c_1 = 0, \\ f_0 = \frac{1}{2}(1 - \beta) & f_1 = 0. \end{cases} \tag{3.26}$$

Hence,

$$u(x) = G_1(\beta x) = \beta G_1(x) + f(x),$$

where

$$f(x) = G_1(\beta x) - \beta G_1(x) = \frac{1}{2}[\mathrm{e}^{-\beta|x|} - \beta \mathrm{e}^{-|x|}].$$

Note that $f \in W^{2,p}(-\infty, +\infty)$, $f(0) = \frac{1}{2}(1-\beta)$, and $f'(0) = 0$, which agrees with (3.26).

REMARK 3.23. Assume that $n = 3$ and $p \in (\frac{3}{2}, 3) = (\frac{n}{n-1}, \frac{n}{n-2})$ or $n = 2$ and $p \in (1, +\infty) = (\frac{n-1}{n-1}, +\infty)$. Then $2 - \frac{n}{p} > 0$ and therefore, by the Sobolev Embedding Theorem, every $f \in W^{2,p}(\mathbb{R}^n)$ is continuous.

Let $u \in W^{\Delta,p}(\mathbb{R}^n \setminus \{0\})$ have the representation

$$u = c_0 G_n + f, \quad c_0 \in \mathbb{C}, \ f \in W^{2,p}(\mathbb{R}^n). \tag{3.27}$$

Then since $G_n(x) \to +\infty$ as $x \to 0$ and f is continuous at 0, we infer that

$$c_0 = \lim_{x \to 0} \frac{u(x)}{G_n(x)}. \tag{3.28}$$

Moreover,

$$f(0) = \lim_{x \to 0} \big[u(x) - c_0 G_n(x)\big]. \tag{3.29}$$

Assume now that $n \geq 4$ and $p < \frac{n}{n-2}$. Then, $2 - \frac{n}{p} < 2 - n\frac{n-2}{n} = 4 - n \leq 0$. The function f in the representation (3.27) of the function u is in general neither bounded nor continuous. In some cases, however, we are able to calculate c_0, see the proof of Lemma 3.28 below.

REMARK 3.24. If $n \geq 2$ and $1 < p < \frac{n}{n-2}$, then the spaces $W^{\Delta,p}(\mathbb{R}^n \setminus \{0\})$ and $W^{2,p}(\mathbb{R}^n \setminus \{0\})$ do not coincide, see Corollary 3.10. On the other hand, if $n \geq 3$ and $p \geq \frac{n}{n-2}$, then by part (iii) of Theorem 3.13 we have

$$W^{2,p}(\mathbb{R}^n) \equiv W^{\Delta,p}\big(\mathbb{R}^n \setminus \{0\}\big).$$

Since obviously,

$$W^{2,p}(\mathbb{R}^n) \subset W^{2,p}\big(\mathbb{R}^n \setminus \{0\}\big) \quad \text{and} \quad W^{2,p}\big(\mathbb{R}^n \setminus \{0\}\big) \subset W^{\Delta,p}\big(\mathbb{R}^n \setminus \{0\}\big),$$

we infer that

$$W^{2,p}(\mathbb{R}^n) \equiv W^{2,p}\big(\mathbb{R}^n \setminus \{0\}\big) \equiv W^{\Delta,p}\big(\mathbb{R}^n \setminus \{0\}\big).$$

Note that if $n = 1$ and $p > 1$, then

$$\begin{aligned} W^{\Delta,p}\big(\mathbb{R}\setminus\{0\}\big) &\equiv W^{\Delta,p}(0,+\infty)\oplus W^{\Delta,p}(-\infty,0)\\ &\equiv W^{2,p}(0,+\infty)\oplus W^{2,p}(-\infty,0)\\ &\equiv W^{2,p}\big(\mathbb{R}\setminus\{0\}\big). \end{aligned}$$

REMARK 3.25. For an arbitrary open set $\mathcal{O}\subset\mathbb{R}^n$ let $W^{2,p}(\mathcal{O})$ be defined as in Section 2 while $\widetilde{W}^{2,p}(\mathcal{O})$ be defined as in [129, Definition 4.2.1/1], i.e. $u\in\widetilde{W}^{2,p}(\mathcal{O})$ if and only if there exists $v\in W^{2,p}(\mathbb{R}^n)$ such that $u = v|_{\mathcal{O}}$. Let us recall that the norm in $\widetilde{W}^{2,p}(\mathcal{O})$ is defined as

$$|u|_{2,p} = \inf\big\{|v|_{W^{2,p}(\mathbb{R}^n)}\colon v\in W^{2,p}(\mathbb{R}^n),\ u = v|_{\mathcal{O}}\big\}.$$

As we have mentioned before, it is well known that if $\mathcal{O}$ is a bounded domain in $\mathbb{R}^n$ satisfying the cone condition, then the two spaces $W^{2,p}(\mathcal{O})$ and $\widetilde{W}^{2,p}(\mathcal{O})$ coincide, see, e.g., [129, Theorem 4.2.4]. However, if $\mathcal{O} = \mathbb{R}^n\setminus\{0\}$, then we only have $\widetilde{W}^{2,p}(\mathcal{O})\subset W^{2,p}(\mathcal{O})$. Indeed, if $n = 1$, then as in Example 3.21 the space $W^{\Delta,p}(\mathbb{R}\setminus\{0\})$ can be identified with $W^{2,p}(-\infty,0)\oplus W^{2,p}(0,+\infty)$ and hence it is strictly larger than $\widetilde{W}^{2,p}(\mathbb{R}\setminus\{0\})$.

DEFINITION 3.26. Given $\lambda > 0$, we set

$$G_{n,\lambda}(x) = G_n(\sqrt{\lambda}x),\quad x\in\mathbb{R}^n\setminus\{0\}. \tag{3.30}$$

REMARK 3.27. Note that $G_{n,\lambda}$ is a solution of equation $(\lambda-\Delta)G_{n,\lambda} = \delta_0$ in $\mathbb{R}^n$ only in dimension $n = 2$. If $n = 3$, the solution of this equation in the distributional sense is associated with the function $(4\pi|x|)^{-1}\mathrm{e}^{-\sqrt{\lambda}|x|}$.

LEMMA 3.28. *The following assertions hold.*

(i) *If $n = 1$, then $G_{1,\lambda} = \sqrt{\lambda}G_1 + f$ with $f\in W^{2,p}(\mathbb{R})$ for every $p\in(1,+\infty)$ and*

$$f(0) = \frac{1-\sqrt{\lambda}}{2}.$$

(ii) *If $n = 2$, then $G_{2,\lambda} = G_2 + f$ with $f\in W^{2,p}(\mathbb{R}^2)$ for every $p\in(1,+\infty)$ and*

$$f(0) = -\frac{\ln\lambda}{4\pi}.$$

(iii) *If $n = 3$, then $G_{3,\lambda} = \lambda^{-\frac{1}{2}}G_3 + f$ with $f\in W^{2,p}(\mathbb{R}^3)$ for every $p\in(1,3)$ and*

$$f(0) = \frac{1-\sqrt{\lambda}}{4\pi\sqrt{\lambda}}.$$

(iv) *If $n\geq 4$, then $G_{n,\lambda} = \lambda^{\frac{2-n}{2}}G_n + f$ with $f\in W^{2,p}(\mathbb{R}^n)$ for every $p\in(1,\frac{n}{n-2})$.*

PROOF OF LEMMA 3.28 (i). Let us choose and fix $\lambda > 0$. Recall from formula (3.15) in Lemma 3.11 that, for $x \in \mathbb{R}$,

$$G_1(x) = \tfrac{1}{2}\,\mathrm{e}^{-|x|}, \quad G_{1,\lambda}(x) = G_1(\sqrt{\lambda}\,x) = \tfrac{1}{2}\,\mathrm{e}^{-\sqrt{\lambda}\,|x|},$$

so that both functions are bounded and continuous on $\mathbb{R}$, decay exponentially at infinity, and belong to $W^{\Delta,p}(\mathbb{R}\setminus\{0\})$ for every $p \in (1,+\infty)$. Set

$$f := G_{1,\lambda} - \sqrt{\lambda}\,G_1.$$

Since the first two terms of the power series expansion of f are

$$f(x) = \frac{1-\sqrt{\lambda}}{2} + \frac{\lambda-\sqrt{\lambda}}{4}\,|x|^2 + \mathrm{O}\big(|x|^3\big),$$

the result follows. ∎

PROOF OF LEMMA 3.28 (iii). Let us choose and fix $\lambda > 0$. By Lemma 3.11, $G_3 \in W^{\Delta,p}(\mathbb{R}^3\setminus\{0\})$ for every $p \in (1,3)$. Hence, by Proposition 3.3, also $G_{3,\lambda} \in W^{\Delta,p}(\mathbb{R}^3\setminus\{0\})$ for every $p \in (1,3)$. Thus, by Theorems 3.13 (ii) and 3.14 there exist a unique $c_0 \in \mathbb{C}$ and a unique f belonging to $W^{2,p}(\mathbb{R}^3)$ for every $p \in (1,3)$ such that

$$G_{3,\lambda} = c_0 G_3 + f.$$

Thus, by formulae (3.28) and (3.29) in Remark 3.23 we infer that

$$c_0 = \lim_{x\to 0}\frac{G_{3,\lambda}(x)}{G_3(x)} = \lim_{x\to 0}\frac{\frac{1}{4\pi\sqrt{\lambda}\,|x|}\,\mathrm{e}^{-\sqrt{\lambda}\,|x|}}{\frac{1}{4\pi\,|x|}\,\mathrm{e}^{-|x|}} = \lambda^{-\frac{1}{2}}.$$

Moreover,

$$\begin{aligned} f(0) &= \lim_{x\to 0}\big[G_{3,\lambda}(x) - \lambda^{-\frac{1}{2}}G_3(x)\big] \\ &= \lim_{x\to 0}\left[\frac{1}{4\pi\sqrt{\lambda}|x|}\mathrm{e}^{-\sqrt{\lambda}|x|} - \frac{1}{4\pi\sqrt{\lambda}|x|}\mathrm{e}^{-|x|}\right] \\ &= \frac{1-\sqrt{\lambda}}{4\pi\sqrt{\lambda}}. \end{aligned}$$

The proof is complete. ∎

PROOF OF LEMMA 3.28 (ii) and (iv). The proof in these cases is similar to the proof in the case (iii). We only need to use Lemma 3.9 instead of Lemma 3.11. ∎

4. A Green formula (or Stokes Theorem) on $\mathbb{R}^n \setminus \{0\}$

Let us now make a small detour. We have defined G_n as the unique element of $\mathcal{S}'(\mathbb{R}^n)$ satisfying

$$\big(G_n, (1-\Delta)\varphi\big) = \varphi(0) \quad \text{for every } \varphi \in \mathcal{S}(\mathbb{R}^n). \tag{4.1}$$

We will now show that the above identity is valid for a larger class of test functions.

Lemma 4.1. *Assume that $p, q \in (1, +\infty)$ are conjugate exponents; i.e., they satisfy*

$$\frac{1}{p} + \frac{1}{q} = 1. \tag{4.2}$$

Then the following hold.

(a) *If $n = 1, 2$ or $n \geq 3$ and $1 < q < \frac{n}{n-2}$, equivalently, $p > \frac{n}{2}$, then $G_n \in L^q(\mathbb{R}^n)$, $W^{2,p}(\mathbb{R}^n) \hookrightarrow \mathcal{C}(\mathbb{R}^n)$, and*

$$\big(G_n, (1-\Delta)\varphi\big) = \varphi(0), \quad \varphi \in W^{2,p}(\mathbb{R}^n). \tag{4.3}$$

(b) *If $n = 1$, then $DG_1 \in L^q(\mathbb{R})$, $W^{2,p}(\mathbb{R}) \hookrightarrow \mathcal{C}^1(\mathbb{R})$ and*

$$\big(DG_1, (1-\Delta)\varphi\big) = -D\varphi(0), \quad \varphi \in W^{2,p}(\mathbb{R}). \tag{4.4}$$

(c) *If $2 \leq n < p$ (which implies that $q < \frac{n}{n-1}$), then $D_i G_n \in L^q(\mathbb{R}^n)$, $i = 1, \ldots, n$, $W^{2,p}(\mathbb{R}^n) \hookrightarrow \mathcal{C}^1(\mathbb{R}^n)$, and*

$$\big(D_i G_n, (1-\Delta)\varphi\big) = -D_i\varphi(0), \quad \varphi \in W^{2,p}(\mathbb{R}^n). \tag{4.5}$$

Proof. Let us choose and fix $p, q \in (1, +\infty)$ such that $\frac{1}{p} + \frac{1}{q} = 1$. We now prove part (a) of the lemma. Let $p > \max\{1, \frac{n}{2}\}$. Then $G_n \in L^q(\mathbb{R}^n)$ by (3.11) in Lemma 3.9. Therefore, the map

$$W^{2,p}(\mathbb{R}^n) \ni \varphi \mapsto \big(G_n, (1-\Delta)\varphi\big) \in \mathbb{C}$$

is continuous and linear. Since $W^{2,p}(\mathbb{R}^n) \hookrightarrow \mathcal{C}(\mathbb{R}^n)$ by the Sobolev Embedding Theorem, the map

$$W^{2,p}(\mathbb{R}^n) \ni \varphi \mapsto \varphi(0) \in \mathbb{C}$$

is also continuous and linear. Since by (4.1) these maps coincide on a dense subset $\mathcal{S}(\mathbb{R}^n)$ of $W^{2,p}(\mathbb{R}^n)$, equality (4.3) follows.

For part (b), assume additionally that $n = 1$. Then, by equality (3.12) in Lemma 3.9, $DG_1 \in L^q(\mathbb{R})$. Moreover, since $2 - \frac{1}{p} > 1$, $W^{2,p}(\mathbb{R}) \hookrightarrow \mathcal{C}^1(\mathbb{R})$ as required. Therefore, equality (4.4) can be proved in exactly the same manner as equality (4.3) was proved above.

We now prove part (c) of the lemma. If $2 \le n < p$, then $q < \frac{n}{n-1}$, and again by equality (3.12) in Lemma 3.9, the function $D_i G_n \in L^q(\mathbb{R}^n)$. Since, by assumption, $2 - \frac{n}{p} > 1$, we infer that $W^{2,p}(\mathbb{R}^n) \hookrightarrow \mathcal{C}^1(\mathbb{R}^n)$ as required. Hence, equality (4.5) can be proved in exactly the same manner as we proved earlier equality (4.3). ∎

Our objective in this section is to prove the following Stokes-type theorem (or Green formula) in the domain $\mathcal{O} = \mathbb{R}^n \setminus \{0\}$. As above let $p, q > 1$ be such that $\frac{1}{p} + \frac{1}{q} = 1$. Let us define the following sesquilinear form:

$$\mathcal{E}(u, v) := (\Delta_{\mathbb{R}^n \setminus \{0\}} u, v) - (u, \Delta_{\mathbb{R}^n \setminus \{0\}} v),$$

for $u \in W^{\Delta,p}(\mathbb{R}^n \setminus \{0\})$ and $v \in W^{\Delta,q}(\mathbb{R}^n \setminus \{0\})$, where, by definition, $\Delta_{\mathbb{R}^n \setminus \{0\}} u \in L^p(\mathbb{R}^n \setminus \{0\})$, $\Delta_{\mathbb{R}^n \setminus \{0\}} v \in L^q(\mathbb{R}^n \setminus \{0\})$, and $(\cdot, \cdot)$ is the pairing between $L^p(\mathbb{R}^n \setminus \{0\})$ and $L^q(\mathbb{R}^n \setminus \{0\})$;

$$(f, g) = \int_{\mathbb{R}^n \setminus \{0\}} f(x)\overline{g(x)}\mathrm{d}x, \quad f \in L^p\big(\mathbb{R}^n \setminus \{0\}\big),\ g \in L^q\big(\mathbb{R}^n \setminus \{0\}\big).$$

Assume that u and v have the following representation:

$$u = a_0 G_n + \sum_{i=1}^{n} a_i D_i G_n + f, \quad v = b_0 G_n + \sum_{i=1}^{n} b_i D_i G_n + g, \tag{4.6}$$

where $f \in W^{2,p}(\mathbb{R}^n)$ and $g \in W^{2,q}(\mathbb{R}^n)$ and $a_i, b_i, i = 0, \dots, n$, are complex numbers.

Since $\Delta_{\mathbb{R}^n \setminus \{0\}} G_n = G_n$, we have

$$\mathcal{E}(G_n, G_n) = (\Delta_{\mathbb{R}^n \setminus \{0\}} G_n, G_n) - (G_n, \Delta_{\mathbb{R}^n \setminus \{0\}} G_n) = 0.$$

Next, for $f \in W^{2,p}(\mathbb{R}^n)$ and $g \in W^{2,q}(\mathbb{R}^n)$, we have

$$\begin{aligned} \mathcal{E}(f, g) &= (\Delta_{\mathbb{R}^n \setminus \{0\}} f, g) - (f, \Delta_{\mathbb{R}^n \setminus \{0\}} g) = (\Delta f, g) - (f, \Delta g) \\ &= \int_{\mathbb{R}^n} \big[\Delta f(x)\overline{g(x)} - f(x)\overline{\Delta g(x)}\big]\mathrm{d}x = 0. \end{aligned}$$

Since

$$\Delta_{\mathbb{R}^n \setminus \{0\}} D_i G_n = D_i G_n,$$

as soon as it is well defined, we have

$$\begin{aligned} \mathcal{E}(D_i G_n, G_n) &= (\Delta_{\mathbb{R}^n \setminus \{0\}} D_i G_n, G_n) - (D_i G_n, \Delta_{\mathbb{R}^n \setminus \{0\}} G_n) \\ &= (D_i G_n, G_n) - (D_i G_n, G_n) = 0. \end{aligned}$$

Similarly, we can prove that

$$\begin{aligned} \mathcal{E}(G_n, D_j G_n) &= 0, \\ \mathcal{E}(D_i G_n, D_j G_n) &= 0. \end{aligned}$$

Therefore, we deduce the following identity:

$$\mathcal{E}(u,v) = a_0\overline{g(0)} - \sum_{i=1}^{n} a_i\,\overline{D_i g(0)} - f(0)\overline{b_0} + \sum_{i=1}^{n} D_i f(0)\overline{b_i}, \tag{4.7}$$

if u and v are of the form (4.6).

This identity implies the following Stokes-type theorem in which we consider all possible cases: the large-p cases follow only by symmetry.

THEOREM 4.2. *Assume that $n \in \mathbb{N}$, $p, q \in (1, +\infty)$ are such that $\frac{1}{p} + \frac{1}{q} = 1$, $u \in W^{\Delta,p}(\mathbb{R}^n \setminus \{0\})$ and $v \in W^{\Delta,q}(\mathbb{R}^n \setminus \{0\})$. Then the following assertions hold.*

(a) *If $n = 1$, then there exists a unique 4-tuple (a_0, a_1, b_0, b_1) of complex numbers such that u and v have the following representations:*

$$u = a_0 G_1 + a_1 DG_1 + f, \quad v = b_0 G_1 + b_1 DG_1 + g,$$

and

$$\mathcal{E}(u,v) = a_0\overline{g(0)} - a_1\overline{Dg(0)} - f(0)\overline{b_0} + Df(0)\overline{b_1}.$$

(b) *If $n = 2$ and $1 < p < \frac{n}{n-1} = 2$, then $q \in (2, +\infty) = (\frac{n}{n-1}, \frac{n}{n-2})$ and there exists a unique 4-tuple $(a_0, a_{1,0}, a_{0,1}, b_0)$ of complex numbers such that*

$$u = a_0 G_2 + a_{1,0} D_1 G_2 + a_{0,1} D_2 G_2 + f, \quad v = b_0 G_2 + g,$$

and

$$\mathcal{E}(u,v) = a_0\overline{g(0)} - a_{1,0}\overline{D_1 g(0)} - a_{0,1}\overline{D_2 g(0)} - f(0)\overline{b_0}.$$

(c) *If $n = 2$ and $p = 2 = q$, then there exists a unique pair (a_0, b_0) of complex numbers such that*

$$u = a_0 G_2 + f, \quad v = b_0 G_2 + g,$$

and

$$\mathcal{E}(u,v) = a_0\overline{g(0)} - f(0)\overline{b_0}.$$

(d) *If $n = 3$ and $1 < p < \frac{n}{n-1} = \frac{3}{2}$, then $\frac{n}{n-2} = 3 < q$ and there exists a unique 4-tuple (a_0, a_1, a_2, a_3) of complex numbers such that*

$$u = a_0 G_3 + \sum_{i=1}^{3} a_i D_i G_3 + f, \quad v = g,$$

and

$$\mathcal{E}(u,v) = a_0\overline{g(0)} - \sum_{i=1}^{3} a_i\,\overline{D_i g(0)}.$$

(e) *If* $n = 3$ *and* $\frac{n}{n-1} = \frac{3}{2} = p$, *then* $3 = q$ *and there exists a unique complex number* a_0 *such that*

$$u = a_0 G_3 + f, \quad v = g,$$

and

$$\mathcal{E}(u, v) = a_0 \overline{g(0)}.$$

(f) *If* $n = 3$ *and* $\frac{n}{n-1} = \frac{3}{2} < p < 3 = \frac{n}{n-2}$, *then* $\frac{n}{n-1} = \frac{3}{2} < q < 3$ *and there exists a unique pair* (a_0, b_0) *of complex numbers such that*

$$u = a_0 G_3 + f, \quad v = b_0 G_3 + g,$$

and

$$\mathcal{E}(u, v) = a_0 \overline{g(0)} - f(0)\overline{b_0}.$$

(g) *If* $n > 3$ *and* $\frac{n}{n-1} \leq p < \frac{n}{n-2}$, *then* $q > n/2 \geq \frac{n}{n-2}$ *and there exists a unique complex number* a_0 *such that*

$$u = a_0 G_n + f, \quad v = g,$$

and

$$\mathcal{E}(u, v) = a_0 \overline{g(0)}.$$

(h) *If* $n > 3$ *and* $1 < p < \frac{n}{n-1}$, *then* $q > n > \frac{n}{n-2}$ *and there exists a unique* $(n+1)$*-tuple* $(a_0, a_1, \ldots, a_n)$ *of complex numbers such that*

$$u = a_0 G_n + \sum_{i=1}^{n} a_i D_i G_n + f, \quad v = g,$$

and

$$\mathcal{E}(u, v) = a_0 \overline{g(0)} - \sum_{i=1}^{n} a_i \overline{D_i g(0)}.$$

PROOF. For every part, from (a) to (h), we apply Theorem 3.13. ∎

5. NON-UNIQUENESS OF EXTENSIONS OF A CERTAIN CLASS

Below we reformulate [107, Theorem 1.33, p. 46] in a slightly more general fashion. Moreover, we provide its full proof. Let us recall that the *resolvent set* $\rho(A)$ of a densely defined linear operator $(A, D(A))$ on a Banach space X is the set of all $\lambda \in \mathbb{C}$ such that $(\lambda I - A)^{-1} \in \mathcal{L}(X)$. Recall from the sign convention fixed in Section 1.2 that the operators A occurring below are minus Laplacians, so that the resolvent of interest is $(\lambda I + A)^{-1}$ with $\lambda > 0$, i.e. $-\lambda \in \rho(A)$. Finally, a set $D_0 \subset D(A)$ is called a *core* of A if D_0 is dense in $D(A)$ endowed with the graph norm.

THEOREM 5.1. *Assume that X is a complex Banach space and $(A, D(A))$ is a closed densely defined operator in X such that $\rho(A) \neq \emptyset$. Assume that $D_0 \subset D(A)$ is a subspace and let us put*

$$D(A_{\min}) := D_0 \quad \text{and} \quad A_{\min} u = Au \quad \text{for } u \in D_0. \tag{5.1}$$

(i) *If D_0 is a core for A, then A is the only extension B of the operator $(A_{\min}, D_0)$ such that*

$$\rho(B) \cap \rho(A) \neq \emptyset. \tag{5.2}$$

(ii) *If D_0 is not a core for A, then there exist infinitely many extensions B of the operator $A_{\min}$ which satisfy condition (5.2).*

PROOF. Assume that D_0 is a core for A. Suppose that B is an extension of the operator $(A_{\min}, D_0)$ for which condition (5.2) holds. Then by [114, Proposition 2.14] the graph of the operator $(\lambda I - B)^{-1}$ is closed (in X^2) and so also the graph of the operator B is closed and hence B is a closed operator.[3] On the other hand, since D_0 is a core for A, we deduce that A is the closure of $A_{\min}$. Hence, $A \subset B$. Since B satisfies condition (5.2), we can choose and fix $\lambda \in \rho(B) \cap \rho(A)$. Then $(\lambda I - A)\colon D(A) \to X$ and $(\lambda I - B)\colon D(B) \to X$ are bijections. Since $A \subset B$, we have $\lambda I - A \subset \lambda I - B$. Since they are bijections, we have $\lambda I - A = \lambda I - B$ and so $A = B$. The proof of (i) is complete.

Our proof of the second part differs from the proof provided in [107]. Let us choose and fix $\lambda \in \rho(A)$. Then $\lambda I - A\colon D(A) \to X$ is a linear bijection. We equip $D(A)$ with the norm induced by the map $\lambda I - A$. Therefore, $\lambda I - A\colon D(A) \to X$ is a linear isomorphism.

Assume that D_0 is not a core for A. Then by the Hahn–Banach theorem, see, e.g., [114, Theorem 3.2], there exists a linear and bounded map

$$\phi\colon D(A) \to \mathbb{C} \quad \text{such that } \phi_{|D_0} = 0 \text{ and } \|\phi\|_{\mathcal{L}(D(A),\mathbb{C})} = 1.$$

Let $f \in D(A)$ be such that

$$0 \neq \phi(f) = |f|_X < \left\|(\lambda I - A)^{-1}\right\|^{-1}_{\mathcal{L}(X,D(A))}.$$

Define the linear operator C_f by

$$C_f\colon D(A) \ni u \mapsto \phi(u) f \in D(A) \subset X.$$

[3] This standard argument implies that if an operator $(A, D(A))$ has non-empty resolvent set, then it is closed.

Let us note that C_f is a linear and bounded (with respect to the graph-norm) operator and

$$\|C_f\|_{\mathcal{L}(D(A),X)} \le |f|_X \|\phi\|_{\mathcal{L}(D(A),\mathbb{C})} < \big\|(\lambda I - A)^{-1}\big\|^{-1}_{\mathcal{L}(X,D(A))}.$$

Therefore, $\lambda I - A - C_f$ is invertible. Indeed, since the operator $I - C_f(\lambda I - A)^{-1}$ is invertible, we have

$$\begin{aligned}(\lambda I - A - C_f)^{-1} &= \big[\big(I - C_f(\lambda I - A)^{-1}\big)(\lambda I - A)\big]^{-1} \\ &= (\lambda I - A)^{-1}\big(I - C_f(\lambda I - A)^{-1}\big)^{-1}.\end{aligned}$$

Put $B_f = C_f + A$, $D(B_f) = D(A)$. Then it follows from the above that

(i) $B_f(u) = A_{\min}u$ if $u \in D_0$ so that $A_{\min} \subset B_f$;

(ii) $B_f \ne A$ since $f \ne 0$;

(iii) $\lambda \in \rho(B_f)$ so that B_f satisfies condition (5.2). ∎

REMARK 5.2. Let us observe that if D_0 is not a core for A, then D_0 may still be dense in X. For example, assume that $p > \frac{n}{2}$, $X = L^p(\mathbb{R}^n)$, $D_0 = \mathcal{C}_0^\infty(\mathbb{R}^n \setminus \{0\})$, and $D(A) = W^{2,p}(\mathbb{R}^n)$, $A := -\Delta$. Then it follows from Theorem B.5 below that the set D_0 is not dense in the Banach space $W^{2,p}(\mathbb{R}^n)$, but obviously it is dense in the Banach space $L^p(\mathbb{R}^n)$.

We think it will be enlightening and useful to formulate [107, Theorem 1.33] in a form closer to the needs of the present paper.

THEOREM 5.3. *Assume that X is a complex Banach space and $(A, D(A))$ is an infinitesimal generator of a C_0-semigroup on X. Assume that $D_0 \subset D(A)$ is a subspace and let us define an operator $A_{\min}$ by* (5.1).

(i) *If D_0 is a core for A, then $(A, D(A))$ is the only extension $(B, D(B))$ of the operator $(A_{\min}, D_0)$ which is an infinitesimal generator of a C_0-semigroup on X.*

(ii) *If D_0 is not a core for A, then there exist infinitely many extensions $(B, D(B))$ of the operator $(A_{\min}, D_0)$ each of which is an infinitesimal generator of a C_0-semigroup on X.*

Let us define a linear unbounded operator $A_{\min}$ by

$$D(A_{\min}) = \mathcal{C}_0^\infty\big(\mathbb{R}^n \setminus \{0\}\big) =: D_0, \quad A_{\min}u := -\Delta u \quad \text{for } u \in D(A_{\min}).$$

Our next theorem deals with the extensions of the operator $A_{\min}$. In fact, the result is a consequence of Theorem 5.1. Note that $A_{\min}$ is the usual minus Laplace operator as in the paper [6]. To simplify the formulation of the result below, let us introduce the following definition. In accordance with the sign convention fixed in Section 1.2, we say

that a linear operator A on a Banach space X is a *negative generator* of a C_0-semigroup on X if and only if the operator $-A$ is the infinitesimal generator of a C_0-semigroup on X.

THEOREM 5.4. *Let $n \in \mathbb{N}$ and $p \in (1, +\infty)$. Assume that $A_{0,p}$ is the Friedrichs extension of the operator $A_{\min}$ in the Banach space $L^p(\mathbb{R}^n)$, i.e.*

$$D(A_{0,p}) = W^{2,p}(\mathbb{R}^n), \quad A_{0,p}u := -\Delta u \quad \text{for } u \in D(A_{0,p}). \tag{5.3}$$

Then the following hold.

- *If $n \geq 3$ and $p \leq \frac{n}{2}$, then $A_{0,p}$ is the unique extension of $A_{\min}$ in the class of negative generators of C_0-semigroups on the Banach space $L^p(\mathbb{R}^n)$.*
- *If $p > \frac{n}{2}$, then, in the class of negative generators of C_0-semigroups on the Banach space $L^p(\mathbb{R}^n)$, there exist infinitely many operators A which are extensions of the operator $A_{\min}$. In particular, if $n = 1, 2$, then $A_{0,p}$ is always a non-unique extension of $A_{\min}$ within the class above.*

REMARK 5.5. The operator $A_{0,p}$ in the Banach space $L^p(\mathbb{R}^n)$ introduced above is also denoted, to emphasize its dependence on the space dimension n, by $-\Delta_{n,p}$; see Section 1.2.

PROOF. Let us begin the proof by the observation that not only $D(A_{0,p}) = W^{2,p}(\mathbb{R}^n)$ but also the standard norm on $W^{2,p}(\mathbb{R}^n)$ and the graph norm on $D(A_{0,p})$ are equivalent.

Assume that $p > \frac{n}{2}$. Then by Theorem B.5 we infer that the set D_0 is not dense in the Banach space $W^{2,p}(\mathbb{R}^n)$. Thus, in view of the definition (5.3) of the operator $A_{0,p}$ we infer that the set D_0 is not dense in the Banach space $D(A_{0,p})$ and therefore the set D_0 is not a core of $A_{0,p}$. Hence, it follows from the second part of Theorem 5.1 that there exist infinitely many extensions of $A_{\min}$, of the form $A_{0,p} + C$, where $C\colon D(A_{0,p}) \to D(A_{0,p})$ is bounded with respect to the graph norm.

Assume next that $p \leq \frac{n}{2}$. Then by Theorem B.5 it follows that the set D_0 is dense in the Banach space $W^{2,p}(\mathbb{R}^n)$. Thus, in view of the definition (5.3) of the operator $A_{0,p}$ we infer that the set D_0 is a core of $A_{0,p}$, and hence, the desired conclusion follows directly from the first part of Theorem 5.1. ∎

REMARK 5.6. If we are interested in the existence of non-trivial extensions of the operator $A_{\min}$ in both spaces $L^p(\mathbb{R}^n)$ and $L^q(\mathbb{R}^n)$, where $p \in (1, +\infty)$ and $q \in (1, +\infty)$ are conjugate exponents, i.e. satisfy condition (4.2), then by the above Theorem 5.4 we would have that

$$\frac{1}{p} < \frac{2}{n} \quad \text{and} \quad \frac{1}{q} < \frac{2}{n}. \tag{5.4}$$

Therefore, in view of (4.2), the dimension n would have to satisfy $1 < \frac{4}{n}$, i.e. $n \leq 3$.

REMARK 5.7. In the notation of Theorem 5.4, assume now that $n \leq 3$ and p and q are conjugate and satisfy (5.4). A natural question arises: can we find non-trivial bounded operators $C_p\colon D(A_{0,p}) = W^{2,p}(\mathbb{R}^n) \to W^{2,p}(\mathbb{R}^n)$ and $C_q\colon D(A_{0,q}) = W^{2,q}(\mathbb{R}^n) \to W^{2,q}(\mathbb{R}^n)$ such that $A_{0,p} + C_p$, respectively, $A_{0,q} + C_q$, is an extension of $A_{\min}$ in $L^p(\mathbb{R}^n)$ and $L^q(\mathbb{R}^n)$, respectively?

REMARK 5.8. Another natural question is whether the second part of Theorem 5.4 remains true when the role of the operator $A_{0,p}$ is played by extensions *other than the Friedrichs extension* of the operator $A_{\min}$? The answers to these questions and to the questions we raised in this and the previous remark are presently not known to us.

6. SOME DENSITY RESULTS

Let us recall that the Banach spaces $W^{\Delta,p}(\mathbb{R}^n \setminus \{0\})$ and $W^{\Delta,p}(\mathbb{R}^n) = W^{2,p}(\mathbb{R}^n)$ have been defined in Definition 2.2 and that, according to (3.1), the following restriction map

$$\pi\colon W^{\Delta,p}(\mathbb{R}^n) \ni u \mapsto u|_{\mathbb{R}^n\setminus\{0\}} \in W^{\Delta,p}\big(\mathbb{R}^n \setminus \{0\}\big)$$

is a well-defined and bounded linear map. Although there are domains $\mathcal{O}$ in $\mathbb{R}^n$ such that the map

$$\pi_{\mathcal{O}}\colon W^{\Delta,p}(\mathbb{R}^n) \ni u \mapsto u|_{\mathcal{O}} \in W^{\Delta,p}(\mathcal{O})$$

is not injective, the map $\pi = \pi_{\mathbb{R}^n\setminus\{0\}}$ is injective. In fact, $\pi_{\mathcal{O}}$ is injective if the Lebesgue measure of the complement set $\mathbb{R}^n \setminus \mathcal{O}$ is equal to zero. The injectivity of the map π can be interpreted as a statement that $W^{\Delta,p}(\mathbb{R}^n) \subset W^{\Delta,p}(\mathbb{R}^n \setminus \{0\})$. In fact, by part (iii) of Theorem 3.13, $W^{\Delta,p}(\mathbb{R}^n) = W^{\Delta,p}(\mathbb{R}^n \setminus \{0\})$ if $n \geq 3$ and $p \geq \frac{n}{n-2}$. Moreover, by parts (i) and (ii) of Theorem 3.13, we can identify the space $W^{\Delta,p}(\mathbb{R}^n)$ with a closed subspace of $W^{\Delta,p}(\mathbb{R}^n \setminus \{0\})$ of codimension 1 or $n+1$, depending on whether $p \in [\frac{n}{n-1}, \frac{n}{n-2})$ or $p \in (1, \frac{n}{n-1})$.

DEFINITION 6.1. Let $W_0^{2,p}(\mathbb{R}^n \setminus \{0\})$ and $W_0^{\Delta,p}(\mathbb{R}^n \setminus \{0\})$ be the closures of the space $\mathcal{C}_0^\infty(\mathbb{R}^n \setminus \{0\})$ in the space $W^{2,p}(\mathbb{R}^n \setminus \{0\})$ and in $W^{\Delta,p}(\mathbb{R}^n \setminus \{0\})$, respectively.

REMARK 6.2. Since, as justified above, the space $W^{\Delta,p}(\mathbb{R}^n)$ is a closed subspace of $W^{\Delta,p}(\mathbb{R}^n \setminus \{0\})$ and $\mathcal{C}_0^\infty(\mathbb{R}^n \setminus \{0\})$ is a subset of the former space, the closure of $\mathcal{C}_0^\infty(\mathbb{R}^n \setminus \{0\})$ in $W^{\Delta,p}(\mathbb{R}^n)$ is equal to the closure of $\mathcal{C}_0^\infty(\mathbb{R}^n \setminus \{0\})$ in $W^{2,p}(\mathbb{R}^n)$. Hence,

$$\begin{aligned} W_0^{\Delta,p}\big(\mathbb{R}^n \setminus \{0\}\big) &= \text{the closure of } \mathcal{C}_0^\infty\big(\mathbb{R}^n \setminus \{0\}\big) \text{ in the space } W^{\Delta,p}(\mathbb{R}^n) \\ &= \text{the closure of } \mathcal{C}_0^\infty\big(\mathbb{R}^n \setminus \{0\}\big) \text{ in the space } W^{2,p}(\mathbb{R}^n) \\ &= W_0^{2,p}\big(\mathbb{R}^n \setminus \{0\}\big). \end{aligned}$$

We have the following explicit characterization of the spaces introduced above.

PROPOSITION 6.3. (i) *If* $2 - \frac{n}{p} \le 0$, *i.e.* $p \le \frac{n}{2}$, *then* $W_0^{2,p}(\mathbb{R}^n \setminus \{0\}) = W^{2,p}(\mathbb{R}^n)$.
(ii) *If* $2 - \frac{n}{p} \in (0, 1]$, *i.e.* $\frac{n}{2} < p \le n$, *then*

$$W_0^{2,p}\big(\mathbb{R}^n \setminus \{0\}\big) = \big\{f \in W^{2,p}(\mathbb{R}^n)\colon f(0) = 0\big\}.$$

(iii) *If* $2 - \frac{n}{p} \in (1, 2)$, *i.e.* $p > n$, *then*

$$W_0^{2,p}\big(\mathbb{R}^n \setminus \{0\}\big) = \big\{f \in W^{2,p}(\mathbb{R}^n)\colon f(0) = 0 = D_i f(0),\ i = 1, \dots, n\big\}.$$

PROOF. Write

$$\mathcal{H} := \begin{cases} W^{2,p}(\mathbb{R}^n) & \text{if } p \le \frac{n}{2}, \\ \big\{u \in W^{2,p}(\mathbb{R}^n) : u(0) = 0\big\} & \text{if } p \in \big(\frac{n}{2}, n\big], \\ \big\{u \in W^{2,p}(\mathbb{R}^n) : u(0) = 0,\ D_i u(0) = 0,\ i = 1, \dots, n\big\} & \text{if } p > n. \end{cases}$$

We will show that $\mathcal{H} = W_0^{2,p}(\mathbb{R}^n \setminus \{0\})$. Obviously, $\mathcal{C}_0^\infty(\mathbb{R}^n \setminus \{0\}) \subset \mathcal{H}$ and by the Sobolev Embedding Theorem, $\mathcal{H}$ is a closed subspace of $W^{2,p}(\mathbb{R}^n)$. Suppose by contradiction that the set $\mathcal{C}_0^\infty(\mathbb{R}^n \setminus \{0\})$ is not dense in $\mathcal{H}$. Then by the Hahn–Banach theorem there exists a non-zero continuous linear functional $\omega\colon W^{2,p}(\mathbb{R}^n) \to \mathbb{C}$ such that $\omega = 0$ on $\mathcal{C}_0^\infty(\mathbb{R}^n \setminus \{0\})$ and $\omega \ne 0$ on $\mathcal{H}$. In other words, $\omega \in [W^{2,p}(\mathbb{R}^n)]' \subset \mathcal{S}'(\mathbb{R}^n) \subset \mathcal{D}'(\mathbb{R}^n)$ and, since $\omega = 0$ on $\mathcal{C}_0^\infty(\mathbb{R}^n \setminus \{0\})$, $\operatorname{supp}(\omega) = \{0\}$. Hence, by [114, Theorems 6.24 (c) and 6.25], in the distributional sense there exists a finite set $\mathcal{A}$ of n-tuples and a family of numbers $c_\alpha \in \mathbb{C}$, $\alpha \in \mathcal{A}$, such that

$$\omega = \sum_{\alpha \in \mathcal{A}} c_\alpha D^\alpha \delta_0.$$

It is enough to show that $\omega = 0$ on $\mathcal{H}$. This follows from Lemma 3.4 as $[W^{2,p}(\mathbb{R}^n)]' = W^{-2,q}(\mathbb{R}^n)$, where p and q are conjugate exponents. Indeed, $p \le \frac{n}{2}$ implies that $q \ge \frac{n}{n-2}$, and $\omega \in W^{-2,q}(\mathbb{R}^n)$ implies $c_\alpha = 0$ for every α. Next, $p \in (\frac{n}{2}, n]$ implies $q \in [\frac{n}{n-1}, \frac{n}{n-2})$ and $\omega \in W^{-2,q}(\mathbb{R}^n)$ implies $c_\alpha = 0$ for every $\alpha\colon |\alpha| \ge 1$. Hence $\omega = 0$ on $\mathcal{H}$, a contradiction. Finally, if $p > n$, then $q \in (1, \frac{n}{n-1})$, and $\omega \in W^{-2,q}(\mathbb{R}^n)$ implies $c_\alpha = 0$ for every $\alpha\colon |\alpha| > 1$. Hence, the claim follows, which completes the proof. ∎

7. SYMMETRIC EXTENSIONS OF THE LAPLACE OPERATOR ON $\mathbb{R}^n \setminus \{0\}$

Recall that $\Delta_{\mathbb{R}^n \setminus \{0\}}$ denotes the distributional Laplacian:

$$\Delta_{\mathbb{R}^n \setminus \{0\}}\colon \mathcal{D}'\big(\mathbb{R}^n \setminus \{0\}\big) \to \mathcal{D}'\big(\mathbb{R}^n \setminus \{0\}\big).$$

Let us observe that in view of our definition of the formally adjoint differential operator, see equality (2.4), the adjoint of $\Delta_{\mathbb{R}^n\setminus\{0\}}$ is equal to $\Delta_{\mathbb{R}^n\setminus\{0\}} \in \mathcal{L}(\mathcal{C}_0^\infty(\mathbb{R}^n \setminus \{0\}))$. Here and below, for a normed vector space Z, by $\mathcal{L}(Z)$ we denote the normed vector space of all bounded linear operators from Z to Z.

In what follows we assume that $p > 1$ and $q = p' > 1$ are conjugate exponents, i.e. satisfying condition (4.2).

Given a densely defined linear operator A with domain $D(A)$ on $L^p(\mathbb{R}^n)$, we denote by A^*, with domain $D(A^*)$, the adjoint operator acting on $L^q(\mathbb{R}^n)$. Since the spaces $L^p(\mathbb{R}^n)$ and resp. $L^q(\mathbb{R}^n)$ are isometrically isomorphic to the spaces $L^p(\mathbb{R}^n \setminus \{0\})$ and resp. $L^q(\mathbb{R}^n \setminus \{0\})$, we may as well consider the operators A and A^* on these latter spaces.

The aim of this section is to characterize all linear operators $(A, D(A))$ on $L^p(\mathbb{R}^n)$ such that

$$\mathcal{C}_0^\infty\big(\mathbb{R}^n \setminus \{0\}\big) \subset D(A) \cap D(A^*), \tag{7.1}$$

$$Au = A^*u = -\Delta_{\mathbb{R}^n\setminus\{0\}}u \quad \text{for } u \in \mathcal{C}_0^\infty\big(\mathbb{R}^n \setminus \{0\}\big), \tag{7.2}$$

$$A = (A^*)^*, \tag{7.3}$$

$$\exists \lambda > 0 \colon (\lambda + A)^{-1} \in \mathcal{L}\big(L^p(\mathbb{R}^n)\big). \tag{7.4}$$

Clearly, if an operator $(A, D(A))$ satisfies conditions (7.1) to (7.3), then it is closed and densely defined. Hence, the adjoint operator A^* is closed, see [34, Theorem 2.17].

We start with the following result.

PROPOSITION 7.1. *Let us assume that A is a closed and densely defined operator in the Banach space $L^p(\mathbb{R}^n)$ satisfying conditions (7.1) and (7.2). Then*

$$W_0^{2,p}\big(\mathbb{R}^n \setminus \{0\}\big) \subset D(A) \subset W^{\Delta,p}\big(\mathbb{R}^n \setminus \{0\}\big) \tag{7.5}$$

and

$$Au = -\widetilde{\Delta_{\mathbb{R}^n\setminus\{0\}}u} \quad \textit{for } u \in D(A). \tag{7.6}$$

Moreover,

$$W_0^{2,q}\big(\mathbb{R}^n \setminus \{0\}\big) \subset D(A^*) \subset W^{\Delta,q}\big(\mathbb{R}^n \setminus \{0\}\big) \tag{7.7}$$

and

$$A^*v = -\widetilde{\Delta_{\mathbb{R}^n\setminus\{0\}}v} \quad \textit{for } v \in D(A^*). \tag{7.8}$$

PROOF. We begin with a proof of the first inclusion in (7.5). For this aim let us choose and fix $u \in W_0^{2,p}(\mathbb{R}^n \setminus \{0\})$. Then by Definition 6.1 there exists a sequence $\{u_k\}_{k=1}^\infty \subset \mathcal{C}_0^\infty(\mathbb{R}^n \setminus \{0\})$, such that $u_k \to u$ in $W^{\Delta,p}(\mathbb{R}^n \setminus \{0\})$ as $k \to \infty$. This implies that

$\Delta_{\mathbb{R}^n\setminus\{0\}}u_k \to \Delta_{\mathbb{R}^n\setminus\{0\}}u$ in $L^p(\mathbb{R}^n)$. On the other hand, $-\Delta_{\mathbb{R}^n\setminus\{0\}}u_k = Au_k$, for every $k \in \mathbb{N}$, by (7.2). Therefore, since A^* is a closed operator, see [34, Theorem 2.17], it follows that $u \in D(A)$ and $Au = -\Delta_{\mathbb{R}^n\setminus\{0\}}u$.

Now we will prove the second inclusion in (7.5). For this aim let us choose and fix $u \in D(A)$ and $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^n \setminus \{0\})$. Thus, $u \in L^p(\mathbb{R}^n)$ and so $u \in \mathcal{D}'(\mathbb{R}^n \setminus \{0\})$. Therefore, $\Delta_{\mathbb{R}^n\setminus\{0\}}u \in \mathcal{D}'(\mathbb{R}^n \setminus \{0\})$ and by Definition 2.5,

$$-(\Delta_{\mathbb{R}^n\setminus\{0\}}u, \varphi) = (u, -\Delta_{\mathbb{R}^n\setminus\{0\}}\varphi).$$

Since $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^n \setminus \{0\})$, by assumptions (7.1) and (7.2) we infer that

$$(u, -\Delta_{\mathbb{R}^n\setminus\{0\}}\varphi) = (u, A^*\varphi),$$

where on the right-hand side we have the duality between the spaces $L^p(\mathbb{R}^n)$ and $L^q(\mathbb{R}^n)$. Next, by the definition of the adjoint operator, we infer that $(u, A^*\varphi) = (Au, \varphi)$. Therefore,

$$(\Delta_{\mathbb{R}^n\setminus\{0\}}u, \varphi) = (-Au, \varphi). \tag{7.9}$$

Therefore, in view of Definition 2.2 we deduce that $u \in W^{\Delta,p}(\mathbb{R}^n \setminus \{0\})$, as claimed. Next we observe that equality (7.6) follows directly from identity (7.9), just proved.

Let us now provide proofs of (7.7) and (7.8). Since for $u \in D(A)$ and $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^n \setminus \{0\})$ we have

$$(Au, \varphi) = (-\Delta_{\mathbb{R}^n\setminus\{0\}}u, \varphi) = (u, -\Delta_{\mathbb{R}^n\setminus\{0\}}\varphi),$$

we infer that $\mathcal{C}_0^\infty(\mathbb{R}^n \setminus \{0\}) \subset D(A^*)$ and $A^*\varphi = -\Delta_{\mathbb{R}^n\setminus\{0\}}\varphi$. This proves the first inclusion in (7.7). To prove the second, let us choose and fix $v \in D(A^*)$. Hence, the map

$$D(A) \ni u \mapsto (Au, v) \in \mathbb{C}$$

is continuous with respect to $L^p(\mathbb{R}^n)$-topology. Thus, by the just proved (7.6) and (7.5) the map

$$W_0^{2,p}\big(\mathbb{R}^n \setminus \{0\}\big) \ni \varphi \mapsto (-\Delta_{\mathbb{R}^n\setminus\{0\}}\varphi, v) \in \mathbb{C}$$

is continuous with respect to $L^p(\mathbb{R}^n)$-topology. Thus, there exists a unique $z \in L^q(\mathbb{R}^n)$ such that

$$(-\Delta_{\mathbb{R}^n\setminus\{0\}}\varphi, v) = (\varphi, z), \quad \text{for every } \varphi \in W_0^{2,p}\big(\mathbb{R}^n \setminus \{0\}\big). \tag{7.10}$$

This, according to Definition 2.2, implies that $v \in W^{\Delta,q}(\mathbb{R}^n \setminus \{0\})$ and $-\Delta_{\mathbb{R}^n\setminus\{0\}}v = z$. This proves the second inclusion in (7.7).

In order to prove identity (7.8) we proceed as follows. We choose and fix $v \in D(A^*)$. Then, since $W_0^{2,p}(\mathbb{R}^n \setminus \{0\}) \subset D(A)$, we have the following train of equalities:

$$(\varphi, A^*v) = (A\varphi, v) = (-\Delta_{\mathbb{R}^n\setminus\{0\}}\varphi, v) = (\varphi, z), \quad \varphi \in W_0^{2,p}\big(\mathbb{R}^n \setminus \{0\}\big),$$

where the last equality is a consequence of (7.10). Therefore $A^*v = z$ and since, by an identity proven earlier in the proof of (7.7), $-\Delta_{\mathbb{R}^n\setminus\{0\}}v = z$, we deduce (7.8). The proof is complete. ∎

REMARK 7.2. Let us observe that equality (7.6) is a natural generalization of assumption (7.2).

Recall our notation, see (3.30), $G_{n,\lambda}(x) = G_n(\sqrt{\lambda}x)$, $\lambda > 0$. Let

$$\mathcal{Y}_{n,p,\lambda} := \begin{cases} \operatorname{linspan}\left\{G_{n,\lambda}, \frac{\partial G_{n,\lambda}}{\partial x_1}, \dots, \frac{\partial G_{n,\lambda}}{\partial x_n}\right\} & \text{if } n \geq 1 \text{ and } 1 < p < \frac{n}{n-1}, \\ \operatorname{linspan}\{G_{n,\lambda}\} & \text{if } n \geq 2 \text{ and } \frac{n}{n-1} \leq p < \frac{n}{n-2}, \\ \{0\} & \text{if } n > 2 \text{ and } p \geq \frac{n}{n-2}. \end{cases}$$

Let us recall that by Proposition 3.20 we have $\mathcal{Y}_{n,p,1} = \mathcal{Y}_{n,p}$. Clearly, if $e \in \mathcal{Y}_{n,p,\lambda}$, then $\Delta_{\mathbb{R}^n\setminus\{0\}}e = \lambda e$, i.e. $-\Delta_{\mathbb{R}^n\setminus\{0\}}e = -\lambda e$. Therefore, we have the following result.

LEMMA 7.3. *Assume that a densely defined and closed linear operator A satisfies conditions (7.1) and (7.2). If $\lambda > 0$ and $-\lambda$ belongs to the resolvent set $\rho(A)$, then $\mathcal{Y}_{n,p,\lambda} \cap D(A) = \{0\}$.*

PROOF. Let us choose and fix $\lambda > 0$ such that $-\lambda \in \rho(A)$, i.e. such that $\lambda I + A$ is a bijection onto $L^p(\mathbb{R}^n)$ and the inverse is bounded. Let us also choose and fix $e \in \mathcal{Y}_{n,p,\lambda} \cap D(A)$. We will show that $e = 0$. For this purpose, note that since $e \in \mathcal{Y}_{n,p,\lambda}$, by the comment preceding the lemma,

$$\Delta_{\mathbb{R}^n\setminus\{0\}}\, e = \lambda e.$$

On the other hand, since $e \in D(A)$ and A satisfies (7.1) and (7.2), Proposition 7.1, in particular identity (7.6), gives

$$Ae = -\Delta_{\mathbb{R}^n\setminus\{0\}}\, e.$$

Hence, $(\lambda I + A)e = 0$. Since, by hypothesis, $-\lambda \in \rho(A)$, the operator $\lambda I + A$ is injective, and therefore $e = 0$. This proves that $\mathcal{Y}_{n,p,\lambda} \cap D(A) = \{0\}$ and the lemma follows. ∎

Let us equip $D(A)$ with the graph norm. Let

$$j_A \colon D(A) \to W^{\Delta,p}\big(\mathbb{R}^n \setminus \{0\}\big)$$

be the natural embedding map, which is well defined because of (7.5). Moreover, by (7.6), it follows that the map j_A is continuous. Similarly, by (7.7), the embedding map

$$j_{A^*}\colon D(A^*) \to W^{\Delta,q}\big(\mathbb{R}^n \setminus \{0\}\big)$$

is well defined and continuous. Let us recall that $\Pi_p\colon W^{\Delta,p}(\mathbb{R}^n \setminus \{0\}) \to W^{2,p}(\mathbb{R}^n)$ and $\xi_p\colon W^{\Delta,p}(\mathbb{R}^n \setminus \{0\}) \to \mathbb{C}$ or $\xi_p\colon W^{\Delta,p}(\mathbb{R}^n \setminus \{0\}) \to \mathbb{C}^{n+1}$, see Definitions 3.18 and 3.19.

Assume that $-1 \in \rho(A)$, i.e. that $(I + A)^{-1} \in \mathcal{L}(L^p(\mathbb{R}^n))$; this is the form taken by (7.4) with $\lambda = 1$ under the sign convention of Section 1.2. It is well known that the linear operator

$$I - \Delta\colon W^{2,p}(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$$

is an isomorphism. By the assumption $-1 \in \rho(A)$, also the linear operator

$$(I + A)^{-1}\colon L^p(\mathbb{R}^n) \to D(A)$$

is an isomorphism. Hence,

$$(I + A^*)^{-1}\colon L^q(\mathbb{R}^n) \to D(A^*)$$

is also a linear isomorphism, see [110, Lemma 10.2]. Therefore, we deduce that the following linear maps

$$\begin{aligned} \gamma_A &:= j_A \circ (I + A)^{-1} \circ (I - \Delta)\colon W^{2,p}(\mathbb{R}^n) \to W^{\Delta,p}\big(\mathbb{R}^n \setminus \{0\}\big), \\ \gamma_{A^*} &:= j_{A^*} \circ (I + A^*)^{-1} \circ (I - \Delta)\colon W^{2,q}(\mathbb{R}^n) \to W^{\Delta,q}\big(\mathbb{R}^n \setminus \{0\}\big) \end{aligned} \tag{7.11}$$

are well defined and bounded.

LEMMA 7.4. *We have*

$$\begin{aligned} u &= \gamma_A \circ \Pi_p u \quad \textit{for every } u \in D(A), \\ v &= \gamma_{A^*} \circ \Pi_q v \quad \textit{for every } v \in D(A^*). \end{aligned}$$

PROOF. It is enough to prove the first part of our lemma since the proof of the second part is fully analogous. For this aim let us choose and fix $u \in D(A) \subset W^{\Delta,p}(\mathbb{R}^n \setminus \{0\})$. We will show that

$$(I + A)^{-1}(I - \Delta)\Pi_p u = u \quad \text{in } L^p(\mathbb{R}^n)$$

or equivalently that

$$(I - \Delta)\Pi_p u = (I + A)u \quad \text{in } L^p(\mathbb{R}^n). \tag{7.12}$$

By Proposition 3.20 we have $u - \Pi_p u \in \mathcal{Y}_{n,p}$. Then (7.12) follows from the following three observations:

(i) According to Proposition 7.1, $Au = -\Delta_{\mathbb{R}^n \setminus \{0\}} u$.

(ii) For any $R \in \mathcal{Y}_{n,p}$, $\Delta_{\mathbb{R}^n\setminus\{0\}} R = R$.

(iii) $\Delta_{\mathbb{R}^n\setminus\{0\}} \Pi_p u = \Delta \Pi_p u$, see identity (3.25) in Proposition 3.20.

In fact, we have

$$\begin{aligned}(I + A)u &= u - \Delta_{\mathbb{R}^n\setminus\{0\}} u = u - \Delta_{\mathbb{R}^n\setminus\{0\}}(u - \Pi_p u + \Pi_p u)\\ &= u - (u - \Pi_p u) - \Delta_{\mathbb{R}^n\setminus\{0\}} \Pi_p u\\ &= \Pi_p u - \Delta \Pi_p u.\end{aligned}$$

∎

As in Section 1.2, $\Delta_{n,p}$ denotes the Friedrichs Laplace operator on $L^p(\mathbb{R}^n)$, i.e. the Laplace operator with domain $W^{2,p}(\mathbb{R}^n)$, so that $A_{0,p} = -\Delta_{n,p}$. Recall that, by definition of the adjoint, $v \in D(A^*)$ if and only if the map $D(A) \ni u \mapsto (Au, v) = -(\Delta u, v) \in \mathbb{C}$ admits a continuous extension to $L^p(\mathbb{R}^n)$, and in that case $A^* v = -\Delta v$. Throughout, we use the Green formula (4.7) or Theorem 4.2, which for $u = a_0 G_n + \sum_{i=1}^n a_i D_i G_n + f$ and $v = b_0 G_n + \sum_{i=1}^n b_i D_i G_n + g$ reads

$$(\Delta u, v) - (u, \Delta v) = a_0 \overline{g(0)} - \sum_{i=1}^n a_i \overline{D_i g(0)} - f(0)\overline{b_0} + \sum_{i=1}^n D_i f(0)\, \overline{b_i}. \tag{7.13}$$

Lemma 7.5. *Assume that $p \geq q$ and $W^{\Delta,p}(\mathbb{R}^n \setminus \{0\}) = W^{2,p}(\mathbb{R}^n)$. If $A = -\Delta_{n,p}$, then $A^* = -\Delta_{n,q}$.*

Proof of Lemma 7.5. The case $W^{\Delta,q}(\mathbb{R}^n \setminus \{0\}) = W^{2,q}(\mathbb{R}^n)$ is obvious. Assume that

$$W^{\Delta,q}\big(\mathbb{R}^n \setminus \{0\}\big) = \mathbb{C} G_n \oplus W^{2,q}(\mathbb{R}^n).$$

Then $\frac{n}{n-1} \leq q < \frac{n}{n-2}$, so $p > \frac{n}{2}$ and hence by the Sobolev Embedding Theorem, $W^{2,p}(\mathbb{R}^n) \hookrightarrow \mathcal{C}(\mathbb{R}^n)$. Let us choose and fix $v = b_0 G_n + g \in W^{\Delta,q}(\mathbb{R}^n \setminus \{0\})$ with $g \in W^{2,q}(\mathbb{R}^n)$, and let $u \in D(A) = W^{2,p}(\mathbb{R}^n)$. Applying (7.13) with u having no singular part and v as above with all remaining coefficients zero, and using $\Delta_{\mathbb{R}^n\setminus\{0\}} G_n = G_n$, we obtain

$$-(\Delta u, v) = -\big(b_0 (u, G_n) + (u, \Delta g)\big) + u(0)\, \overline{b_0}.$$

Since $G_n \in L^q(\mathbb{R}^n)$ and $\Delta g \in L^q(\mathbb{R}^n)$, the maps $u \mapsto b_0 (u, G_n)$ and $u \mapsto (u, \Delta g)$ are continuous on $L^p(\mathbb{R}^n)$. Hence, the map $u \mapsto -(\Delta u, v)$ extends continuously to $L^p(\mathbb{R}^n)$ if and only if the map

$$D(A) = W^{2,p}(\mathbb{R}^n) \ni u \mapsto b_0\, \overline{u(0)} \in \mathbb{C}$$

does, that is, if and only if $b_0 = 0$. Thus, $v \in D(A^*)$ forces $v \in W^{2,q}(\mathbb{R}^n)$, and then $A^* v = -\Delta v$.

Consider now the last case $1 < q < \frac{n}{n-1}$ and $p > n$. Then $W^{2,p}(\mathbb{R}^n) \hookrightarrow \mathcal{C}^1(\mathbb{R}^n)$ and

$$W^{\Delta,q}\big(\mathbb{R}^n \setminus \{0\}\big) = \mathbb{C}G_n \oplus \bigoplus_{k=1}^{n} \mathbb{C}D_k G_n \oplus W^{2,q}(\mathbb{R}^n).$$

Let us now choose and fix $v = b_0 G_n + \sum_{k=1}^n b_k D_k G_n + g$ with $g \in W^{2,q}(\mathbb{R}^n)$ and let $u \in D(A) = W^{2,p}(\mathbb{R}^n)$. Applying (7.13) as above and using $\Delta_{\mathbb{R}^n\setminus\{0\}} D_i G_n = D_i G_n$, we obtain

$$-(\Delta u, v) = -\bigg(b_0(u, G_n) + \sum_{k=1}^{n} b_k(u, D_k G_n) + (u, \Delta g)\bigg)$$
$$+ u(0)\,\bar{b}_0 - \sum_{k=1}^{n} \bar{b}_k\, D_k u(0).$$

As above, the three inner-product terms are continuous in $u \in L^p(\mathbb{R}^n)$. The functionals

$$u \mapsto u(0), \quad u \mapsto D_k u(0) \quad (k = 1, \dots, n)$$

are linearly independent and none admits a continuous extension to $L^p(\mathbb{R}^n)$. Consequently, no non-trivial linear combination of them does. Hence, $u \mapsto -(\Delta u, v)$ extends continuously to $L^p(\mathbb{R}^n)$ if and only if

$$b_0 = b_1 = \cdots = b_n = 0,$$

that is, if and only if $v \in W^{2,q}(\mathbb{R}^n)$, and then $A^* v = -\Delta v$. This proves $A^* = -\Delta_{n,q}$. ∎

Definition 7.6. Given $\beta \in \mathbb{C} \cup \{\infty\}$, we define a linear map

$$\tau_\beta \colon W^{\Delta,p}\big(\mathbb{R}^n \setminus \{0\}\big) \ni u = c_0 G_n + f \mapsto \tau_\beta u \in \mathbb{C},$$

$f \in W^{2,p}(\mathbb{R}^n)$ as above, by

$$\tau_\beta u := \begin{cases} c_0 - \beta f(0) & \text{if } \beta \neq \infty, \\ -f(0) & \text{if } \beta = \infty. \end{cases} \tag{7.14}$$

The map τ_β will be called a boundary operator.

Theorem 7.7. *Assume that $(A, D(A))$ satisfies (7.1) to (7.4). Assume that $-1 \in \rho(A)$, equivalently that $(I + A)^{-1} \in \mathcal{L}(L^p(\mathbb{R}^n))$. The following assertions hold true:*

(i) *If $n \geq 4$ or $n = 3$ and $p \notin (\frac{3}{2}, 3)$, then*

$$D(A) = W^{2,p}(\mathbb{R}^n) \quad \text{and} \quad D(A^*) = W^{2,q}(\mathbb{R}^n),$$

and $Au = -\Delta u$ if $u \in D(A)$ and $A^* v = -\Delta v$ if $v \in D(A^*)$.

(ii) *If* $n = 3$ *and* $p \in (\frac{3}{2}, 3)$, *or* $n = 2$, *then there is a* $\beta \in \mathbb{C}$ *such that*

$$D(A) = \{u \in W^{\Delta,p}(\mathbb{R}^n \setminus \{0\}) : \tau_\beta u = 0\},$$
$$D(A^*) = \{v \in W^{\Delta,q}(\mathbb{R}^n \setminus \{0\}) : \tau_{\overline{\beta}} v = 0\}.$$

(iii) *If* $n = 1$, *then there are complex numbers* β_j^i, $i, j = 0, 1$ *such that*

$$D(A) = \{(\beta_0^0 f(0) + \beta_1^0 f'(0))G_1 + (\beta_0^1 f(0) + \beta_1^1 f'(0))G_1' + f : f \in W^{2,p}(\mathbb{R})\},$$
$$D(A^*) = \{(\overline{\beta_0^0}\, g(0) - \overline{\beta_0^1}\, g'(0))G_1 + (-\overline{\beta_1^0}\, g(0) + \overline{\beta_1^1}\, g'(0))G_1' + g : g \in W^{2,q}(\mathbb{R})\}.$$

Proof. The first part of the theorem follows from Lemma 7.5. We now prove the second part. Assume that $q \le p$. Then

$$W^{\Delta,p}(\mathbb{R}^n \setminus \{0\}) = \mathbb{C}G_n \oplus W^{2,p}(\mathbb{R}^n).$$

By Theorem 3.13, the map ξ_p defined by

$$\xi_p : W^{\Delta,p}(\mathbb{R}^n \setminus \{0\}) \ni c_0 G_n + f \mapsto c_0 \in \mathbb{C}$$

is bounded and linear.

On the other hand, by Lemma 7.4, $\gamma_A \circ \Pi_p$ is the identity operator on $D(A)$, where γ_A is the linear map defined in (7.11). Hence, we infer that

$$u = c_0 G_n + f \in D(A) \Longrightarrow c_0 = \xi_p \circ \gamma_A \circ \Pi_p(u) = \xi_p \circ \gamma_A(f). \tag{7.15}$$

Recall, see (7.1), that $\mathcal{C}_0^\infty(\mathbb{R}^n \setminus \{0\}) \subset D(A)$. Therefore, by (7.15), $\xi_p \circ \gamma_A(f) = 0$ for every $f \in \mathcal{C}_0^\infty(\mathbb{R}^n \setminus \{0\})$. Since $\mathcal{C}_0^\infty(\mathbb{R}^n \setminus \{0\})$ is dense in $W_0^{2,p}(\mathbb{R}^n \setminus \{0\})$, we have, by continuity, that $\xi_p \circ \gamma_A(f) = 0$ for every $f \in W_0^{2,p}(\mathbb{R}^n \setminus \{0\})$. Recall that either $n = 2$ and $p \ge 2$ or $n = 3$ and $2 \le p < 3$. Therefore, $W^{2,p}(\mathbb{R}^n) \hookrightarrow \mathcal{C}(\mathbb{R}^n)$, and by Proposition 6.3, $\xi_p \circ \gamma_A(f) = 0$ for every $f \in W^{2,p}(\mathbb{R}^n)$ such that $f(0) = 0$. Hence, we have the following fact:

$$\exists\, \beta \in \mathbb{C} : \forall\, f \in W^{2,p}(\mathbb{R}^n), \quad \xi_p \circ \gamma_A(f) = \beta f(0).$$

We are going to show that

$$D(A) = \{u = c_0 G_n + f \in W^{\Delta,p}(\mathbb{R}^n \setminus \{0\}) : c_0 = \beta f(0)\},$$
$$D(A^*) = \{v = b_0 G_n + g \in W^{\Delta,q}(\mathbb{R}^n \setminus \{0\}) : b_0 = \overline{\beta} g(0)\}.$$

Recall that either $n = 2$ and $p \ge 2$ or $n = 3$ and $2 \le p < 3$, $\frac{3}{2} < q \le 2$. Therefore, $W^{2,q}(\mathbb{R}^n) \hookrightarrow \mathcal{C}(\mathbb{R}^n)$, and the right-hand side of the last identity above is well defined.

So far we know that if $c_0 G_n + f \in D(A)$, then $c_0 = \beta f(0)$. Therefore, we have to exclude the case where $D(A) = W_0^{2,p}(\mathbb{R}^n \setminus \{0\})$. Suppose, by contradiction, that $D(A) = W_0^{2,p}(\mathbb{R}^n \setminus \{0\})$. Since $G_n \in L^q(\mathbb{R}^n)$, the Green formula (4.3) yields

$$W_0^{2,p}(\mathbb{R}^n \setminus \{0\}) \ni u \mapsto (Au, G_n) = (-\Delta u, G_n) = -(u, G_n).$$

This implies that $G_n \in D(A^*)$ which contradicts Lemma 7.3, applied to A^*, with $\lambda = 1$. By definition $v \in W^{\Delta,q}(\mathbb{R}^n \setminus \{0\})$ belongs to $D(A^*)$ if and only if there is a constant C such that

$$\big|(Au, v)\big| \le C|u|_{L^p(\mathbb{R}^n)}, \quad \forall u \in D(A).$$

We have by assumption that $u = f$, where $f \in W^{2,p}(\mathbb{R}^n)$ is such that $f(0) = 0$. Applying the Green formula, we obtain

$$\big|(Au, G_n)\big| = \big|(\Delta f, G_n)\big| = \big|(f, G_n)\big| \le |G_n|_{L^q(\mathbb{R}^n)}|f|_{L^p(\mathbb{R}^n)}$$

since, by Lemma 3.9, $G_n \in L^q(\mathbb{R}^n)$.

We have shown that there exists a vector space $\mathcal{X}$ such that

$$W_0^{2,p}(\mathbb{R}^n \setminus \{0\}) \subsetneqq \mathcal{X} \subset W^{2,p}(\mathbb{R}^n),$$

and

$$D(A) = \big\{u = \beta f(0) G_n + f : f \in \mathcal{X}\big\}. \tag{7.16}$$

To see that

$$D(A^*) \supset \big\{v = \bar{\beta} g(0) G_n + g : g \in W^{2,q}(\mathbb{R}^n)\big\} \tag{7.17}$$

take $g \in W^{2,q}(\mathbb{R}^n)$, $v = \bar{\beta} g(0) G_n + g$. Then for every $u = \beta f(0) G_n + f$, $f \in \mathcal{X}$, we get

$$\begin{aligned}(Au, v) &= -(\Delta_{\mathbb{R}^n \setminus \{0\}} u, v) = -(u, \Delta_{\mathbb{R}^n \setminus \{0\}} v) - \big(c_0 \overline{g(0)} - f(0)\overline{b_0}\big) \\ &= \big(u, -\bar{\beta} g(0) G_n - \Delta g\big)\end{aligned}$$

because

$$c_0 \overline{g(0)} - f(0)\overline{b_0} = \beta f(0)\overline{g(0)} - f(0)\beta \overline{g(0)} = 0.$$

Since $\bar{\beta} g(0) G_n + \Delta g \in L^q(\mathbb{R}^n)$, we have

$$\big|(Au, v)\big| \le \big|\bar{\beta} g(0) G_n + \Delta g\big|_{L^q(\mathbb{R}^n)} |u|_{L^p(\mathbb{R}^n)}.$$

Therefore, we also have (7.17). In fact, if $n = 3$, then we have equality in (7.17). We have

$$W^{\Delta,q}\big(\mathbb{R}^3 \setminus \{0\}\big) = \mathbb{C} G_3 \oplus W^{2,q}(\mathbb{R}^3)$$

and using arguments leading to (7.16), we can find a subspace $\mathcal{Z} \subset W^{2,q}(\mathbb{R}^3)$ and a constant ρ such that

$$D(A^*) \subset \{v = \rho g(0)G_3 + g : g \in \mathcal{Z}\}.$$

Taking into account assertion (7.17), we conclude that $\rho = \overline{\beta}$ and $\mathcal{Z} = W^{2,q}(\mathbb{R}^3)$.

It remains to consider the case where $n = 2$ and $p \geq 2$. From the proof of the second part we know already that (7.16) and (7.17) hold. Our aim is to show the reverse inclusion to the one in (7.17). For this aim we adapt our earlier argument from the case of dimension $n = 3$. Let us choose and fix $v \in D(A^*)$. By Lemma 7.4, we infer that $v = \gamma_{A^*}\Pi_q v$. Let, for $i = 0, 1, 2$, $\xi_{i,q}$ be the linear map defined by

$$\xi_{i,q} : W^{\Delta,q}(\mathbb{R}^2 \setminus \{0\}) \ni c_0 G_2 + c_1 D_1 G_2 + c_2 D_2 G_2 + g \mapsto c_i \in \mathbb{C}.$$

Because by (7.7), $D(A^*) \subset W^{\Delta,q}(\mathbb{R}^2 \setminus \{0\})$, v can be written in the form

$$v = c_0 G_2 + c_1 D_1 G_2 + c_2 D_2 G_2 + g.$$

Thus, we infer that

$$c_i = \xi_{i,q} \circ \gamma_{A^*}(g), \quad i = 0, 1, 2.$$

By Lemma 7.3, $\xi_{i,q} \circ \gamma_{A^*}(g) = 0$ for $g \in W_0^{2,q}(\mathbb{R}^2 \setminus \{0\})$. Therefore, by Proposition 6.3, for each i there are $\beta_0^i, \beta_1^i, \beta_2^i$ such that

$$\xi_{i,q} \circ \gamma_{A^*}(g) = \beta_0^i g(0) + \beta_1^i D_1 g(0) + \beta_2^i D_2 g(0).$$

Therefore,

$$v = c_0 G_2 + c_1 D_1 G_2 + c_2 D_2 G_2 + g \in D(A^*)$$

implies that

$$c_i = \beta_0^i g(0) + \beta_1^i D_1 g(0) + \beta_2^i D_2 g(0).$$

This proves that

$$D(A^*) = \{v = \overline{\beta} g(0)G_2 + g : g \in W^{2,q}(\mathbb{R}^2)\}.$$

Since $A = (A^*)^*$, we see by the Green formula that

$$D(A) = \{u = \beta f(0)G_2 + f : f \in W^{2,p}(\mathbb{R}^2)\}.$$

In the proof of the last part of the theorem, one can adopt arguments from the proofs of the second part. Namely, by Proposition 6.3 and Lemmata 7.3 and 7.4, there are complex numbers β_j^i and $\widetilde{\beta}_j^i$ such that

$$u = c_0 G_1 + c_1 G_1' + f \in D(A) \quad \text{and} \quad v = b_0 G_1 + b_1 G_1' + g \in D(A^*)$$

imply that

$$c_i = \beta_0^i f(0) + \beta_1^i f'(0) \quad \text{and} \quad b_i = \widetilde{\beta}_0^i g(0) + \widetilde{\beta}_1^i g'(0).$$

The desired conclusion follows from the Green formula:

$$\begin{aligned}(u, A^*v) - (Au, v) &= c_0\overline{g(0)} - c_1\overline{g'(0)} - f(0)\overline{b_0} + f'(0)\overline{b_1} \\ &= \big(\beta_0^0 f(0) + \beta_1^0 f'(0)\big)\overline{g(0)} - \big(\beta_0^1 f(0) + \beta_1^1 f'(0)\big)\overline{g'(0)} \\ &\quad - f(0)\overline{\widetilde{\beta}_0^0 g(0) + \widetilde{\beta}_1^0 g'(0)} + f'(0)\overline{\widetilde{\beta}_0^1 g(0) + \widetilde{\beta}_1^1 g'(0)} \\ &= f(0)\overline{g(0)}(\beta_0^0 - \overline{\widetilde{\beta}_0^0}) + f'(0)\overline{g(0)}(\beta_1^0 + \overline{\widetilde{\beta}_0^1}) \\ &\quad + f(0)\overline{g'(0)}(-\beta_0^1 - \overline{\widetilde{\beta}_1^0}) + f'(0)\overline{g'(0)}(-\beta_1^1 + \overline{\widetilde{\beta}_1^1}).\end{aligned}$$

This leads to the following identities:

$$\beta_0^0 = \overline{\widetilde{\beta}_0^0}, \quad \beta_1^0 = -\overline{\widetilde{\beta}_0^1}, \quad \beta_0^1 = -\overline{\widetilde{\beta}_1^0}, \quad \beta_1^1 = \overline{\widetilde{\beta}_1^1},$$

or equivalently

$$\widetilde{\beta}_0^0 = \overline{\beta_0^0}, \quad \widetilde{\beta}_1^0 = -\overline{\beta_0^1}, \quad \widetilde{\beta}_0^1 = -\overline{\beta_1^0}, \quad \widetilde{\beta}_1^1 = \overline{\beta_1^1}.$$

Therefore, if we put $B = (\beta_j^i)_{i,j=0,1}$, $\widetilde{B} = (\widetilde{\beta}_j^i)_{i,j=0,1}$ and $J = \operatorname{diag}(1, -1)$, then $\widetilde{B} = JB^*J$, where B^* is the conjugate transpose of B. Since $J^2 = I$, the map $B \mapsto \widetilde{B}$ is an involution, in accordance with $A = (A^*)^*$. Since the matrix $B \in \mathbb{C}^{2\times 2}$ is arbitrary, the family is genuinely 4-parametric. ∎

8. Case of $n = 2, 3$

In this section, we consider $n = 2$ with $p \in [2, +\infty)$ or $n = 3$ with $p \in (\frac{3}{2}, 3)$, where $W^{\Delta,p}(\mathbb{R}^n \setminus \{0\}) = \mathbb{C}G_n \oplus W^{2,p}(\mathbb{R}^n)$ and the boundary operator τ_β (Definition 7.6) is well defined (Theorems 3.13 and 7.7 (ii)). Since p is fixed in this section and in Section 9, we write A_0 for the operator $A_{0,p} = -\Delta_{n,p}$, see Remark 5.5. While Sections 5–7 implicitly classified all operators $(A, D(A))$ on $L^p(\mathbb{R}^n)$ extending $-\Delta_{\mathbb{R}^n\setminus\{0\}}$ under (7.1)–(7.4) via a single boundary condition $\tau_\beta u = 0$ at the origin, this description depends on the choice of fundamental solution G_n and lacks spectral information. We make this classification explicit and computationally applicable. We also furnish spectral information.

- Lemmata 8.1–8.2 quantify the transformation $\beta \mapsto \beta_\lambda$ under $G_n \to G_{n,\lambda}$ (8.4), confirming that the domains $\ker \tau_\beta$ are intrinsic. The pole of this Möbius map highlights the missing operator A_∞ (Lemma 8.5), establishing $\mathbb{C} \cup \{\infty\}$ as the natural parameter space.

- Definition 8.3 introduces $(A_\beta)_{\beta\in\mathbb{C}\cup\{\infty\}}$ on $L^p(\mathbb{R}^n)$ with $D(A_\beta) = \ker \tau_\beta$ and $A_\beta u = -\Delta u$, recalling the definition of A_∞, interpolating between the minus Friedrichs Laplacian $A_0 = -\Delta_{n,p}$ and A_∞.
- Theorem 8.6 establishes that each A_β satisfies (7.1)–(7.4) and has at most one negative eigenvalue $-\lambda_\beta$ (with eigenfunction $e_{n,\beta} = G_{n,\lambda_\beta}$) and that $(A_\beta)_{\beta\in\mathbb{C}\cup\{\infty\}}$ contains all admissible extensions. The resolvent is evaluated via λ-adapted parameters $\tau^\lambda_{\beta_\lambda} u = 0$.

For $n \geq 4$ or $n = 3$ with $p \notin (\frac{3}{2}, 3)$, Theorem 7.7 (i) yields only minus the Friedrichs Laplacian $A_0 := -\Delta_{n,p}$, while $n = 1$ requires a four-parameter family. The constructed family (A_β) underpins Section 9 (analytic semigroups S_β, heat kernels $\mathcal{G}_\beta$, and boundary value problems) and Appendix A (where for $p = 2$, β-transformations match coupling constant renormalization in point interaction theory [6, 12]).

LEMMA 8.1. *Assume that $\lambda > 0$. Then every $u \in W^{\Delta,p}(\mathbb{R}^n \setminus \{0\})$ having the representation*

$$u = c_0 G_n + f, \quad f \in W^{2,p}(\mathbb{R}^n), \tag{8.1}$$

can be uniquely written in the following form:

$$u = c_0^\lambda G_{n,\lambda} + f^\lambda, \quad f^\lambda \in W^{2,p}(\mathbb{R}^n). \tag{8.2}$$

Moreover,

$$c_0^\lambda = \begin{cases} c_0 & \text{if } n = 2, \\ \sqrt{\lambda} c_0 & \text{if } n = 3 \end{cases}$$

and

$$f^\lambda(0) = \begin{cases} f(0) + c_0 \frac{\ln \lambda}{4\pi} & \text{if } n = 2, \\ f(0) - c_0 \frac{1}{4\pi}[1 - \sqrt{\lambda}] & \text{if } n = 3. \end{cases}$$

PROOF. Let us choose and fix $\lambda > 0$. Assume first that $n = 2$. Then, see Lemma 3.28 (ii),

$$G_{2,\lambda} = G_2 + g^\lambda, \quad g^\lambda \in W^{2,p}(\mathbb{R}^2).$$

We have

$$g^\lambda(0) = [G_{2,\lambda} - G_2](0) = \frac{1}{2\pi} \lim_{x\to 0} \left[\ln \frac{1}{|\sqrt{\lambda} x|} - \ln \frac{1}{|x|} \right] = -\frac{1}{4\pi} \ln \lambda .$$

Therefore, if u has representation (8.1), then

$$u = c_0 G_{2,\lambda} - c_0 g^\lambda + f.$$

Hence, we infer that equality (8.2) holds with

$$f^\lambda = f - c_0 g^\lambda, \quad f^\lambda(0) = f(0) + c_0 \frac{\ln \lambda}{4\pi}.$$

Assume next that $n = 3$. Then, see Lemma 3.28 (iii),

$$G_{3,\lambda} = a_\lambda G_3 + g^\lambda, \quad g^\lambda \in W^{2,p}(\mathbb{R}^3).$$

Since $G_3(x) = (4\pi|x|)^{-1}\mathrm{e}^{-|x|}$ and

$$a_\lambda = \lim_{x\to 0} \frac{G_{3,\lambda}(x)}{G_3(x)} = |\lambda|^{-\frac{1}{2}} \lim_{x\to 0} \mathrm{e}^{-|\sqrt{\lambda}x|+|x|} = \frac{1}{\sqrt{\lambda}},$$

we have

$$G_{3,\lambda} = \frac{1}{\sqrt{\lambda}} G_3 + g^\lambda,$$

and

$$g^\lambda(0) = \lim_{x\to 0}\left[G_{3,\lambda}(x) - \frac{1}{\sqrt{\lambda}} G_3(x)\right] = \lim_{x\to 0} \left(4\pi|\sqrt{\lambda}x|\right)^{-1}(\mathrm{e}^{-|\sqrt{\lambda}x|} - \mathrm{e}^{-|x|})$$
$$= \frac{1}{4\pi} \lim_{x\to 0}\left[\frac{\mathrm{e}^{-|\sqrt{\lambda}x|}-1}{|\sqrt{\lambda}x|} + \frac{1-\mathrm{e}^{-|x|}}{|x|}\frac{1}{\sqrt{\lambda}}\right] = \frac{1}{4\pi}\left[\frac{1}{\sqrt{\lambda}} - 1\right].$$

Therefore, if u has representation (8.1), then

$$u = \sqrt{\lambda} c_0 G_{3,\lambda} - \sqrt{\lambda} c_0 g^\lambda + f = \sqrt{\lambda} c_0 G_{3,\lambda} + f^\lambda,$$

and

$$f^\lambda(0) = -\sqrt{\lambda} c_0 g^\lambda(0) + f(0) = f(0) - c_0 \frac{1}{4\pi}[1 - \sqrt{\lambda}]. \qquad \blacksquare$$

LEMMA 8.2. *Assume that $\lambda \in (0, +\infty) \setminus \{1\}$ and $\beta \in \mathbb{C}$. Assume that*

$$\beta \neq \begin{cases} \frac{4\pi}{\ln \lambda} & \text{if } n = 2, \\ -4\pi\sqrt{\lambda}(1-\sqrt{\lambda})^{-1} & \text{if } n = 3 \end{cases} \tag{8.3}$$

and define, for a function $u \in W^{\Delta,p}(\mathbb{R}^n \setminus \{0\})$ having representation (8.2), a number

$$\tau_\beta^\lambda u = c_0^\lambda - \beta f^\lambda(0).$$

Then

$$\tau_\beta^\lambda u = 0 \iff \tau_{\beta_\lambda} u = 0,$$

where τ_{β_λ} has been defined in Definition 7.6 and

$$\beta_\lambda = \begin{cases} \beta\big(1-\beta\frac{\ln\lambda}{4\pi}\big)^{-1} & \text{if } n=2,\\ \beta\big[\sqrt{\lambda}+\frac{\beta}{4\pi}(1-\sqrt{\lambda})\big]^{-1} & \text{if } n=3 \end{cases} \tag{8.4}$$

or equivalently

$$\beta = \begin{cases} \beta_\lambda\big(1+\beta_\lambda\frac{\ln\lambda}{4\pi}\big)^{-1} & \text{if } n=2,\\ \sqrt{\lambda}\beta_\lambda\big[1-\frac{\beta_\lambda}{4\pi}(1-\sqrt{\lambda})\big]^{-1} & \text{if } n=3. \end{cases}$$

Proof. Let us choose and fix $\lambda>0$. Assume that $n=2$. By Lemma 8.1, we have

$$\begin{aligned}\tau_\beta^\lambda u &= c_0^\lambda-\beta f^\lambda(0)=c_0-\beta\Big[f(0)+c_0\frac{\ln\lambda}{4\pi}\Big]=c_0\Big(1-\beta\frac{\ln\lambda}{4\pi}\Big)-\beta f(0)\\ &=\Big(1-\beta\frac{\ln\lambda}{4\pi}\Big)\Big[c_0-\beta\Big(1-\beta\frac{\ln\lambda}{4\pi}\Big)^{-1}f(0)\Big].\end{aligned}$$

Assume that $n=3$. Then

$$\begin{aligned}\tau_\beta^\lambda u &= c_0^\lambda-\beta f^\lambda(0)=\sqrt{\lambda}c_0-\beta\Big[f(0)-c_0\frac{1}{4\pi}(1-\sqrt{\lambda})\Big]\\ &=\Big[\sqrt{\lambda}+\frac{\beta}{4\pi}(1-\sqrt{\lambda})\Big]\Big\{c_0-\beta\Big[\sqrt{\lambda}+\frac{\beta}{4\pi}(1-\sqrt{\lambda})\Big]^{-1}f(0)\Big\}.\end{aligned}$$

The proof is complete. ∎

Given $\lambda>0$ and $\beta\in\mathbb{C}$, let us define

$$\begin{aligned} D(A_{\lambda,\beta}) &:= \big\{u=\beta f^\lambda(0)G_{n,\lambda}+f^\lambda\in W^{2,p}(\mathbb{R}^n)\big\}\\ &= \big\{u\in W^{\Delta,p}\big(\mathbb{R}^n\setminus\{0\}\big)\colon \tau_\beta^\lambda u=0\big\},\\ A_{\lambda,\beta}u &:= -\lambda\beta f^\lambda(0)G_{n,\lambda}-\Delta_{n,p}f^\lambda. \end{aligned} \tag{8.5}$$

Definition 8.3. We define $(A_\infty, D(A_\infty))$ and the minus Friedrichs Laplacian, A_0, by putting

$$\begin{aligned} D(A_\infty) &= \big\{c_0G_n+f\colon c_0\in\mathbb{C},\ f\in W_0^{2,p}\big(\mathbb{R}^n\setminus\{0\}\big)\big\},\\ A_\infty u &= -c_0G_n-\Delta f,\quad u\in D(A_\infty),\\ D(A_0) &= W^{2,p}(\mathbb{R}^n),\\ A_0u &= -\Delta_{n,p}u,\quad u\in W^{2,p}(\mathbb{R}^n). \end{aligned}$$

Moreover, if $\beta\in\mathbb{C}$, then we define an operator A_β by

$$\begin{aligned} D(A_\beta) &= \big\{u=\beta f(0)G_n+f\colon f\in W^{2,p}(\mathbb{R}^n)\big\},\\ A_\beta u &= -\beta f(0)G_n-\Delta_{n,p}f. \end{aligned} \tag{8.6}$$

REMARK 8.4. Using the boundary operators τ_β defined in Definition 7.6, we can write the above definition of A_β more compactly as follows:

$$D(A_\beta) := \{u \in W^{\Delta,p}(\mathbb{R}^n \setminus \{0\}) : \tau_\beta(u) = 0\},$$
$$A_\beta u = -\Delta u, \quad u \in D(A_\beta),$$

where Δu is the distributional Laplacian in $\mathbb{R}^n \setminus \{0\}$ of a distribution $u \in \mathcal{D}'(\mathbb{R}^n \setminus \{0\})$.

LEMMA 8.5. *Assume that $\lambda \in (0, +\infty) \setminus \{1\}$ and $\beta \in \mathbb{C}$ satisfies*

$$\beta = \begin{cases} \frac{4\pi}{\ln \lambda} & \text{if } n = 2, \\ -4\pi\sqrt{\lambda}(1-\sqrt{\lambda})^{-1} & \text{if } n = 3. \end{cases} \tag{8.7}$$

Then for $n = 2, 3$

$$D(A_{\lambda,\beta}) = \{c_0 G_n + f : c_0 \in \mathbb{C},\ f \in W_0^{2,p}(\mathbb{R}^n \setminus \{0\})\},$$
$$A_{\lambda,\beta} u = -c_0 G_n - \Delta f.$$

In other words, if $\beta \in \mathbb{C}$ satisfies (8.7), then $A_{\lambda,\beta} = A_\infty$, with A_∞ defined in Definition 8.3.

PROOF. Assume that $n = 2$. Then

$$u = c_0 G_2 + f = c_0 G_{2,\lambda} + c_0(G_2 - G_{2,\lambda}) + f \in D(A_{\lambda, \frac{4\pi}{\ln\lambda}})$$

if and only if

$$c_0 = \frac{4\pi}{\ln\lambda} c_0 [G_2 - G_{2,\lambda}](0) + \frac{4\pi}{\ln\lambda} f(0) = c_0 + \frac{4\pi}{\ln\lambda} f(0).$$

This happens if and only if $f(0) = 0$. Summing up, we proved, as required, that

$$D(A_{\lambda, \frac{4\pi}{\ln\lambda}}) = \mathbb{C} G_2 \oplus W_0^{2,p}(\mathbb{R}^2 \setminus \{0\}).$$

The same arguments work in the case where $n = 3$. ∎

In the formulation of the theorem below, K_0 is the Macdonald function defined earlier in formula (3.7).

THEOREM 8.6. *Assume that $\beta \in \mathbb{C} \cup \{\infty\}$ and that $n = 2$ or $n = 3$. Then the following hold:*

(i) *$(A_\beta, D(A_\beta))$ satisfies (7.1) to (7.4), and*

$$(-\infty, 0) \setminus -\Lambda_\beta \subset \rho(A_\beta), \tag{8.8}$$

where either $\Lambda_\beta = \emptyset$ *or, in the cases listed below in* (8.9), $\Lambda_\beta = \{\lambda_\beta\}$, *where*

$$\lambda_\beta = \begin{cases} \mathrm{e}^{-\frac{4\pi}{\beta}} & \text{if } n = 2, \beta \in \mathbb{R} \setminus \{0\}, \\ \left(1 - \frac{4\pi}{\beta}\right)^2 & \text{if } n = 3 \text{ and } \beta \in (-\infty, 0) \cup (4\pi, +\infty). \end{cases} \tag{8.9}$$

Moreover, in the latter cases, we have

$$A_\beta e_{n,\beta} = -\lambda_\beta e_{n,\beta},$$

where

$$e_{3,\beta}(x) = \sqrt{\lambda_\beta} G_{3,\lambda_\beta}(x) = \frac{1}{4\pi|x|} \mathrm{e}^{-\sqrt{\lambda_\beta}|x|}, \quad x \in \mathbb{R}^3,$$
$$e_{2,\beta}(x) = G_{2,\lambda_\beta}(x) = \frac{1}{2\pi} K_0\big(\mathrm{e}^{-\frac{2\pi}{\beta}}|x|\big), \quad x \in \mathbb{R}^2.$$

(ii) *Any operator satisfying* (7.1) *to* (7.4) *is equal to* $(A_\beta, D(A_\beta))$ *for some* $\beta \in \mathbb{C} \cup \{\infty\}$.

(iii) *The resolvent sets of* A_0 *and* A_∞ *satisfy*

$$\rho(A_0) \supset (-\infty, 0) \quad \text{and} \quad \rho(A_\infty) \supset (-\infty, 0) \setminus \{-1\}.$$

Indeed, $A_0 = -\Delta_{n,p}$ *is non-negative, while* $A_\infty G_n = -G_n$, *so that* -1 *is an eigenvalue of* A_∞; *this is the case* $\lambda_\beta = 1$ *of* (8.9).

Proof.

Step 1. Proof of (8.8). Let us choose and fix $\beta \in \mathbb{C} \setminus \{0\}$ and $\lambda \in (0, +\infty) \setminus \{1\}$ satisfying (8.3). Recall that $(A_{\lambda,\beta}, D(A_{\lambda,\beta}))$ is defined by (8.5). Let us choose and fix $u \in D(A_{\lambda,\beta})$. Then by equality (8.5),

$$\begin{aligned} u &= \beta f^\lambda(0) G_{n,\lambda} + f^\lambda, \\ (\lambda I + A_{\lambda,\beta})u &= \lambda\beta f^\lambda(0) G_{n,\lambda} + \lambda f^\lambda - \lambda\beta f^\lambda(0) G_{n,\lambda} - \Delta_{n,p} f^\lambda \\ &= (\lambda I - \Delta_{n,p}) f^\lambda. \end{aligned}$$

Therefore, by Definition 3.18 of the map Π_p, we infer that

$$\lambda I + A_{\lambda,\beta} = (\lambda I - \Delta_{n,p}) \circ \Pi_p.$$

Hence, we infer that $-\lambda \in \rho(A_{\lambda,\beta})$ since $\lambda I - \Delta_{n,p}$ is a bijection from $W^{2,p}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ and its inverse is bounded. Moreover, since $\Pi_p \colon D(A_{\lambda,\beta}) \to W^{2,p}(\mathbb{R}^n)$ is bijective, we have

$$\Pi_p^{-1} f = \beta f(0) G_{n,\lambda} + f, \quad f \in W^{2,p}(\mathbb{R}^n),$$

and, by the Sobolev Embedding Theorem, we can find a constant c such that

$$|\Pi_p^{-1} f|_{L^p(\mathbb{R}^n)} \le \big(c|\beta||G_{n,\lambda}|_{L^p(\mathbb{R}^n)} + 1\big)\|f\|_{W^{2,p}(\mathbb{R}^n)}.$$

Moreover, see Lemma 8.2,

$$A_\beta = \begin{cases} A_{\lambda,\beta(1+\beta\frac{\ln\lambda}{4\pi})^{-1}} & \text{if } n=2,\\ A_{\lambda,\sqrt{\lambda}\beta\left[1-\frac{\beta}{4\pi}(1-\sqrt{\lambda})\right]^{-1}} & \text{if } n=3,\end{cases}$$

provided condition (8.3) is satisfied. From this we see that $-\lambda \in \rho(A_\beta)$ provided $\lambda > 0$ and $\lambda \neq \lambda_\beta$. This completes Step 1.

Step 2. Let us choose and fix $\lambda \in (0,+\infty)\setminus\{1\}$. We will show now that $G_{n,\lambda_\beta} \in D(A_\beta)$. Assume first that $n = 2$. Then by Lemma 3.28 (ii),

$$G_{2,\lambda} = G_2 + f, \quad \text{for some } f \in W^{2,p}(\mathbb{R}^2). \tag{8.10}$$

Since

$$f(0) = [G_{2,\lambda} - G_2](0) = \frac{1}{2\pi}\lim_{x\to 0}\left[\ln\frac{1}{|\sqrt{\lambda}x|} - \ln\frac{1}{|x|}\right] = -\frac{1}{4\pi}\ln\lambda,$$

by Definition 7.6 of the map τ_β, we infer that

$$\tau_\beta G_{2,\lambda} = 1 - \beta f(0) = 1 + \beta\frac{\ln\lambda}{4\pi}.$$

Hence, $\tau_\beta G_{2,\lambda} = 0$ if and only if $\ln\lambda = -\frac{4\pi}{\beta}$, i.e. if and only if $\lambda = \mathrm{e}^{-\frac{4\pi}{\beta}} = \lambda_\beta$. Thus, we deduce $G_{2,\lambda_\beta} \in D(A_\beta)$ and, with notation from (8.10),

$$\begin{aligned}(\lambda_\beta + A_\beta)G_{2,\lambda_\beta} &= \lambda_\beta G_{2,\lambda_\beta} - \beta f(0)G_2 - \Delta_{n,p} f\\ &= \lambda_\beta G_{2,\lambda_\beta} - G_2 - \Delta(G_{2,\lambda_\beta} - G_2) = 0.\end{aligned}$$

The case where $n = 3$ can be treated in the same way. This proves the first part of the theorem. ∎

Proof of the second part of the theorem. Let $\gamma > 0$. Then, see the proof of Theorem 7.7, any operator A satisfying (7.1) to (7.4) is equal to $A_{\gamma,\beta}$ for some $\beta \in \mathbb{C}\cup\{\infty\}$ or $-A$ is the Friedrichs Laplacian, that is, $A = A_0$. Therefore, either $A = A_0$, or $A = A_\beta$ for some $\beta \in \mathbb{C}\setminus\{0\}$, or, see Lemma 8.5, $A = A_\infty$ in the last occurring case. ∎

Proof of the third part of the theorem. This proof is short. Indeed, $A_\infty = A_{\gamma,\beta}$ for $\gamma \neq 1$, and hence $\rho(A_\infty) \supset (-\infty,0)\setminus\{-1\}$. Since $G_n \in D(A_\infty)$, we have $A_\infty G_n = -G_n$, so that -1 is indeed an eigenvalue of A_∞ and cannot be removed from the exceptional set. The case where $A_0 = -\Delta_{n,p}$ is obvious since A_0 is non-negative and self-adjoint on $L^2(\mathbb{R}^n)$, with $\sigma(A_0) = [0,+\infty)$. ∎

REMARK 8.7. If $n = 2$ or $n = 3$, then for every $\lambda > 0$, $G_{n,\lambda}$ is a fundamental solution of the operator $\lambda - \Delta$, i.e.,

$$\lambda G_{n,\lambda} - \Delta G_{n,\lambda} = \delta_0.$$

Therefore, for every $f \in L^p(\mathbb{R}^n)$, with $*$ meaning the convolution, we have

$$G_{n,\lambda} * f = (\lambda - \Delta_{n,p})^{-1} f.$$

9. HEAT EVOLUTIONS AND BOUNDARY PROBLEMS

In this section, we set $n = 2, 3$ and consider the open set

$$\mathcal{O} = \mathbb{R}^n \setminus \{0\}.$$

Recall that $(A_\beta, D(A_\beta))$, $\beta \in \mathbb{C} \cup \{\infty\}$, are operators defined in the previous sections, more precisely in (8.6). Each A_β is an operator on $L^p(\mathbb{R}^n)$ where $p \in (1, +\infty)$ if $n = 2$ and $p \in (\frac{3}{2}, 3)$ if $n = 3$. Let us recall that if $\beta \in \mathbb{R} \cup \{\infty\}$, then A_β is self-adjoint on $L^2(\mathbb{R}^n)$ and bounded from below, with

$$\sigma(A_\beta) \subset \{-\lambda_\beta\} \cup [0, +\infty), \quad \text{equivalently } \rho(A_\beta) \supset (-\infty, 0) \setminus \{-\lambda_\beta\}.$$

Recall the sign convention of Section 1.2: A_β is minus the Laplacian, so its spectrum lies to the right of $-\lambda_\beta$. Here λ_β was defined in (8.9), see also Appendix A for comments on the relation with previous work.

Moreover, $-A_\beta$ generates an analytic C_0 semigroup S_β on $L^2(\mathbb{R}^n)$. We will see later in (9.8) that S_β is given by a kernel $\mathcal{G}_\beta(t, x, y)$, that is,

$$S_\beta(t)u(x) = \int_{\mathbb{R}^n} \mathcal{G}_\beta(t, x, y)u(y)\mathrm{d}y.$$

Obviously, as $-A_0$ is the Friedrichs Laplacian $\Delta_{n,p}$, it is well defined on every $L^p(\mathbb{R}^n)$-space with $p \in (1, +\infty)$ also in dimension $n = 3$.

Let us define the following *generalized Dirichlet map*:

$$\mathcal{D}: \mathbb{C} \ni \gamma \mapsto \gamma G_n \in W^{\Delta,p}(\mathcal{O}). \tag{9.1}$$

Note that this map $\mathcal{D}$ is independent of $\beta \in \mathbb{C}$. Let us recall that τ_β has been defined in (7.14). The following result justifies the name we have used for the map $\mathcal{D}$.

PROPOSITION 9.1. *Assume that $\beta \in \mathbb{C}$. Then for every $\gamma \in \mathbb{C}$, the function $u = \mathcal{D}\gamma = \gamma G_n$ solves the boundary problem*

$$\Delta_{\mathcal{O}} u(x) = u(x) \quad \text{for } x \in \mathcal{O},$$
$$\tau_\beta u = \gamma$$

in the class $W^{\Delta,p}(\mathcal{O})$.

PROOF. Obviously, we have $\tau_\beta \gamma G_n = \gamma$. The fact that $\Delta_{\mathcal{O}} G_n = G_n$ follows from Lemma 3.9 or more directly from Lemma 3.11. ∎

PROPOSITION 9.2. *Let $\beta \in \mathbb{R} \setminus \{0\}$. Assume the following:*

(i) *For every $t > 0$ there exists an integrable function $\eta\colon \mathcal{O} \to (0, +\infty)$ such that*

$$0 \le \frac{\mathcal{G}_\beta(t, x, y)}{G_n(x)} \le \eta(y) \quad \text{for all } x, y \in \mathcal{O} \text{ such that } |x| \le 1.$$

(ii) *For every $t > 0$ and every $y \in \mathcal{O}$ the following limit exists:*

$$R_\beta(t, y) := \lim_{x \to 0} \frac{\mathcal{G}_\beta(t, x, y)}{G_n(x)}.$$

Then for every $t > 0$ we have

$$(I + A_\beta) S_\beta(t) \mathcal{D}\gamma = \frac{\gamma}{\beta} R_\beta(t, \cdot),$$

where $S_\beta(t)$, $\beta \in \mathbb{R}$, was defined at the beginning of this section.

PROOF. Let $v \in L^2(\mathbb{R}^n)$. Since the semigroup S_β is analytic and self-adjoint on $L^2(\mathbb{R}^n)$, and since $\Delta_{\mathcal{O}} G_n = G_n$, the Green formula yields

$$\begin{aligned}
\big((I + A_\beta) S_\beta(t) \mathcal{D}\gamma, v\big) &= \gamma\big(G_n, (I + A_\beta) S_\beta(t) v\big) \\
&= \gamma\big(G_n, S_\beta(t) v\big) - \gamma\big(G_n, \Delta_{\mathcal{O}} S_\beta(t) v\big) \\
&= \gamma\big(\Delta_{\mathcal{O}} G_n, S_\beta(t) v\big) - \gamma\big(G_n, \Delta_{\mathcal{O}} S_\beta(t) v\big) \\
&= \gamma \overline{g(t, 0)},
\end{aligned}$$

where we used that $A_\beta S_\beta(t) = -\Delta_{\mathcal{O}} S_\beta(t)$ and where $g(t) \in W^{2,2}(\mathbb{R}^n)$ is such that

$$S_\beta(t) v(y) = b_0(t) G_n(y) + g(t, y).$$

The proof will be complete as soon as we show that

$$g(t, 0) = \frac{1}{\beta} \int_{\mathbb{R}^n} R_\beta(t, y) v(y)\, dy.$$

We have

$$b_0(t) = \lim_{x \to 0} \frac{S_\beta(t) v(x)}{G_n(x)}.$$

Again, the semigroup S_β being analytic, we have $S_\beta(t) v \in D(A_\beta)$. Consequently, we get

$$b_0(t) = \beta g(t, 0).$$

Therefore,

$$g(t,0) = \frac{1}{\beta} \lim_{x\to 0} \int_{\mathbb{R}^n} \frac{\mathcal{G}_\beta(t,x,y)}{G_n(x)} v(y)\mathrm{d}y$$

and the desired conclusion follows from the Lebesgue dominated convergence theorem. ∎

Let us now briefly discuss the case where $\beta = 0$. In this case, by (8.6) we have $D(A_0) = W^{2,p}(\mathbb{R}^n)$. Therefore, $-A_0$ is the Friedrichs Laplacian $\Delta_{n,p}$ on $\mathbb{R}^n$. In this case, $p \in (1,+\infty)$ also if $n = 3$. The corresponding semigroup $S := S_0$ is the heat semigroup on $\mathbb{R}^n$ with Green kernel

$$G_0(t,x,y) := P(t,x-y), \quad x,y \in \mathbb{R}^n,\ t > 0,$$

where

$$P(t,x) := (4\pi t)^{-\frac{n}{2}} \mathrm{e}^{-|x|^2/4t}, \quad t > 0,\ x \in \mathbb{R}^n. \tag{9.2}$$

We will also use the following notation, beware of potential ambiguity,

$$P(t,r) := (4\pi t)^{-\frac{n}{2}} \mathrm{e}^{-r^2/4t}, \quad t > 0,\ r \geq 0.$$

Proposition 9.3. *For every $t > 0$ we have*

$$\big[(I + A_0)S_0(t)\mathcal{D}\gamma\big](x) = \gamma P(t,x) = \gamma S_0(t)\delta_0(x), \quad x \in \mathcal{O}.$$

Proof. We will follow the proof of the previous proposition. Let $v \in L^2(\mathbb{R}^n)$. Since the semigroup is analytic and self-adjoint, and since $\Delta_{\mathcal{O}} G_n = G_n$, the Green formula yields

$$\begin{aligned}
\big((I + A_0)S_0(t)\mathcal{D}\gamma, v\big) &= \gamma\big(G_n, (I + A_0)S_0(t)v\big) \\
&= \gamma\big(G_n, S_0(t)v\big) - \gamma\big(G_n, \Delta_{\mathcal{O}} S_0(t)v\big) \\
&= \gamma\big(\Delta_{\mathcal{O}} G_n, S_0(t)v\big) - \gamma\big(G_n, \Delta_{\mathcal{O}} S_0(t)v\big) \\
&= \gamma\overline{g(t,0)},
\end{aligned}$$

where $g(t) \in W^{2,2}(\mathbb{R}^n)$ is such that

$$S_0(t)v(y) = b_0(t)G_n(y) + g(t,y).$$

Clearly, if $\beta = 0$, then $b_0 \equiv 0$, and $g(t,y) = S_0(t)v(y)$. Therefore,

$$g(t,0) = \int_{\mathbb{R}^n} P(t,y)v(y)\mathrm{d}y.$$ ∎

9.1. *Stochastic boundary problem*

Let $W = (W(t) : t \geq 0)$ be a real-valued Wiener process defined on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, satisfying the usual assumptions, where $\mathbb{F} = (\mathcal{F}_t : t \geq 0)$ is the filtration. We are concerned with the boundary problem

$$\begin{aligned} \frac{\partial u}{\partial t} &= \Delta u, \\ \tau_\beta u &= \frac{\mathrm{d}W}{\mathrm{d}t}, \\ u(0) &= u_0, \end{aligned} \tag{9.3}$$

where we assume that $\beta \in \mathbb{R}$. Let us recall that τ_β is the boundary operator defined earlier in (7.14). We assume in this section that β is real because W is a real Wiener process, and we want the solutions to be real Gaussian random fields.

REMARK 9.4. As far as the initial datum u_0 is concerned, the following convention is used throughout this section. Each of the well-posedness results below is stated in a Banach space E: one of $L^p(\mathbb{R}^n)$, $H^{-\ell}(\mathbb{R}^n)$ or $H^{-\ell}_\beta(\mathcal{O})$, on which $-A_\beta$ generates a C_0-semigroup S_β, and in each of them the initial datum is assumed to satisfy

$$u_0 \in E, \tag{9.4}$$

u_0 being deterministic (or, more generally, $\mathcal{F}_0$-measurable and $\mathbb{P}$-almost surely E-valued). No further regularity is needed: since S_β is a C_0-semigroup on E, the first term $S_\beta(\cdot)u_0$ of the mild formulation below is automatically a continuous E-valued function on $[0, +\infty)$, so that the well-posedness of (9.3) in E is decided exclusively by the stochastic convolution term. This is why the conditions on n and p appearing in the statements below involve only that second term. If u_0 is random, it is in addition assumed to be independent of W, which is what makes the resulting family Markovian.

Taking into account [35, 72], the mild formulation of (9.3) takes the following form:

$$\begin{aligned} u(t) &= S_\beta(t)u_0 + (I + A_\beta)\int_0^t S_\beta(t-s)\mathcal{D}\mathrm{d}W(s) \\ &= S_\beta(t)u_0 + \int_0^t (I + A_\beta)S_\beta(t-s)G_n\mathrm{d}W(s), \quad t \geq 0, \end{aligned} \tag{9.5}$$

where we used the definition (9.1).

COROLLARY 9.5. *Assume that $\beta \neq 0$. Then under the assumptions of Proposition 9.2 we have*

$$u(t,y) = \big[S_\beta(t)u_0\big](y) + \frac{1}{\beta}\int_0^t R_\beta(t-s,y)\mathrm{d}W(s), \quad t \geq 0,\ y \in \mathbb{R}^n \setminus \{0\}.$$

9.2. *Case $\beta = 0$ for $n = 2, 3$*

Recall that, by Proposition 9.3, in this case we have

$$(I + A_0)\int_0^t S_0(t-s)\mathcal{D}\,\mathrm{d}W(s) = \int_0^t S_0(t-s)\delta_0\mathrm{d}W(s),$$

where S_0 is the heat semigroup with the kernel P defined by (9.2) (the heat semigroup kernel).

PROPOSITION 9.6. *Assume that $\beta = 0$ and u_0 satisfies condition (9.4) with $E = L^p(\mathbb{R}^n)$. If $n = 2$, then for every $p \in (1,2)$ the stochastic boundary problem (9.3) is well posed and defines a Gaussian Markov family in the $L^p(\mathbb{R}^2)$-space. If $n = 3$, then the stochastic boundary problem (9.3) is well posed and defines a Gaussian Markov family in the $L^p(\mathbb{R}^3)$-space for every $p \in (1, \frac{3}{2})$.*

PROOF. Assume that $p \in (1, +\infty)$. Since for $t > s \geq 0$,

$$\big(S_0(t-s)\delta_0\big)(y) = \big(4\pi(t-s)\big)^{-\frac{n}{2}}\mathrm{e}^{-\frac{|y|^2}{4(t-s)}}, \quad y \in \mathbb{R}^n \setminus \{0\},$$

the problem is well posed in $L^p(\mathbb{R}^n)$ if and only if, see [36], for all, or equivalently some, $t > 0$, $I(t,p) < +\infty$, where

$$I(t,p) := \int_{\mathbb{R}^n}\left(\int_0^t (4\pi s)^{-n}\mathrm{e}^{-2\frac{|y|^2}{4s}}\mathrm{d}s\right)^{p/2}\mathrm{d}y.$$

Note that $I(t,p) < +\infty$ if and only if $J(p) < +\infty$ where

$$\begin{aligned} J(p) &:= \int_{\{y\in\mathbb{R}^n:|y|\leq 1\}}\left(\int_0^{+\infty}(4\pi s)^{-n}\mathrm{e}^{-s-\frac{|\sqrt{2}y|^2}{4s}}\mathrm{d}s\right)^{p/2}\mathrm{d}y \\ &= \int_{\{y\in\mathbb{R}^n:|y|\leq 1\}}\big(G_{2n}(\sqrt{2}y)\big)^{p/2}\mathrm{d}y. \end{aligned}$$

Since, see (3.14), for small y,

$$G_{2n}(y) \approx \frac{\Gamma(n-1)}{4\pi^n}|y|^{-2n+2},$$

and, for some constant $C > 0$,

$$\int_{\{y\in\mathbb{R}^n:|y|\le 1\}} |y|^{-np+p}\mathrm{d}y = C\int_0^1 r^{-np+p+n-1}\mathrm{d}r,$$

we infer that $I(t,p) < +\infty$ if and only if $-np + p + n > 0$. ∎

REMARK 9.7. Since for some constant $C > 0$,

$$\sup_{t>0} I(t,p) = C\int_{\mathbb{R}^n}\left(\int_0^{+\infty} s^{-n}\mathrm{e}^{-\frac{|y|^2}{4s}}\mathrm{d}s\right)^{p/2}\mathrm{d}y = C\int_{\mathbb{R}^n}|y|^{-p(n-1)}\,\mathrm{d}y = +\infty,$$

there does not exist an invariant measure for the Markov family associated with the solutions to (9.3) with $\beta = 0$.

REMARK 9.8. Let

$$\begin{aligned} X(t,x) &:= \int_0^t S_0(t-s)\delta_0\mathrm{d}W(s)(x)\\ &= \int_0^t P(t-s,x)\,\mathrm{d}W(s), \quad t\ge 0,\ x\in\mathbb{R}^n\setminus\{0\}. \end{aligned}$$

Clearly, $X(t,\cdot)$, for $t\ge 0$, is a Gaussian random field on $\mathcal{O}$, with mean zero and covariance

$$\begin{aligned} \Gamma(t,x,y) &:= \mathbb{E}\,X(t,x)X(t,y) = \int_0^t P(s,x)P(s,y)\mathrm{d}s\\ &= (4\pi)^{-n}\int_0^t s^{-n}\mathrm{e}^{-\frac{|x|^2+|y|^2}{4s}}\mathrm{d}s,\ x,y\in\mathbb{R}^n\setminus\{0\}. \end{aligned}$$

The limiting "covariance" Γ as $t\to\infty$, is given by

$$\Gamma(x,y) = (4\pi)^{-n}\int_0^{+\infty} s^{-n}\mathrm{e}^{-\frac{|x|^2+|y|^2}{4s}}\mathrm{d}s = C_n\big(|x|^2+|y|^2\big)^{-n+1},$$

where

$$C_n := (4\pi)^{-n}\int_0^{+\infty} s^{-n}\mathrm{e}^{-\frac{1}{4s}}\mathrm{d}s.$$

PROPOSITION 9.9. *Assume that $\beta = 0$ and $\ell > n/2$. Then for every u_0 satisfying condition (9.4) with $E = H^{-\ell}(\mathbb{R}^n)$, the stochastic boundary problem (9.3) is well posed and defines a Gaussian Markov family in the space $H^{-\ell}(\mathbb{R}^n)$. Moreover, this family has an invariant (Gaussian) measure μ on $H^{-\ell}(\mathbb{R}^n)$ if and only if $n = 3$.*

PROOF. Since $(I-\Delta)^{-\ell/2}$ is an isometric isomorphism between the Hilbert spaces $H^{-\ell}(\mathbb{R}^n)$ and $L^2(\mathbb{R}^n)$, the desired conclusion follows because there exist constants

$C, C' > 0$ such that for every $t > 0$

$$\begin{aligned}
&\mathbb{E}\left|(I-\Delta)^{-\ell/2}\int_0^t S(t-s)\delta_0 \mathrm{d}W(s)\right|^2_{L^2(\mathbb{R}^n)} \\
&\quad = \int_0^t \int_{\mathbb{R}^n} \left((I-\Delta)^{-\ell/2} P(s,\cdot)(x)\right)^2 \mathrm{d}s\,\mathrm{d}x \\
&\quad = C\int_{\mathbb{R}^n}\int_0^t \left(1+|x|^2\right)^{-\ell} \mathrm{e}^{-2s|x|^2}\mathrm{d}s\,\mathrm{d}x \\
&\quad \le Ct\int_{\mathbb{R}^n}\left(1+|x|^2\right)^{-\ell}\mathrm{d}x \\
&\quad \le C't\int_0^{+\infty}(1+r^2)^{-\ell} r^{n-1}\,\mathrm{d}r < +\infty.
\end{aligned}$$

Next

$$\begin{aligned}
&\sup_{t>0}\mathbb{E}\left|(I-\Delta)^{-\ell/2}\int_0^t S(t-s)\delta_0 \mathrm{d}W(s)\right|^2_{L^2(\mathbb{R}^n)} \\
&\quad = C\int_{\mathbb{R}^n}\int_0^{+\infty}\left(1+|x|^2\right)^{-\ell}\mathrm{e}^{-2s|x|^2}\mathrm{d}s\,\mathrm{d}x \\
&\quad = C\int_{\mathbb{R}^n}\left(1+|x|^2\right)^{-\ell}\frac{1}{2|x|^2}\mathrm{d}x \\
&\quad = C'\int_0^{+\infty}(1+r^2)^{-\ell} r^{-2+n-1}\,\mathrm{d}r.
\end{aligned}$$

The integral converges outside the unit ball and it converges inside it if and only if $n > 2$. ∎

9.3. *The case of $\beta > 0$, the first approach*

Throughout this subsection, we assume that $\beta > 0$. Then either $\Lambda_\beta = \emptyset$ or $\lambda_\beta < 1$, see (8.9). Hence, by Theorem 8.6, the self-adjoint operator $I + A_\beta$ on $L^2(\mathbb{R}^n)$ is bounded from below by a positive constant, and its fractional powers $(I + A_\beta)^s$, $s \in \mathbb{R}$, are well defined. Let us define the norm $|\cdot|_{H_\beta^{-\ell}(\mathcal{O})}$ by

$$|\psi|^2_{H_\beta^{-\ell}(\mathcal{O})} := \left|(I+A_\beta)^{-\ell/2}\psi\right|^2_{L^2(\mathcal{O})}, \quad \psi \in L^2(\mathcal{O}),$$

and let $H_\beta^{-\ell}(\mathcal{O})$ be the completion of $L^2(\mathcal{O})$ with respect to this norm. Then $(I+A_\beta)^{-\ell/2}$ extends to an isometric isomorphism of $H_\beta^{-\ell}(\mathcal{O})$ onto $L^2(\mathcal{O})$.

PROPOSITION 9.10. *Assume that $\beta > 0$ and $\ell > 1$. Then for every u_0 satisfying condition (9.4) with $E = H_\beta^{-\ell}(\mathcal{O})$, the stochastic boundary value problem (9.3) is well posed and defines a Gaussian Markov family in the space $H_\beta^{-\ell}(\mathcal{O})$.*

PROOF. We have, for a constant $C > 0$,

$$\mathbb{E}\left|\int_0^t (I + A_\beta)S_\beta(t-s)\mathcal{D}\mathrm{d}W(s)\right|^2_{H_\beta^{-\ell}(\mathcal{O})}$$
$$= \int_0^t \left|(I + A_\beta)^{1-\ell/2}S_\beta(s)G_n\right|^2_{L^2(\mathcal{O})}\mathrm{d}s \le C\int_0^t s^{-2+\ell}\mathrm{d}s < +\infty. \qquad \blacksquare$$

REMARK 9.11. Let us note that the Markov family defined by the boundary value problem (9.3) has an invariant probability measure in $H_\beta^{-\ell}(\mathcal{O})$ if and only if

$$\sup_{t>0}\mathbb{E}\left|\int_0^t (I + A_\beta)S_\beta(t-s)\mathcal{D}\mathrm{d}W(s)\right|^2_{H_\beta^{-\ell}(\mathcal{O})} < +\infty. \tag{9.6}$$

Let us explain this. Put $E := H_\beta^{-\ell}(\mathcal{O})$ and define a process Z by

$$Z(t) := \int_0^t (I + A_\beta)S_\beta(t-s)\mathcal{D}\mathrm{d}W(s), \quad t \ge 0.$$

Let us observe that $Z(t)$ is a centered Gaussian random variable in E. Then, by formula (9.5), the unique solution u of equation (9.3) starting at $u_0 \in E$ is given by $u(t) = S_\beta(t)u_0 + Z(t)$, $t \ge 0$. Thus, u is an E-valued Ornstein–Uhlenbeck process with the transition semigroup $(P_t : t \ge 0)$ on E defined by

$$(P_t\varphi)(u_0) = \mathbb{E}\,\varphi\big(S_\beta(t)u_0 + Z(t)\big), \quad u_0 \in E,\ t \ge 0.$$

If $\Lambda_\beta = \emptyset$, then $-A_\beta$ generates a uniformly bounded analytic semigroup on the space E, and hence $S_\beta(t)u_0 \to 0$ weakly as $t \to +\infty$, and therefore an invariant measure, if it exists, must be the weak limit of the laws $\mathcal{L}(Z(t))$. But the covariance operator Q_t of the associated family $\mathcal{L}(Z(t))$ of centered Gaussian measures is increasing with respect to t. Indeed, by the Itô isometry and the change of variable $s \mapsto t - s$,

$$\langle Q_t h, h\rangle_E = \int_0^t \left|\big\langle (I + A_\beta)S_\beta(s)\mathcal{D}, h\big\rangle_E\right|^2 \mathrm{d}s.$$

But a family of centered Gaussian measures on a separable Hilbert space with increasing covariances converges weakly if and only if the traces $\operatorname{tr} Q_t = \mathbb{E}|Z(t)|_E^2$ remain bounded; i.e., (9.6) holds. The reader may consult [46, Chapter 11] for this criterion in a general setting. Conversely, if condition (9.6) holds, then the invariant measure is the centered Gaussian measure with covariance $Q_\infty = \lim_{t\to\infty} Q_t$. Obviously, it is the unique one. If condition (9.6) fails, no invariant probability measure exists because the second moments of $Z(t)$ diverge. Note also that the criterion does not involve u_0, in accordance with (9.4).

REMARK 9.12. We have

$$\begin{aligned}\sup_{t>0}\mathbb{E}\left|\int_0^t (I+A_\beta)S_\beta(t-s)\mathcal{D}\,\mathrm{d}W(s)\right|^2_{H_\beta^{-\ell}(\mathcal{O})} &= \frac{1}{\beta^2}\sup_{t>0}\mathbb{E}\left|\int_0^t R_\beta(t-s)\,\mathrm{d}W(s)\right|^2_{H_\beta^{-\ell}(\mathcal{O})}\\ &= \frac{1}{\beta^2}\int_{\mathbb{R}^n}\int_0^{+\infty}\left[(I+A_\beta)^{-\ell/2}R_\beta(s,\cdot)(x)\right]^2\mathrm{d}s\,\mathrm{d}x.\end{aligned}$$

These identities rewrite the abstract criterion (9.6) of Remark 9.11 as a concrete integrability condition on the kernel R_β. We can justify them as follows.

The first one is simply Proposition 9.2 together with the identity $\mathcal{D}\gamma = \gamma G_n$. Indeed, if $\gamma = 1$, then $(I+A_\beta)S_\beta(t)\mathcal{D} = \frac{1}{\beta}R_\beta(t,\cdot)$. Thus, up to the factor β^{-1}, the integrand in the stochastic convolution is the function $y \mapsto R_\beta(t-s,y)$. In other words, the operator integrand $(I+A_\beta)S_\beta(t-s)\mathcal{D}$, which a priori takes values in a space of distributions, is in fact represented by a genuine function $R_\beta(t-s,\cdot)$. This is precisely what follows from Proposition 9.2.

The second equality combines the Itô isometry with the change of variable $s \mapsto t-s$ and the Monotone Convergence Theorem. For every t, we have

$$\mathbb{E}\left|\int_0^t R_\beta(t-s,\cdot)\mathrm{d}W(s)\right|^2_{H_\beta^{-\ell}(\mathcal{O})} = \int_0^t \left|R_\beta(s,\cdot)\right|^2_{H_\beta^{-\ell}(\mathcal{O})}\mathrm{d}s,$$

and the right-hand side increases to the corresponding integral over the entire half-line as $t \to \infty$.

Finally, the $H_\beta^{-\ell}(\mathcal{O})$ norm is expressed using the isometric isomorphism

$$(I+A_\beta)^{-\ell/2}\colon H_\beta^{-\ell}(\mathcal{O}) \to L^2(\mathbb{R}^n),$$

see the definition of the norm $|\cdot|_{H_\beta^{-\ell}(\mathcal{O})}$ at the beginning of Section 9.3. The Fubini Theorem is then applied to swap the order of integration, as both integrands are non-negative.

The following remark is important. The resulting criterion is essentially a statement about the short-time behavior of R_β. Indeed, by part (iii) of Lemma 9.13, $R_\beta(s,y) \asymp \frac{8\pi s}{|y|}P(s,y)$ for s in a bounded interval. Consequently, the existence of an invariant measure is determined by the convergence of the double integral near $s=0$, exactly as in the case $\beta = 0$ examined in Proposition 9.3.

From now on we will assume that $n = 3$ and collect, for later use, the explicit form of the heat kernel of S_β and the estimates on the associated boundary kernel R_β. The correspondence $\alpha \leftrightarrow \beta$ between the parametrization of [6] and ours, which is used in the formulae below, is established in Lemma A.1. We also assume that

$$0 < \beta \le 4\pi \tag{9.7}$$

and define α via formula (A.2) below. Let us emphasize here that although the case $\alpha = \frac{-1}{4\pi}$ plays a special role in (A.1) and (A.2), this is no longer the case for formula (9.8) below.

Let us denote by $S_\beta = (S_\beta(t))_{t\geq 0}$ the C_0-analytic semigroup generated by $-A_\beta$. It is known, see [7], that S_β is C_0 on the space $L^p(\mathbb{R}^3)$ if and only if $p \in (\frac{3}{2}, 3)$. Moreover, $S_\beta(t)$ for $t > 0$ is an integral operator with the integral kernel $\mathcal{G}_\beta(t, x, y)$, given by

$$\mathcal{G}_\beta(t,x,y) = P(t,x-y) + \frac{2t}{|x||y|}P\big(t,|x|+|y|\big) - \frac{8\pi\alpha t}{|x||y|}\int_0^{+\infty} \mathrm{e}^{-4\pi\alpha u} P\big(t,u+|x|+|y|\big)\mathrm{d}u, \quad x,y\in\mathbb{R}^3. \tag{9.8}$$

Note that

$$\lim_{\alpha\to+\infty} \frac{8\pi\alpha t}{|x||y|}\int_0^{+\infty} \mathrm{e}^{-4\pi\alpha u} P\big(t,u+|x|+|y|\big)\mathrm{d}u = \frac{2t}{|x||y|}P\big(t,|x|+|y|\big),$$

and hence

$$\lim_{\beta\to 0+} \mathcal{G}_\beta(t,x,y) = \mathcal{G}_0(t,x,y) = P(t,x-y).$$

LEMMA 9.13. *Assume that (9.7) holds. The following assertions hold for the open subset* $\mathcal{O} = \mathbb{R}^3 \setminus \{0\}$.

(i) *For every* $t > 0$ *there exists an integrable function* $\eta\colon \mathcal{O} \to (0, +\infty)$ *such that*

$$0 \leq \frac{\mathcal{G}_\beta(t,x,y)}{G_3(x)} \leq \eta(y) \quad \text{for all } x\colon |x| \leq 1 \text{ and } y \in \mathcal{O}.$$

(ii) *For every* $t > 0$, *the following limit exists:*

$$R_\beta(t,y) := \lim_{x\to 0} \frac{\mathcal{G}_\beta(t,x,y)}{G_3(x)} = \frac{8\pi t}{|y|}P(t,y) - \frac{32\pi^2\alpha t}{|y|}\int_0^{+\infty} \mathrm{e}^{-4\pi\alpha u} P\big(t,u+|y|\big)\mathrm{d}u. \tag{9.9}$$

(iii) *For every* $T \in (0, +\infty)$ *there is a constant* $c = c(\alpha, T) > 0$ *such that*

$$c\frac{8\pi t}{|y|}P(t,y) \leq R_\beta(t,y) \leq \frac{8\pi t}{|y|}P(t,y), \quad t \in (0,T],\ y \in \mathcal{O}. \tag{9.10}$$

PROOF OF PART (i). Let us choose and fix $t > 0$ and $x, y \in \mathcal{O}$. Since

$$\frac{8\pi\alpha t}{|x||y|}\int_0^{+\infty} \mathrm{e}^{-4\pi\alpha u} P\big(t,u+|x|+|y|\big)\mathrm{d}u \leq \frac{8\pi\alpha t}{|x||y|}\int_0^{+\infty} \mathrm{e}^{-4\pi\alpha u}\mathrm{d}u P\big(t,|x|+|y|\big) \leq \frac{2t}{|x||y|}P\big(t,|x|+|y|\big),$$

we infer that

$$\mathcal{G}_\beta(t,x,y) \geq P(t,x-y) \geq 0.$$

Next let us assume that additionally $|x| \leq 1$ and $y \in \mathcal{O}$. Then, by (9.8) we infer that

$$\begin{aligned} 0 \leq \frac{\mathcal{G}_\beta(t,x,y)}{G_3(x)} &\leq 4\pi|x|\mathrm{e}^{|x|}\left(P(t,x-y)+\frac{2t}{|x||y|}P\big(t,|x|+|y|\big)\right) \\ &\leq 4\pi\mathrm{e}\times\left(\sup_{z:|z|\leq 1}P(t,z-y)+\frac{2t}{|y|}P(t,y)\right) \\ &\leq 4\pi\mathrm{e}\times\left((4\pi t)^{-\frac{3}{2}}\mathbb{1}_{\{z:|z|\leq 1\}}(y)+P\big(t,|y|-1\big)+\frac{2t}{|y|}P(t,y)\right), \end{aligned}$$

which completes the proof of part (i). ∎

The proof of part (ii) is obvious.

Proof of part (iii). In view of (9.9) (and since $P(t,\cdot)>0$), the right-hand inequality in (9.10) is obvious. In order to prove the first inequality in (9.10), we choose and fix $T>0$ and then we note that

$$0 \leq P\big(t,u+|y|\big) \leq P(t,y)\mathrm{e}^{-\frac{u^2}{4T}},$$

and hence

$$\int_0^{+\infty}\mathrm{e}^{-4\pi\alpha u}P\big(t,u+|y|\big)\mathrm{d}u \leq P(t,y)\int_0^{+\infty}\mathrm{e}^{-4\pi|\alpha|u-\frac{u^2}{4T}}\mathrm{d}u.$$

Consequently, it is enough to observe that

$$c(\alpha,T) := 1-4\pi\alpha\int_0^{+\infty}\mathrm{e}^{-4\pi|\alpha|u-\frac{u^2}{4T}}\mathrm{d}u > 0.$$ ∎

9.4. *Stochastic boundary problem on $\mathbb{R}^3\setminus\{0\}$; Case $0<\beta\leq 4\pi$*

Taking into account Corollary 9.5, Proposition 9.2, and Lemma 9.13, we can rewrite (9.3) as follows:

$$u(t,y) = S_\beta(t)u_0(y)+\frac{1}{\beta}\int_0^t R_\beta(t-s,y)\mathrm{d}W(s), \quad t\geq 0. \tag{9.11}$$

Proposition 9.14. *Assume that $0<\beta\leq 4\pi$, $p\in(\frac{3}{2},3)$. Then for every u_0 satisfying condition (9.4) with $E=L^p(\mathbb{R}^3)$, the stochastic boundary value problem (9.3) is well posed and defines a Gaussian Markov family in the space $L^p(\mathbb{R}^3)$.*

Proof. It is known, see [7, 38], that if $p\in(\frac{3}{2},3)$, then S_β is a C_0-semigroup on the space $L^p(\mathbb{R}^3)$.

Given $p > 1$, by applying the Burkholder inequality, we infer that

$$\mathbb{E}\int_{\mathbb{R}^3}\left|\int_0^t R_\beta(t-s,y)\mathrm{d}W(s)\right|^p \mathrm{d}y < +\infty$$

if and only if

$$\int_{\mathbb{R}^3}\left(\int_0^t R_\beta^2(t-s,y)\mathrm{d}s\right)^{p/2}\mathrm{d}y < +\infty. \tag{9.12}$$

Finally, the estimate from part (iii) of Lemma 9.13 yields that (9.12) holds if and only if

$$\begin{aligned}&\int_{\mathbb{R}^3}\left(\int_0^t \frac{(8\pi s)^2}{|y|^2}P^2(s,y)\mathrm{d}s\right)^{p/2}\mathrm{d}y\\ &\quad=\int_{\mathbb{R}^3}\left(\int_0^t \frac{(8\pi s)^2}{|y|^2}(4\pi s)^{-3}\mathrm{e}^{-\frac{|y|^2}{2s}}\,\mathrm{d}s\right)^{p/2}\mathrm{d}y := I(p,t) < +\infty.\end{aligned}$$

Clearly, $I(p,t) < +\infty$ for some, equivalently for all, t if and only if $J(p) < +\infty$, where

$$\begin{aligned}J(p) &:= \int_{\mathbb{R}^3}|y|^{-p}\left(\int_0^{+\infty}(4\pi s)^{-1}\mathrm{e}^{-s-\frac{|\sqrt{2}y|^2}{4s}}\mathrm{d}s\right)^{p/2}\mathrm{d}y\\ &= \int_{\mathbb{R}^3}|y|^{-p}\big(G_2\big(\sqrt{2}|y|\big)\big)^{p/2}\,\mathrm{d}y,\end{aligned}$$

where G_2 has been defined in (3.6), and, since G_2 is radial, $G_2(r)$ denotes its value at any point of $\mathbb{R}^2$ of norm r. Since by (3.6), $G_2(x) = (2\pi)^{-1}K_0(|x|)$, and the Macdonald function K_0, see [93], decays exponentially at infinity and

$$G_2(r) \sim \frac{1}{2\pi}\ln\frac{1}{r} \quad \text{as } r \to 0,$$

we have $J(p) < +\infty$ if and only if $-p+2 > -1$, that is, $p < 3$. Obviously, $p > 3/2$ by assumption, so the proposition is proved. ∎

Remark 9.15. Proposition 9.14 shows that, for $0 < \beta \le 4\pi$, the Lebesgue spaces $L^p(\mathbb{R}^3)$ with $p \in (\frac{3}{2}, 3)$, including the Hilbert space $L^2(\mathbb{R}^3)$, are natural state spaces for SPDEs of type (9.3). This observation is related to the papers [118] by Schnaubelt and Veraar and [25] by Alòs and Bonaccorsi.

A. Comparison with earlier approaches, especially the one described in [6]

Throughout this appendix we assume that $n = 2$ or $n = 3$ and $\mathcal{O} := \mathbb{R}^n \setminus \{0\}$, see [7]. In the case $n = 3$, we assume in addition that $\frac{3}{2} < p < 3$, in which case the conjugate exponent q, determined by $\frac{1}{p}+\frac{1}{q} = 1$, also satisfies $\frac{3}{2} < q < 3$. For $n = 2$

any $p \in (1, +\infty)$ is allowed. The appendix has two purposes. In Lemma A.1, we identify the family $(A_\beta)_\beta$ constructed in Section 8 with the family $(A_\alpha)_\alpha$ of [6], by exhibiting an explicit change of parameter, and correct four misprints in [6], which we came across while doing so. In Section A.1, we then compare the results obtained here with those of [6] and with the other approaches to point interactions available in the literature and explain what is gained by the present one.

LEMMA A.1. *Assume that $\beta \in \mathbb{C} \cup \{\infty\}$. Let $(A_\beta, D(A_\beta))$ be the operator introduced in Definition 8.3. Let $\alpha \in (-\infty, +\infty]$ and let $(A_\alpha, D(A_\alpha))$ be the operator introduced in [6], whose resolvent is given by formula (2.10) therein. Then $A_\alpha = A_\beta$ if and only if α and β are related via conditions (A.1)–(A.2) when $n = 3$, and via condition (A.3) when $n = 2$. In particular, the two families of operators coincide, the parametrizations differing by an explicit Möbius change of parameter. The value $\beta = 0$, i.e. minus the Friedrichs Laplacian, corresponds to $\alpha = +\infty$, and the value $\beta = \infty$, i.e. the operator A_∞ of Definition 8.3, corresponds to $\alpha = -\frac{1}{4\pi}$ when $n = 3$, and to $\alpha = 0$ when $n = 2$.*

PROOF OF LEMMA A.1 IN THE CASE $n = 3$. Using the notation from the paper [6], if $\alpha \in (-\infty, +\infty]$, then the operator A_α from [6] is equal to the operator A_β from the present paper if and only if

$$\beta = \Gamma^\alpha(1) = \begin{cases} 4\pi(1 + 4\pi\alpha)^{-1} & \text{if } \alpha \in (-\infty, +\infty) \setminus \{-\frac{1}{4\pi}\}, \\ +\infty & \text{if } \alpha = -\frac{1}{4\pi}, \\ 0 & \text{if } \alpha = +\infty; \end{cases} \tag{A.1}$$

i.e.

$$\alpha = \begin{cases} \frac{1}{\beta} - \frac{1}{4\pi} & \text{if } \beta \in (-\infty, +\infty) \setminus \{0\}, \\ +\infty & \text{if } \beta = 0, \\ -\frac{1}{4\pi} & \text{if } \beta = +\infty. \end{cases} \tag{A.2}$$

To prove this, let us choose and fix $u \in D(A_\alpha)$. Thus we can find a unique $h \in L^2(\mathbb{R}^3)$ such that $u = (\lambda + A_\alpha)^{-1}h$. Thus, by formula (2.10) from [6], with $\gamma^\alpha = \Gamma^\alpha$ given by formula (2.7) therein, we infer that

$$u(x) = [R_\lambda * h](x) + \Gamma^\alpha(\lambda) R_\lambda(x) \int_{\mathbb{R}^3} R_\lambda(y) h(y)\, \mathrm{d}y, \quad x \in \mathbb{R}^3 \setminus \{0\},$$

where R_λ is the solution to $(\lambda - \Delta)R_\lambda = \delta_0$ in $\mathbb{R}^3$. Note that $R_1 = G_3$.

We also remark that $R_\lambda * h \in W^{2,2}(\mathbb{R}^3)$ and, since R_λ is a symmetric function,

$$\int_{\mathbb{R}^3} R_\lambda(y) h(y) \mathrm{d}y = \int_{\mathbb{R}^3} R_\lambda(0 - y) h(y) \mathrm{d}y = [R_\lambda * h](0).$$

Therefore, for $x \in \mathbb{R}^3 \setminus \{0\}$

$$u(x) = [R_\lambda * h](x) + \Gamma^\alpha(\lambda) R_\lambda(x)[R_\lambda * h](0).$$

Taking $\lambda = 1$ and putting $f := G_3 * h \in W^{2,2}(\mathbb{R}^3)$, we have

$$u(x) = [G_3 * h](x) + \Gamma^\alpha(1) G_3(x)[G_3 * h](0) = f(x) + \Gamma^\alpha(1) f(0) G_3(x).$$

Hence, by the definition (8.6) of the operator A_β, we infer that $u \in D(A_\beta)$ with β given by the first part of (A.1). Thus $D(A_\alpha) \subset D(A_\beta)$. Both operators are restrictions of the adjoint of the operator $-\Delta$ with domain $\mathcal{C}_0^\infty(\mathcal{O})$, i.e. both act as minus the distributional Laplacian on $\mathcal{O}$, see Remark 8.4. Hence $A_\alpha \subset A_\beta$. Since α, and hence β, is real, both operators are self-adjoint on $L^2(\mathbb{R}^3)$, see [6] and the beginning of Section 9. Since a self-adjoint operator has no proper self-adjoint extension, we conclude that $A_\alpha = A_\beta$.

In particular, $\beta = 0$ corresponds to $\alpha = \infty$, $\beta = \infty$ corresponds to $\alpha = -\frac{1}{4\pi}$, and vice versa. The choice we made that $\beta = \infty$ when $\alpha = -\frac{1}{4\pi}$ is dictated by

$$\lim_{\alpha \to -\frac{1}{4\pi}+} 4\pi(1 + 4\pi\alpha)^{-1} = +\infty.$$

Note also that the eigenvalue $-\lambda_\beta$, see (8.9), satisfies

$$-\lambda_\beta = -(4\pi\alpha)^2, \quad \text{if } \alpha < 0, \text{ i.e. } \beta \in (-\infty, 0) \cup (4\pi, +\infty).$$

The proof is now complete for the case $n = 3$. ∎

THE CASE $n = 2$. Let $\alpha \in \mathbb{R} \setminus \{0\}$. As in the case $n = 3$, let us choose and fix $u \in D(A_\alpha)$ and let $h \in L^2(\mathbb{R}^2)$ be the unique function such that $u = (1 + A_\alpha)^{-1} h$. By formula (2.10) from [6] with $\lambda = 1$, and with $f = G_2 * h$, we have for $x \in \mathbb{R}^2 \setminus \{0\}$,

$$u(x) = [G_2 * h](x) + \Gamma^\alpha(1) G_2(x)[G_2 * h](0) = f(x) + \Gamma^\alpha(1) f(0) G_2(x).$$

Hence, by the definition (8.6) of the operator A_β, we infer that $u \in D(A_\beta)$ with

$$\beta = \Gamma^\alpha(1),$$

where, by formula (2.7) in [6], $\Gamma^\alpha(\lambda) = 4\pi(\alpha + \ln\lambda)^{-1}$. Hence

$$\beta = \begin{cases} \frac{4\pi}{\alpha} & \text{if } \alpha \in (-\infty, +\infty) \setminus \{0\}, \\ +\infty & \text{if } \alpha = 0, \\ 0 & \text{if } \alpha = +\infty. \end{cases} \tag{A.3}$$

As in the case $n = 3$, the inclusion $D(A_\alpha) \subset D(A_\beta)$ implies that $A_\alpha = A_\beta$. The value $\alpha = +\infty$ corresponds to minus the Friedrichs Laplacian, i.e. to $\beta = 0$, and the choice $\beta = \infty$ when $\alpha = 0$ is dictated by $\lim_{\alpha \to 0} |4\pi/\alpha| = +\infty$. We conclude that using the notation from [6], if $n = 2$ and $\alpha \in (-\infty, +\infty]$, then the operator A_α from [6] is equal to the operator A_β from the present paper if and only if condition (A.3) holds. ∎

REMARK A.2. While proving Lemma A.1 we came across four misprints in [6], which we record here. Firstly, in [6, formula (2.13)], one should have $-\mathrm{e}^{-\alpha}$ and not, as it has been printed, $-4\mathrm{e}^{-2\alpha}$; i.e., the correct version is

$$\sigma_p(A^\alpha) = \begin{cases} \{-\mathrm{e}^{-\alpha}\} & \text{for } \alpha \in \mathbb{R},\ n = 2, \\ \{-(4\pi\alpha)^2\} & \text{for } \alpha < 0,\ n = 3, \\ \emptyset & \text{otherwise.} \end{cases}$$

In [6, formula (2.15)] instead of $G_{4\mathrm{e}^{-2\alpha}}$ we should have $G_{\mathrm{e}^{-\alpha}}$. Thus, the correct version of [6, formula (2.15)] is

$$\Psi^\alpha(x) = G_{\mathrm{e}^{-\alpha}}(x) \qquad \text{for } n = 2,\ \alpha \in \mathbb{R},$$
$$\Psi^\alpha(x) = G_{(4\pi\alpha)^2}(x) \qquad \text{for } n = 3,\ \alpha < 0.$$

There is also another misprint in [6]. Formula (2.7) therein states that for $n = 3$

$$\Gamma^\alpha(\lambda) = 4\pi(\sqrt{\pi} + 4\pi\alpha)^{-1}.$$

The correct version of that formula should be

$$\Gamma^\alpha(\lambda) = 4\pi(\sqrt{\lambda} + 4\pi\alpha)^{-1}. \tag{A.4}$$

The correct formula is used later on, at the bottom of p. 227 of [6], and again in [7], formula (1.3). Finally, formula (2.2) in [6] is written with the normalised functions $g_\lambda = G_\lambda / \|G_\lambda\|$, see (2.5) therein, whereas the constant $\Gamma^\alpha(\lambda)$ given in (2.7) is the coefficient of $G_\lambda(x)G_\lambda(y)$ in formula (2.10) therein. Thus formula (2.10), and not formula (2.2) combined with (2.5) and (2.7), gives the resolvent kernel.

A.1. A comparison of results and of techniques

Lemma A.1 states that as *families of operators* we obtain nothing new here: for $n = 2, 3$, the operators A_β are precisely the operators A_α from paper [6], which in turn are the point interactions from [12]. What differs, and what is the subject of the present paper, is the *category* in which these operators are constructed, and the *information* that this construction provides. We list these differences below.

(a) The space in which the extension problem is posed. The classical treatment of point interactions, [6, 12], is a Hilbert space theory. One starts from the symmetric operator $-\Delta|_{\mathcal{C}_0^\infty(\mathbb{R}^n\setminus\{0\})}$ on $L^2(\mathbb{R}^n)$, observes that for $n = 2, 3$ it has deficiency indices $(1, 1)$, and applies von Neumann theory or boundary triplet theory, [51], and Krein-type resolvent formulae, [111], to parameterize its self-adjoint extensions by a point on the circle, equivalently a point of the set $\mathbb{R} \cup \{\infty\}$. This approach is not available on the Lebesgue spaces $L^p(\mathbb{R}^n)$ for $p \neq 2$ as the adjoint operators act on different spaces, there is no notion of deficiency indices, and there is no von Neumann theory.

The classification obtained here, Theorem 7.7, replaces the deficiency index with the finite-dimensional space $\mathcal{Y}_{n,p}$ from Proposition 3.20, and the von Neumann theory with the representation theorem, Theorem 3.13 together with the Green formula from Theorem 4.2. The result is the same one-parameter family when $p = 2$, but it is now accessible for every p in the admissible range, and this range is itself determined—a phenomenon that depends essentially on p and is invisible in the Hilbertian case.

(b) Resolvent formulae versus domain description. The starting point of paper [6] is an explicit Krein-type resolvent formula, formula (2.10) therein, from which everything else is deduced. The operator there is defined via its resolvent. Here, the operators are defined by a boundary condition, $D(A_\beta) = \ker \tau_\beta$, Remark 8.4, and the formula for the resolvent is a consequence, Theorem 8.6 (i). This second approach makes the boundary value problem (9.3) meaningful. Indeed, the boundary datum in (9.3) is prescribed through τ_β, so describing $D(A_\beta)$ as the kernel of a boundary operator is not a matter of taste but a necessity. This is precisely why Sections 3–4 occupy the space they do in the present paper and have no counterpart in [6].

(c) The role of the chart. In [6], the parameter α is assigned once and for all to the resolvent at a fixed spectral point. Here, the dependence on this choice is made explicit. Indeed, Lemmata 8.1–8.2 quantify what happens when the fundamental solution G_n is replaced by $G_{n,\lambda}$ and show that the parameter is transformed via a Möbius transformation. Consequently, the natural parameter space is the one-dimensional Riemann sphere rather than the line, and the operator A_∞, the one invisible in any single chart, is identified, Lemma 8.5. The change of parameter (A.1)–(A.2) between α and β is a manifestation of the same phenomenon.

(d) What is essentially new. Two things appear neither in [6] nor in the earlier literature. The first is the description of extensions for $p \neq 2$, which is not a formal generalization. Indeed, the admissible range of p, and along with it the dimension of the space $\mathcal{Y}_{n,p}$ of singular directions as well as the number of traces available on $W^{2,p}(\mathbb{R}^n \setminus \{0\})$, change with p, see Table 1 and Proposition 6.3. None of this is visible in the Hilbertian framework, where $p = q = 2$ throughout. The second is the application to the stochastic problem (9.3) from Section 9: the well-posedness of this problem is governed by the L^p-integrability of the boundary kernel R_β, so an L^p-theory rather than merely an L^2-theory is required, as is the boundary operator from point (b) above.

(e) Other approaches. Point interactions have also been constructed using quadratic forms, [16, 69], non-standard analysis and limits in the norm resolvent sense of scaled potentials, [12, 95], Γ-convergence, [50], and boundary triplet theory, [51]. The quadratic form approach yields an L^2-theory together with an associated Dirichlet

form, hence also a probabilistic interpretation, but it is restricted to $p = 2$ and real β. The boundary triplets approach is the closest in spirit to what we do here, as it also proceeds via an abstract Green identity, but it too is a Hilbert space theory. Semigroup approaches in L^p exist for related Wentzell-type boundary conditions, [21,42]. What distinguishes the present situation from those is the fact that the "boundary" here is a single point and thus has codimension n, so the trace theory must be built from scratch, which forms the content of Sections 3 and 4.

B. Density in Sobolev spaces

This appendix is based on the fifth section of Chapter 3 of the monograph [1, pp. 70–77].

Definition B.1. Assume that $n \in \mathbb{N}^*$, $m \in \mathbb{N}^*$, and $p \in (1, +\infty)$. Let $q = p^*$ be the conjugate exponent to p, i.e. $q \in (1, +\infty)$ such that $\frac{1}{p} + \frac{1}{q} = 1$. A closed subset $F \subset \mathbb{R}^n$ is called (m, q)-*polar* if and only if

$$W^{-m,q}(\mathbb{R}^n) \cap \mathcal{D}'(F) = \{0\},$$

where $\mathcal{D}'(F)$ is the set of all distributions on $\mathbb{R}^n$ whose support is contained in the set F, i.e.

$$\mathcal{D}'(F) := \{\Lambda \in \mathcal{D}'(\mathbb{R}^n) \colon \operatorname{supp} \Lambda \subset F\}$$

and the space $W^{-m,q}(\mathbb{R}^n)$, usually defined as the dual space to $W^{m,p}(\mathbb{R}^n)$, is the set of all distributions $\Lambda \in \mathcal{D}'(\mathbb{R}^n)$ of the form

$$\Lambda = \sum_{0 \le |\alpha| \le m} (-1)^{|\alpha|} D^\alpha v_\alpha, \quad v_\alpha \in L^q(\mathbb{R}^n) \tag{B.1}$$

equipped with the norm

$$\|\Lambda\|^q_{W^{-m,q}} = \inf \Big\{ \sum_{0 \le |\alpha| \le m} |v_\alpha|^q_{L^q(\mathbb{R}^n)} \colon \text{(B.1) holds} \Big\}.$$

In particular, a closed subset $F \subset \mathbb{R}^n$ is called $(1, p)$-*polar* if and only if

$$W^{-1,p}(\mathbb{R}^n) \cap \mathcal{D}'(F) = \{0\}.$$

Equivalently, a closed subset $F \subset \mathbb{R}^n$ is called $(1, p)$-polar if and only if for every distribution Λ of the following form

$$\Lambda = v_0 + \sum_{j=1}^{n} D_j v_j, \quad v_i \in L^p(\mathbb{R}^n),\ i = 0, \dots, n,$$

if $\operatorname{supp} \Lambda \subset F$, then $\Lambda = 0$.

We would like to emphasize here that in the symbol (m,q)-polar the second entry is the exponent of the *dual* space $W^{-m,q}(\mathbb{R}^n)$. Hence, in $(1,p)$-polar it is p that plays the role of the dual exponent, the primal one being q. This is worth keeping in mind when reading Corollary B.4 below.

The following result is obvious.

PROPOSITION B.2. *Assume that $n \in \mathbb{N}^*$, $m,k \in \mathbb{N}^*$, and $p_1, p_2 \in (1,+\infty)$. Let p_1^*, p_2^* be the conjugate exponents of p_1 and p_2, respectively. If*

$$W^{m,p_1}(\mathbb{R}^n) \subset W^{k,p_2}(\mathbb{R}^n),$$

then every set which is (m,p_1^*)-polar is also (k,p_2^*)-polar.

EXAMPLE B.3. Assume that F is a singleton, i.e. $F=\{a\}$, for some $a \in \mathbb{R}^n$. If $\Lambda \in \mathcal{D}'(F)$, then by [114, Theorem 6.25], there exists a finite set of numbers c_α, $\alpha \in \mathbb{N}^n$, such that

$$\Lambda = \sum_\alpha c_\alpha D^\alpha \delta_a,$$

where δ_a is the Dirac delta distribution at a. Hence, F is not an (m,q)-polar set if and only if $\delta_a \in W^{-m,q}(\mathbb{R}^n)$. But, by the Sobolev Embedding Theorem (at least when m is a natural number), the latter condition holds if and only if the space $W^{m,p}(\mathbb{R}^n)$ is a subset of $\mathcal{C}_b(\mathbb{R}^n)$, i.e. if and only if $m - \frac{n}{p} > 0$.

Therefore, the following holds.

COROLLARY B.4. *Assume that $n \in \mathbb{N}^*$, $m \in \mathbb{N}^*$. Let $p,q \in (1,+\infty)$ be the conjugate exponents. A singleton subset of $\mathbb{R}^n$ is an (m,q)-polar set if and only if $m - \frac{n}{p} \le 0$, i.e. if and only if $m + \frac{n}{q} \le n$. In particular, a singleton subset of $\mathbb{R}^n$ is a $(1,p)$-polar set if and only if $1 + \frac{n}{p} \le n$. Thus, if $n=1$, this condition reads $1 + \frac{1}{p} \le 1$, i.e. $\frac{1}{p} \le 0$, and it is therefore satisfied by no $p \in (1,+\infty)$. Hence, no singleton subset of $\mathbb{R}^1$ is a $(1,p)$-polar set. A singleton subset of $\mathbb{R}^n$, $n \ge 2$, is a $(1,p)$-polar set if and only if*

$$p \ge \frac{n}{n-1} = 1 + \frac{1}{n-1}.$$

The two statements are consistent. Indeed, if n is formally regarded as a real parameter, the right-hand side $\frac{n}{n-1}$ converges to $+\infty$ as $n \searrow 1$, so the range of admissible p becomes empty in the limit. They are also consistent with Theorem B.5 below, which for $n=m=1$ gives density only for $p \le 1$, and with Proposition 6.3, according to which for $n=1$ the trace $f \mapsto f(0)$ never vanishes identically on $W^{2,p}(\mathbb{R})$, i.e. $W_0^{2,p}(\mathbb{R}\setminus\{0\}) \neq W^{2,p}(\mathbb{R})$.

The following result is a consequence of Corollary B.4 and [1, Theorem 3.28].

THEOREM B.5. *Assume that* $n \in \mathbb{N}^*$, $m \in \mathbb{N}^*$, $p \in (1, +\infty)$ *and* $a \in \mathbb{R}^n$. *Then the set* $\mathcal{C}_0^\infty(\mathbb{R}^n \setminus \{a\})$ *is dense in* $W^{m,p}(\mathbb{R}^n)$ *if and only if* $p \leq \frac{n}{m}$.

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