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A Convolution Inequality Concerning Cantor-Lebesgue Measures

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It is known that there exist positive measures μ on the circle group $T = \mathbb{R}/2\pi Z$, totally singular with respect to Lebesgue measure, for which there exist exponents $1 < p < q < \infty$ such that $\|f*\mu\|_q \leqslant C\|f\|_p$ for all f. Thus convolution with μ is smoothing in a weak sense. For instance, Stein [4] has pointed out that any measure satisfying $|\hat{\mu}(n)| = 0(|n|^{-\epsilon})$, $\epsilon > 0$, has this property, and Bonami [2] has shown that certain Riesz products, whose Fourier coefficients do not tend to zero, do also. Let μ_λ denote the Cantor-Lebesgue measure associated with the Cantor set of constant ratio of dissection $\lambda > 2$. Then Oberlin [3] proved that μ_3 has the property in question, and by building in part on his work Beckner, Janson and Jerison [1] proved the same for all rational $\lambda > 2$. In the present note a simple technique for the treatment of questions of this type will be introduced and applied to the μ_λ, for irrational λ as well. The technique is rather imprecise but flexible, and applies to Riesz products as well as to a certain class of multiplier operators. It rests on Littlewood-Paley theory and iteration, together with knowledge of the Fourier coefficients of the μ_λ.

To fix the notation let dx denote Lebesgue measure on T, normalized to be a probability measure. $\|f\|_p = \left(\int |f(x)|^p \, dx\right)^{1/p}$. If $m: Z \to \mathbb{C}$ and $1 < p \leqslant q < \infty$, its multiplier norm is defined to be

$$\|m\|_{p,q} = \sup_f \|(m\hat{f})^\vee\|_q / \|f\|_p.$$

Both the function and the associated operator will be denoted by m. Thus mf denotes $(m\hat{f})^\vee$ and $m_1 m_2$ denotes both the product of two functions and the composition of two operators. Let $2 < \lambda \in \mathbb{R}$. The Cantor set E_λ of constant ratio of dissection λ is the subset of $T = [-\pi, \pi)$ defined as follows: Delete from T the interval of length $2\pi(1 - 2\lambda^{-1})$ centered at 0. From each of the two intervals remaining delete a centered interval whose length is $(1 - 2\lambda^{-1})$ times the length of the interval. Continue indefinitely and let E_λ be the set of all points not eventually deleted. Associated to E_λ in a natural way is a totally singular probability measure μ_λ. We refer to Zygmund [5] for the precise definition and for the formula

$$\hat{\mu}(n) = (-1)^n \prod_{j=1}^{\infty} \cos(\pi(\lambda - 1)\lambda^{-j} n).$$

Theorem. *For any real $\lambda > 2$ and any $p \in (1, \infty)$ there exists $q(p, \lambda) > p$ such that $\|f * \mu_\lambda\|_q \leqslant \|f\|_q$ for all $f \in L^p$.*

It suffices to demonstrate the existence of $q > 2$ and $B < \infty$ such that $\|f * \mu_\lambda\|_q \leqslant B\|f\|_2$ for all f. For the general result of Beckner, Janson and Jerison then implies the existence of $r(B, q) > 2$ for which convolution with μ_λ is actually a contraction from L^2 to L^r. Alternatively, our argument could be refined slightly to yield $B = 1$ directly. Since convolution with any probability measure of mass one is a contraction on L^1 and L^∞, the Riesz-Thörin interpolation theorem then establishes our therorem for all p. One advantage of the case $p = 2$ is the next remark, taken from [1]: If $m_1, m_2 : Z \to \mathbb{C}$ are multipliers, $q \geqslant 2$ and $|m_1(n)| \leqslant |m_2(n)|$ for all $n \in Z$, then $\|m_1\|_{2,q} \leqslant \|m_2\|_{2,q}$. For m_1 may be expressed as $m_0 m_2$ where $\|m_0\|_{l^\infty} \leqslant 1$, and hence $\|m_1\|_{2,q} \leqslant$ $\leqslant \|m_0\|_{2,2} \; \|m_2\|_{2,q} \leqslant \|m\|_{2,q}$.

We say that a strictly increasing sequence $\{n_j : j \geqslant 0\} \subset Z$ is σ-lacunary if $\sigma > 1$ and $(n_{j+1} - n_j) \geqslant \sigma(n_j - n_{j-1})$ for all $j \geqslant 1$. Given such a sequence define multiplier operators Δ_j by

$$(\Delta_j f)^\wedge(n) = \begin{cases} \hat{f}(n) & \text{if} \quad n_j \leqslant n < n_{j+1} \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 1. *If $1 < p \leqslant 2 \leqslant q < \infty$ and $\sigma > 1$, there exists $A_1(p, q, \sigma) < \infty$ such that for any σ-lacunary sequence $\{n_j\}$ and any $m : Z \to \mathbb{C}$ satisfying $m(n) = 0$ for all $n < n_0$,*

$$\|m\|_{p,q} \leqslant A_1 \sup_j \|\Delta_j m\|_{p,q}.$$

If σ is fixed then $A_1 \to 1$ as $p, q \to 2$.

PROOF. By Littlewood-Paley theory $\|(\Sigma|\Delta_j f|^2)^{1/2}\|_p \leqslant C_1\|f\|_p$ for all $p \in (1, \infty)$, where $C_1 = C_1(p, \sigma) \to 1$ as $p \to 2$. Moreover if $\hat{f}(n) = 0$ for all $n < n_0$ then $\|f\|_p \leqslant C_2\|(\Sigma|\Delta_j f|^2)^{1/2}\|_p$, where again $C_2 \to 1$ as $p \to 2$. Hence

$$\begin{aligned}\|mf\|_q &\leqslant C_2\|(\Sigma|\Delta_j mf|^2)^{1/2}\|_q \\ &\leqslant C_2(\Sigma\|\Delta_j mf\|_q^2)^{1/2} \\ &\leqslant C_2 \sup \|\Delta_j m\|_{p,q} \cdot (\Sigma\|\Delta_j f\|_p^2)^{1/2} \\ &\leqslant C_2 \sup \|\Delta_j m\|_{p,q}\|(\Sigma|\Delta_j f|^2)^{1/2}\|_p \\ &\leqslant C_1 C_2 \sup \|\Delta_j m\|_{p,q}\|f\|_p.\end{aligned}$$

Minkowski's inequality plus the hypotheses $p \leqslant 2 \leqslant q$ imply the second and fourth inequalities.

This clarification of the author's original proof is due to E. M. Stein. The lemma fails for all other pairs of exponents p, q. An elementary variant will also be useful below. Suppose $\{I_j: 1 \leqslant j \leqslant N\}$ are disjoint intervals. Let $m_j = m \cdot \chi_{I_j}$ and suppose that $m = \Sigma m_j$.

Lemma 2. *For any $1 < p \leqslant q < \infty$ there exists $A_2(p, q, N) < \infty$ such that $\|m\|_{p,q} \leqslant A_2 \max \|m_j\|_{p,q}$. If N is fixed then $A_2 \to 1$ as $p, q \to 2$.*

The proof involves only the boundedness of the Hilbert transform and the Riesz-Thörin theorem.

Fix λ and let $\delta > 0$ be a small number, depending on λ, to be specified momentarily. By an interval we henceforth mean a subinterval I of $\mathbb{R}$, neither of whose endpoints lie in Z. Though only the intersection of I with Z will actually be relevant, it is convenient to work in $\mathbb{R}$.

Lemma 3. *For any $\lambda > 1$ there exists $\delta > 0$ such that for any $k \geqslant 1$ and any interval I of length $|I| \leqslant \lambda^k/2(\lambda - 1)$, there exists a subinterval $J \subset I$ so that*

$$|J| \leqslant \frac{\lambda^{k-1}}{2(\lambda - 1)}$$

$$|I \backslash J| \leqslant \frac{(\lambda^k - \lambda^{k-1})}{2(\lambda - 1)} = \frac{\lambda^{k-1}}{2}$$

$$|\cos(\pi(\lambda - 1)\lambda^{-k}\xi)| \leqslant 1 - \delta \quad \textit{for} \quad \xi \in I \backslash J$$

and so that each endpoint of J either coincides with an endpoint of I or lies at distance greater than $\delta\lambda^k$ from the boundary of I.

This holds by homogeneity and the fact that $\cos(\pi\xi)$ has at most one quarter of a full period on any interval of length $1/2$, hence has absolute value equal to one at most once. This final conclusion is purely technical in significance.

Finally we turn to the Cantor-Lebesgue measures. Let $q = q(\lambda)$ be slightly larger than two. Set $m_k(\xi) = \prod_{j=1}^{k} \cos(\pi(\lambda - 1)\lambda^{-k}\xi)$.

We show by induction that $\|m_k\chi_I\|_{2,q} \leqslant B$ for any interval I of length at most $\lambda^k/2(\lambda - 1)$, with q and B indpendent of I and k. Since $|\mu_\lambda(\xi)| \leqslant |m_k(\xi)|$, the theorem then follows via the remark preceding Lemma 1 and an easy passage to the limit.

Given such an I, fix a subinterval $J_k \subset I$ satisfying the conclusions of Lemma 3. Partition $I \backslash J_k$ into at most $\lambda + 3$ subintervals of lengths at most $\lambda^{k-1}/2(\lambda - 1)$. By induction on k the multiplier norm of the restriction of m_{k-1} to each subinterval is at most B, and hence $\|m_k\chi_{I\backslash J_k}\|_{2,q} \leqslant (1 - \delta)\|m_{k-1}\chi_{I\backslash J_k}\|_{2,q} \leqslant$ $\leqslant (1 - \delta)A_2B$ by Lemma 2 and the remark preceding Lemma 1.

Since $|J_k| \leqslant \lambda^{k-1}/2(\lambda - 1)$, Lemma 3 may be applied repeatedly to construct $J_1 \subset J_2 \subset \cdots \subset J_k$ where $|J_i| \leqslant \lambda^{i-1}/2(\lambda - 1)$, so that all conclusions of that lemma hold at each step. By induction and the reasoning of the last paragraph $\|m_k\chi_{J_{i+1}\backslash J_i}\|_{2,q} \leqslant (1 - \delta)A_2B$. Let $\{n_j\}$ denote the finite sequence of distinct right endpoints of the intervals J_i, in ascending order, and let R and L be those portions of I lying to the right and left of J_1, respectively. If it were true that $\{n_j\}$ must be σ-lacunary, then we could conclude by Lemma 1 that $\|m_k\chi_R\|_{2,q} \leqslant (1 - \delta)A_1A_2B$. Since J_1 contains at most one integer and $\|m_k\|_{l^\infty} \leqslant 1$, certainly $\|m_k\chi_{J_1}\|_{2,q} \leqslant 1$. Treating L in the same fashion as R and applying Lemma 2 yields $\|m_k\chi_I\|_{2,q} \leqslant A_2 \max(1, (1 - \delta)A_1A_2B)$. Fix any B strictly larger than one. Then $A_2 \max(1, (1 - \delta)A_1A_2B) \leqslant B$ provided q is sufficiently close to two. Thus the inductive step would be complete.

Unfortunately $\{n_j\}$ need not quite be lacunary. But let N be the least integer such that $\delta\lambda^N \leqslant 1$. Then $\{n_j: j \equiv 0 \bmod N\}$ is σ-lacunary, with $\sigma = 2(\lambda - 1) > 1$, provided δ is small. Indeed the worst case occurs when a large number of the J_i share one right endpoint, so that $n_{Nj} - n_{N(j-1)} = \lambda^k/2(\lambda - 1)$ for some large k. But if $n_{N(j+1)} > n_{Nj}$ then by the final clause of Lemma 3 $n_{N(j+1)} - n_{Nj} \geqslant$ $\geqslant \delta\lambda^{k+N} \geqslant \lambda^k$, so $\sigma \geqslant \lambda^k/(\lambda^k/2(\lambda - 1)) = 2(\lambda - 1)$. By first Lemma 2 and then Lemma 1, $\|m_k\chi_R\|_{2,q} \leqslant (1 - \delta)A_1A_2^2B$, and the proof is concluded as above.

Remarks

1. The Theorem holds for Cantor-Lebesgue measures with variable ratios of dissection as well. Suppose $2 < A < \infty$ and let $2 < \lambda_j \leqslant A$ for each $j \geqslant 1$. Then

$$\hat{\mu}(n) = (-1)^n \prod_{j=1}^{\infty} \cos\left(\pi(\lambda_j - 1) \prod_{i \leq j} \lambda_i^{-1} n\right)$$

is the sequence of Fourier coefficients of a probability measure μ [5], and the above arguments apply equally well to μ.

2. If $\lambda \leqslant 2$ then the construction of the Cantor-Lebesgue measure μ_λ breaks down. But the formula for $\hat{\mu}_\lambda$ still makes sense, and by the same reasoning defines a bounded multiplier from L^2 to L^q for some $q > 2$, provided $\lambda > 1$.

3. Our techniques produce examples of weighted norm inequalities for Fourier series which fall outside the scope of the general theory presently known. If convolution with μ is bounded from L^p to L^2 then $(\Sigma|\hat{f}(n)|^2 w(n))^{1/2} \leqslant C\|f\|_p$ where $w(n) = |\hat{\mu}(n)|^2$. More general sequences w may be constructed by iterating Littlewood-Paley decompositions of Z as in our proof.

4. The simplest examples [4] of singular measures μ with the property in question are those for which $\hat{\mu}(n) \to 0$ at a geometric rate as $|n| \to \infty$. Riesz products and Cantor-Lebesgue measures are interesting in part because their Fourier coefficients do not tend to zero. However the main point in our argument is that their Fourier coefficients actually do tend to zero, as n «tends to infinity» in a rather nonstandard sense reminiscent of p-adic analysis.

5. Our argument is closely related to the theory of $\Lambda(p)$ sets.

That μ_λ has the L^p-improving property for all rational $\lambda > 2$ was established independently and almost simultaneously by Ritter [6] and Beckner, Janson and Jerison; Ritter's proof appears to have been the first.

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