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Quantum limits of the Laplacian perturbed along a geodesic on $\mathbb{S}^2$

Santiago Verdasco

Abstract. The study of the transition between integrability and chaos in quantum systems has been the subject of much attention in the last forty years. This article studies a generalization of a model introduced for this purpose by Šeba: he considered perturbations of the free Hamiltonian in an integrable billiard by point scatterers, obtained as the limit of solid scatterers as the radius tends to zero, which render the billiard chaotic. The model considered here is the Laplace–Beltrami operator on the two-sphere $\mathbb{S}^2$ perturbed by a measure supported on an equator, which is formally the limit of a solid scatterer supported in a small neighborhood of a geodesic. We analyze quantum limits and semiclassical measures of sequences of high-frequency eigenfunctions, which describe how their L^2-masses are distributed in phase space. A consequence of the complete integrability of the geodesic flow of the sphere is that the family of all possible semiclassical measures associated with sequences of eigenfunctions for the free Laplacian coincides with the set of positive measures on the unit cosphere bundle $S^*\mathbb{S}^2$ that are invariant under geodesic flow. If a bounded perturbation is added, it is known that this equality may become a strict inclusion. In this article, we show that the presence of a singular delta potential on a closed geodesic results in the existence of sequences of eigenfunctions whose semiclassical measure is not invariant under geodesic flow. In particular, one can find a sequence of eigenfunctions whose energy asymptotically concentrates on the hemisphere bounded by the equator on which the potential is concentrated.

1. Introduction

Understanding the properties of eigenfunctions of a self-adjoint Schrödinger operator has been a topic of intensive research from different areas of mathematics (Analysis, Differential Geometry, Number Theory, and Representation theory, among others) over the last sixty years. They arise originally as the basic building blocks for analyzing the dynamics of classical and quantum waves, heat propagation, etc.; see the monograph [49] or the survey [48] for a detailed introduction to the field. A particularly relevant aspect of the study of eigenfunctions is understanding how the global dynamical properties of the underlying classical Hamiltonian system, such as complete integrability or chaotic behavior, determine their structure. When the classical system under consideration is the geodesic flow on a compact Riemannian manifold without boundary (M, g), the corresponding quantum systems are bounded perturbations of the Laplace–Beltrami operator. In that setting,

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eigenfunctions encode local and global geometric properties of (M, g): for example, the first eigenfunctions provide information about the curvature of (M, g) and on Cheeger's constant [12]; high-frequency eigenfunctions, on the other hand, are related to the long-time dynamics of the geodesic flow. In this article, we are interested in the latter regime; in particular, we are interested in understanding the geometry of those regions on M where sequences of high-frequency eigenfunctions may concentrate. A relevant conjecture in that direction was formulated by Rudnick and Sarnak [35]; roughly speaking, it states that sequences of high-frequency eigenfunctions on a manifold with chaotic geodesic flow should equidistribute on the manifold.

The precise definition of *high-frequency* eigenfunctions (that is, eigenfunctions corresponding to large eigenvalues) of Schrödinger operators on (M, g) is as follows: let Δ be the positive Laplace–Beltrami operator on (M, g), and for a real-valued $V \in L^\infty(M)$ form the operator $\Delta + V$. Eigenfunctions of $\Delta + V$ are solutions $u \in L^2(M)$, $u \neq 0$, to the equation

$$[\Delta + V]u = \eta u \quad \eta \in \mathbb{R}. \tag{1.1}$$

Since M is compact, (1.1) has a non-zero solution only for an increasing sequence of real numbers $(\eta_n)_{n\in\mathbb{N}}$, $\eta_n \nearrow +\infty$, called eigenvalues; we write $\mathrm{Spec}(\Delta + V) = \{\eta_n\}_{n\in\mathbb{N}}$, this is the spectrum of $\Delta + V$.

A well-established approach to the study of high-frequency eigenfunctions consists of considering sequences of probability measures $(|u_n|^2\,\mathrm{d}x)_{n\in\mathbb{N}}$ on M, where, for every $n \in \mathbb{N}$, u_n satisfies

$$(\Delta + V)u_n = \lambda_n^2 u_n, \quad \|u_n\|_{L^2(M)} = 1, \quad \lim_{n\to\infty} \lambda_n^2 \to \infty. \tag{1.2}$$

Observe that the sequence of measures $(|u_n|^2\,\mathrm{d}x)_{n\in\mathbb{N}}$ is bounded in $(\mathcal{C}(M))^*$; thus, thanks to the Banach–Alaoglu theorem, there exist a subsequence $(|u_{n_j}|^2\,\mathrm{d}x)_{j\in\mathbb{N}}$ and a positive measure ν such that

$$\lim_{j\to\infty} \int_M a(x)\left|u_{n_j}(x)\right|^2 \mathrm{d}x = \int_M a(x)\nu(\mathrm{d}x) \quad \forall a \in \mathcal{C}(M).$$

Measures ν arising this way are called *quantum limits*. Several results support—see a small account on related literature below—the idea that the structure of the set of all quantum limits is related to the dynamical properties of the geodesic flow of (M, g). In the language of quantum limits, Rudnick and Sarnak's aforementioned *Quantum Unique Ergodicity conjecture* states that the only possible quantum limit on a constant negatively curved surface, whose geodesic flow has strong ergodic properties [7, Appendix], is the Riemann volume measure.

To bring the connection between quantum limits and geodesic flow to light, it is useful to study them in a common setting. The geodesic flow of (M, g) can be defined naturally as a Hamiltonian flow on phase space T^*M, the cotangent bundle of M. On the other hand, we can also lift the probability measures $|u_{n_j}(x)|^2\,\mathrm{d}x$ to T^*M. One way to do this is to consider (semiclassical) *Wigner distributions*, which are defined for $u \in L^2(M)$ and

$h > 0$ by

$$W_h[u](a) := \left\langle u \mid \mathrm{Op}_h(a)u \right\rangle_{L^2(M)}, \quad a \in \mathcal{C}_c^\infty(T^*M), \tag{1.3}$$

where $\mathrm{Op}_h(a)$ is the semiclassical pseudodifferential operator with symbol a obtained by the semiclassical Weyl Quantization; see [51, Chapter 14] for a definition of the operators $\mathrm{Op}_h(a)$.

If one chooses a sequence of *semiclassical parameters* $h_n \to 0^+$ and swaps the sequence of measures $(|u_n|^2 \mathrm{d}x)_{n\in\mathbb{N}}$ with the sequence of Wigner distributions $(W_{h_n}[u_n])_{n\in\mathbb{N}}$, a weak compactness argument implies the existence of a subsequence $(W_{h_{n_k}}[u_{n_k}])_{k\in\mathbb{N}}$ and a positive measure μ on T^*M such that

$$\lim_{k\to\infty} \left\langle u_{n_k} \mid \mathrm{Op}_{h_{n_k}}(a)u_{n_k} \right\rangle_{L^2(M)} = \int_{T^*M} a(x,\xi)\mu(\mathrm{d}x,\mathrm{d}\xi) \quad \forall a \in \mathcal{C}_c^\infty(T^*M).$$

These measures μ are called *semiclassical measures*; they simultaneously detect concentration effects *and* oscillations of order h_n^{-1} developed by the sequence $(u_n)_{n\in\mathbb{N}}$. When studying eigenfunctions of a second order differential operator, such as $\Delta + V$, one usually makes the choice $h_n := \lambda_n^{-1}$,[1] and equation (1.2) can be rewritten in semiclassical form:

$$[h_n^2\Delta + h_n^2 V]u_n = u_n, \quad \|u_n\|_{L^2(M)} = 1, \quad \lim_{n\to\infty} h_n = 0. \tag{1.4}$$

With this choice of $(h_n)_{n\in\mathbb{N}}$, semiclassical measures are always probability measures on T^*M. If a sequence $(u_n)_{n\in\mathbb{N}}$ satisfying (1.4) has a unique quantum limit ν and a unique semiclassical measure μ then

$$\int_{T^*M} a(x)\mu(\mathrm{d}x,\mathrm{d}\xi) = \int_M a(x)\nu(\mathrm{d}x) \quad \forall a \in \mathcal{C}(M). \tag{1.5}$$

Let us denote by $\mathcal{M}_{\mathrm{sc}}(\Delta + V)$ the family of all possible semiclassical measures μ arising from a sequence of eigenfunctions $(u_n)_{n\in\mathbb{N}}$ as in (1.4). Using the symbolic calculus for pseudodifferential operators, one can show that every $\mu \in \mathcal{M}_{\mathrm{sc}}(\Delta + V)$ belongs to $\mathcal{P}_{\mathrm{inv}}(S^*M)$, the family of all measures that possess the following two properties:

(1) μ is a probability measure, and $\mathrm{supp}\,\mu \subseteq S^*M$,

(2) μ is invariant under geodesic flow ϕ_t on T^*M, that is, $(\phi_t)_*\mu = \mu$ for all $t \in \mathbb{R}$.

The problem of characterizing the set $\mathcal{M}_{\mathrm{sc}}(\Delta + V)$ for a given geometry is a very difficult one, even for $V = 0$, and has been completely solved in very few cases.

In compact rank-one symmetric spaces, a setting with a large number of symmetries and periodic geodesic flow, for example, the spheres $\mathbb{S}^d$, it was proved that $\mathcal{M}_{\mathrm{sc}}(\Delta) = \mathcal{P}_{\mathrm{inv}}(S^*M)$ [6, 22, 27]. However, this is a highly unusual situation: many of the known results show that under certain conditions in the geometry or potential, semiclassical measures enjoy additional properties that force $\mathcal{M}_{\mathrm{sc}}(\Delta + V) \subsetneq \mathcal{P}_{\mathrm{inv}}(S^*M)$. For example, if

[1] In fact, any other sequence $(h_n)_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty} h_n\lambda_n = 1$ will produce the same measure μ.

$V \neq 0$ and either (M, g) is a Zoll manifold [27, 28, 30], or (M, g) has completely integrable geodesic flow [2–4, 29], it has been shown that any $\mu \in \mathcal{M}_{\mathrm{sc}}(\Delta + V)$ is invariant under an additional Hamiltonian flow on S^*M derived from V.

Meanwhile, if $V = 0$ and (M, g) has negative sectional curvature, the results of [1, 5, 34] imply that every $\mu \in \mathcal{M}_{\mathrm{sc}}(\Delta)$ has positive entropy, which prevents μ from being supported inside a closed geodesic. Additionally, if $\dim(M) = 2$, then every $\mu \in \mathcal{M}_{\mathrm{sc}}(\Delta)$ has full support on S^*M, that is, $\mu(U) > 0$ for every open subset of S^*M [15, 16]. These results are partial advances towards the Quantum Unique Ergodicity Conjecture [35], whose precise statement is that on a compact closed surface with constant negative curvature $\mathcal{M}_{\mathrm{sc}}(\Delta) = \{\mathfrak{L}\}$, where $\mathfrak{L}$ stands for the Liouville measure on S^*M induced by the symplectic structure on T^*M. This conjecture was based on the works [14, 39, 47], whose combined efforts led to what is currently known as the Quantum Ergodicity Theorem: given a closed Riemannian manifold (M, g) with ergodic geodesic flow and an orthonormal basis of eigenfunctions of Δ, there exists a density-one subsequence whose single semiclassical measure is $\mathfrak{L}$.

In [37], Šeba introduced a different model with classical integrable dynamics but conjectured to possess quantum chaotic features. This model is very different from surfaces of negative curvature, as it is obtained by perturbing any Laplacian Δ with a Dirac delta potential, δ_q. From a classical point of view, such short-range perturbation only affects a zero measure set of trajectories; hence it is not meaningful from a dynamical standpoint; however, from a quantum point of view, this point-perturbation should produce diffraction phenomena similar to those produced by corners. What is now known as the Šeba billiard is obtained as the limiting dynamics of the billiard flow in a rectangle minus a disk of radius $r > 0$ (a Sinai billiard) as r tends to zero. The quantizations of the Šeba billiard can be realized as a one-parameter family of operators, which can be written formally as

$$\Delta + \alpha |\delta_q\rangle\langle\delta_q| \tag{1.6}$$

for some constant $\alpha \in \mathbb{R}$ that represents the intensity of the perturbation. Here $|\delta_q\rangle\langle\delta_q|$ is short for the finite rank-one projector

$$|\delta_q\rangle\langle\delta_q| \colon \mathcal{C}^\infty(M) \to \mathcal{D}'(M), \quad |\delta_q\rangle\langle\delta_q| u = u(q)\, \delta_q. \tag{1.7}$$

This projector can be extended continuously to a map from $H^2(M)$ to $H^{-2}(M)$, only when $\dim(M) \leq 3$. This is why these perturbations are considered for this range of dimensions.

The Šeba billiard motivated the study of point-perturbations on tori. The eigenvalue statistics of these perturbed operators have drawn much attention from the mathematical physics community; see [9, 18, 21, 33, 36, 41] and the references therein. Understanding the properties of the set of semiclassical measures for high-energy eigenstates has been a problem of interest in the last 20 years. For example, a Quantum Ergodicity theorem was proved for one point perturbation in the square torus in dimension 2 [25] and in dimension 3 [46]. However, concentration in the momentum variable for irrational tori has been

observed in dimensions 2 and 3 [23, 24, 26]. These results depend on fine spectral estimates coming from Analytic Number Theory, exploiting the lattice structure underlying the spectrum of Δ in tori.

In general, point perturbations of Δ on a finite set Q may be considered; these are formally represented as

$$\Delta + \sum_{p,q\in Q} a_{pq}|\delta_q\rangle\langle\delta_p| \tag{1.8}$$

for a Hermitian matrix $\mathbb{A} = (a_{pq})_{p,q\in Q}$. The non-diagonal entries of the matrix $\mathbb{A}$ represent non-local interactions between the points of Q. In the semiclassical regime, it makes sense to make the perturbation parameter $\mathbb{A}$ depend on h_n; see [38, 42] for a physical heuristic supporting this claim. In this regime, in the spheres $\mathbb{S}^2$ and $\mathbb{S}^3$, the author proved the existence of sequences of eigenfunctions of point-perturbations of Δ on $Q = \{-q, q\}$ whose semiclassical measure is not invariant under geodesic flow [44]. This phenomenon is specific to settings where the subset Q can "trap" part of the geodesic flow. More precisely, on a general closed Riemannian manifold (M, g) with $\dim(M) = 2, 3$, as soon as the set Q is *non-self-focal*, that is, for every $q \in Q$ the set of unit-speed co-vectors for which the geodesic flow returns to Q in finite time has zero measure on S_q^*M, then every semiclassical measure is invariant under the geodesic flow [43].

In this article, we study the concentration and high-oscillation properties of sequences of high-energy eigenfunctions of singular perturbations of Δ on $\mathbb{S}^2$ by the measure δ_γ, where γ is a closed geodesic on $\mathbb{S}^2$ and

$$\delta_\gamma a := \int_0^{2\pi} a\big(\gamma(s)\big)\frac{\mathrm{d}s}{2\pi}, \quad a \in \mathcal{C}(\mathbb{S}^2).$$

These are operators that formally look like (see Section 2 for a precise definition)

$$\Delta + \alpha|\delta_\gamma\rangle\langle\delta_\gamma|, \quad \alpha \in \mathbb{R}, \tag{1.9}$$

where $|\delta_\gamma\rangle\langle\delta_\gamma|$ is defined analogously to (1.7).

Remark 1.1. Singular perturbations of Δ by a measure such as δ_q or δ_γ are usually denoted in the literature as (1.6) and (1.9) respectively; however, this notation can be misleading because the convention is that $\alpha = 0$ does not correspond to the positive Laplace–Beltrami operator, but rather $\alpha = \pm\infty$. Moreover, this notation gives the intuition that there are perturbations that are stronger—more singular—than others, but that is not the case because the space of all possible singular perturbations of Δ by δ_γ is parametrized by $\theta \in \mathbb{R}/(\pi\mathbb{Z})$ (see Proposition 2.1), which is compact and does not admit a total order. In Corollary 2.8, we provide an alternative identification between $\alpha \in \mathbb{R}$ and the family of singular perturbations of Δ by δ_γ with the additional property that $\alpha = 0$ corresponds to Δ.

Given a sequence of singular perturbations of Δ on $\mathbb{S}^2$ by δ_γ, $(\Delta + \alpha_n|\delta_\gamma\rangle\langle\delta_\gamma|)_{n\in\mathbb{N}}$, for every $n \in \mathbb{N}$ let $u_n \in L^2(\mathbb{S}^2)$ and $h_n > 0$ be such that

$$h_n^2\big[\Delta + \alpha_n|\delta_\gamma\rangle\langle\delta_\gamma|\big]u_n = u_n, \quad \|u_n\|_{L^2(\mathbb{S}^2)} = 1, \quad \lim_{n\to\infty} h_n = 0. \tag{1.10}$$

Let us denote

$$\mathcal{M}_{\mathrm{sc}}\big(\big(\Delta+\alpha_n|\delta_\gamma\rangle\langle\delta_\gamma|\big)_{n\in\mathbb{N}}\big)$$

as the set of all possible weak-$\star$ accumulation points of the sequence of Wigner distributions $(W_{h_n}[u_n])_{n\in\mathbb{N}}$ where u_n and h_n satisfy (1.10). The main results in this article state that $\mathcal{M}_{\mathrm{sc}}((\Delta+\alpha_n|\delta_\gamma\rangle\langle\delta_\gamma|)_{n\in\mathbb{N}})$ always contains the set of invariant probability measures supported on $S^*\mathbb{S}^2$ and that, in some situations, the inclusion is strict.

Theorem 1.2. *Let γ be a closed geodesic in $\mathbb{S}^2$, and fix a sequence $(\Delta+\alpha_n|\delta_\gamma\rangle\langle\delta_\gamma|)_{n\in\mathbb{N}}$ of singular perturbations of Δ by the distribution δ_γ. Then*

$$\mathcal{P}_{\mathrm{inv}}(S^*\mathbb{S}^2)\subseteq\mathcal{M}_{\mathrm{sc}}\big(\big(\Delta+\alpha_n|\delta_\gamma\rangle\langle\delta_\gamma|\big)_{n\in\mathbb{N}}\big). \tag{1.11}$$

Moreover, there exist infinitely many sequences $(\Delta+\alpha_n|\delta_\gamma\rangle\langle\delta_\gamma|)_{n\in\mathbb{N}}$ for which

$$\mathcal{P}_{\mathrm{inv}}(S^*\mathbb{S}^2)=\mathcal{M}_{\mathrm{sc}}\big(\big(\Delta+\alpha_n|\delta_\gamma\rangle\langle\delta_\gamma|\big)_{n\in\mathbb{N}}\big). \tag{1.12}$$

Remark 1.3. In fact, if we restrict ourselves to the *weak coupling regime*, that is $\alpha_n=\alpha\in\mathbb{R}$ for all $n\in\mathbb{N}$, then (1.12) holds for all but a countable family of $\alpha\in\mathbb{R}$, which amounts to a full measure subset of $\theta\in\mathbb{R}/(\pi\mathbb{Z})$ with respect to the Lebesgue measure (see Corollary 2.8 and Remark 4.4). Furthermore, we believe that in the weak coupling regime, (1.12) holds for all $\alpha\in\mathbb{R}$, but that requires a fine analysis of the spectrum of $\Delta+\alpha|\delta_\gamma\rangle\langle\delta_\gamma|$ in the high-energy regime.

Theorem 1.4. *Let γ be a closed geodesic in $\mathbb{S}^2$ and let Ω_+ and Ω_- be the two connected components of $\mathbb{S}^2\setminus\gamma$. Consider the probability measure on $\mathbb{S}^2$,*

$$\nu_\gamma(\mathrm{d}x):=\frac{2}{\pi}\frac{1}{\cos\big(d(x,\gamma)\big)}\frac{\mathrm{d}x}{\mathrm{vol}(\mathbb{S}^2)}.$$

There exists a sequence $(\Delta+\alpha_n|\delta_\gamma\rangle\langle\delta_\gamma|)_{n\in\mathbb{N}}$ such that for every $m\in[0,2]$ there exist $(u_n)_{n\in\mathbb{N}}\subseteq L^2(\mathbb{S}^2)$ and $(h_n)_{n\in\mathbb{N}}\subseteq(0,1)$ satisfying (1.10) such that for all $b\in\mathcal{C}(\mathbb{S}^2)$,

$$\lim_{n\to\infty}\int_{\mathbb{S}^2}b(x)\big|u_n(x)\big|^2\,\mathrm{d}x=\int_{\mathbb{S}^2}b(x)\big[m\mathbb{1}_{\Omega_+}(x)+(2-m)\mathbb{1}_{\Omega_-}(x)\big]\nu_\gamma(\mathrm{d}x). \tag{1.13}$$

For $m\neq1$, these quantum limits cannot be the projection onto the base manifold $\mathbb{S}^2$ of a measure μ on $S^*\mathbb{S}^2$ that is invariant under geodesic flow because on that case $\pi_*\mu(\Omega_+)=\pi_*\mu(\Omega_-)$ for the projection map $\pi(x,\xi):=x$.

The non-invariance can be understood as an outcome of a reflection phenomenon on the hypersurface γ in the semiclassical limit. Reflection phenomena due to codimension-one perturbations of a Schrödinger operator in the semiclassical limit ($h\to0^+$) have been studied in the literature, mostly for Schrödinger operators $H_h:=h^2\Delta+V$ with potentials having a finite-order conical singularity on some embedded hypersurface $Y\subseteq M$: $V\in\mathcal{C}^\infty(M\setminus Y;\mathbb{R})$, but $V\in\mathcal{C}^k(M)$ for some $k\in\mathbb{N}$. If $M=\mathbb{R}$ is equipped with

the canonical metric, [8] describes a connection through the WKB method between the regularity of V on $Y = \{y_0\}$ and the size of the reflection term. More recently, [11,17,20] extended these results to higher dimensions and curved geometries. In [10,19], invariance under classical dynamics was studied for semiclassical measures arising from sequences of eigenfunctions of H_h in the semiclassical limit. Reflection phenomena also contribute to the spectral properties of these operators in that limit; see, for instance, [45, 50]. The perturbations analyzed in this article can be interpreted as a limiting case for these type of potentials.

1.1. Outline of the proof

Proving Theorems 1.4 and 1.2 requires precise knowledge about the eigenfunctions of singular perturbations of Δ by δ_γ. Section 2 is devoted to proving that these operators have compact resolvent, characterizing their spectrum, and computing their eigenspaces. Eigenfunctions u can be classified into three categories: (a) old eigenfunctions, which are eigenfunctions of Δ with the added condition $\delta_\gamma u = 0$; (b) new eigenfunctions, which have a singularity along γ; and (c) linear combinations of old eigenfunctions and new eigenfunctions.

The family of old eigenfunctions is rich enough in the high-energy regime to produce any invariant probability measure; an application of [44, Theorem 1.3] gives (1.11). On the other hand, new eigenfunctions have a very rigid structure: they are infinite linear combinations of zonal harmonics. First, in Section 3 we show that a proper truncation of the series produces a quasimode approximation. In the proof of Theorem 4.1, we use this quasimode approximation, ladder operators for zonal harmonics, which are h-ΨDO, together with symbolic calculus to eventually approximate a new eigenfunction by a h-ΨDO acting on a single zonal harmonic z_{ℓ_n}, obtaining an asymptotic of the form

$$\begin{aligned}&\left\langle u_n \mid \operatorname{Op}_{h_n}(a) u_n \right\rangle_{L^2(\mathbb{S}^2)} \\ &\quad = \left\langle z_{\ell_n} \mid \operatorname{Op}_{h_n}\left(a|\Gamma^\Upsilon|^2\right) z_{\ell_n} \right\rangle_{L^2(\mathbb{S}^2)} + o(1_n) + \Upsilon^{-\frac{1}{2}} \mathcal{O}(1) \quad \text{as } n \to \infty,\end{aligned}$$

for a certain symbol Γ^Υ and a large parameter $\Upsilon \in \mathbb{N}$ that comes from the truncation procedure. The family of zonal harmonics $\{z_\ell^q\}_{\ell \in \mathbb{N}}$ have a unique semiclassical defect measure, μ_q, which is uniform along the geodesic flow-out from q [43, Theorem 3.1]. Taking the limit $n \to \infty$ first, and then analyzing $|\Gamma^\Upsilon|^2$ on $\operatorname{supp} \mu_q$, we find that $|\Gamma^\Upsilon|^2 \to 1$ on $\operatorname{supp} \mu_q$ as $\Upsilon \to \infty$, thus any semiclassical measure arising from a sequence of high-energy new eigenfunctions is invariant under the geodesic flow. For most singular perturbations of Δ by δ_γ, old and new eigenfunctions are the only eigenfunctions of the perturbed operator, thus (1.12) holds.

In this setting, a non-invariant semiclassical measure can arise only from a sequence of linear combinations of old and new eigenfunctions. To show that, in the proof of Theorem 4.5, we consider a sequence of linear combinations of zonal harmonics and new eigenfunctions

$$u_n := \varphi z_{\ell_n} + \psi g_n,$$

and using the approximation described above, one obtains an asymptotic of the form

$$\begin{aligned}&\big\langle u_n \mid \mathrm{Op}_{h_n}(a)u_n\big\rangle_{L^2(\mathbb{S}^2)}\\&\quad= \big\langle z_{\ell_n} \mid \mathrm{Op}_{h_n}\big(a|\varphi+\psi\Gamma^{\Upsilon}|^2\big)z_{\ell_n}\big\rangle_{L^2(\mathbb{S}^2)} + o(1_n) + \Upsilon^{-\frac{1}{2}}\mathcal{O}(1) \quad \text{as } n\to\infty.\end{aligned}$$

This time, after taking $n\to\infty$, the symbol $|\varphi+\psi\Gamma^{\Upsilon}|^2$ converges to a non-constant function on $\operatorname{supp}\mu_q$ as $\Upsilon\to\infty$, hence obtaining a non-invariant semiclassical measure.

2. Singular perturbations of the Laplacian

2.1. Description of singular perturbations of Δ by δ_γ

Let Δ denote the positive Laplace–Beltrami operator on $(\mathbb{S}^2, \text{can.})$, and let $\mathrm{d}x$ denote the Riemannian measure on $\mathbb{S}^2$. Consider the Hilbert space of complex-valued functions $L^2(\mathbb{S}^2,\mathrm{d}x)$ with the inner product

$$\langle u \mid v\rangle_{L^2(\mathbb{S}^2)} := \int_{\mathbb{S}^2} \overline{u(x)}v(x)\,\mathrm{d}x.$$

As an operator on $L^2(\mathbb{S}^2)$, let Δ be defined on $\operatorname{dom}(\Delta) := H^2(\mathbb{S}^2)$; then Δ is a self-adjoint operator on $L^2(\mathbb{S}^2)$.

Let γ be a closed geodesic on $\mathbb{S}^2$ and define the distribution

$$\delta_\gamma u := \int_0^{2\pi} u\big(\gamma(s)\big)\frac{\mathrm{d}s}{2\pi}, \quad u\in\mathcal{C}^\infty(\mathbb{S}^2). \tag{2.1}$$

Thanks to the Trace Theorem for Sobolev space, we find that for all $u\in\mathcal{C}^\infty(\mathbb{S}^2)$,

$$|\delta_\gamma u| \le \|u\|_{L^\infty(\gamma)} \le \sqrt{2\pi}\|u\|_{L^2(\gamma)} \le C\|u\|_{H^1(\mathbb{S}^2)}, \tag{2.2}$$

hence the distribution δ_γ can be extended continuously to $H^1(\mathbb{S}^2)$. Naturally, δ_γ is not bounded on $L^2(\mathbb{S}^2)$. We are interested in operators that formally look like

$$\Delta + \alpha|\delta_\gamma\rangle\langle\delta_\gamma| \tag{2.3}$$

for some $\alpha\in\mathbb{R}$. These singular perturbations are defined via the von Neumann theory of symmetric extensions (see the appendix): they are self-adjoint extensions of the operator $T := \Delta|_{\operatorname{dom}(T)}$,[2] where

$$\operatorname{dom}(T) := \big\{u\in\operatorname{dom}(\Delta)\colon \delta_\gamma u = 0\big\}. \tag{2.4}$$

[2] In general, a symmetric operator may have no self-adjoint extensions, such as the operator $\frac{1}{i}\partial_x$ defined on $\mathcal{C}_c^\infty(0,\infty)\subseteq L^2(0,\infty)$. See [31, Lemma 10.1] for a geometric explanation of this example.

Proposition 2.1. *Let γ be a closed geodesics in $\mathbb{S}^2$, and let Ω_+ and Ω_- be the two connected components of $\mathbb{S}^2 \setminus \gamma$. Define the following two H^1-functions:*

$$u_{\gamma 0}(x) := 1, \quad u_{\gamma 1}(x) := -\tfrac{1}{2}\sin\big(d(x,\gamma)\big), \quad x \in \mathbb{S}^2. \tag{2.5}$$

The self-adjoint extensions of T are the operators $\{\Delta_{\gamma,\theta}\}_{\theta\in\Theta}$, $\Theta = \mathbb{R}/(\pi\mathbb{Z})$, where

$$\mathrm{dom}(\Delta_{\gamma,\theta}) := \mathrm{dom}(T) + \mathrm{Span}_{\mathbb{C}}\{\cos\theta\, u_{\gamma 0} + \sin\theta\, u_{\gamma 1}\} \tag{2.6}$$

and for $u_0 \in \mathrm{dom}(T)$, $c \in \mathbb{C}$,

$$\Delta_{\gamma,\theta}\big[u_0 + c(\cos\theta\, u_{\gamma 0} + \sin\theta\, u_{\gamma 1})\big] := \Delta u_0 + c(2\sin\theta\, u_{\gamma 1}). \tag{2.7}$$

In other words, $\Delta_{\gamma,\theta} := T^*|_{\mathrm{dom}(\Delta_{\gamma,\theta})}$, where T^* is the adjoint operator of T (see Definition A.2 and Proposition 2.3). Moreover, $\mathrm{dom}(\Delta_{\gamma,0}) = \mathrm{dom}(\Delta)$, hence $\Delta_{\gamma,0} = \Delta$.

We leave the proof of Proposition 2.1 to the end of the section, as we need some auxiliary lemmas.

Lemma 2.2. *The operator T is a closed symmetric operator in $L^2(\mathbb{S}^2)$.*

Proof. This is a direct consequence of Lemma A.1 for $A = \Delta$ and $f = \delta_\gamma$ and (2.2) after the observation that the graph-norm of Δ

$$\|u\|_\Delta = \big(\|u\|^2_{L^2(\mathbb{S}^2)} + \|\Delta u\|^2_{L^2(\mathbb{S}^2)}\big)^{\frac{1}{2}}$$

is equivalent to the H^2-norm. ∎

In order to find all self-adjoint extensions of T, one starts considering the adjoint operator T^*. We refer the reader to Definition A.2 for the definition of T^* on the Hilbert space $\mathcal{H} = L^2(\mathbb{S}^2)$.

Proposition 2.3. *Let γ be a closed geodesic in $\mathbb{S}^2$ and consider $u_{\gamma 0}$, $u_{\gamma 1}$ as defined in (2.5). If T is defined as in (2.4), then $\dim(\mathrm{dom}(T^*)/\mathrm{dom}(T)) = 2$ and*

$$\mathrm{dom}(T^*) = \mathrm{dom}(T) + \mathrm{Span}_{\mathbb{C}}\{u_{\gamma 0}, u_{\gamma 1}\},$$

and

$$T^* u_{\gamma 0} = 0, \quad T^* u_{\gamma 1} = 2u_{\gamma 1}. \tag{2.8}$$

Remark 2.4. In general, one cannot expect $\mathrm{dom}(T^*) \subseteq H^1(\mathbb{S}^2)$ for a singular perturbation of Δ, or the unbounded functional $f = \delta_\gamma$ to be defined on $\mathrm{dom}(T^*)$. For example, if $f = \delta_q$ [44, Section 2], u_{q1} would have a logarithmic singularity on q, thus u_{q1} would not belong to $H^1(\mathbb{S}^2)$ and $\delta_q u_{q1}$ would not even make sense—we would need $H^{1+\varepsilon}$-regularity! However, in the present case we have $u_{\gamma 1} \in H^1(\mathbb{S}^2)$ and $\delta_\gamma u_{\gamma 1} = 0$.

Proof of Proposition 2.3. Thanks to Lemma 2.2 we know that T is a closed symmetric operator. Observe that Δ is a self-adjoint extension of T and that $\dim(\mathrm{dom}(\Delta)/\mathrm{dom}(T)) = 1$ because $\mathrm{dom}(T)$ is the kernel of a linear functional on $\mathrm{dom}(\Delta)$, thus Lemma A.6 (3) implies $\dim(\mathrm{dom}(T^*)/\mathrm{dom}(T)) = 2$. It is clear that

$$\mathrm{dom}(\Delta) = \mathrm{dom}(T) + \mathrm{Span}_{\mathbb{C}}\{u_{\gamma 0}\}, \quad \text{and} \quad u_{\gamma 1} \notin \mathrm{dom}(\Delta). \tag{2.9}$$

In addition, using that

$$[\Delta u_{\gamma 1}](x) = 2u_{\gamma 1}(x) \quad \forall x \in \Omega_+ \cup \Omega_-,$$

and an integration by parts argument on Ω_+ and Ω_-, we find that for all $w \in \mathrm{dom}(\Delta)$,

$$\langle u_{\gamma 1} \mid \Delta w\rangle_{L^2(\mathbb{S}^2)} = \delta_\gamma w + \langle 2u_{\gamma 1} \mid w\rangle_{L^2(\mathbb{S}^2)}.$$

Therefore, $u_{\gamma 1} \in \mathrm{dom}(T^*)$ and $T^* u_{\gamma 1} = 2u_{\gamma 1}$. ∎

Proof of Proposition 2.1. Let A be a self-adjoint extension of T. Since A is self-adjoint, then $A = A^* \subset T^*$, thus $A = T^*|_{\mathrm{dom}(A)}$ for some subspace $\mathrm{dom}(T) \subseteq \mathrm{dom}(A) \subseteq \mathrm{dom}(T^*)$. Thanks to Proposition A.5, we know that $\mathrm{dom}(A)$ must be a Lagrangian subspace for the skew-Hermitian form

$$\omega(u, v) := \langle u \mid T^* v\rangle_{L^2(\mathbb{S}^2)} - \langle T^* u \mid v\rangle_{L^2(\mathbb{S}^2)}, \quad u, v \in \mathrm{dom}(T^*). \tag{2.10}$$

Thanks to Proposition 2.3, we know that

$$\mathrm{dom}(T^*) = \mathrm{dom}(T) + \mathrm{Span}_{\mathbb{C}}\{u_{\gamma 0}, u_{\gamma 1}\},$$

thus

$$\mathrm{dom}(A) = \mathrm{dom}(T) + \mathrm{Span}_{\mathbb{C}}\{v\}$$

for some $v \in \mathrm{Span}_{\mathbb{C}}\{u_{\gamma 0}, u_{\gamma 1}\}$, due to Lemma A.6 (3). Thanks to Lemma A.3, it is necessary and sufficient for $\mathrm{dom}(A)$ to be Lagrangian that $\omega(v, v) = 0$.

Note that

$$\begin{aligned} &\omega(u_{\gamma 0}, u_{\gamma 0}) = 0, \\ &\omega(u_{\gamma 1}, u_{\gamma 1}) = 0, \\ &\omega(u_{\gamma 1}, u_{\gamma 0}) = 0 - \langle 2u_{\gamma 1} \mid u_{\gamma 0}\rangle_{L^2(\mathbb{S}^2)} = \int_{\Omega_+} \sin\big(d(x, \gamma)\big)\, \mathrm{d}x = 2\pi, \end{aligned} \tag{2.11}$$

therefore, if we write $v = a u_{\gamma 0} + b u_{\gamma 1}$ for $a, b \in \mathbb{C}$, we see that

$$\omega(v, v) = 0 \iff \bar{b}a - \bar{a}b = 0 \iff \Im(\bar{b}a) = 0.$$

Without loss of generality, we may assume that $a \in \mathbb{R}$ and that $|a|^2 + |b|^2 = 1$, hence $b \in \mathbb{R}$ and the only solutions are

$$\begin{cases} a = \cos\theta, \\ b = \sin\theta, \end{cases} \quad \theta \in \mathbb{R}.$$

This gives that $\mathrm{dom}(A)$ must be of the form (2.6). (2.7) is a direct consequence of (2.8) and $A = T^*|_{\mathrm{dom}(A)}$.

The fact that setting $\theta = 0$ lets you recover the Laplacian Δ is equivalent to the identity

$$\mathrm{dom}(\Delta) = \mathrm{dom}(T) + \mathrm{Span}_{\mathbb{C}}\{u_{\gamma 0}\} = \mathrm{dom}(\Delta_{\gamma,0}). \qquad \blacksquare$$

2.2. Spectrum of singular perturbations of Δ by δ_γ

The operator Δ has compact resolvent, therefore Δ has no continuous spectrum, and every $\lambda \in \mathrm{Spec}(\Delta)$ is an eigenvalue with finite multiplicity. In fact, one has (see, for instance, [40, Section 2 of Chapter IV])

$$\mathrm{Spec}(\sqrt{\Delta}) = \{\lambda_\ell\}_{\ell\in\mathbb{N}} \subseteq [0,\infty), \quad \lambda_\ell^2 = \ell(\ell+1).$$

In this section, we will prove that operators $\Delta_{\gamma,\theta}$ have compact resolvent, among other of their spectral properties.

For $z \in \mathbb{C} \setminus \mathrm{Spec}(\Delta)$, let $G_z^\gamma \in L^2(\mathbb{S}^2)$ be such that

$$\left\langle G_z^\gamma \,|\, (\Delta - \bar{z})u \right\rangle_{L^2(\mathbb{S}^2)} = \delta_\gamma u \tag{2.12}$$

for all $u \in \mathrm{dom}(\Delta) = H^2(\mathbb{S}^2)$. The existence and uniqueness of the function G_z^γ for $z \in \mathbb{C} \setminus \mathrm{Spec}(\Delta)$ is granted by the Riesz representation theorem: for every $z \in \mathbb{C} \setminus \mathrm{Spec}(\Delta)$, the functional

$$F_{\gamma,z}\colon L^2(\mathbb{S}^2) \to \mathbb{C}, \quad F_{\gamma,z} w = \delta_\gamma\big[(\Delta - \bar{z})^{-1} w\big]$$

is bounded on $L^2(\mathbb{S}^2)$, hence there exists a unique $G_z^\gamma \in L^2(\mathbb{S}^2)$ such that

$$F_{\gamma,z} w = \langle G_z^\gamma \,|\, w\rangle_{L^2(\mathbb{S}^2)} \quad \forall w \in L^2(\mathbb{S}^2).$$

Note that for all $z \in \mathbb{C} \setminus \mathrm{Spec}_{\mathrm{even}}(\Delta)$, $G_z^\gamma \in \mathrm{dom}(T^*)$ because for all $u \in \mathrm{dom}(T)$

$$\langle G_z^\gamma \,|\, \Delta u\rangle_{L^2(\mathbb{S}^2)} = \delta_\gamma u + \langle G_z^\gamma \,|\, \bar{z} u\rangle_{L^2(\mathbb{S}^2)} = \langle z G_z^\gamma \,|\, u\rangle_{L^2(\mathbb{S}^2)},$$

which additionally shows that $T^* G_z^\gamma = z G_z^\gamma$. In particular, $G_z^\gamma \in H^1(\mathbb{S}^2)$.

In order to introduce the main result of this section, Theorem 2.6, we first need to state a technical proposition that extends the definition of G_z^γ from $\mathbb{C} \setminus \mathrm{Spec}(\Delta)$ to $\mathbb{C} \setminus \mathrm{Spec}_{\mathrm{even}}(\Delta)$, where

$$\mathrm{Spec}_{\mathrm{even}}(\Delta) := \{\lambda_\ell^2 \colon \ell \in \mathbb{N} \text{ even}\}. \tag{2.13}$$

Proposition 2.5. *Let γ be a closed geodesic in $\mathbb{S}^2$. The holomorphic map*

$$\mathcal{G}^\gamma\colon \mathbb{C} \setminus \mathrm{Spec}(\Delta) \to L^2(\mathbb{S}^2), \quad \mathcal{G}^\gamma(z) = G_z^\gamma, \tag{2.14}$$

admits a (unique) holomorphic extension to $\mathbb{C} \setminus \mathrm{Spec}_{\mathrm{even}}(\Delta)$ such that

$$\left\langle G_z^\gamma \,|\, (\Delta - \bar{z})u \right\rangle_{L^2(\mathbb{S}^2)} = \delta_\gamma u \quad \forall u \in \mathrm{dom}(\Delta) \tag{2.15}$$

holds for all $z \in \mathbb{C} \setminus \mathrm{Spec}_{\mathrm{even}}(\Delta)$.[3] *Consequently,* $G_z^\gamma \in \mathrm{dom}(T^*)$ *and* $T^* G_z^\gamma = z G_z^\gamma$ *for all* $z \in \mathbb{C} \setminus \mathrm{Spec}_{\mathrm{even}}(\Delta)$. *See* (2.24) *for an eigenfunction expansion of* G_z^γ.

Theorem 2.6. *Let* γ *be a closed geodesic in* $\mathbb{S}^2$, *and for* $\theta \in \Theta = \mathbb{R}/(\pi\mathbb{Z})$, $\theta \neq 0$, *let* $\Delta_{\gamma,\theta}$ *be a self-adjoint perturbation of* Δ *on* $\mathbb{S}^2$ *by* δ_γ ($\Delta_{\gamma,\theta} \neq \Delta$, *see Proposition* 2.1). *For every* $z \in \mathbb{C} \setminus (\mathrm{Spec}(\Delta) \cup \mathrm{Spec}(\Delta_{\gamma,\theta}))$, *there exists* $\alpha(\theta, z) \in \mathbb{C}$ (2.34) *such that*

$$\left[(\Delta_{\gamma,\theta} - z)^{-1} - (\Delta - z)^{-1}\right] f = \left[\alpha(\theta, z) \langle G_{\bar{z}}^\gamma \mid f \rangle_{L^2(\mathbb{S}^2)}\right] G_z^\gamma \quad \forall f \in L^2(\mathbb{S}^2), \tag{2.16}$$

thus the resolvent of $\Delta_{\gamma,\theta}$ *is compact. Moreover, for* $\eta \in \mathbb{R} \setminus \mathrm{Spec}_{\mathrm{even}}(\Delta)$ *we have*

$$\ker(\Delta_{\gamma,\theta} - \eta) = \begin{cases} \ker(\Delta - \eta) + \mathrm{Span}_{\mathbb{C}}\{G_\eta^\gamma\} & \text{if } \eta \text{ is a solution to (2.17)}, \\ \ker(\Delta - \eta) & \text{otherwise}, \end{cases}$$

where (see Lemma 2.11 *below)*

$$\delta_\gamma G_\eta^\gamma = \frac{1}{2\pi} \operatorname{cotan} \theta. \tag{2.17}$$

Meanwhile, for every $\ell \in \mathbb{N}$,

$$\ker(\Delta_{\gamma,\theta} - \lambda_{2\ell}^2) = \left\{u \in \ker(\Delta - \lambda_{2\ell}^2) \colon \delta_\gamma u = 0\right\}.$$

Remark 2.7. For every $\ell \geq 1$ and $\theta \in (0, \pi)$, equation (2.17) has a unique solution $\eta_{\ell,\theta} \in (\lambda_{2\ell}^2, \lambda_{2\ell+2}^2)$—see Lemma 2.12 at the end of the section. Additionally, for every $\theta \in \Theta$, $\theta \neq 0$, and every $\ell \in \mathbb{N}$,

$$\dim\left(\ker(\Delta_{\gamma,\theta} - \lambda_{2\ell}^2)\right) = \dim\left(\ker(\Delta - \lambda_{2\ell}^2)\right) - 1,$$

$$\dim\left(\ker(\Delta_{\gamma,\theta} - \lambda_{2\ell+1}^2)\right) = \begin{cases} \dim\left(\ker(\Delta - \lambda_{2\ell+1}^2)\right) + 1 & \text{if } \lambda_{2\ell+1}^2 \text{ is a solution to (2.17)}, \\ \dim\left(\ker(\Delta - \lambda_{2\ell+1}^2)\right) & \text{otherwise.} \end{cases}$$

Corollary 2.8. *Let* γ *be a closed geodesic in* $\mathbb{S}^2$ *and let* $z \in \mathbb{R} \setminus \mathrm{Spec}(\Delta)$. *For any* $\theta \in \Theta = \mathbb{R}/(\pi\mathbb{Z})$, $\theta \neq 0$, *such that* $z \notin \mathrm{Spec}(\Delta_{\gamma,\theta})$, *let* $\alpha(\theta, z)$ *be given by Theorem* 2.6. *Set* $P_\theta := (\Delta_{\gamma,\theta} - z)^{-1}(\Delta - z)$. *Then for all* $u \in \mathrm{dom}(\Delta)$,

$$\left\langle P_\theta u \mid (\Delta_{\gamma,\theta} - z) P_\theta u \right\rangle_{L^2(\mathbb{S}^2)} = \langle u \mid \Delta u \rangle_{L^2(\mathbb{S}^2)} + \alpha(\gamma, \theta) |\delta_\gamma u|^2.$$

Proof. From Theorem 2.6 we see that

$$P_\theta u = u + \alpha(\theta, z)[\delta_\gamma u] G_z^\gamma \quad \forall u \in \mathrm{dom}(\Delta).$$

Therefore, using $\Delta_{\gamma,\theta} = T^*|_{\mathrm{dom}(\Delta_{\gamma,\theta})}$, we get that for all $u \in \mathrm{dom}(\Delta)$,

$$\begin{aligned} \left\langle P_\theta u \mid (\Delta_{\gamma,\theta} - z) P_\theta u \right\rangle_{L^2(\mathbb{S}^2)} &= \left\langle u + \alpha(\theta, z)[\delta_\gamma u] G_z^\gamma \mid (\Delta - z) u \right\rangle_{L^2(\mathbb{S}^2)} \\ &= \left\langle u \mid (\Delta - z) u \right\rangle_{L^2(\mathbb{S}^2)} + \overline{\alpha(\theta, z)} |\delta_\gamma u|^2. \end{aligned}$$

Lastly, since $z \in \mathbb{R}$, then $\alpha(\theta, z) \in \mathbb{R}$ (see (2.34)) and the result follows.[4] ∎

[3] We make a little abuse of notation by writing G_z^γ instead of $\mathcal{G}^\gamma(z)$.

[4] Right from the definition of $\alpha(\theta, z)$ (2.34), one can see that $\alpha(\theta, z) \to 0$ when $\theta \to 0$.

We postpone the proof of Theorem 2.6 to the end of the section; we first state and prove some auxiliary results. The first connects δ_γ to the eigenspaces of Δ, revealing the reason why $\mathcal{G}_z^\gamma$ (2.14) can be extended to $\mathbb{C} \setminus \mathrm{Spec}_{\mathrm{even}}(\Delta)$.

For every $\ell \in \mathbb{N}$, let Y_ℓ^γ be the unique eigenfunction in $\ker(\sqrt{\Delta} - \lambda_\ell)$ such that

$$\langle Y_\ell^\gamma \mid u \rangle_{L^2(\mathbb{S}^2)} = \delta_\gamma u \quad \forall u \in \ker(\sqrt{\Delta} - \lambda_\ell). \tag{2.18}$$

Existence and uniqueness of Y_ℓ^γ is granted by the Riesz representation theorem in the finite-dimensional space $\ker(\sqrt{\Delta} - \lambda_\ell)$. These eigenfunctions Y_ℓ^γ play the same role as zonal harmonics on $\mathbb{S}^2$ for the distribution δ_γ instead of δ_q. The ℓ-th zonal harmonic on $q \in \mathbb{S}^2$, Z_ℓ^q, is defined as the unique eigenfunction $Z_\ell^q \in \ker(\sqrt{\Delta} - \lambda_\ell)$ such that

$$\langle Z_\ell^q \mid u \rangle_{L^2(\mathbb{S}^2)} = u(q) \quad \forall u \in \ker(\sqrt{\Delta} - \lambda_\ell). \tag{2.19}$$

In fact, if $d(\gamma, q) = \frac{\pi}{2}$, eigenfunctions Z_ℓ^q and Y_ℓ^γ are linearly independent.

Lemma 2.9. *Let γ be a closed geodesic in $\mathbb{S}^2$, and take $q \in \mathbb{S}^2$ such that $d(q, \gamma) = \frac{\pi}{2}$. For every $\ell \in \mathbb{N}$*

$$\begin{cases} Y_{2\ell}^\gamma = (-1)^\ell \dfrac{\Gamma\left(\ell + \frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma(\ell + 1)} Z_{2\ell}^q, \\ Y_{2\ell+1}^\gamma = 0. \end{cases}$$

Proof. Let L_q be the Killing field[5] on $\mathbb{S}^2$ such that $L_q(q) = 0$ and $L_q(\gamma(s)) = \gamma'(s)$. The subspace $\mathrm{Span}_{\mathbb{C}}\{Z_\ell^q\}$ is characterized in $L^2(\mathbb{S}^2)$ by the properties [40, Corollary 2.13]

$$[\sqrt{\Delta} - \lambda_\ell]u = 0, \quad L_q u = 0, \tag{2.20}$$

that is, $u \in \mathrm{Span}_{\mathbb{C}}\{Z_\ell^q\}$ if and only if $u \in L^2(\mathbb{S}^2)$ satisfies (2.20). We will show that Y_ℓ^γ enjoys (2.20) and then find the appropriate multiplicative constant.

Fix some $\ell \in \mathbb{N}$. Y_ℓ^γ enjoys property (1) by definition. Y_ℓ^γ satisfies property (2) due to the fact $[\Delta, L_q] = 0$. If Π_ℓ is the eigenprojector onto $\ker(\sqrt{\Delta} - \lambda_\ell)$ and $u \in \mathcal{C}^\infty(\mathbb{S}^2)$,

$$\langle L_q Y_\ell^\gamma \mid u \rangle_{L^2(\mathbb{S}^2)} = -\langle Y_\ell^\gamma \mid L_q u \rangle_{L^2(\mathbb{S}^2)} = -\langle Y_\ell^\gamma \mid \Pi_\ell L_q u \rangle_{L^2(\mathbb{S}^2)} = -\langle Y_\ell^\gamma \mid L_q \Pi_\ell u \rangle_{L^2(\mathbb{S}^2)}.$$

Thus, we may assume without loss of generality that $u \in \ker(\sqrt{\Delta} - \lambda_\ell)$. Now, for $u \in \ker(\sqrt{\Delta} - \lambda_\ell)$,

$$\langle Y_\ell^\gamma \mid L_q u \rangle_{L^2(\mathbb{S}^2)} = \delta_\gamma(L_q u) = \int_0^{2\pi} [L_q u]\big(\gamma(s)\big) \frac{\mathrm{d}s}{2\pi} = \int_0^{2\pi} [u \circ \gamma]'(s) \frac{\mathrm{d}s}{2\pi} = 0,$$

thus $L_q Y_\ell^\gamma = 0$.

[5] A *Killing field* is a vector field X on a Riemannian manifold (M, g) such that the Lie derivative of g with respect to X vanishes, that is, $\mathcal{L}_X g = 0$.

Since Y_ℓ^γ satisfies properties (1) and (2), there exists $\alpha_\ell \in \mathbb{C}$ such that

$$Y_\ell^\gamma = \alpha_\ell Z_\ell^q. \tag{2.21}$$

If we know how to compute $\langle Y_\ell^\gamma \mid Z_\ell^q \rangle_{L^2(\mathbb{S}^2)}$ in two different ways, we will find α_ℓ. For that, we recall that

$$Z_\ell^q(x) = \frac{2\ell+1}{\mathrm{vol}(\mathbb{S}^2)} P_\ell(\cos r), \quad r = d(x,q),$$

where $\{P_\ell\}_{\ell\in\mathbb{N}}$ is the family of Legendre polynomials with the normalization $P_\ell(1) = 1$ (see [44, Proposition B.1], for instance). Note that for every $s \in \mathbb{R}$,

$$Z_\ell^q\big(\gamma(s)\big) = \frac{2\ell+1}{\mathrm{vol}(\mathbb{S}^2)} P_\ell(0),$$

therefore

$$\langle Y_\ell^\gamma \mid Z_\ell^q \rangle_{L^2(\mathbb{S}^2)} = \int_0^{2\pi} Z_\ell^q\big(\gamma(s)\big) \frac{\mathrm{d}s}{2\pi} = \frac{2\ell+1}{\mathrm{vol}(\mathbb{S}^2)} P_\ell(0).$$

Meanwhile

$$\langle Y_\ell^\gamma \mid Z_\ell^q \rangle_{L^2(\mathbb{S}^2)} = \overline{\alpha_\ell} \langle Z_\ell^q \mid Z_\ell^q \rangle_{L^2(\mathbb{S}^2)} = \overline{\alpha_\ell} \frac{2\ell+1}{\mathrm{vol}(\mathbb{S}^2)} P_\ell(1),$$

thus

$$\alpha_\ell = P_\ell(0). \tag{2.22}$$

Values of Legendre polynomials at 0 are well known [32, Table 18.6.1]

$$\begin{cases} P_{2\ell}(0) = (-1)^\ell \dfrac{\Gamma\left(\ell + \frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma(\ell+1)}, \\ P_{2\ell+1}(0) = 0. \end{cases} \tag{2.23}$$

Combining (2.21), (2.22), and (2.23), the result is concluded. ∎

Proof of Proposition 2.5. Recall that the map

$$\mathcal{G}^\gamma \colon \mathbb{C} \setminus \mathrm{Spec}_{\mathrm{even}}(\Delta) \to L^2(\mathbb{S}^2), \quad \mathcal{G}^\gamma(z) = G_z^\gamma$$

is holomorphic on $\mathbb{C} \setminus \mathrm{Spec}(\Delta)$ because the resolvent $(\Delta - z)^{-1}$ is holomorphic on $z \in \mathbb{C} \setminus \mathrm{Spec}(\Delta)$ and δ_γ is linear and continuous on $\mathrm{dom}(\Delta)$.

Let $z \in \mathbb{C} \setminus \mathrm{Spec}(\Delta)$ and write G_z^γ as a linear combination of eigenfunctions of Δ. Thanks to the defining property (2.12), one finds that

$$\mathcal{G}^\gamma(z) = G_z^\gamma = \sum_{\ell\in\mathbb{N}} \frac{1}{\lambda_\ell^2 - z} Y_\ell^\gamma. \tag{2.24}$$

We observe now that $Y^\gamma_{2\ell+1} = 0$ for all $\ell \in \mathbb{N}$, thanks to Lemma 2.9; therefore, every isolated singularity $z = \lambda^2_{2\ell+1}$ is a removable singularity, and thus $\mathcal{G}^\gamma$ extends holomorphically to $\mathbb{C} \setminus \mathrm{Spec}_{\mathrm{even}}(\Delta)$ following the formula (2.24). With formula (2.24) at hand, it is very easy to prove that (2.15) holds for all $z \in \mathbb{C} \setminus \mathrm{Spec}_{\mathrm{even}}(\Delta)$ since

$$\delta_\gamma = \sum_{\ell\in\mathbb{N}} Y^\gamma_{2\ell} \quad \text{in } H^{-2}(\mathbb{S}^2). \qquad \blacksquare$$

The following proposition characterizes the eigenfunctions of the adjoint operator T^*, where T was defined back in (2.4). Recall that the self-adjoint perturbations of Δ by δ_γ are restrictions of T^*.

Proposition 2.10. *Let γ be a closed geodesic on $\mathbb{S}^2$, and let $T := \Delta|_{\mathrm{dom}(T)}$ where $\mathrm{dom}(T)$ was defined back in (2.4). For every $z \in \mathbb{C}$,*

$$\ker(T^* - z) = \begin{cases} \ker(\Delta - z) + \mathrm{Span}_{\mathbb{C}}\{G^\gamma_z\} & \text{if } z \in \mathbb{C} \setminus \mathrm{Spec}_{\mathrm{even}}(\Delta) \\ \ker(\Delta - z) & \text{if } z \in \mathrm{Spec}_{\mathrm{even}}(\Delta). \end{cases} \tag{2.25}$$

Proof. We will use the spectral decomposition given by Δ: given $u \in H^s(\mathbb{S}^2)$, there exists $(u_\ell)_{\ell\in\mathbb{N}}$ such that $u_\ell \in \ker(\Delta - \lambda^2_\ell)\ \forall\, \ell \in \mathbb{N}$,

$$\left((1+\lambda^2_\ell)^{\frac{s}{2}} \|u_\ell\|_{L^2(\mathbb{S}^2)}\right)_{\ell\in\mathbb{N}} \in \ell^2(\mathbb{N}) \quad \text{and} \quad u = \sum_{\ell\in\mathbb{N}} u_\ell \quad \text{in the } H^s\text{-topology.} \tag{2.26}$$

For example, the spectral decomposition of δ_γ is

$$\delta_\gamma = \sum_{\ell\in\mathbb{N}} Y^\gamma_\ell = \sum_{\ell\in\mathbb{N}} Y^\gamma_{2\ell},$$

but we can also use it to characterize the subspace $\mathrm{dom}(T)$: $v \in \mathrm{dom}(T)$ if and only if its spectral decomposition $(v_\ell)_{\ell\in\mathbb{N}}$ satisfies (2.26) for $s = 2$ and also

$$\sum_{\ell\in\mathbb{N}} \langle Y^\gamma_{2\ell} \mid v_{2\ell}\rangle_{L^2(\mathbb{S}^2)} = 0. \tag{2.27}$$

By definition, $(T^* - z)u_z = 0$ if and only if $u_z \in L^2(\mathbb{S}^2)$ and $\langle u_z \mid (\Delta - \bar z)v\rangle_{L^2(M)} = 0$ for all $v \in \mathrm{dom}(T)$. Translating this into the notation above, there exists $(u_{z,\ell})_{\ell\in\mathbb{N}}$ such that (2.26) holds for $s = 0$ and

$$\sum_{\ell\in\mathbb{N}} \left\langle u_{z,\ell} \mid (\lambda^2_\ell - \bar z)v_\ell\right\rangle_{L^2(\mathbb{S}^2)} = 0 \quad \forall v \in \mathrm{dom}(T).$$

From this identity we infer that $f := \sum_{\ell\in\mathbb{N}}(\lambda^2_\ell - z)u_{z,\ell} \in H^{-2}(\mathbb{S}^2)$ and $f(v) = 0$ for all $v \in \mathrm{dom}(T) = \{v \in H^2(\mathbb{S}^2) : \delta_\gamma v = 0\}$. Therefore

$$\sum_{\ell\in\mathbb{N}} (\lambda^2_{2\ell} - z)u_{z,2\ell} \in \mathrm{Span}_{\mathbb{C}}\{\delta_\gamma\},$$

that is, there exists $\beta \in \mathbb{C}$ such that

$$(\lambda_\ell^2 - z)u_{z,\ell} = \beta Y_\ell^\gamma \quad \forall \ell \in \mathbb{N}.$$

If $z \in \mathrm{Spec}_{\mathrm{even}}(\Delta)$, say $z = \lambda_{2k}^2$, then for $\ell = 2k$ we have the equation

$$0 = (\lambda_{2k}^2 - z)u_{z,2k} = \beta Y_{2k}^\gamma,$$

from which we read that $\beta = 0$, but $u_{z,2k}$ can be chosen freely. Therefore,

$$u_z = w_z$$

for some $w_z \in \ker(\Delta - z)$ arbitrary.

If $z \in \mathrm{Spec}_{\mathrm{odd}}(\Delta)$, say $z = \lambda_{2k+1}^2$, the equation for $\ell = 2k+1$

$$0 = (\lambda_{2k+1}^2 - z)u_{z,2k+1} = \beta Y_{2k+1}^\gamma = 0,$$

imposes no restriction on β nor on $u_{z,2k+1} \in \ker(\Delta - \lambda_{2k+1}^2)$. Nonetheless, from the equations for $\ell \neq 2k+1$ we find that

$$u_{z,\ell} = \frac{\beta}{\lambda_\ell^2 - z} Y_\ell^\gamma,$$

therefore

$$u_z = w_z + \beta G_z^\gamma,$$

for some $w_z \in \ker(\Delta - z)$ arbitrary.

Lastly, if $z \in \mathbb{C} \setminus \mathrm{Spec}(\Delta)$, then

$$u_{z,\ell} = \frac{\beta}{\lambda_\ell^2 - z} Y_\ell^\gamma \quad \forall \ell \in \mathbb{N},$$

and thus,

$$u_z = \beta G_z^\gamma. \qquad \blacksquare$$

Next we prove a simple but useful connection between the functions G_z^γ for $z \in \mathbb{C} \setminus \mathrm{Spec}_{\mathrm{even}}(\Delta)$ and $u_{\gamma 1}$.

Lemma 2.11. *Let $z \in \mathbb{C} \setminus \mathrm{Spec}_{\mathrm{even}}(\Delta)$. Then*

$$G_z^\gamma - c u_{\gamma 1} \in \mathrm{dom}(\Delta) \iff c = \frac{1}{2\pi}. \tag{2.28}$$

Moreover, $G_z^\gamma \in \mathrm{dom}(\Delta_{\gamma,\theta})$ if and only if

$$\delta_\gamma G_z^\gamma = \frac{1}{2\pi} \operatorname{cotan} \theta. \tag{2.29}$$

Proof. We use the following characterization of $\mathrm{dom}(\Delta)$: if $w \in \mathrm{dom}(T^*)$,

$$w \in \mathrm{dom}(\Delta) \iff \omega(u_{\gamma 0}, w) = 0.$$

Let us first prove this. Fix $w \in \mathrm{dom}(T^*)$. Proposition 2.3 says there exist $w_0 \in \mathrm{dom}(T)$ and $a, b \in \mathbb{C}$ such that

$$w = w_0 + a u_{\gamma 0} + b u_{\gamma 1}.$$

Thanks to (2.11) and Lemma A.3 we see

$$\omega(u_{\gamma 0}, w) = -2\pi b.$$

Therefore, $\omega(u_{\gamma 0}, w) = 0$ if and only if $b = 0$, which is equivalent to $w \in \mathrm{dom}(\Delta)$ due to (2.9).

Let us prove now that

$$\omega(u_{\gamma 0}, G_z^\gamma - c u_{\gamma 1}) = 0 \iff c = \frac{1}{2\pi}. \tag{2.30}$$

For that, using the definition of ω (2.10), $u_{\gamma 0} \in \mathrm{dom}(\Delta)$, $T^* G_z^\gamma = z G_z^\gamma$ (see Proposition 2.10), and (2.15),

$$\begin{aligned} \omega(u_{\gamma 0}, G_z^\gamma) &= \langle u_{\gamma 0} \,|\, z G_z^\gamma \rangle_{L^2(\mathbb{S}^2)} - \langle \Delta u_{\gamma 0} \,|\, G_z^\gamma \rangle_{L^2(\mathbb{S}^2)} \\ &= \big\langle (\bar{z} - \Delta) u_{\gamma 0} \,|\, G_z^\gamma \big\rangle_{L^2(\mathbb{S}^2)} = -\overline{\delta_\gamma u_{\gamma 0}} = -1. \end{aligned}$$

Meanwhile, we know from (2.11) that

$$\omega(u_{\gamma 0}, u_{\gamma 1}) = -2\pi.$$

Combining these two identities, we obtain (2.30).

Lastly, $G_z^\gamma \in \mathrm{dom}(\Delta_{\gamma,\theta})$ if and only if there exist $g_0 \in \mathrm{dom}(T)$ and $c \in \mathbb{C}$ such that

$$G_z^\gamma = g_0 + c[\cos\theta\, u_{\gamma 0} + \sin\theta\, u_{\gamma 1}].$$

We rewrite it as follows:

$$G_z^\gamma - c \sin\theta\, u_{\gamma 1} = g_0 + c \cos\theta\, u_{\gamma 0} \in \mathrm{dom}(\Delta).$$

Thanks to (2.28), we get that $G_z^\gamma \in \mathrm{dom}(\Delta_{\gamma,\theta})$ if and only if (observe that $\sin\theta \neq 0$)

$$G_z^\gamma - \frac{1}{2\pi} u_{\gamma 1} = g_0 + \frac{1}{2\pi} \operatorname{cotan}\theta\, u_{\gamma 0},$$

or as a scalar equation,

$$\delta_\gamma \Big(G_z^\gamma - \frac{1}{2\pi} u_{\gamma 1} \Big) = \frac{1}{2\pi} \operatorname{cotan}\theta.$$

Recalling that $G_z^\gamma, u_{\gamma 1} \in \mathrm{dom}(T^*) \subseteq H^1(\mathbb{S}^2)$, and $\delta_\gamma u_{\gamma 1} = 0$, we get (2.29). ∎

Proof of Theorem 2.6. Let $z \in \mathbb{C} \setminus (\mathrm{Spec}(\Delta_{\gamma,\theta}) \cup \mathrm{Spec}(\Delta))$ and fix some $f \in L^2(\mathbb{S}^2)$. Since Δ and $\Delta_{\gamma,\theta}$ are self-adjoint, there exist unique $u \in H^2(\mathbb{S}^2)$ and $w \in \mathrm{dom}(T^*)$ such that

$$(\Delta - z)u = f, \quad (\Delta_{\gamma,\theta} - z)[u + w] = f.$$

This directly implies that

$$\left[(\Delta_{\gamma,\theta} - z)^{-1} - (\Delta - z)^{-1}\right] f = w. \tag{2.31}$$

Since both Δ and $\Delta_{\gamma,\theta}$ are restrictions of the operator T^*, we may apply $(T^* - z)$ on both sides of (2.31) and obtain $w \in \ker(T^* - z)$. Now Proposition 2.10 implies that

$$w = \beta G_z^\gamma$$

for some $\beta \in \mathbb{C}$ depending on f. This proves that

$$(\Delta_{\gamma,\theta} - z)^{-1} - (\Delta - z)^{-1} \in \mathrm{Span}_{\mathbb{C}}\{G_z^\gamma\}$$

that is, it is a rank-one operator, thus $\Delta_{\gamma,\theta}$ has compact resolvent and $z \in \mathrm{Spec}(\Delta_{\gamma,\theta})$ if and only if z is an eigenvalue of $\Delta_{\gamma,\theta}$.

To compute β, we use that $u + w \in \mathrm{dom}(\Delta_{\gamma,\theta})$, which means that there exist $u_0 \in \mathrm{dom}(T)$ and $c \in \mathbb{C}$ such that

$$u + \beta G_z^\gamma = u_0 + c[\cos\theta\, u_{\gamma 0} + \sin\theta\, u_{\gamma 1}], \tag{2.32}$$

where $u_{\gamma 0}, u_{\gamma 1}$ were defined in (2.5). Let us rewrite (2.32) as follows:

$$\beta G_z^\gamma - c\sin\theta\, u_{\gamma 1} = u_0 + c\cos\theta\, u_{\gamma 0} - u \in \mathrm{dom}(\Delta).$$

Due to (2.28), we get that $c = \frac{\beta}{2\pi\sin\theta}$ (recall that we assumed $\theta \neq 0$). Therefore, applying δ_γ on both sides of (2.32),

$$\delta_\gamma u + \beta\delta_\gamma G_z^\gamma = \frac{\beta}{2\pi}\cot\theta,$$

and solving for β,

$$\beta = \frac{1}{\frac{1}{2\pi}\cot\theta - \delta_\gamma G_z^\gamma}\delta_\gamma u, \tag{2.33}$$

where we used that $\frac{1}{2\pi}\cot\theta - \delta_\gamma G_z^\gamma \neq 0$ because $z \notin \mathrm{Spec}(\Delta_{\gamma,\theta})$ (see Lemma 2.11). Lastly, using that

$$\delta_\gamma u = \left\langle G_{\bar{z}}^\gamma \mid (\Delta - z)u \right\rangle_{L^2(\mathbb{S}^2)} = \langle G_{\bar{z}}^\gamma \mid f \rangle_{L^2(\mathbb{S}^2)},$$

and combining it with (2.33), we obtain

$$\beta = \frac{1}{\frac{1}{2\pi}\cot\theta - \delta_\gamma G_z^\gamma}\langle G_{\bar{z}}^\gamma \mid f \rangle_{L^2(\mathbb{S}^2)},$$

thus proving identity (2.16) for

$$\alpha(\theta, z) := \frac{1}{\frac{1}{2\pi}\operatorname{cotan}\theta - \delta_\gamma G_z^\gamma}. \tag{2.34}$$

The rest of the identities in Theorem 2.6 are consequences of the more general identity

$$\ker(\Delta_{\gamma,\theta} - \eta) = \ker(T^* - \eta) \cap \operatorname{dom}(\Delta_{\gamma,\theta}) \quad \forall \eta \in \mathbb{R}, \tag{2.35}$$

in combination with Proposition 2.10. Let us start considering the case $\eta \in \mathbb{R} \setminus \operatorname{Spec}_{\text{even}}(\Delta)$. Proposition 2.10 gives

$$\ker(T^* - \eta) = \ker(\Delta - \eta) + \operatorname{Span}_{\mathbb{C}}\{G_\eta^\gamma\}.$$

We see that for every $w_\eta \in \ker(\Delta - \eta)$, $\delta_\gamma w_\eta = 0$, because either $\omega_\eta = 0$ since $\eta \in \mathbb{R} \setminus \operatorname{Spec}(\Delta)$, or because $\eta \in \operatorname{Spec}_{\text{odd}}(\Delta)$ but $\delta_\gamma \equiv 0$ on this case due to Lemma 2.9. Therefore, we find that $\ker(\Delta - \eta) \subseteq \operatorname{dom}(\Delta_{\gamma,\theta})$. If η is not a solution to (2.17), then $G_\eta^\gamma \notin \operatorname{dom}(\Delta_{\gamma,\theta})$ due to (2.29) and thus

$$\ker(\Delta_{\gamma,\theta} - \eta) = \ker(\Delta - \eta).$$

If η is a solution to (2.17), then $G_\eta^\gamma \in \operatorname{dom}(\Delta_{\gamma,\theta})$, therefore

$$\ker(\Delta_{\delta,\gamma} - \eta) = \ker(\Delta - \eta) + \operatorname{Span}_{\mathbb{C}}\{G_z^\gamma\}.$$

If we consider now $\eta \in \operatorname{Spec}_{\text{even}}(\Delta)$, say $\eta = \lambda_{2\ell}^2$, Proposition 2.10 implies that

$$\ker(T^* - \eta) = \ker(\Delta - \eta).$$

Since $Y_{2\ell}^\gamma \in \ker(\Delta - \eta) \setminus \ker(\Delta_{\gamma,\theta} - \eta)$, we immediately obtain

$$\ker(\Delta_{\gamma,\theta} - \lambda_{2\ell}^2) = \{u \in \ker(\Delta - \lambda_{2\ell}^2) \colon \delta_\gamma u = 0\}. \qquad \blacksquare$$

Lemma 2.12. *Fix $m \in \mathbb{N}$, and consider the interval $J_m := (\lambda_{2m-2}^2, \lambda_{2m}^2)$ if $m \geq 1$, and $J_0 := (-\infty, 0)$ if $m = 0$. The function*

$$F \colon J_m \to \mathbb{R}, \quad F(\eta) := \delta_\gamma G_\eta^\gamma$$

is strictly increasing, and if $m \geq 1$, it is bijective.

Proof. From the eigenfunction expansion of G_η^γ for $\eta \in \mathbb{R}$ (2.24), we see that G_η^γ is real-valued (because Y_ℓ^γ is real-valued for all $\ell \in \mathbb{N}$), therefore $F(\eta) \in \mathbb{R}$ for all $\eta \in J_m$. Moreover, Proposition 2.5 gives that $\mathcal{G}^\gamma$ is analytic on $\mathbb{C} \setminus \operatorname{Spec}_{\text{even}}(\Delta)$, therefore F is analytic on J_m, and thus differentiable. We compute $F'(\eta)$, for $\eta \in J_m$.

$$\begin{aligned} F'(\eta) &= \delta_\gamma\left(\frac{\mathrm{d}}{\mathrm{d}\tau}(G_\tau^\gamma)\big|_{\tau=\eta}\right) = \delta_\gamma\left(\sum_{\ell\in\mathbb{N}} \frac{1}{(\lambda_{2\ell}^2 - \eta)^2} Y_{2\ell}^\gamma\right) \\ &= \sum_{\ell\in\mathbb{N}} \frac{1}{(\lambda_{2\ell}^2 - \eta)^2} \delta_\gamma Y_{2\ell}^\gamma \overset{(2.18)}{=} \sum_{\ell\in\mathbb{N}} \frac{1}{(\lambda_{2\ell}^2 - \eta)^2} \|Y_{2\ell}^\gamma\|_{L^2(\mathbb{S}^2)}^2 = \|G_\eta^\gamma\|_{L^2(\mathbb{S}^2)}^2 > 0. \end{aligned}$$

This proves that F is strictly increasing on J_m. To show that it is bijective, it suffices to show that $\lim_{\eta\to\lambda_{2m}^2} F(\eta) = +\infty$ and $\lim_{\eta\to\lambda_{2m-2}^2} F(\eta) = -\infty$ for $m \geq 1$.

Let us define the function

$$H^\gamma(z) := G_z^\gamma - \frac{1}{\lambda_{2m}^2 - z} Y_{2m}^\gamma, \quad z \in \mathbb{C} \setminus \mathrm{Spec}_{\mathrm{even}}(\Delta).$$

It is a $H^1(\mathbb{S}^2)$-valued analytic function with a removable singularity at $z = \lambda_{2m}^2$. Therefore, for $\eta \in (\lambda_{2m-2}^2, \lambda_{2m}^2)$, $m \geq 1$,

$$\begin{aligned}\lim_{\eta\to(\lambda_{2m}^2)^-} F(\eta) &= \lim_{\eta\to(\lambda_{2m}^2)^-} \delta_\gamma\left(H^\gamma(\eta) + \frac{1}{\lambda_{2m}^2 - \eta} Y_{2m}^\gamma\right) \\ &= \delta_\gamma\big(H^\gamma(\lambda_{2m}^2)\big) + \|Y_{2m}^\gamma\|_{L^2(\mathbb{S}^2)}^2 \lim_{\eta\to(\lambda_{2m}^2)^-} \frac{1}{\lambda_{2m}^2 - \eta} = +\infty.\end{aligned}$$

The statement $\lim_{\eta\to(\lambda_{2m-2}^2)^+} F(\eta) = -\infty$ is proven similarly. ∎

One could follow the same steps as presented in this section to make a singular perturbation of a self-adjoint operator A with a certain unbounded functional f, provided Lemma A.1 holds for the pair (A, f). One such an example are point-perturbations of Δ on a Riemannian manifold (M, g), that is, singular perturbations of Δ by the functional δ_q for some point $q \in M$. For simplicity, let us assume that $\dim(M) = 2$. Then

$$\mathrm{dom}(T) = \big\{u \in \mathrm{dom}(\Delta) \colon u(q) = 0\big\},$$

and there exist $u_{q0} = \mathbb{1}_M$, $u_{q1} \in L^2(M) \setminus H^1(M)$ such that

$$\mathrm{dom}(T^*) = \mathrm{dom}(T) + \mathrm{Span}_{\mathbb{C}}\{u_{q0}, u_{q1}\}.$$

The family of all point-perturbations is parameterized by $\theta \in \mathbb{R}/(\pi\mathbb{Z})$ too. Then, we would define the functions G_z^q for $z \in \mathbb{C} \setminus \mathrm{Spec}(\Delta)$, which are precisely the Green's functions of Δ at the point q. A resolvent identity as (2.16) still holds, which implies that any point-perturbation of Δ has compact resolvent if and only if Δ does. The main difference from the present case, as reported in Remark 2.4, is that functions G_z^q and u_{q1} both have a logarithmic singularity on q, thus we cannot make the evaluations $\delta_q u_{q1}$ nor $\delta_q G_z^q$, but a lemma like Lemma 2.11 tells us that $G_z^q - c u_{q1}$ belongs to $\mathrm{dom}(\Delta)$ for some $c \in \mathbb{R}$, allowing us to evaluate this function on q.

3. Quasimodes for new eigenfunctions

In this section, we are going to find quasimodes of Δ that approximate G_η^γ in the high-energy limit. This is the semiclassical limit; thus, we instead introduce a semiclassical parameter $h > 0$ and write everything in terms of it. We are interested in understanding the family of new eigenfunctions for any singular perturbation, the parameter $\theta \in \Theta$ will play no role in this section.

Given $h > 0$ such that $h^{-2} \notin \mathrm{Spec}_{\mathrm{even}}(\Delta)$, we will denote it throughout the rest of the paper

$$G_h^\gamma := \sum_{\ell \in \mathbb{N}} \frac{1}{\lambda_{2\ell}^2 - h^{-2}} Y_{2\ell}^\gamma, \tag{3.1}$$

and

$$g_h^\gamma := \frac{1}{\|G_h^\gamma\|_{L^2(\mathbb{S}^2)}} G_h^\gamma. \tag{3.2}$$

We look for quasimodes of g_h^γ of the form

$$\frac{1}{\|G_h^\gamma\|_{L^2(\mathbb{S}^2)}} \sum_{\ell \in I_h} \frac{1}{\lambda_{2\ell}^2 - h^{-2}} Y_{2\ell}^\gamma,$$

for some finite interval I_h depending on h. For that, we need to understand the asymptotics of

$$\frac{1}{\|G_h^\gamma\|_{L^2(\mathbb{S}^2)}} \quad \text{as } h \to 0^+,$$

and of the L^2-norm of the terms

$$\frac{1}{\lambda_{2\ell}^2 - h^{-2}} Y_{2\ell}^\gamma \quad \text{as } h \to 0^+ \text{ and } \ell \text{ depends on } h.$$

Therefore, given a sequence $(h_n)_{n\in\mathbb{N}} \subseteq (0, \infty)$ such that $h_n^{-2} \notin \mathrm{Spec}_{\mathrm{even}}(\Delta)$, for every $n \in \mathbb{N}$ we introduce $\ell_n \in \mathbb{N}$ and $\sigma_n \in (0, 1)$ defined by the following identity:

$$(h_n)^{-2} = \big(2(\ell_n + \sigma_n)\big)\big(2(\ell_n + \sigma_n) + 1\big). \tag{3.3}$$

One has $(h_n)^{-2} \in (\lambda_{2\ell_n}^2, \lambda_{2\ell_n+2}^2)$, and the parameter σ_n measures the proximity of $(h_n)^{-2}$ to the ends of this interval, which will help us control the coefficients $(\lambda_{2\ell}^2 - h^{-2})^{-1}$. Lastly, for $\ell \in \mathbb{N}$, define

$$y_{2\ell}^\gamma := \frac{1}{\|Y_{2\ell}^\gamma\|_{L^2(\mathbb{S}^2)}} Y_{2\ell}^\gamma. \tag{3.4}$$

Theorem 3.1. *Let $(h_n)_{n\in\mathbb{N}} \subseteq (0, 1)$ such that $h_n^{-2} \notin \mathrm{Spec}_{\mathrm{even}}(\Delta)$, and let $\ell_n \in \mathbb{N}$ and $\sigma_n \in (0, 1)$ be defined as in (3.3). Assume that $\lim_{n\to\infty} h_n = 0$ and that $\lim_{n\to\infty} \sigma_n = \sigma \in [0, 1]$.*

(1) *If $\sigma \in (0, 1)$, set*

$$C_\sigma := \left(\sum_{k \in \mathbb{Z}} \frac{1}{(k - \sigma)^2} \right)^{\frac{1}{2}}. \tag{3.5}$$

There exists $C > 1$ and $\Upsilon_0 \in \mathbb{N}$ such that for every $\Upsilon \geq \Upsilon_0$ there exists $N(\Upsilon) \in \mathbb{N}$ such that for all $n \geq N(\Upsilon)$,

$$\left\| g_{h_n}^\gamma - \frac{1}{C_\sigma} \sum_{|k-\sigma| \leq \Upsilon} \frac{1}{k - \sigma} y_{2(\ell_n+k)}^\gamma \right\|_{L^2(\mathbb{S}^2)} \leq C\big[\Upsilon h_n + |\sigma_n - \sigma| + \Upsilon^{-\frac{1}{2}}\big].$$

(2) *If $\sigma \in \{0,1\}$, there exists $C > 1$ and $N \in \mathbb{N}$ such that for all $n \geq N$,*

$$\left\| g^{\gamma}_{h_n} - (-1)^{1-\sigma} y^{\gamma}_{2(\ell_n+\sigma)} \right\|_{L^2(\mathbb{S}^2)} \leq C\big[h_n + |\sigma_n - \sigma|\big].$$

Theorem 3.1 will be a consequence of the asymptotics provided in Propositions 3.6 and 3.7, which themselves depend on several simpler asymptotics, which we collect in lemmas below.

Lemma 3.2. *Let γ be a closed geodesic in $\mathbb{S}^2$, and let Y^{γ}_{ℓ} be the eigenfunction defined in (2.18). There exists $C > 0$ such that for all $\ell \in \mathbb{N}$*

$$\left| \|Y^{\gamma}_{2\ell}\|^2_{L^2(\mathbb{S}^2)} - \frac{1}{\pi^2} \right| \leq C(1+\ell)^{-1}. \tag{3.6}$$

Proof. Lemma 2.9 implies that for all $\ell \in \mathbb{N}$

$$Y^{\gamma}_{2\ell} = (-1)^{\ell} \frac{\Gamma\big(\ell + \frac{1}{2}\big)}{\Gamma\big(\frac{1}{2}\big)\Gamma(\ell+1)} Z^{q}_{2\ell}.$$

Moreover, recall that

$$\|Z^{q}_{\ell}\|^2_{L^2(\mathbb{S}^2)} = Z^{q}_{\ell}(q) = \frac{m_{\ell}}{\mathrm{vol}(\mathbb{S}^2)} = \frac{2\ell+1}{\mathrm{vol}(\mathbb{S}^2)}.$$

Putting both things together, we find that

$$\|Y^{\gamma}_{2\ell}\|^2_{L^2(\mathbb{S}^2)} = \frac{1}{\Gamma\big(\frac{1}{2}\big)^2 \mathrm{vol}(\mathbb{S}^2)} \frac{\Gamma\big(\ell+\frac{1}{2}\big)^2 (4\ell+1)}{\Gamma(\ell+1)^2}.$$

Lastly, we may invoke the following asymptotic estimate for the Gamma function [32, (5.11.3)]:

$$\frac{\Gamma(x+a)}{\Gamma(x+b)} = x^{a-b}\big(1 + \mathcal{O}(x^{-1})\big) \quad \text{as } x \to +\infty,$$

and the identities $\Gamma(1/2) = \sqrt{\pi}$ and $\mathrm{vol}(\mathbb{S}^2) = 4\pi$, to obtain

$$\left| \|Y^{\gamma}_{2\ell}\|^2_{L^2(\mathbb{S}^2)} - \frac{1}{\pi^2} \right| = \mathcal{O}(\ell^{-1}) \quad \text{as } \ell \to \infty.$$

Since $\mathbb{N}$ is discrete in $\mathbb{R}$, one can find $C > 0$ such that (3.6) holds for all $\ell \in \mathbb{N}$. ∎

Lemma 3.3. *For $h > 0$ such that $h^{-2} \notin \mathrm{Spec}_{\mathrm{even}}(\Delta)$, set $\ell \in \mathbb{N}$ and $\sigma \in (0,1)$ such that*

$$h^{-2} = \big(2(\ell+\sigma)\big)\big(2(\ell+\sigma)+1\big) \in (\lambda^2_{2\ell}, \lambda^2_{2\ell+2}).$$

Then, for every $0 < h < 1$

$$\big|h^{-1} - \big[2(\ell+\sigma) + \tfrac{1}{2}\big]\big| \leq \frac{h}{4}, \quad \left| h - \frac{1}{2(\ell+\sigma)+\frac{1}{2}} \right| \leq \frac{h^3}{4}.$$

Proof. The first inequality follows from a direct computation:

$$\left|h^{-1} - \left(2(\ell+\sigma)+\tfrac{1}{2}\right)\right| = \frac{\left|h^{-2} - \left[2(\ell+\sigma)+\frac{1}{2}\right]^2\right|}{h^{-1} + \left[2(\ell+\sigma)+\frac{1}{2}\right]} \leq \frac{\frac{1}{4}}{h^{-1}} \leq \frac{h}{4}.$$

The second inequality is a consequence of the first. We see that

$$2(\ell+\sigma)+\tfrac{1}{2} \geq h^{-1},$$

therefore,

$$\left|h - \frac{1}{2(\ell+\sigma)+\frac{1}{2}}\right| = \frac{\left|h^{-1} - \left[2(\ell+\sigma)+\frac{1}{2}\right]\right|}{h^{-1}\left[2(\ell+\sigma)+\frac{1}{2}\right]} \leq \frac{h/4}{h^{-2}} = \frac{h^3}{4}. \qquad \blacksquare$$

Lemma 3.4. *For $h > 0$ such that $h^{-2} \notin \mathrm{Spec}_{\mathrm{even}}(\Delta)$, set $\ell \in \mathbb{N}$ and $\sigma \in (0,1)$ such that*

$$h^{-2} = \big(2(\ell+\sigma)\big)\big(2(\ell+\sigma)+1\big) \in (\lambda_{2\ell}^2, \lambda_{2\ell+2}^2).$$

Then, for every $0 < h < 1$ and every $k \in \mathbb{Z}$, $k \geq -\ell$,

$$\left|\lambda_{2(\ell+k)}^2 - \left[h^{-2} + 4(k-\sigma)h^{-1} + 4(k-\sigma)^2\right]\right| \leq |k-\sigma|h. \tag{3.7}$$

Proof. Using the identity $x(x+1) - y(y+1) = (x-y)(x+y+1)$, we find that

$$\begin{aligned}\lambda_{2(\ell+k)}^2 - h^{-2} &= \big(2(\ell+k)\big)\big(2(\ell+k)+1\big) - \big(2(\ell+\sigma)\big)\big(2(\ell+\sigma)+1\big)\\ &= 2(k-\sigma)[4\ell + 2k + 2\sigma + 1]\\ &= 4(k-\sigma)\left[2(\ell+\sigma)+\tfrac{1}{2}\right] + 4(k-\sigma)^2.\end{aligned}$$

Therefore,

$$\lambda_{2(\ell+k)}^2 - h^{-2} - 4(k-\sigma)h^{-1} - 4(k-\sigma)^2 = 4(k-\sigma)\left[2(\ell+\sigma)+\tfrac{1}{2} - h^{-1}\right].$$

Now Lemma 3.3 implies

$$\left|\lambda_{2(\ell+k)}^2 - h^{-2} - 4(k-\sigma)h^{-1} - 4(k-\sigma)^2\right| \leq |k-\sigma|h. \qquad \blacksquare$$

Lemma 3.5. *For $h > 0$ such that $h^{-2} \notin \mathrm{Spec}_{\mathrm{even}}(\Delta)$, set $\ell \in \mathbb{N}$ and $\sigma \in (0,1)$ such that*

$$h^{-2} = \big(2(\ell+\sigma)\big)\big(2(\ell+\sigma)+1\big) \in (\lambda_{2\ell}^2, \lambda_{2\ell+2}^2).$$

There exists $C > 0$ and $h_0 > 0$ such that for all $0 < h < h_0$ and all $k \geq -\ell$

$$\begin{aligned}&\left|\frac{1}{\left|\lambda_{2(\ell+k)}^2 - h^{-2}\right|^2}\left\|Y_{2(\ell+k)}^{\gamma}\right\|_{L^2(\mathbb{S}^2)}^2 - \frac{1}{16(k-\sigma)^2}\frac{1}{(h^{-1}+k-\sigma)^2}\frac{1}{\pi^2}\right|\\ &\qquad \leq \frac{C}{|k-\sigma|^2}\left[h^2|1+\ell+k|^{-1} + h^4 + h^5|k-\sigma|\right]. \end{aligned}\tag{3.8}$$

Proof. This is a consequence of the triangular inequality, after adding and subtracting

$$\frac{1}{\big[4(k-\sigma)(h^{-1}+k-\sigma)\big]^2}\|Y^{\gamma}_{2(\ell+k)}\|^2_{L^2(\mathbb{S}^2)}.$$

On the one hand, thanks to Lemma 3.2,

$$\left|\frac{1}{|\lambda^2_{2(\ell+k)}-h^{-2}|^2}\|Y^{\gamma}_{2(\ell+k)}\|^2_{L^2(\mathbb{S}^2)}-\frac{1}{\big[4(k-\sigma)(h^{-1}+k-\sigma)\big]^2}\|Y^{\gamma}_{2(\ell+k)}\|^2_{L^2(\mathbb{S}^2)}\right| \le C\left|\frac{[\lambda^2_{2(\ell+k)}-h^{-2}]^2-\big[4(k-\sigma)(h^{-1}+k-\sigma)\big]^2}{\big[(\lambda^2_{2(\ell+k)}-h^{-2})4(k-\sigma)(h^{-1}+k-\sigma)\big]^2}\right|.$$

For the numerator, we may apply identity $x^2-y^2=(x-y)(x+y)$ and Lemma 3.4 to obtain

$$\begin{aligned}\big|[\lambda^2_{2(\ell+k)}-h^{-2}]^2-\big[4(k-\sigma)(h^{-1}+k-\sigma)\big]^2\big| &\le |k-\sigma|h\big|\lambda^2_{2(\ell+k)}-h^{-2}+4(k-\sigma)h^{-1}+4(k-\sigma)^2\big|\\ &\le C|k-\sigma|h\big[|k-\sigma|h^{-1}+|k-\sigma|^2\big].\end{aligned}$$

For the denominator, adding and subtracting $4(k-\sigma)(h^{-1}+k-\sigma)$ and thanks to Lemma 3.4,

$$\begin{aligned}|\lambda^2_{2(\ell+k)}-h^{-2}| &\ge \big|4(k-\sigma)(h^{-1}+k-\sigma)\big|\\ &\quad-\big|\lambda^2_{2(\ell+k)}-\big[h^{-2}+4(k-\sigma)h^{-1}+4(k-\sigma)^2\big]\big|\\ &\ge 4|k-\sigma|(h^{-1}+k-\sigma)-|k-\sigma|h\\ &= 4|k-\sigma|\big(h^{-1}+k-\sigma-\tfrac{1}{4}h\big),\end{aligned}$$

thus, using the fact that there exists $h_0>0$ such that for all $k\ge-\ell$ and all $0<h<h_0$

$$h^{-1}+k-\sigma-\tfrac{1}{4}h\ge\tfrac{1}{2}h^{-1}-\tfrac{1}{4}h\ge\tfrac{1}{4}h^{-1}, \tag{3.9}$$

we find that

$$|\lambda^2_{2(\ell+k)}-h^{-2}|\ge|k-\sigma|h^{-1}\quad\forall 0<h<h_0. \tag{3.10}$$

Therefore,

$$\big|(\lambda^2_{2(\ell+k)}-h^{-2})4(k-\sigma)(h^{-1}+k-\sigma)\big|^2\ge|k-\sigma|^4h^{-4}.$$

Putting everything together,

$$\left|\frac{1}{|\lambda^2_{2(\ell+k)}-h^{-2}|^2}\|Y^{\gamma}_{2(\ell+k)}\|^2_{L^2(\mathbb{S}^2)}-\frac{1}{\big[4(k-\sigma)(h^{-1}+k-\sigma)\big]^2}\|Y^{\gamma}_{2(\ell+k)}\|^2_{L^2(\mathbb{S}^2)}\right| \le\frac{C|k-\sigma|h}{|k-\sigma|^4h^{-4}}\big[|k-\sigma|h^{-1}+|k-\sigma|^2\big]\le C\frac{1}{|k-\sigma|^2}\big[h^4+h^5|k-\sigma|\big].$$

On the other hand, Lemma 3.2 and inequality (3.9) let us obtain

$$\left|\frac{1}{\big[4(k-\sigma)(h^{-1}+k-\sigma)\big]^2}\|Y^{\gamma}_{2(\ell+k)}\|^2_{L^2(\mathbb{S}^2)} - \frac{1}{16(k-\sigma)^2}\frac{1}{(h^{-1}+k-\sigma)^2}\frac{1}{\pi^2}\right| \leq \frac{C}{4(k-\sigma)^2}h^2|1+\ell+k|^{-1}.$$

The triangular inequality gives the result. ∎

For a subset $X \subseteq \mathbb{N}$, let Π_X be the orthogonal projection in $L^2(\mathbb{S}^2)$ onto the subspace $\bigoplus_{\ell\in X}\ker(\Delta-\lambda_\ell^2)$, and for $s\in\mathbb{R}$, $r\geq 0$, let $I(s,r) := [s-r, s+r]\cap\mathbb{N}$. For example, for $h_n>0$ and $\ell_n\in\mathbb{N}$ as defined above, and $\Upsilon\in\mathbb{N}$,

$$\Pi_{I(2\ell_n,2\Upsilon)}G^{\gamma}_{h_n} = \sum_{|k|\leq\Upsilon}\frac{1}{\lambda^2_{2(\ell_n+k)}-h_n^{-2}}Y^{\gamma}_{2(\ell_n+k)}. \tag{3.11}$$

Proposition 3.6. *For $h>0$ such that $h^{-2}\notin \mathrm{Spec}_{\mathrm{even}}(\Delta)$, set $\ell\in\mathbb{N}$ and $\sigma\in(0,1)$ such that*

$$h^{-2} = \big(2(\ell+\sigma)\big)\big(2(\ell+\sigma)+1\big) \in (\lambda^2_{2\ell},\lambda^2_{2\ell+2}).$$

There exists $C>0$ and $h_0>0$ such that for all integers $\Upsilon\geq 1$, and all $0<h<h_0$

$$\|G^{\gamma}_h - \Pi_{I(2(\ell+\sigma),2\Upsilon)}G^{\gamma}_h\|^2_{L^2(\mathbb{S}^2)} \leq C\frac{h^2}{\Upsilon}.$$

Proof. Note that since $\sigma\in(0,1)$, then

$$I\big(2(\ell+\sigma),2\Upsilon\big)\cap 2\mathbb{N} = \big\{2(\ell+k)\in\mathbb{N}: -(\Upsilon-1)\leq k\leq\Upsilon\big\}.$$

Therefore, we have to estimate

$$\|G^{\gamma}_h - \Pi_{I(2(\ell+\sigma),2\Upsilon)}G^{\gamma}_h\|^2_{L^2(\mathbb{S}^2)} = \left[\sum_{k=-\ell}^{-\Upsilon} + \sum_{k=\Upsilon+1}^{\infty}\right]\frac{1}{|\lambda^2_{2(\ell+k)}-h^{-2}|^2}\|Y^{\gamma}_{2(\ell+k)}\|^2_{L^2(\mathbb{S}^2)}$$

for small $h>0$. For that, let $h_0>0$ be given by (3.9). Therefore, (3.10) holds and we get that for all $0<h<h_0$

$$\frac{1}{|\lambda^2_{2(\ell+k)}-h^{-2}|^2} \leq C\frac{1}{|k|^2}h^2, \qquad \text{if } -\ell\leq k\leq-\Upsilon,$$
$$\frac{1}{|\lambda^2_{2(\ell+k)}-h^{-2}|^2} \leq C\frac{1}{|k-1|^2}h^2, \quad \text{if } k\geq\Upsilon+1.$$

Thanks to Lemma 3.2, we obtain that for all $0<h<h_0$ and all integers $\Upsilon\geq 1$,

$$\|G^{\gamma}_h - \Pi_{I(2(\ell+\sigma),2\Upsilon)}G^{\gamma}_h\|^2_{L^2(\mathbb{S}^2)} \leq C\left(\sum_{k\geq\Upsilon}\frac{1}{k^2}\right)h^2 \leq C\frac{h^2}{\Upsilon}.$$ ∎

Proposition 3.7. *Let $(h_n)_{n\in\mathbb{N}} \subseteq (0,1)$ be such that $h_n^{-2} \notin \mathrm{Spec}_{\mathrm{even}}(\Delta)$ for all $n \in \mathbb{N}$. For $n \in \mathbb{N}$, set $\ell_n \in \mathbb{N}$ and $\sigma_n \in (0,1)$ such that (3.3)*

$$(h_n)^{-2} = \big(2(\ell_n + \sigma_n)\big)\big(2(\ell_n + \sigma_n) + 1\big).$$

Assume that $\lim_{n\to\infty} h_n = 0$ and that $\lim_{n\to\infty} \sigma_n = \sigma \in [0,1]$.

(1) *If $\sigma \in (0,1)$, set*

$$C_\sigma := \Big(\sum_{k\in\mathbb{Z}} \frac{1}{(k-\sigma)^2}\Big)^{\frac{1}{2}}. \tag{3.12}$$

There exists $C > 1$ such that for every $\Upsilon \geq 1$ there exists $N(\Upsilon) \in \mathbb{N}$ such that for all $n \geq N(\Upsilon)$,

$$\Big| \| \Pi_{I(2(\ell_n+\sigma),2\Upsilon)} G^\gamma_{h_n} \|^2_{L^2(\mathbb{S}^2)} - \frac{(C_\sigma)^2}{16\pi^2}(h_n)^2 \Big| \leq C(h_n)^2 \big[\Upsilon h_n + |\sigma_n - \sigma| + \Upsilon^{-1}\big]. \tag{3.13}$$

(2) *If $\sigma \in \{0,1\}$, there exists $C > 1$ and $N \in \mathbb{N}$ such that for all $n \geq N$,*

$$\Big| \| \Pi_{I(2(\ell_n+\sigma),0)} G^\gamma_{h_n} \|^2_{L^2(\mathbb{S}^2)} - \frac{1}{16\pi^2}\frac{(h_n)^2}{|\sigma_n - \sigma|^2} \Big| \leq C \frac{(h_n)^3}{|\sigma_n - \sigma|^2}. \tag{3.14}$$

Proof. (1) First assume that $\sigma \in (0,1)$. We have

$$\| \Pi_{I(2(\ell_n+\sigma),2\Upsilon)} G^\gamma_{h_n} \|^2_{L^2(\mathbb{S}^2)} = \sum_{k=-\Upsilon+1}^{\Upsilon} \frac{1}{|\lambda^2_{2(\ell_n+k)} - h_n^{-2}|^2} \| Y^\gamma_{2(\ell_n+k)} \|^2_{L^2(\mathbb{S}^2)}$$

therefore,

$$\begin{aligned}
&\Big| \| \Pi_{I(2(\ell_n+\sigma),2\Upsilon)} G^\gamma_{h_n} \|^2_{L^2(\mathbb{S}^2)} - \frac{(C_\sigma)^2}{16\pi^2}(h_n)^2 \Big| \\
&\quad \leq \sum_{k=-\Upsilon+1}^{\Upsilon} \Big| \frac{1}{|\lambda^2_{2(\ell_n+k)} - h_n^{-2}|^2} \| Y^\gamma_{2(\ell_n+k)} \|^2_{L^2(\mathbb{S}^2)} - \frac{(h_n)^2}{16(k-\sigma)^2}\frac{1}{\pi^2} \Big| \\
&\qquad + \frac{(h_n)^2}{16\pi^2} \Big[\sum_{k=-\ell_n}^{-\Upsilon} + \sum_{k=\Upsilon+1}^{\infty} \Big] \frac{1}{|k-\sigma|^2} \\
&\quad =: \sum_{k=-\Upsilon+1}^{\Upsilon} A_k + B_\Upsilon.
\end{aligned}$$

It is immediate to prove that for all $\Upsilon \geq 1$,

$$B_\Upsilon \leq C \frac{(h_n)^2}{\Upsilon}.$$

To estimate A_k for $-\Upsilon+1\leq k\leq\Upsilon$, we combine triangular inequality with Lemma 3.5: there exists $C>0$ and $h_0>0$ such that for all $0<h_n<h_0$ and $-\Upsilon+1\leq k\leq\Upsilon$,

$$A_k\leq\frac{C}{|k-\sigma_n|^2}\big[\Upsilon^{-1}(h_n)^2+\Upsilon(h_n)^5\big]+\frac{1}{16\pi^2}\left|\frac{1}{|k-\sigma_n|^2}\frac{1}{|h_n^{-1}+k-\sigma_n|^2}-\frac{(h_n)^2}{|k-\sigma|^2}\right|.$$

We estimate the second term invoking triangular inequality once again, adding and subtracting

$$\frac{1}{(k-\sigma)^2}\frac{1}{|h_n^{-1}+k-\sigma_n|^2}.$$

Take $N_1\in\mathbb{N}$ such that $h_n<h_0$ and $|\sigma_n-\sigma|\leq\frac{1}{2}d(\sigma,\mathbb{Z})$ for all $n\geq N_1$. One finds

$$\begin{aligned}\left|\frac{1}{(k-\sigma_n)^2}-\frac{1}{(k-\sigma)^2}\right|&=\frac{1}{(k-\sigma)^2}\left|\frac{1}{\big(1+\frac{\sigma-\sigma_n}{k-\sigma}\big)^2}-1\right|\\&=\frac{1}{(k-\sigma)^2}\frac{\big|\frac{\sigma-\sigma_n}{k-\sigma}\big|\big|2+\frac{\sigma-\sigma_n}{k-\sigma}\big|}{\big(1+\frac{\sigma-\sigma_n}{k-\sigma}\big)^2}\leq 10\frac{|\sigma-\sigma_n|}{|k-\sigma|^3},\end{aligned}\tag{3.15}$$

thus, taking $N_2\in\mathbb{N}$ such that $h_n^{-1}\geq 3\Upsilon$ for all $n\geq N_2$, triangular inequality and (3.9) gives that for all $n\geq\max\{N_1,N_2\}$ and all $-\Upsilon+1\leq k\leq\Upsilon$,

$$\begin{aligned}&\left|\frac{1}{|k-\sigma_n|^2}\frac{1}{|h_n^{-1}+k-\sigma_n|^2}-\frac{(h_n)^2}{|k-\sigma|^2}\right|\\&\qquad\leq C\frac{|\sigma-\sigma_n|}{|k-\sigma|^3}(h_n)^2+\frac{1}{(k-\sigma)^2}\left|\frac{1}{|h_n^{-1}+k-\sigma_n|^2}-(h_n)^2\right|.\end{aligned}$$

Moreover, for all $n\geq\max\{N_1,N_2\}$ and all $-\Upsilon+1\leq k\leq\Upsilon$,

$$\left|\frac{1}{|h_n^{-1}+k-\sigma_n|^2}-(h_n)^2\right|\leq C\Upsilon(h_n)^3.$$

Putting everything together, we have for all $n\geq N:=\max\{N_1,N_2\}$, and all $-\Upsilon+1\leq k\leq\Upsilon$,

$$A_k\leq C(h_n)^2\left[\frac{\Upsilon^{-1}}{|k-\sigma|^2}+\frac{|\sigma-\sigma_n|}{|k-\sigma|^3}+\frac{h_n\Upsilon}{|k-\sigma|^2}\right],$$

where we applied (3.15) to change $|k-\sigma_n|$ for $|k-\sigma|$ in the denominator when needed. Combining the estimates for A_k and B_Υ, and using that

$$\sum_{k\in\mathbb{Z}}\frac{1}{|k-\sigma|^2}<\infty\quad\text{and}\quad\sum_{k\in\mathbb{Z}}\frac{1}{|k-\sigma|^3}<\infty,$$

we obtain (3.13).

(2) Let us assume now that $\sigma\in\{0,1\}$. We have

$$\|\Pi_{I(2(\ell_n+\sigma),0)}G^\gamma_{h_n}\|^2_{L^2(\mathbb{S}^2)}=\frac{1}{\big|\lambda^2_{2(\ell_n+\sigma)}-(h_n)^{-2}\big|^2}\|Y^\gamma_{2(\ell_n+\sigma)}\|^2_{L^2(\mathbb{S}^2)}.$$

A direct application of Lemma 3.5 for $k = \sigma$ and triangular inequality gives

$$\left| \|\Pi_{I(2(\ell_n+\sigma),0)} G^{\gamma}_{h_n}\|^2_{L^2(\mathbb{S}^2)} - \frac{1}{16\pi^2}\frac{(h_n)^2}{|\sigma_n-\sigma|^2} \right| \leq \frac{C}{|\sigma-\sigma_n|^2}(h_n)^3 + \frac{1}{16\pi^2|\sigma-\sigma_n|^2}\left| \frac{1}{|h_n^{-1}+\sigma-\sigma_n|^2} - (h_n)^2 \right|.$$

Since $\sigma_n, \sigma \in [0,1]$, we have

$$\left| \frac{1}{|h_n^{-1}+\sigma-\sigma_n|^2} - (h_n)^2 \right| \leq C(h_n)^3,$$

hence, we have shown that

$$\left| \|\Pi_{I(2(\ell_n+\sigma),0)} G^{\gamma}_{h_n}\|^2_{L^2(\mathbb{S}^2)} - \frac{1}{16\pi^2}\frac{(h_n)^2}{|\sigma_n-\sigma|^2} \right| \leq C\frac{(h_n)^3}{|\sigma-\sigma_n|^2}. \qquad \blacksquare$$

Proof of Theorem 3.1. (1) Assume that $\sigma \in (0,1)$ and let $\Upsilon \geq 1$. Note that

$$I\big(2(\ell_n+\sigma_n), 2\Upsilon\big) = I\big(2(\ell_n+\sigma), 2\Upsilon\big).$$

We can take $\Upsilon_0 \in \mathbb{N}$ large enough, and for every $\Upsilon \geq \Upsilon_0$ take $N_1(\Upsilon) \in \mathbb{N}$ large enough so that the right-hand side of (3.13) is small enough and the following lower bound for $\|G^{\gamma}_{h_n}\|_{L^2(\mathbb{S}^2)}$ holds for all $n \geq N_1(\Upsilon)$:

$$\|G^{\gamma}_{h_n}\|^2_{L^2(\mathbb{S}^2)} \geq \|\Pi_{I(2(\ell_n+\sigma),2\Upsilon)} G^{\gamma}_{h_n}\|^2_{L^2(\mathbb{S}^2)} \geq \frac{(C_\sigma)^2}{64\pi^2}(h_n)^2. \tag{3.16}$$

As a first approximation for $g^{\gamma}_{h_n}$ we have

$$\|g^{\gamma}_{h_n} - \Pi_{I(2(\ell_n+\sigma),2\Upsilon)} g^{\gamma}_{h_n}\|_{L^2(\mathbb{S}^2)} \leq C\,\Upsilon^{-\frac{1}{2}}, \tag{3.17}$$

thanks to Proposition 3.6 and equation (3.16). We use Pythagorean theorem to estimate the remaining term in the approximation,

$$\left\| \Pi_{I(2(\ell_n+\sigma),2\Upsilon)} g^{\gamma}_{h_n} - \frac{1}{C_\sigma}\sum_{|k-\sigma|\leq\Upsilon} \frac{1}{k-\sigma} y^{\gamma}_{2(\ell_n+k)} \right\|_{L^2(\mathbb{S}^2)} = \left(\sum_{|k-\sigma|\leq\Upsilon} \left| \frac{\|Y^{\gamma}_{2(\ell_n+k)}\|_{L^2(\mathbb{S}^2)}}{\|G^{\gamma}_{h_n}\|_{L^2(\mathbb{S}^2)}(\lambda^2_{2(\ell_n+k)} - h_n^{-2})} - \frac{1}{C_\sigma}\frac{1}{k-\sigma} \right|^2 \right)^{\frac{1}{2}}. \tag{3.18}$$

In order to upper bound

$$\left| \frac{\|Y^{\gamma}_{2(\ell_n+k)}\|_{L^2(\mathbb{S}^2)}}{\|G^{\gamma}_{h_n}\|_{L^2(\mathbb{S}^2)}(\lambda^2_{2(\ell_n+k)} - h_n^{-2})} - \frac{1}{C_\sigma}\frac{1}{k-\sigma} \right|$$

for $|k-\sigma| \leq \Upsilon$ we are going to make repeated use of triangular inequality, making partial approximations on each step, until we get the simplified term $\frac{1}{C_\sigma}\frac{1}{k-\sigma}$. First, for $\Upsilon \geq \Upsilon_0$, take $N_2(\Upsilon) \in \mathbb{N}$ so that $h_n < \min\{h_0, (3\Upsilon)^{-1}\}$ for all $n \geq N_2(\Upsilon)$, where h_0 is given in Lemma 3.5. Therefore, for all $n \geq \max\{N_1(\Upsilon), N_2(\Upsilon)\}$, and all $-\Upsilon+1 \leq k \leq \Upsilon$, Lemma 3.5 and equation (3.16), in combination with the identity

$$|x-y| = \frac{|x^2-y^2|}{|x+y|},$$

give

$$\left| \frac{\|Y^\gamma_{2(\ell_n+k)}\|_{L^2(\mathbb{S}^2)}}{\|G^\gamma_{h_n}\|_{L^2(\mathbb{S}^2)}(\lambda^2_{2(\ell_n+k)} - h_n^{-2})} - \frac{1}{\|G^\gamma_{h_n}\|_{L^2(\mathbb{S}^2)}} \frac{1}{4\pi(k-\sigma_n)} \frac{1}{h_n^{-1}+k-\sigma_n} \right| \leq C(h_n)^{-1} \frac{(h_n)^3 + \Upsilon(h_n)^5}{|k-\sigma|^2} \frac{|k-\sigma_n|}{h_n} \leq C \frac{\Upsilon h_n}{|k-\sigma|^2},$$

where we used the estimate for $-\Upsilon+1 \leq k \leq \Upsilon$, $(h_n)^{-1} \geq 3\Upsilon$,

$$\left| \frac{\|Y^\gamma_{2(\ell_n+k)}\|_{L^2(\mathbb{S}^2)}}{(\lambda^2_{2(\ell_n+k)} - h_n^{-2})} + \frac{1}{4\pi(k-\sigma_n)} \frac{1}{h_n^{-1}+k-\sigma_n} \right| \geq \left| \frac{1}{4\pi(k-\sigma_n)} \frac{1}{h_n^{-1}+k-\sigma_n} \right| \geq \frac{1}{C} \frac{h_n}{|k-\sigma_n|},$$

which is due to $(\lambda^2_{2(\ell_n+k)} - h_n^{-2})$ and $(k-\sigma_n)$ having the same sign and $h_n^{-1}+k-\sigma_n > 0$ for all k.

Next, taking $N_3 \in \mathbb{N}$ such that $d(\sigma_n, \mathbb{Z}) \geq \frac{1}{2} d(\sigma, \mathbb{Z})$ for all $n \geq N_3$, and thanks to (3.16), we get for all $n \geq \max\{N_1(\Upsilon), N_2(\Upsilon), N_3\}$, and all $-\Upsilon+1 \leq k \leq \Upsilon$,

$$\left| \frac{1}{\|G^\gamma_{h_n}\|_{L^2(\mathbb{S}^2)}} \frac{1}{4\pi(k-\sigma_n)} \frac{1}{h_n^{-1}+k-\sigma_n} - \frac{1}{\|G^\gamma_{h_n}\|_{L^2(\mathbb{S}^2)}} \frac{h_n}{4\pi(k-\sigma)} \right| \leq C(h_n)^{-1} \left| \frac{h_n^{-1}(\sigma_n-\sigma) + (k-\sigma_n)^2}{(k-\sigma_n)(k-\sigma)h_n^{-1}(h_n^{-1}+k-\sigma_n)} \right| \leq C \left[\frac{|\sigma_n-\sigma|}{|k-\sigma|} + \frac{\Upsilon h_n}{|k-\sigma|} \right].$$

Finally, we arrive to the last approximation

$$\left| \frac{1}{\|G^\gamma_{h_n}\|_{L^2(\mathbb{S}^2)}} \frac{h_n}{4\pi(k-\sigma)} - \frac{1}{C_\sigma} \frac{1}{k-\sigma} \right| \leq \frac{1}{C_\sigma |k-\sigma| \|G^\gamma_{h_n}\|_{L^2(\mathbb{S}^2)}} \left| \|G^\gamma_{h_n}\|_{L^2(\mathbb{S}^2)} - \frac{C_\sigma}{4\pi} h_n \right|.$$

On the one hand, thanks to (3.16), we have for all $n \geq N_1(\Upsilon)$ and all $-\Upsilon+1 \leq k \leq \Upsilon$,

$$\frac{1}{C_\sigma |k-\sigma| \|G^\gamma_{h_n}\|_{L^2(\mathbb{S}^2)}} \leq \frac{C}{|k-\sigma|} (h_n)^{-1}.$$

On the other hand, adding and subtracting $\|\Pi_{I(2\ell_n+\sigma,2\Upsilon)}G^\gamma_{h_n}\|_{L^2(\mathbb{S}^2)}$ and applying triangular inequality,

$$\begin{aligned}&\Big|\|G^\gamma_{h_n}\|_{L^2(\mathbb{S}^2)}-\frac{C_\sigma}{4\pi}h_n\Big|\\&\le\big|\|G^\gamma_{h_n}\|_{L^2(\mathbb{S}^2)}-\|\Pi_{I(2\ell_n+\sigma,2\Upsilon)}G^\gamma_{h_n}\|_{L^2(\mathbb{S}^2)}\big|+\Big|\|\Pi_{I(2\ell_n+\sigma,2\Upsilon)}G^\gamma_{h_n}\|_{L^2(\mathbb{S}^2)}-\frac{C_\sigma}{4\pi}h_n\Big|.\end{aligned}$$

Triangular inequality on $L^2(\mathbb{S}^2)$ and Proposition 3.6 give

$$\begin{aligned}\big|\|G^\gamma_{h_n}\|_{L^2(\mathbb{S}^2)}-\|\Pi_{I(2\ell_n+\sigma,2\Upsilon)}G^\gamma_{h_n}\|_{L^2(\mathbb{S}^2)}\big|&\le\|G^\gamma_{h_n}-\Pi_{I(2\ell_n+\sigma,2\Upsilon)}G^\gamma_{h_n}\|_{L^2(\mathbb{S}^2)}\\&\le C\,\Upsilon^{-1}(h_n)^2.\end{aligned}$$

Meanwhile, Proposition 3.7 and inequality $|x-y|\le\frac{|x^2-y^2|}{y}$ for $x,y>0$ imply that for all $n\ge N_1(\Upsilon)$,

$$\begin{aligned}\Big|\|\Pi_{I(2\ell_n+\sigma,2\Upsilon)}G^\gamma_{h_n}\|_{L^2(\mathbb{S}^2)}-\frac{C_\sigma}{4\pi}h_n\Big|&\le C(h_n)^2\big[\Upsilon h_n+|\sigma_n-\sigma|+\Upsilon^{-1}\big]\frac{1}{\frac{C_\sigma}{4\pi}h_n}\\&\le Ch_n\big[\Upsilon h_n+|\sigma_n-\sigma|+\Upsilon^{-1}\big].\end{aligned}$$

Collecting all the last estimates, we find that for all $n\ge N_1(\Upsilon)$ and all $-\Upsilon+1\le k\le\Upsilon$,

$$\Big|\frac{1}{\|G^\gamma_{h_n}\|_{L^2(\mathbb{S}^2)}}\frac{h_n}{4\pi(k-\sigma)}-\frac{1}{C_\sigma}\frac{1}{k-\sigma}\Big|\le C\,\frac{1}{|k-\sigma|}\big[\Upsilon h_n+|\sigma_n-\sigma|+\Upsilon^{-1}\big].$$

Putting everything together, we have shown that for all $n\ge\max\{N_1(\Upsilon),N_2(\Upsilon),N_3\}$ and all $-\Upsilon+1\le k\le\Upsilon$,

$$\Big|\frac{\|Y^\gamma_{2(\ell_n+k)}\|_{L^2(\mathbb{S}^2)}}{\|G^\gamma_{h_n}\|_{L^2(\mathbb{S}^2)}(\lambda^2_{2(\ell_n+k)}-h_n^{-2})}-\frac{1}{C_\sigma}\frac{1}{k-\sigma}\Big|\le\frac{C}{|k-\sigma|}\big[\Upsilon h_n+|\sigma_n-\sigma|+\Upsilon^{-1}\big].$$

Inserting this into (3.18), we get that for all $n\ge\max\{N_1(\Upsilon),N_2(\Upsilon),N_3\}$,

$$\begin{aligned}&\Big\|\Pi_{I(2(\ell_n+\sigma),2\Upsilon)}g^\gamma_{h_n}-\frac{1}{C_\sigma}\sum_{|k-\sigma|\le\Upsilon}\frac{1}{k-\sigma}y^\gamma_{2(\ell_n+k)}\Big\|_{L^2(\mathbb{S}^2)}\\&\qquad\le C\big[\Upsilon h_n+|\sigma_n-\sigma|+\Upsilon^{-1}\big].\end{aligned}\tag{3.19}$$

Combining equations (3.17) and (3.19) we get the desired estimate for all $\Upsilon\ge\Upsilon_0$ and all $n\ge\max\{N_1(\Upsilon),N_2(\Upsilon),N_3\}$ in the case $\sigma\in(0,1)$.

(2) Assume that $\sigma\in\{0,1\}$. This time we can take $N_1\in\mathbb{N}$ large enough so that the right-hand side of (3.14) is small enough and the following lower bound for $\|G^\gamma_{h_n}\|_{L^2(\mathbb{S}^2)}$ holds for all $n\ge N_1$:

$$\|G^\gamma_{h_n}\|^2_{L^2(\mathbb{S}^2)}\ge\|\Pi_{I(2(\ell_n+\sigma),0)}G^\gamma_{h_n}\|^2_{L^2(\mathbb{S}^2)}\ge\frac{1}{64\pi^2}\frac{(h_n)^2}{|\sigma_n-\sigma|^2}.\tag{3.20}$$

Additionally, let $h_0 > 0$ be given by Lemma 3.5 and take $N_2 \in \mathbb{N}$ so that $h_n < h_0$ for all $n \geq N_2$, hence for all $n \geq N_2$,

$$\frac{\|Y^{\gamma}_{2(\ell_n+\sigma\pm1)}\|^2_{L^2(\mathbb{S}^2)}}{|\lambda^2_{2(\ell_n+\sigma\pm1)} - h_n^{-2}|^2} \leq C h_n^2. \tag{3.21}$$

Therefore, thanks to Proposition 3.6 for $\Upsilon = 1$, and equations (3.21) and (3.20), we have

$$\|g^{\gamma}_{h_n} - \Pi_{I(2(\ell_n+\sigma),0)} g^{\gamma}_{h_n}\|_{L^2(\mathbb{S}^2)} \leq C\,|\sigma_n - \sigma| \tag{3.22}$$

for all $n \geq \max\{N_1, N_2\}$. Essentially, in this case, we have that the term $k = \sigma$ possesses almost all the mass of the function $g^{\gamma}_{h_n}$.

Next, we estimate

$$\begin{aligned}\big\|\Pi_{I(2(\ell_n+\sigma),0)} g^{\gamma}_{h_n} - (-1)^{1-\sigma} y^{\gamma}_{2(\ell_n+\sigma)}\big\|_{L^2(\mathbb{S}^2)} \\ = \frac{1}{\|G^{\gamma}_{h_n}\|_{L^2(\mathbb{S}^2)}} \left| \frac{\|Y^{\gamma}_{2(\ell_n+\sigma)}\|_{L^2(\mathbb{S}^2)}}{\lambda^2_{2(\ell_n+\sigma)} - h_n^{-2}} - (-1)^{1-\sigma} \|G^{\gamma}_{h_n}\|_{L^2(\mathbb{S}^2)} \right|.\end{aligned}$$

Triangular inequality gives

$$\begin{aligned}\left| \frac{\|Y^{\gamma}_{2(\ell_n+\sigma)}\|_{L^2(\mathbb{S}^2)}}{\lambda^2_{2(\ell_n+\sigma)} - h_n^{-2}} - (-1)^{1-\sigma} \|G^{\gamma}_{h_n}\|_{L^2(\mathbb{S}^2)} \right| \leq \left| \frac{\|Y^{\gamma}_{2(\ell_n+\sigma)}\|_{L^2(\mathbb{S}^2)}}{\lambda^2_{2(\ell_n+\sigma)} - h_n^{-2}} - \frac{1}{4\pi} \frac{h_n}{\sigma - \sigma_n} \right| \\ + \left| \frac{1}{4\pi} \frac{h_n}{\sigma - \sigma_n} - (-1)^{1-\sigma} \|G^{\gamma}_{h_n}\|_{L^2(\mathbb{S}^2)} \right|.\end{aligned}$$

Note that $(\lambda^2_{2(\ell_n+\sigma)} - h_n^{-2})$ and $(\sigma - \sigma_n)$ have the same sign, thus we may apply the inequality $|x - y| \leq \frac{|x^2 - y^2|}{y}$ for $x, y > 0$ to obtain for all $n \geq N_2$

$$\left| \frac{\|Y^{\gamma}_{2(\ell_n+\sigma)}\|_{L^2(\mathbb{S}^2)}}{\lambda^2_{2(\ell_n+\sigma)} - h_n^{-2}} - \frac{1}{4\pi} \frac{h_n}{\sigma - \sigma_n} \right| \leq C \frac{h_n^3}{|\sigma - \sigma_n|^2} \frac{4\pi|\sigma - \sigma_n|}{h_n} \leq C \frac{h_n^2}{|\sigma - \sigma_n|},$$

where we used Lemma 3.5 for $k = \sigma$. On the other hand, $(\sigma - \sigma_n)$ and $(-1)^{1-\sigma}$ have the same sign, so we may write

$$\begin{aligned}\left| \frac{1}{4\pi} \frac{h_n}{\sigma - \sigma_n} - (-1)^{1-\sigma} \|G^{\gamma}_{h_n}\|_{L^2(\mathbb{S}^2)} \right| \leq \left| \|G^{\gamma}_{h_n}\|_{L^2(\mathbb{S}^2)} - \|\Pi_{I(2(\ell_n+\sigma),0)} G^{\gamma}_{h_n}\|_{L^2(\mathbb{S}^2)} \right| \\ + \left| \|\Pi_{I(2(\ell_n+\sigma),0)} G^{\gamma}_{h_n}\|_{L^2(\mathbb{S}^2)} - \frac{1}{4\pi} \frac{h_n}{|\sigma - \sigma_n|} \right|.\end{aligned}$$

For the first term, we combine triangular inequality in $L^2(\mathbb{S}^2)$ with Proposition 3.6 with $\Upsilon = 1$ and equation (3.21) to obtain for all $n \geq N_2$

$$\left| \|G^{\gamma}_{h_n}\|_{L^2(\mathbb{S}^2)} - \|\Pi_{I(2(\ell_n+\sigma),0)} G^{\gamma}_{h_n}\|_{L^2(\mathbb{S}^2)} \right| \leq \|G^{\gamma}_{h_n} - \Pi_{I(2(\ell_n+\sigma),0)} G^{\gamma}_{h_n}\|_{L^2(\mathbb{S}^2)} \leq C h_n.$$

For the second term, we just invoke Proposition 3.7 in combination with the inequality $|x-y| \leq \frac{|x^2-y^2|}{y}$ for $x, y > 0$ to obtain for all $n \geq N_1$,

$$\left| \|\Pi_{I(2(\ell_n+\sigma),0)} G^\gamma_{h_n}\|_{L^2(\mathbb{S}^2)} - \frac{1}{4\pi} \frac{h_n}{|\sigma-\sigma_n|} \right| \leq C \frac{(h_n)^3}{|\sigma-\sigma_n|^2} \frac{4\pi|\sigma-\sigma_n|}{h_n} \leq C \frac{(h_n)^2}{|\sigma-\sigma_n|}.$$

Collecting all the previous identities, we have proved that for all $n \geq \max\{N_1, N_2\}$,

$$\left| \frac{\|Y^\gamma_{2(\ell_n+\sigma)}\|_{L^2(\mathbb{S}^2)}}{\lambda^2_{2(\ell_n+\sigma)} - h_n^{-2}} - (-1)^{1-\sigma} \|G^\gamma_{h_n}\|_{L^2(\mathbb{S}^2)} \right| \leq C \frac{h_n}{|\sigma-\sigma_n|} \big[h_n + |\sigma_n - \sigma|\big],$$

thus, thanks to (3.20), we get that for all $n \geq \max\{N_1, N_2\}$,

$$\big\| \Pi_{I(2(\ell_n+\sigma),0)} g^\gamma_{h_n} - (-1)^{1-\sigma} y^\gamma_{2(\ell_n+\sigma)} \big\|_{L^2(\mathbb{S}^2)} \leq C \big[h_n + |\sigma_n - \sigma|\big]. \tag{3.23}$$

Combining (3.22) and (3.23) we obtain the desired estimate for all $n \geq \max\{N_1, N_2\}$ in the case $\sigma \in \{0, 1\}$. ∎

4. Proofs of main theorems

In this section, we state and prove the results related to the semiclassical measures of high-energy eigenfunctions of singular perturbations of Δ by δ_γ. In the introduction, these were denoted by

$$\Delta + \alpha |\delta_\gamma\rangle\langle\delta_\gamma|, \quad \alpha \in \mathbb{R},$$

but we saw in Proposition 2.1 that these are parameterized instead by $\theta \in \mathbb{R}/(\pi\mathbb{Z})$, and we denote them by $\Delta_{\gamma,\theta}$ (see Corollary 2.8, where a connection between these two parameterizations is proposed). Theorem 1.2 is a consequence of Theorem 4.3, and Theorem 1.4 is a consequence of Theorem 4.5 and Lemma 4.7.

Recall the definition of Y^γ_ℓ (2.18), Z^q_ℓ (2.19), and G^γ_h (3.1). In this section, we will work mainly with their normalized versions; those are

$$y^\gamma_{2\ell} := \frac{1}{\|Y^\gamma_{2\ell}\|_{L^2(\mathbb{S}^2)}} Y^\gamma_{2\ell}, \quad z^q_\ell := \frac{1}{\|Z^q_\ell\|_{L^2(\mathbb{S}^2)}} Z^q_\ell, \quad g^\gamma_h := \frac{1}{\|G^\gamma_h\|_{L^2(\mathbb{S}^2)}} G^\gamma_h. \tag{4.1}$$

Note that if q and γ are such that $d(q,\gamma) = \frac{\pi}{2}$, then Proposition 2.9 implies that

$$y^\gamma_{2\ell} = (-1)^\ell z^q_{2\ell} \quad \forall \ell \in \mathbb{N}. \tag{4.2}$$

For $q \in \mathbb{S}^2$, let us define the probability measure μ_q on $T^*\mathbb{S}^2$,

$$\int_{T^*\mathbb{S}^2} a(x,\xi)\mu_q(\mathrm{d}x, \mathrm{d}\xi) := \int_{S^*_q\mathbb{S}^2} \int_0^{2\pi} a\big(\phi_t(q,\xi)\big) \frac{\mathrm{d}t}{2\pi} \frac{\mathrm{d}\xi}{2\pi}, \quad \forall a \in \mathcal{C}^\infty_c(T^*\mathbb{S}^2). \tag{4.3}$$

Note that $\mu_q \in \mathcal{P}_{\mathrm{inv}}(S^*\mathbb{S}^2)$, that is, it is supported inside $S^*\mathbb{S}^2$ and is invariant under geodesic flow; in fact, μ_q is the unique semiclassical measure associated to the sequence of normalized zonal harmonics on $q \in \mathbb{S}^2$ (see [44, Theorem 3.1]).

Theorem 4.1. *Let γ be a closed geodesic on $\mathbb{S}^2$, and let $q \in \mathbb{S}^2$ be such that $d(q,\gamma) = \frac{\pi}{2}$. Let $(h_n)_{n\in\mathbb{N}} \subseteq (0,1)$ be such that $\lim_{n\to\infty} h_n = 0$ and $(h_n)^{-2} \in \mathbb{R} \setminus \mathrm{Spec}_{\mathrm{even}}(\Delta)$. For every $a \in \mathcal{C}_c^\infty(T^*\mathbb{S}^2)$,*

$$\lim_{n\to\infty} \langle g_{h_n}^\gamma \mid \mathrm{Op}_{h_n}(a) g_{h_n}^\gamma \rangle_{L^2(\mathbb{S}^2)} = \int_{T^*\mathbb{S}^2} a\mu_q(\mathrm{d}x, \mathrm{d}\xi).$$

Proof. We prove the sequence of Wigner distributions

$$W_{h_n}[g_{h_n}^\gamma](a) := \langle g_{h_n}^\gamma \mid \mathrm{Op}_{h_n}(a) g_{h_n}^\gamma \rangle_{L^2(\mathbb{S}^2)},$$

converges to the measure μ_q in the weak-$\star$ topology of $\mathcal{D}'(T^*\mathbb{S}^2)$ by showing that every subsequence of $(W_{h_n}[g_{h_n}^\gamma])_{n\in\mathbb{N}}$ has a further subsequence that converges to μ_q.

Given any subsequence of Wigner distributions as described above, there exists a further subsequence, which we do not relabel, and a Radon measure μ on $T^*\mathbb{S}^2$ such that (see [51, Theorem 5.2])

$$\lim_{n\to\infty} \langle g_{h_n}^\gamma \mid \mathrm{Op}_{h_n}(a) g_{h_n}^\gamma \rangle_{L^2(\mathbb{S}^2)} = \int_{T^*\mathbb{S}^2} a\mu \quad \forall a \in \mathcal{C}_c^\infty(T^*\mathbb{S}^2). \tag{4.4}$$

Up to extraction of a further subsequence, we may assume that $\sigma_n \to \sigma \in [0,1]$. We will show that $\mu = \mu_q$ under this additional convergence of the sequence $(\sigma_n)_{n\in\mathbb{N}}$.

(A) *Case $\sigma \in \{0,1\}$.* Theorem 3.1 (2) says that there exists $C > 0$ and $N \in \mathbb{N}$ such that for all $n \geq N$,

$$\left\| g_{h_n}^\gamma - (-1)^{1-\sigma} y_{2(\ell_n+\sigma)}^\gamma \right\|_{L^2(\mathbb{S}^2)} \leq C\big[h_n + |\sigma_n - \sigma|\big].$$

If we choose $q \in \mathbb{S}^2$ such that $d(q,\gamma) = \frac{\pi}{2}$, then (4.2) implies that for all $n \geq N$

$$\left\| g_{h_n}^\gamma - (-1)^{1+\ell_n} z_{2(\ell_n+\sigma)}^q \right\|_{L^2(\mathbb{S}^2)} \leq C\big[h_n + |\sigma_n - \sigma|\big].$$

Consequently, we have the following approximation as $n \to \infty$:

$$\langle g_{h_n}^\gamma \mid \mathrm{Op}_{h_n}(a) g_{h_n}^\gamma \rangle_{L^2(\mathbb{S}^2)} = \langle z_{2(\ell_n+\sigma)}^q \mid \mathrm{Op}_{h_n}(a) z_{2(\ell_n+\sigma)}^q \rangle_{L^2(\mathbb{S}^2)} + \mathcal{O}(h_n) + \mathcal{O}\big(|\sigma_n - \sigma|\big).$$

We now take limits $n \to \infty$ and invoke Theorem 3.1 from [44] to obtain

$$\int_{T^*\mathbb{S}^2} a\mu = \lim_{n\to\infty} \langle g_{h_n}^\gamma \mid \mathrm{Op}_{h_n}(a) g_{h_n}^\gamma \rangle_{L^2(\mathbb{S}^2)} = \int_{T^*\mathbb{S}^2} a\mu_q,$$

as we wanted to show.

(B) *Case $\sigma \in (0,1)$.* Let $\Upsilon_0 \in \mathbb{N}$ be given by Theorem 3.1 and take $\Upsilon \geq \Upsilon_0$. Thanks to Theorem 3.1, there exist $C_\sigma > 0$ (3.5), $C > 0$, and $N_1(\Upsilon) \in \mathbb{N}$ such that for all $n \geq N(\Upsilon)$,

$$\left\| g_{h_n}^\gamma - \frac{1}{C_\sigma} \sum_{|k-\sigma|\leq\Upsilon} \frac{1}{k-\sigma} y_{2(\ell_n+k)}^\gamma \right\|_{L^2(\mathbb{S}^2)} \leq C\big[\Upsilon h_n + |\sigma_n - \sigma| + \Upsilon^{-\frac{1}{2}}\big].$$

Let us fix $q \in \mathbb{S}^2$ such that $d(q,\gamma) = \frac{\pi}{2}$, thus (4.2) implies that for all $n \geq N_1(\Upsilon)$,

$$\left\| g^{\gamma}_{h_n} - (-1)^{\ell_n} \frac{1}{C_\sigma} \sum_{|k-\sigma|\leq \Upsilon} \frac{(-1)^k}{k-\sigma} z^q_{2(\ell_n+k)} \right\|_{L^2(\mathbb{S}^2)} \leq C\big[\Upsilon h_n + |\sigma_n - \sigma| + \Upsilon^{-\frac{1}{2}}\big]. \quad (4.5)$$

For $r = d(x,q)$, let

$$\kappa^q(x,\xi) := \cos r, \quad \varsigma^q(x,\xi) := \xi(\sin r \cdot \partial_r), \quad (x,\xi) \in T^*\mathbb{S}^2, \quad (4.6)$$

and define the smooth symbol on $T^*\mathbb{S}^2$,

$$\Gamma^{\sigma,\Upsilon} := \frac{1}{C_\sigma}\Bigg[\sum_{k=-\Upsilon+1}^{0} \frac{(-1)^k}{k-\sigma} (\kappa^q - i\varsigma^q)^{-(2k-1)} + \sum_{k=1}^{\Upsilon} \frac{(-1)^k}{k-\sigma} (\kappa^q + i\varsigma^q)^{2k-1} \Bigg]. \quad (4.7)$$

Theorem 5.1 from [44] for $x_n = X \in \ell_2(\mathbb{Z})$,

$$X = (X_j)_{j\in\mathbb{Z}} := \begin{cases} \frac{1}{C_\sigma}\frac{(-1)^k}{k-\sigma} & j = 2k, \\ 0 & j = 2k+1, \end{cases} \qquad S = S(\Upsilon) := [-2\Upsilon+1, 2\Upsilon-1] \cap \mathbb{N},$$

and $\rho_n := 2\ell_n + 1$ implies that there exist $C(\Upsilon) > 0$ and $N_2(\Upsilon) \in \mathbb{N}$ such that for all $n \geq N_2(\Upsilon)$

$$\left\| \frac{1}{C_\sigma} \sum_{|k-\sigma|\leq\Upsilon} \frac{(-1)^k}{k-\sigma} z^q_{2(\ell_n+k)} - \mathrm{Op}_{h_n}(\Gamma^{\sigma,\Upsilon}) z^q_{2\ell_n+1} \right\|_{L^2(\mathbb{S}^2)} \leq C\,\Upsilon^{-\frac{1}{2}} + C(\Upsilon) h_n. \quad (4.8)$$

Combining (4.5) and (4.8) we obtain that for all $n \geq \max\{N_1(\Upsilon), N_2(\Upsilon)\}$,

$$\big\| g^{\gamma}_{h_n} - (-1)^{\ell_n} \mathrm{Op}_{h_n}(\Gamma^{\sigma,\Upsilon}) z^q_{2\ell_n+1} \big\|_{L^2(\mathbb{S}^2)} \leq C\big[C(\Upsilon)h_n + |\sigma_n - \sigma| + \Upsilon^{-\frac{1}{2}}\big]. \quad (4.9)$$

Consequently, for every $n \geq \max\{N_1(\Upsilon), N_2(\Upsilon)\}$,

$$\begin{aligned} \big| \langle g^{\gamma}_{h_n} \mid \mathrm{Op}_{h_n}(a) g^{\gamma}_{h_n} \rangle_{L^2(\mathbb{S}^2)} - \big\langle \mathrm{Op}_{h_n}(\Gamma^{\sigma,\Upsilon}) z^q_{2\ell_n+1} \mid \mathrm{Op}_{h_n}(a)\,\mathrm{Op}_{h_n}(\Gamma^{\sigma,\Upsilon}) z^q_{2\ell_n+1} \big\rangle_{L^2(\mathbb{S}^2)} \big| \\ \leq C\big[C(\Upsilon)h_n + |\sigma_n - \sigma| + \Upsilon^{-\frac{1}{2}}\big]. \end{aligned}$$

Symbolic Calculus for semiclassical pseudodifferential operators (see [51], for instance), namely for every $a, b \in \mathcal{C}^\infty_c(T^*M)$ there exists $C_a, C_{a,b} > 0$ such that for all $h \in (0,1)$,

$$\begin{aligned} \big\| \mathrm{Op}_h(a)^* - \mathrm{Op}_h(\bar{a}) \big\|_{L^2\to L^2} &\leq C_a h, \\ \big\| \mathrm{Op}_h(a)\,\mathrm{Op}_h(b) - \mathrm{Op}_h(ab) \big\|_{L^2\to L^2} &\leq C_{a,b} h, \end{aligned}$$

give that there exists $C_{a,\Upsilon} > 0$ such that

$$\big\| \mathrm{Op}_{h_n}(\Gamma^{\sigma,\Upsilon})^*\,\mathrm{Op}_{h_n}(a)\,\mathrm{Op}_{h_n}(\Gamma^{\sigma,\Upsilon}) - \mathrm{Op}_{h_n}\big(a|\Gamma^{\sigma,\Upsilon}|^2\big) \big\|_{L^2\to L^2} \leq C_{a,\Upsilon} h_n.$$

Combining these estimates we find that for every $n \geq \max\{N_1(\Upsilon), N_2(\Upsilon)\}$,

$$\left|\left\langle g^{\gamma}_{h_n} \mid \mathrm{Op}_{h_n}(a) g^{\gamma}_{h_n}\right\rangle_{L^2(\mathbb{S}^2)} - \left\langle z^q_{2\ell_n+1} \mid \mathrm{Op}_{h_n}\left(a|\Gamma^{\sigma,\Upsilon}|^2\right) z^q_{2\ell_n+1}\right\rangle_{L^2(\mathbb{S}^2)}\right| \leq C\left[C(\Upsilon)h_n + |\sigma_n - \sigma| + \Upsilon^{-\frac{1}{2}}\right] + C_{a,\Upsilon} h_n.$$

We may take $n \to \infty$ in the last estimate thanks [44, Theorem 3.1] to find that for every $\Upsilon \geq \Upsilon_0$,

$$\left|\int_{T^*\mathbb{S}^2} a\mu - \int_{T^*\mathbb{S}^2} a \cdot |\Gamma^{\sigma,\Upsilon}|^2 \mu_q\right| \leq C\,\Upsilon^{-\frac{1}{2}}. \tag{4.10}$$

We would like to take $\Upsilon \to \infty$ in this last identity, but for that we first need to study the weak-$\star$ limits of sequence of measures $(\left|\Gamma^{\sigma,\Upsilon}\right|^2 \mu_q)_{\Upsilon\in\mathbb{N}}$.

For every $a \in \mathcal{C}^\infty_c(T^*\mathbb{S}^2)$ we have

$$\int_{T^*\mathbb{S}^2} a \cdot |\Gamma^{\sigma,\Upsilon}|^2 \mu_q = \int_{S^*_q\mathbb{S}^2} \int_0^{2\pi} a\big(\phi_t(q,\xi)\big)\left|\Gamma^{\sigma,\Upsilon}\big(\phi_t(q,\xi)\big)\right|^2 \frac{\mathrm{d}t}{2\pi} \frac{\mathrm{d}\xi}{\mathrm{vol}(S^*_q\mathbb{S}^2)},$$

where ϕ_t denote the geodesic flow at time $t \in \mathbb{R}$. From the definition (4.6) we see that

$$\kappa^q\big(\phi_t(q,\xi)\big) = \cos t, \quad \varsigma^q\big(\phi_t(q,\xi)\big) = \sin t \quad \forall \xi \in S^*_q\mathbb{S}^2,\ t \in \mathbb{R}, \tag{4.11}$$

therefore,

$$\Gamma^{\sigma,\Upsilon}\big(\phi_t(q,\xi)\big) = \frac{e^{-it}}{C_\sigma} \sum_{k=-\Upsilon+1}^{\Upsilon} \frac{(-1)^k}{k-\sigma} e^{i2kt}.$$

Lemma 4.2. *For $\sigma \in (0,1)$, let $\zeta^{\sigma,1} \in L^2(\mathbb{R}/(2\pi\mathbb{Z}))$ be defined by*

$$\zeta^{\sigma,1}(t) := \frac{1}{C_\sigma} \frac{2\pi i}{1 - e^{i2\pi\sigma}} e^{i\sigma t}, \quad t \in [0, 2\pi),$$

which is discontinuous at $t = 0 \in \mathbb{R}/(2\pi\mathbb{Z})$. Then the function $\zeta^{\sigma,2} \in L^2(\mathbb{R}/(2\pi\mathbb{Z}))$ given by

$$\zeta^{\sigma,2}(t) := \begin{cases} \zeta^{\sigma,1}(2t+\pi) & t \in \left[0, \frac{\pi}{2}\right), \\ \zeta^{\sigma,1}(2t-\pi) & t \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right), \\ \zeta^{\sigma,1}(2t-3\pi) & t \in \left[\frac{3\pi}{2}, 2\pi\right) \end{cases} \tag{4.12}$$

has Fourier series

$$\zeta^{\sigma,2}(t) = \frac{1}{C_\sigma} \sum_{k\in\mathbb{Z}} \frac{(-1)^k}{k-\sigma} e^{i2kt}.$$

Proof of Lemma 4.2. One first proves that $\zeta^{\sigma,1}$ has Fourier series

$$\zeta^{\sigma,1}(t) = \frac{1}{C_\sigma} \sum_{k\in\mathbb{Z}} \frac{1}{k-\sigma} e^{ikt}.$$

One obtains as a consequence that

$$C_\sigma = \frac{2\pi}{|1 - e^{i2\pi\sigma}|}. \tag{4.13}$$

Now, the Fourier series of $\zeta^{\sigma,2}$ can be computed from the Fourier series of $\zeta^{\sigma,1}$ because $\zeta^{\sigma,2} = D_2 S_{-\pi} \zeta^{\sigma,1}$, where D_2 and $S_{-\pi}$ are the unitary maps on $L^2(\mathbb{R}/(2\pi\mathbb{Z}))$

$$[D_2 f](t) \coloneqq f(2t), \quad [S_{-\pi} f](t) \coloneqq f(t + \pi). \qquad \blacksquare$$

From Lemma 4.2 we infer that $\Gamma^{\sigma,\Upsilon}(\phi_t(q,\xi))$ is the partial Fourier series of $e^{-it}\zeta^{\sigma,2}(t)$ (4.12). Thanks to Carleson's theorem we get that

$$\Gamma^{\sigma,\Upsilon}\big(\phi_t(q,\xi)\big) \xrightarrow[\Upsilon\to\infty]{} e^{-it}\zeta^{\sigma,2}(t) \quad \text{a.e. } t \in \mathbb{R}/(2\pi\mathbb{Z}),$$

therefore, due to identity (4.13),

$$\big|\Gamma^{\sigma,\Upsilon}\big(\phi_t(q,\xi)\big)\big|^2 \xrightarrow[\Upsilon\to\infty]{} \big|\zeta^{\sigma,2}(t)\big|^2 = 1 \quad \text{a.e. } t \in \mathbb{R}/(2\pi\mathbb{Z}).$$

Now we take $\Upsilon \to \infty$ in (4.10) and apply the dominated convergence theorem to get

$$\int_{T^*\mathbb{S}^2} a\mu = \int_{T^*\mathbb{S}^2} a\mu_q. \qquad \blacksquare$$

Let γ be a closed geodesic on $\mathbb{S}^2$, and for a given $\theta \in \Theta = \mathbb{R}/(\pi\mathbb{Z})$, let $\Delta_{\gamma,\theta}$ be the singular perturbation of Δ by δ_γ as defined in (2.6). Consider the equation for $\eta \in \mathbb{R} \setminus \mathrm{Spec}_{\mathrm{even}}$,

$$\delta_\gamma G_\eta^\gamma = \frac{1}{2\pi} \operatorname{cotan}\theta. \tag{4.14}$$

For a fixed $\theta \in \mathbb{R}/(\pi\mathbb{Z})$, we define the set of solutions η of (4.14) as

$$\Lambda_\theta \coloneqq \big\{\eta \in \mathbb{R} \setminus \mathrm{Spec}_{\mathrm{even}}(\Delta) \colon \eta \text{ satisfies (4.14)}\big\}. \tag{4.15}$$

The set Λ_θ is countable as a consequence of Lemma 2.12.

Given a sequence of singular perturbations of Δ by δ_γ, $(\Delta_{\gamma,\theta_n})_{n\in\mathbb{N}}$, we denote

$$\mathcal{M}_{\mathrm{sc}}\big((\Delta_{\gamma,\theta_n})_{n\in\mathbb{N}}\big)$$

as the set of all possible weak-$\star$ accumulation points of sequences of Wigner distributions $(W_{h_n}[u_n])_{n\in\mathbb{N}}$ where $u_n \in L^2(\mathbb{S}^2)$ and $h_n > 0$ satisfy

$$(h_n)^2 \Delta_{\gamma,\theta_n} u_n = u_n, \quad \|u_n\|_{L^2(\mathbb{S}^2)} = 1, \quad \lim_{n\to\infty} h_n = 0. \tag{4.16}$$

Theorem 4.3. *Let γ be a closed geodesic on $\mathbb{S}^2$, and let $(\Delta_{\gamma,\theta_n})_{n\in\mathbb{N}}$ be a sequence of singular perturbations of Δ by δ_γ. Then*

$$\mathcal{P}_{\mathrm{inv}}(S^*\mathbb{S}^2) \subseteq \mathcal{M}_{\mathrm{sc}}\big((\Delta_{\gamma,\theta_n})_{n\in\mathbb{N}}\big).$$

If one further assumes that for every $n \in \mathbb{N}$ (see (4.15)),

$$\Lambda_{\theta_n} \cap \mathrm{Spec}_{\mathrm{odd}}(\Delta) = \varnothing, \tag{4.17}$$

then

$$\mathcal{P}_{\mathrm{inv}}(S^*\mathbb{S}^2) = \mathcal{M}_{\mathrm{sc}}\big((\Delta_{\gamma,\theta_n})_{n\in\mathbb{N}}\big).$$

Remark 4.4. Condition

$$\Lambda_\theta \cap \mathrm{Spec}_{\mathrm{odd}}(\Delta) = \varnothing$$

is satisfied for all but a countable collection of $\theta \in \Theta$. This is because $\mathrm{Spec}_{\mathrm{odd}}(\Delta)$ is countable and if $\lambda^2_{2\ell+1} \in \Lambda_\theta$, then $\lambda^2_{2\ell+1} \notin \Lambda_{\theta'}$ for $\theta' \neq \theta$ (see Lemma 2.12).

Proof. (A) We prove

$$\mathcal{P}_{\mathrm{inv}}(S^*\mathbb{S}^2) \subseteq \mathcal{M}_{\mathrm{sc}}\big((\Delta_{\gamma,\theta_n})_{n\in\mathbb{N}}\big).$$

Let $\mu \in \mathcal{P}_{\mathrm{inv}}(S^*\mathbb{S}^2)$. Take $q \in \mathbb{S}^2$ such that $d(q,\gamma) = \frac{\pi}{2}$, and let $(\ell_n)_{n\in\mathbb{N}} \subseteq \mathbb{N}$ and $(u_n)_{n\in\mathbb{N}} \subseteq L^2(\mathbb{S}^2)$ be given by [44, Theorem 1.3], which enjoy the following properties:

$$\Delta u_n = \lambda^2_{\ell_n} u_n, \quad \|u_n\|_{L^2(\mathbb{S}^2)} = 1, \quad u_n(q) = 0,$$

and additionally, if $h_n := \lambda^{-1}_{\ell_n}$, then $h_n \to 0^+$ and for all $a \in \mathcal{C}^\infty_c(T^*\mathbb{S}^2)$,

$$\lim_{n\to\infty} \big\langle u_n \mid \mathrm{Op}_{h_n}(a) u_n \big\rangle_{L^2(\mathbb{S}^2)} = \int_{T^*\mathbb{S}^2} a\mu.$$

If we prove $\delta_\gamma u_n = 0$ for all $n \in \mathbb{N}$, then $(h_n)_{n\in\mathbb{N}}$ and $(u_n)_{n\in\mathbb{N}}$ would satisfy (4.16), thus proving that $\mu \in \mathcal{M}_{\mathrm{sc}}((\Delta_{\gamma,\theta_n})_{n\in\mathbb{N}})$. Fix $n \in \mathbb{N}$. Since $u_n \in \ker(\Delta - \lambda^2_{\ell_n})$, (2.18) gives that

$$\delta_\gamma u_n = \langle Y^\gamma_{\ell_n} \mid u_n \rangle.$$

Next, Lemma 2.9 implies that $Y^\gamma_{\ell_n} = \alpha_n Z^q_{\ell_n}$ for some $\alpha_n \in \mathbb{R}$, therefore

$$\delta_\gamma u_n = \alpha_n \langle Z^q_{\ell_n} \mid u_n \rangle_{L^2(\mathbb{S}^2)}.$$

Lastly, since $u_n \in \ker(\Delta - \lambda^2_{\ell_n})$, (2.19) gives

$$\delta_\gamma u_n = \alpha_n u_n(q) = 0.$$

(B) We prove

$$\mathcal{M}_{\mathrm{sc}}\big((\Delta_{\gamma,\theta_n})_{n\in\mathbb{N}}\big) \subseteq \mathcal{P}_{\mathrm{inv}}(S^*\mathbb{S}^2)$$

under the additional hypothesis (4.17). Fix $\mu \in \mathcal{M}_{\mathrm{sc}}((\Delta_{\gamma,\theta_n})_{n\in\mathbb{N}})$. Let $(u_n)_{n\in\mathbb{N}} \subseteq L^2(\mathbb{S}^2)$ and $(h_n)_{n\in\mathbb{N}} \subseteq (0,1)$, be sequences satisfying (4.16) and additionally, for all $a \in \mathcal{C}^\infty_c(T^*\mathbb{S}^2)$,

$$\lim_{n\to\infty} \big\langle u_n \mid \mathrm{Op}_{h_n}(a) u_n \big\rangle_{L^2(\mathbb{S}^2)} = \int_{T^*\mathbb{S}^2} a\mu.$$

Since (4.17) holds for all $n \in \mathbb{N}$, Theorem 2.6 implies that for every $n \in \mathbb{N}$, one of the following two mutually exclusive events occurs:

(1) $h_n^{-2} \in \mathrm{Spec}(\Delta)$ and $u_n \in \ker(\Delta - h_n^{-2})$, $\delta_\gamma u_n = 0$.

(2) $h_n^{-2} \in \mathbb{R} \setminus \mathrm{Spec}(\Delta)$ and $u_n = c g^\gamma_{h_n}$, $c \in \mathbb{C}$, $|c| = 1$.

Up to extraction of a subsequence, we may assume that (1) holds for all $n \in \mathbb{N}$, or that (2) holds for all $n \in \mathbb{N}$. If (1) holds for all $n \in \mathbb{N}$, then $\mu \in \mathcal{P}_{\mathrm{inv}}(S^*\mathbb{S}^2)$. If (2) holds for all $n \in \mathbb{N}$, Theorem 4.1 implies that $\mu = \mu_q \in \mathcal{P}_{\mathrm{inv}}(S^*\mathbb{S}^2)$. ∎

For $q \in \mathbb{S}^2$ and a Lebesgue-measurable set $J \subseteq \mathbb{R}/(2\pi\mathbb{Z})$, let $\mu_{q,J}$ be the positive Radon measure on $T^*\mathbb{S}^2$,

$$\int_{T^*\mathbb{S}^2} a\mu_{q,J} := \int_{S^*_q\mathbb{S}^2} \int_J a\big(\phi_t(q,\xi)\big) \frac{\mathrm{d}t}{2\pi} \frac{\mathrm{d}\xi}{2\pi}, \quad a \in \mathcal{C}(T^*\mathbb{S}^2). \tag{4.18}$$

Note that $\mu_{q,J}$ is a probability measure if and only if J is of full measure in $\mathbb{R}/(2\pi\mathbb{Z})$; moreover, $\mu_{q,J}$ is invariant under geodesic flow if and only if J is of full measure in $\mathbb{R}/(2\pi\mathbb{Z})$. Additionally, if J and K are two disjoint Lebesgue measurable subsets of $\mathbb{R}/(2\pi\mathbb{Z})$, then $\mu_{q,J} + \mu_{q,K} = \mu_{q,J\cup K}$.

Theorem 4.5. *Let γ be a closed geodesic on $\mathbb{S}^2$, and let $(\Delta_{\gamma,\theta_n})_{n\in\mathbb{N}}$ be a sequence of singular perturbations of Δ by δ_γ. Assume that there exists $(\ell_n)_{n\in\mathbb{N}} \subseteq \mathbb{N}$ such that (see (4.15))*

$$\lambda^2_{2\ell_n+1} \in \Lambda_{\theta_n} \quad \text{and} \quad \lim_{n\to\infty} \ell_n = \infty. \tag{4.19}$$

Set $q \in \mathbb{S}^2$ such that $d(q,\gamma) = \frac{\pi}{2}$, and intervals $J := [\frac{\pi}{2}, \frac{3\pi}{2})$ and $K := \mathbb{R}/(2\pi\mathbb{Z}) \setminus J$. Take $m_J, m_K \in [0,2]$ such that $m_J + m_K = 2$. Moreover, set $h_n := \lambda^{-1}_{2\ell_n+1}$. There exist $(u_n)_{n\in\mathbb{N}} \subseteq L^2(\mathbb{S}^2)$ satisfying (4.16) such that for all $a \in \mathcal{C}^\infty_c(T^*\mathbb{S}^2)$,

$$\lim_{n\to\infty} \big\langle u_n \mid \mathrm{Op}_{h_n}(a) u_n \big\rangle_{L^2(\mathbb{S}^2)} = \int_{T^*\mathbb{S}^2} a[m_J\mu_{q,J} + m_K\mu_{q,K}].$$

Remark 4.6. Under the notation of Theorem 4.5, if $\Omega_\pm$ are the two open connected components of $\mathbb{S}^2 \setminus \gamma$, and $q \in \Omega_+$, then $\mathrm{supp}\,\mu_{q,J} \subseteq S^*\Omega_-$ and $\mathrm{supp}\,\mu_{q,K} \subseteq S^*\Omega_+$.

The additional hypothesis (4.19) on the sequence $(\Delta_{\gamma,\theta_n})_{n\in\mathbb{N}}$ implies that

$$\delta_\gamma G^\gamma_{\lambda^2_{2\ell_n+1}} = \frac{1}{2\pi} \cot\theta_n \quad \forall n \in \mathbb{N},$$

therefore,

$$\ker(\Delta_{\gamma,\theta_n} - \lambda^2_{2\ell_n+1}) = \ker(\Delta - \lambda^2_{2\ell_n+1}) + \mathrm{Span}_{\mathbb{C}} \big\{ G^\gamma_{\lambda^2_{2\ell_n+1}} \big\} \quad \forall n \in \mathbb{N},$$

thanks to Theorem 2.6. This increase in multiplicity means that linear combinations of spherical harmonics and new eigenfunction, $g^\gamma_{\lambda^2_{2\ell_n+1}}$, are eigenfunction of the perturbed operators Δ_{γ,θ_n}. Such cooked up eigenfunctions are the source of non-invariant semiclassical measures.

Proof of Theorem 4.5. Set $\varphi, \psi \in \mathbb{C}$ such that $m_J = |\varphi + \psi|^2$ and $m_K = |\varphi - \psi|^2$. Due to hypothesis (4.19), Theorem 2.6 implies that

$$u_n := \varphi z^q_{2\ell_n+1} + \psi g^\gamma_{h_n} \in \ker(\Delta_{\gamma,\theta_n} - \lambda^2_{2\ell_n+1}) \quad \forall n \in \mathbb{N}. \tag{4.20}$$

Moreover, since $z^q_{2\ell_n+1}$ and $g^\gamma_{h_n}$ are mutually orthogonal (see (2.24) and Lemma 2.9), one has

$$\|u_n\|^2_{L^2(\mathbb{S}^2)} = |\varphi|^2 + |\psi|^2 = \frac{1}{2}\big(|\varphi+\psi|^2 + |\varphi-\psi|^2\big) = \frac{1}{2}(m_J + m_K) = 1,$$

therefore $(u_n)_{n\in\mathbb{N}} \subseteq L^2(\mathbb{S}^2)$ satisfies (4.16). We claim that $(u_n)_{n\in\mathbb{N}}$ is the sequence we were looking for.

Up to extraction of a subsequence, we may assume there exists a unique measure μ on $T^*\mathbb{S}^2$ such that

$$\lim_{n\to\infty} \langle u_n \mid \mathrm{Op}_{h_n}(a) u_n \rangle_{L^2(\mathbb{S}^2)} = \int_{T^*\mathbb{S}^2} a\mu \quad \forall a \in \mathcal{C}^\infty_c(T^*\mathbb{S}^2).$$

Fix $a \in \mathcal{C}^\infty_c(T^*\mathbb{S}^2)$. Note that if for $n \in \mathbb{N}$ we set $\sigma_n \in (0,1)$ such that

$$\lambda^2_{2\ell_n+1} = (h_n)^{-2} = \big(2(\ell_n+\sigma_n)\big)\big(2(\ell_n+\sigma_n)+1\big),$$

then $\sigma_n = \frac{1}{2}$ for all $n \in \mathbb{N}$, thus certainly $\sigma_n \to \sigma = \frac{1}{2}$. Let $\Upsilon \in \mathbb{N}$ be a large parameter. Using the same reasoning as in the proof of Theorem 4.1, part B, we infer that there exist a smooth symbol $\Gamma^{\frac{1}{2},\Upsilon}$ (4.7), constants $C, C(\Upsilon) > 0$, and $N = N(\Upsilon) \in \mathbb{N}$ such that for all $n \geq N$ (see (4.9)),

$$\big\| g^\gamma_{h_n} - \mathrm{Op}_{h_n}(\Gamma^{\frac{1}{2},\Upsilon}) z^q_{2\ell_n+1} \big\|_{L^2(\mathbb{S}^2)} \leq C\big[C(\Upsilon)h_n + \Upsilon^{-\frac{1}{2}}\big].$$

Consequently, we find that for all $\Upsilon \in \mathbb{N}$ and all $n \geq N(\Upsilon)$,

$$\big\| u_n - \mathrm{Op}_{h_n}(\varphi + \psi\Gamma^{\frac{1}{2},\Upsilon}) z^q_{2\ell_n+1} \big\|_{L^2(\mathbb{S}^2)} \leq C\big[C(\Upsilon)h_n + \Upsilon^{-\frac{1}{2}}\big].$$

Using symbolic calculus, we get that for all $\Upsilon \in \mathbb{N}$ and all $n \geq N(\Upsilon)$,

$$\Big| \langle u_n \mid \mathrm{Op}_{h_n}(a) u_n \rangle_{L^2(\mathbb{S}^2)} - \big\langle z^q_{2\ell_n+1} \mid \mathrm{Op}_{h_n}\big(a|\varphi + \psi b \Gamma^{\frac{1}{2},\Upsilon}|^2\big) z^q_{2\ell_n+1} \big\rangle_{L^2(\mathbb{S}^2)} \Big| \leq C\big[C(\Upsilon)h_n + \Upsilon^{-\frac{1}{2}}\big].$$

Thanks to [44, Theorem 3.1], we may take limit $n \to \infty$ and obtain for all $\Upsilon \in \mathbb{N}$,

$$\left| \int_{T^*\mathbb{S}^2} a\mu - \int_{T^*\mathbb{S}^2} a|\varphi + \psi\Gamma^{\frac{1}{2},\Upsilon}|^2 \mu_q \right| \leq C\,\Upsilon^{-\frac{1}{2}}. \tag{4.21}$$

We would like to take $\Upsilon \to \infty$ now, but first we need to compute the weak-$\star$ limit of the sequence of measures $(|\varphi + \psi\Gamma^{1/2,\Upsilon}|^2 \mu_q)_{\Upsilon\in\mathbb{N}}$. We see that

$$\int_{T^*\mathbb{S}^2} a|\varphi+\psi\Gamma^{\frac{1}{2},\Upsilon}|^2\mu_q = \int_{S^*_q\mathbb{S}^2} \int_0^{2\pi} a\big(\phi_t(q,\xi)\big)\big|\varphi+\psi\Gamma^{\frac{1}{2},\Upsilon}\big(\phi_t(q,\xi)\big)\big|^2 \frac{\mathrm{d}t}{2\pi} \frac{\mathrm{d}\xi}{\mathrm{vol}(S^*_q\mathbb{S}^2)}.$$

Due to the definition of $\Gamma^{\frac{1}{2},\Upsilon}$ (4.7) and the properties (4.11),

$$\varphi + \psi\Gamma^{\frac{1}{2},\Upsilon}\big(\phi_t(q,\xi)\big) = \varphi + \psi\frac{e^{-it}}{C_{\frac{1}{2}}}\sum_{k=-\Upsilon+1}^{\Upsilon}\frac{(-1)^k}{k-\frac{1}{2}}e^{ikt}. \tag{4.22}$$

Therefore, Lemma 4.2 for $\sigma = \frac{1}{2}$ implies that (4.22) is the partial Fourier series of

$$[\varphi + \psi e^{-it}\zeta^{\frac{1}{2},2}](t) = \varphi + \psi\big[\mathbb{1}_J(t) - \mathbb{1}_K(t)\big],$$

where $J := [\frac{\pi}{2}, \frac{3\pi}{2})$ and $K := \mathbb{R}/(2\pi\mathbb{Z}) \setminus J$. Carleson's theorem gives that for all $\xi \in S_q^*\mathbb{S}^2$

$$\big|\varphi + \psi\Gamma^{\frac{1}{2},\Upsilon}\big(\phi_t(q,\xi)\big)\big|^2 \xrightarrow[\Upsilon\to\infty]{} |\varphi+\psi|^2\mathbb{1}_J(t) + |\varphi-\psi|^2\mathbb{1}_K(t) \quad \text{a.e. } t \in \mathbb{R}/(2\pi\mathbb{Z}).$$

The dominated convergence theorem let us take $\Upsilon \to \infty$ in (4.21) and conclude

$$\int_{T^*\mathbb{S}^2} a\mu = \int_{T^*\mathbb{S}^2} a[m_J\mu_{q,J} + m_K\mu_{q,K}]. \qquad \blacksquare$$

Lemma 4.7. *Let γ be a closed geodesic in $\mathbb{S}^2$ and let $q \in \mathbb{S}^2$ be such that $d(q,\gamma) = \frac{\pi}{2}$. Consider the following smooth map:*

$$f\colon \mathbb{R}/(2\pi\mathbb{Z}) \times S_q^*\mathbb{S}^2 \to \mathbb{S}^2, \quad f(t,\xi) := \exp_q(t\xi),$$

and the measures μ_q on $\mathbb{R}/(2\pi\mathbb{Z}) \times S_q^*\mathbb{S}^2$ and ν_γ on $\mathbb{S}^2$,

$$\mu_q(\mathrm{d}t,\mathrm{d}\xi) := \frac{\mathrm{d}t}{2\pi}\frac{\mathrm{d}\xi}{\mathrm{vol}(S_q^*\mathbb{S}^2)},$$
$$\nu_\gamma(\mathrm{d}x) := \frac{2}{\pi}\frac{1}{\cos\big(d(x,\gamma)\big)}\frac{\mathrm{d}x}{\mathrm{vol}(\mathbb{S}^2)} = \frac{2}{\pi}\frac{1}{\sin\big(d(x,q)\big)}\frac{\mathrm{d}x}{\mathrm{vol}(\mathbb{S}^2)}.$$

Assume that U is an open connected subset of $\mathbb{R}/(2\pi\mathbb{Z}) \times S_q^*\mathbb{S}^2$ such that $U \subseteq \{(t,\xi)\colon t \neq 0,\pi\}$. Then $f|_U$ is a diffeomorphism onto its image and the pushforward of μ_q by $f|_U$ is

$$(f|_U)_*\mu_q = \frac{1}{2}\nu_\gamma.$$

Lemma 4.7 shows that the pushforward of the measure on $T^*\mathbb{S}^2$ obtained in Theorem 4.5 by the map $\pi\colon T^*\mathbb{S}^2 \to \mathbb{S}^2$, $\pi(x,\xi) := x$, has the form of the measure on $\mathbb{S}^2$ in Theorem 1.4, proving the latter result.

Appendix: The von Neumann theory of symmetric extensions

In this appendix, we collect and prove some abstract results on the von Neumann theory of symmetric extensions that we invoked in the article.

Let $\mathcal{H}$ be a complex Hilbert space with inner product $\langle u \mid v\rangle$, $u, v \in \mathcal{H}$. We will assume that it is conjugate-linear in the first argument:

$$\langle zu + w \mid v\rangle = \bar{z}\langle u \mid v\rangle + \langle w \mid v\rangle.$$

Let $T\colon \mathrm{dom}(T) \to \mathcal{H}$ be a closed symmetric operator, where $\mathrm{dom}(T)$ is a dense subspace of $\mathcal{H}$.

Since T is closed, $\mathrm{dom}(T)$ inherits a Hilbert space structure given by the graph-inner product

$$\langle u \mid v\rangle_T := \langle u \mid v\rangle + \langle Tu \mid Tv\rangle \quad \forall u, v \in \mathrm{dom}(T). \tag{A.1}$$

We first present a lemma that shows that the operator T considered throughout the paper is in fact a densely defined symmetric closed operator.

Lemma A.1. *Let A be a closed symmetric operator and let $f\colon \mathrm{dom}(A) \to \mathbb{C}$ be a linear functional such that*

$$\begin{aligned} &(1)\quad \exists C > 0 \quad \text{s.t.} \quad \big|f(x)\big| \le C\|x\|_A \quad \forall x \in \mathrm{dom}(A), \\ &(2)\quad \forall M > 0 \quad \exists x_M \in \mathrm{dom}(A) \quad \text{s.t.} \quad \|x_M\|_{\mathcal{H}} = 1 \quad \text{and} \quad f(x_M) \ge M. \end{aligned} \tag{A.2}$$

Then $T := A|_{\mathrm{dom}(T)}$ for

$$\mathrm{dom}(T) := \big\{x \in \mathrm{dom}(A)\colon f(x) = 0\big\},$$

is a (densely defined) closed symmetric operator on $\mathcal{H}$.

Proof. We show first that $\mathrm{dom}(T)$ is dense in $\mathcal{H}$. Fix $v \in \mathcal{H}$. Since $\mathrm{dom}(A)$ is dense, there exists $(v_n)_{n\in\mathbb{N}} \subseteq \mathrm{dom}(A)$ such that $v_n \to v$ in $\mathcal{H}$. For every $n \in \mathbb{N}$, let

$$M_n := \max\big\{n\big|f(v_n)\big|, n\big\} > 0$$

and take $x_{M_n} \in \mathrm{dom}(A)$ given by (A.2), and define

$$w_n := v_n - \frac{f(v_n)}{f(x_{M_n})} x_{M_n}.$$

It is clear that $w_n \in \mathrm{dom}(T)$ for all $n \in \mathbb{N}$, and in addition, $w_n \to v$ as $n \to \infty$ because

$$\|w_n - v\|_{\mathcal{H}} \le \|v_n - x\|_{\mathcal{H}} + \frac{\big|f(v_n)\big|}{f(x_{M_n})} \le \|v_n - x\|_{\mathcal{H}} + \frac{1}{n} \quad \forall n \in \mathbb{N}.$$

Next, T is symmetric because it is the restriction of the symmetric operator A: for all $x, y \in \mathrm{dom}(T)$,

$$\langle x \mid Ty\rangle = \langle x \mid Ay\rangle = \langle Ax \mid y\rangle = \langle Tx \mid y\rangle.$$

Moreover, T is a closed operator because $\mathrm{dom}(T) = \ker(f)$ and f is a continuous functional with respect to the graph-inner product of A: if $(v_n)_{n\in\mathbb{N}} \subseteq \mathrm{dom}(T)$ is such that

$v_n \to V$ and $Tv_n = Av_n \to W$ in $\mathcal{H}$, then $V \in \mathrm{dom}(A)$ and $W = AV$ because A is closed and

$$\begin{aligned} \big|f(V)\big|^2 = \big|f(V - v_n)\big|^2 &\leq C^2 \|V - v_n\|_A^2 \\ &= C^2\big(\|V - v_n\|_{\mathcal{H}}^2 + \|AV - Av_n\|_{\mathcal{H}}^2\big) \xrightarrow{n\to\infty} 0. \end{aligned}$$

■

Definition A.2. The adjoint operator $T^*\colon \mathrm{dom}(T^*) \to \mathcal{H}$ is the operator given by

$$\mathrm{dom}(T^*) := \big\{u \in \mathcal{H}\colon \exists\, C_u > 0 \text{ s.t. } \big|\langle u \,|\, Tw\rangle\big| \leq C_u \|w\| \ \forall w \in \mathrm{dom}(T)\big\},$$

and for $u \in \mathrm{dom}(T^*)$, $T^*u \in \mathcal{H}$ is defined via the Riesz representation theorem: T^*u is the unique vector such that

$$\langle T^*u \,|\, w\rangle = \langle u \,|\, Tw\rangle \quad \forall w \in \mathrm{dom}(T).$$

Then, we consider two additional structures on $\mathrm{dom}(T^*)$. First, since T^* is a closed operator (see [13, Proposition 2.47]) $\mathrm{dom}(T^*)$ is a Hilbert space for its graph-inner product on $\mathrm{dom}(T^*)$, $\langle u \,|\, v\rangle_{T^*}$ (A.1). Since $T \subset T^*$ is closed, $\mathrm{dom}(T)$ is a T^*-closed subspace of $\mathrm{dom}(T^*)$. Additionally, we also define a skew-Hermitian form on $\mathrm{dom}(T^*)$:

$$\omega(u, v) := \langle u \,|\, T^*v\rangle - \langle T^*u \,|\, v\rangle, \quad u, v \in \mathrm{dom}(T^*). \tag{A.3}$$

Lemma A.3. *Given a closed symmetric operator T on $\mathcal{H}$ and the skew-Hermitian form ω on $\mathrm{dom}(T^*)$, one has*

$$u \in \mathrm{dom}(T) \iff \omega(u, w) = 0 \quad \forall w \in \mathrm{dom}(T^*).$$

Proof. If $u \in \mathrm{dom}(T)$, then $T^*u = Tu$ because $T \subset T^*$. Therefore, for every $w \in \mathrm{dom}(T^*)$, one has

$$\omega(u, w) = \langle u \,|\, T^*w\rangle - \langle T^*u \,|\, w\rangle = \langle Tu \,|\, w\rangle - \langle Tu \,|\, w\rangle = 0.$$

If $u \in \mathrm{dom}(T^*)$ is such that $\omega(u, w) = 0$ for all $w \in \mathrm{dom}(T^*)$, then $u \in \mathrm{dom}((T^*)^*) = \mathrm{dom}(\overline{T}) = \mathrm{dom}(T)$ because T is closed. This is a direct computation: for every $w \in \mathrm{dom}(T^*)$,

$$\langle u \,|\, T^*w\rangle = \omega(u, w) + \langle T^*u \,|\, w\rangle = \langle T^*u \,|\, w\rangle.$$

■

Definition A.4. For a given subspace $L \subseteq \mathrm{dom}(T^*)$, we define the ω-orthogonal subspace of L as follows:

$$L^\omega := \big\{u \in \mathrm{dom}(T^*)\colon \omega(u, w) = 0 \ \forall w \in L\big\}.$$

Thus, we say that a linear subspace L is isotropic if and only if $\mathrm{dom}(T) \subseteq L \subseteq L^\omega$,[6] and L is Lagrangian if and only if $L = L^\omega$.

[6] Lemma A.3 implies that $\mathrm{dom}(T) \subseteq L^\omega$ for any linear subspace L.

Proposition A.5. *Let T be a closed symmetric operator, and let ω be the skew-Hermitian form* (A.3) *on* $\mathrm{dom}(T^*)$. *For a subspace* $\mathrm{dom}(T) \subseteq L \subseteq \mathrm{dom}(T^*)$, *define the operator* $T_L := T^*|_L$.

(1) $(T_L)^* = T_{L^\omega}$.

(2) T_L *is symmetric if and only if* L *is isotropic.*

(3) T_L *is self-adjoint if and only if* L *is Lagrangian.*

Proof. Let L be a subspace as described above. To prove item (1), it suffices to show that $\mathrm{dom}((T_L)^*) = L^\omega$. If that were the case, for all $u \in L^\omega$ and all $w \in L$,

$$\big\langle (T_L)^* u \,|\, w \big\rangle = \langle u \,|\, T_L w \rangle = \langle u \,|\, T^* w \rangle = \omega(u, w) + \langle T^* u \,|\, w \rangle = \langle T^* u \,|\, w \rangle.$$

Assume first that $u \in \mathrm{dom}((T_L)^*)$. We are going to show that $\omega(u, w) = 0$ for all $w \in L$. First, by the definition of $\mathrm{dom}((T_L)^*)$, there exists $C_u > 0$ such that

$$\big|\langle u \,|\, T^* w \rangle\big| \leq C_u \|w\| \quad \forall w \in L.$$

Since $\mathrm{dom}(T) \subseteq L$, we infer that $u \in \mathrm{dom}(T^*)$. This implies that

$$\omega_u(w) := \omega(u, w) = \langle u \,|\, T^* w \rangle - \langle T^* u \,|\, w \rangle, \quad w \in \mathrm{dom}(T^*),$$

is a bounded functional over L in the $\mathcal{H}$ norm. Since $\omega_u(w) = 0$ for all $w \in \mathrm{dom}(T)$ and $\mathrm{dom}(T)$ is a dense subspace of $\mathcal{H}$, continuity implies that $\omega_u \equiv 0$, thus $u \in L^\omega$.

Assume now that $u \in L^\omega$. Therefore, for all $w \in L$,

$$\langle u \,|\, T_L w \rangle = \langle u \,|\, T^* w \rangle = \omega(u, w) + \langle T^* u \,|\, w \rangle = \langle T^* u \,|\, w \rangle,$$

which implies that $u \in \mathrm{dom}((T_L)^*)$, and $(T_L)^* u = T^* u$ in addition.

Items (2) and (3) are direct consequences of item (1). ∎

In fact, the inner product structure and the skew-Hermitian structure are related to each other through the operator T^*.

Lemma A.6. *Let T be a closed symmetric operator on $\mathcal{H}$ and let E be the orthogonal complement of* $\mathrm{dom}(T)$ *inside* $\mathrm{dom}(T^*)$. *Define the operator $J := T^*|_E$.*

(1) *For all $u \in E$, $T^* u \in \mathrm{dom}(T^*)$ and $T^*(T^* u) = -u$.*

(2) *If $u \in E$, then $\omega(T^* u, w) = \langle u \,|\, w \rangle_{T^*}$ for all $w \in \mathrm{dom}(T^*)$.*

(3) *Assume that A is a self-adjoint extension of T and that* $\dim(\mathrm{dom}(A)/\mathrm{dom}(T)) < \infty$. *Then*

$$\dim\big(\mathrm{dom}(T^*)/\,\mathrm{dom}(T)\big) = 2\dim\big(\mathrm{dom}(A)/\,\mathrm{dom}(T)\big). \tag{A.4}$$

Proof. Let $u \in E$, that is, such that $\langle u \,|\, w \rangle_{T^*} = 0$ for all $w \in \mathrm{dom}(T)$. To show that $Ju = T^* u \in E$ we study $\langle T^* u \,|\, T w \rangle$ for $w \in \mathrm{dom}(T)$.

$$\langle T^* u \,|\, T w \rangle = \langle T^* u \,|\, T^* w \rangle = \langle u \,|\, w \rangle_{T^*} - \langle u \,|\, w \rangle = \langle -u \,|\, w \rangle.$$

From this identity, we read $T^*u \in \mathrm{dom}(T^*)$ and $T^*(T^*u) = -u$. Item (2) is a direct computation from item (1).

Assume that A is a self-adjoint extension of T. Since T is symmetric, then $A = A^* \subset T^*$, thus $A = T^*|_L$ for some Lagrangian subspace L of $\mathrm{dom}(T^*)$ (Proposition A.5). Let us write $L = \mathrm{dom}(T) + S$, where $S \subseteq E$. We will prove that

$$\mathrm{dom}(T^*) = \mathrm{dom}(T) + S + T^*[S], \tag{A.5}$$

and that $T^*[S]$ and S are T^*-orthogonal; if $\dim(\mathrm{dom}(A)/\mathrm{dom}(T)) < \infty$, using that $T^*|_E$ is an isomorphism of E, one obtains (A.4).

To prove (A.5) and that $T^*[S]$ and S are T^*-orthogonal, it suffices to show

$$L^\perp := \{u \in \mathrm{dom}(T^*) \colon \langle u \mid w\rangle_{T^*} = 0 \ \forall w \in L\} = J[S].$$

If $u \in L^\perp$, then $u \in E$ because $\langle u \mid w\rangle_{T^*} = 0$ for all $w \in \mathrm{dom}(T)$. Due to item (1), $T^*u \in E$ too. In addition, for every $w \in L$, due to item (2),

$$0 = \langle u \mid w\rangle_{T^*} = \omega(T^*u, w),$$

thus $T^*u \in L^\omega = L$. Since $T^*u \in E \cap L$, we have $T^*u \in S$, therefore $u = T^*[-T^*u] \in J[S]$.

Now, let $v \in S$ and take $u = T^*v \in E$. We have for every $w \in L$,

$$\langle u \mid w\rangle_{T^*} = \langle T^*v \mid w\rangle_{T^*} = \omega(v, w) = 0,$$

because $v \in S \subseteq L = L^\omega$, hence $u \in L^\perp$. ∎

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Santiago Verdasco
M^{2}ASAI, ETSI Navales, Universidad Politécnica de Madrid, Avda. de la Memoria, 4,
28040 Madrid, Spain; santiago.verdasco@upm.es
Author IDs: zbMATH verdasco.santiago ORCID 0009-0007-3915-2836

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