

# Special First Order Systems in Clifford Analysis and Resolutions

I. Sabadini and F. Sommen

**Abstract.** In this paper we present and discuss to some extent a number of first order systems of partial differential operators with constant coefficients which arise naturally within the language of Clifford analysis. We also present resolutions for certain examples.

**Keywords:** *Abstract vector variables, monogenic functions of higher spin, matrix variables, combinatorial systems, resolutions, syzygies*

**AMS subject classification:** 15A66, 30G35, 35A25, 35A27

## 1. Introduction

Clifford analysis deals primarily with the function theory of the Dirac operator, called theory of monogenic functions. To that end, let  $\{e_1, \dots, e_m\}$  be an orthonormal basis of  $\mathbb{R}^m$  and let  $\mathcal{C}_m$  be the real Clifford algebra with defining relations  $e_i e_j + e_j e_i = -2\delta_{ij}$ . Then nullsolutions of the Dirac operator (or vector derivative [13])  $\partial_{\underline{x}} = \sum e_j \partial_{x_j}$  are called monogenic functions and for the function theory (without claiming completeness) we refer to [8, 11]. But Clifford analysis contains much more than only the theory of the Dirac operator, and already in our Seiffen paper [23] we gave a survey of operators and systems (some of which unknown at that time while several of them were already studied by many people and research groups). Also, part of this paper is a continuation of [23] in which we present and discuss systems which in our opinion deserve to be called "classical systems in Clifford analysis"; a second part of the paper is devoted to the minimal free resolutions of some of these systems, computed with the aid of the special computer software called CoCoA

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I. Sabadini: Politecnico di Milano, Dip. di Mate., Via Bonardi 9, 20133 Milano, Italy

F. Sommen: State Univ. of Ghent, Dept. Math. Anal., Galglaan 2, B-9000 Gent, Belgium

sabadini@mate.polimi.it and fs@cage.rug.ac.be

<sup>1)</sup> But already at the present time, Clifford analysis includes way more than only the theory of the Dirac operator and we first present a short summary of [23] and other recent developments. A first extension of Clifford analysis has to do with the consideration of several Dirac operators, i.e. one considers several  $m$ -tuples of coordinates such as  $(x_1, \dots, x_m)$ ,  $(y_1, \dots, y_m)$ ,  $(u_1, \dots, u_m)$ ,  $\dots$ , and several Dirac operators

$$\partial_{\underline{x}} = \sum e_j \partial_{x_j}, \quad \partial_{\underline{y}} = \sum e_j \partial_{y_j}, \quad \partial_{\underline{u}} = \sum e_j \partial_{u_j}, \quad \dots$$

of which one can investigate simultaneous nullsolutions, but also the inhomogeneous systems and resolutions (syzygies). The function theory for the nullsolutions was developed to some extent in the thesis [7] and further in our recent paper [24]. In the case of several quaternion variables we first of all should mention the pioneering work by Pertici in [17, 18] and the later works [1, 16, 19] in which not only the homogeneous system but also the inhomogeneous system as well as resolutions and syzygies were considered in the quaternion setting. This theory was recently generalized to the case of several Dirac operators in our paper [21], where it became clear that the algebra of abstract vector variables and derivatives (see [26]) plays an important role. For the earlier work on several Dirac or Fueter operators we refer to [4, 14, 26]. In any case, the theory of Dirac or Fueter operators in several variables may nowadays be called a classical topic of Clifford analysis. Much less classical is the construction of so-called monogenic functions of higher spin we presented in [23] and earlier in [27] although similar (though not identical) constructions are well known in the theory of Dirac operators for higher spin fields, where symmetric tensor products of spinor bundles are considered. Moreover, in connection with higher spin fields we also mention the papers [31, 32], which are based on abstract representation theory, as well as the papers [6, 30] in which polynomial-valued functions are considered. In our paper [27] we started from a Clifford basis denoted as

$$e_{j.k} \quad (j = 1, \dots, m; k = 1, \dots, n)$$

satisfying the defining relations

$$e_{j.k} e_{i.l} + e_{i.l} e_{j.k} = -2\delta_{ij} \delta_{kl}$$

and we produce the canonical higher spin vector derivatives

$$\partial_{\underline{x}.j} = \sum \partial_{x_k} e_{k.j}$$

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<sup>1)</sup> CoCoA is a special computer system for doing computations in commutative algebra. It is freely available by anonymous ftp from <http://cocoa.dima.unige.it> ■

leading to the definition of higher spin monogenics as simultaneous nullsolutions of  $\partial_{\underline{x}.j}f(x) = 0$  ( $j = 1, \dots, n$ ). The function theory for these solutions was elaborated in [5]. But as we already discussed in [23], one may also consider higher spin systems in several vector variables, i.e. starting from several  $m$ -tuples  $(x_1, \dots, x_m), (u_1, \dots, u_m), \dots$  and producing the operators  $\partial_{\underline{x}.j}, \partial_{\underline{u}.j}, \dots$  or

$$\partial_{\underline{x}_l.k} = \sum \partial_{x_{il}} e_{i.k},$$

and in a recent paper [28] we considered the system

$$D_{l,k} \partial_{\underline{x}_l.k} f(x_{il}) = 0$$

whereby  $D_{l,k}$  is an incidence matrix. As we will point out in this paper, it is even more interesting to consider systems of the form

$$\partial_i = \sum \Lambda_{ijk} \partial_{\underline{x}.j.k}$$

whereby  $\Lambda_{ijk}$  has entries in the set  $\{0, 1, -1\}$ . In [23] we also introduced Dirac operators with respect to a matrix variable  $\partial_A = \sum e_{i.1} e_{j.2} \partial_{A_{ij}}$  and, as already mentioned in [13], any Dirac-type operator defined in terms of vector variables or vector indices can always be extended to the spaces of bivectors, trivectors, multivectors.

In Section 2 of this paper we present certain classes of systems of linear homogeneous partial differential equations treated in a systematic way when possible. Of special importance to us are systems that may be expressed in Clifford algebra language, although there seem to exist natural systems outside the boundaries of Clifford analysis, for which a more intensive study is required though not included in this paper. In section 2 we also discuss so called "Seiffen-type systems" which are expressed in terms of the higher spin Dirac operators  $\partial_{\underline{x}.k}, \partial_{\underline{u}.k}$  in several vector variables. Finally, we present an entirely new class of systems that arise from the consideration of finite geometries and have to do with colorings of bipartite graphs. In Section 3 we present several special examples of systems that arise from Section 2, for which we compute and discuss the resolutions and syzygies with the help of CoCoA.

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## 2. Linear homogeneous systems of partial differential equations

A general homogeneous linear partial differential operator with constant coefficients is an aggregate of the form

$$\partial_{a|x} = \sum a_j \partial_{x_j}$$

where  $x_1, \dots, x_m$  is a collection of coordinates and  $a_1, \dots, a_m$  are elements of a certain algebra. In general one hereby has in mind a matrix algebra while in Clifford analysis one usually considers  $a_j$  to belong to a certain Clifford algebra. At the moment we only assume associativity, i.e. that the elements  $a_1, \dots, a_m$  belong to some unspecified associative algebra, algebra which is itself often determined by a certain requirement about the operator or system of operators. Some examples of this have to do with the square of the operator  $\partial_{a|x}$ :

$$\partial_{a|x}^2 = \frac{1}{2} \sum (a_j a_k + a_k a_j) \partial_{x_j} \partial_{x_k}.$$

**Example 2.1** (Clifford algebra). If one assumes that  $\partial_{a|x}^2 = -\sum g_{ij} \partial_{x_i} \partial_{x_j}$  for some fixed metric tensor  $g_{ij}$ , one arrives at the Clifford algebra defining relations  $a_j a_k + a_k a_j = -2g_{jk}$ .

**Example 2.2** (Radial algebra). If one merely assumes that the operator  $\partial_{a|x}^2$  is commutative it is meant that the  $\partial_{a|x}^2$  also commutes with the elements  $a_j$ , which leads to the identities  $a_j (a_k a_l + a_l a_k) = (a_k a_l + a_l a_k) a_j$  on which is based the definition of algebra of abstract vector variables (radial algebra). This algebra and its endomorphisms were considered in [26] as an alternative approach to the "geometric calculus" developed in [13].

More in general one could start with two general operators  $\partial_{a|x}$  and  $\partial_{b|x}$  with values in an associative algebra to define the following

**Example 2.3** (Habetha relations). The Habetha defining relations (see [12]) follow from the assumption that the operator  $\partial_{b|x} \partial_{a|x}$  is equal to a given fixed scalar operator  $-\sum g_{kl} \partial_{x_k} \partial_{x_l}$ , and they are given by

$$b_j a_k + b_k a_j = -2g_{jk}.$$

This leads to an algebra in which function theoretic results are still possible to some extent. However, the free algebra with these relations is necessarily infinite dimensional.

**Example 2.4** (Several Habetha-type relations). A more restricted algebra which we discussed together with M. Shapiro and N. Vasilevski, arises from the assumption that both  $\partial_{b|x} \partial_{a|x}$  and  $\partial_{a|x} \partial_{b|x}$  are given fixed scalar

operators  $-\sum g_{jk}\partial_{x_j}\partial_{x_k}$  and  $-\sum h_{jk} = \partial_{x_j}\partial_{x_k}$ . This leads to the defining relations

$$\left. \begin{aligned} b_j a_k + b_k a_j &= -2g_{jk} \\ a_j b_k + a_k b_j &= -2h_{jk} \end{aligned} \right\}$$

but also in this case the free algebra itself remains infinite dimensional. From the evaluation of  $\partial_{b|x}\partial_{a|x}\partial_{b|x}$  it follows that  $g_{jk} = h_{jk}$ . Moreover, in the case  $g_{jk} = \delta_{jk}$  and  $j \neq k, l$

$$b_j a_k b_j a_l + b_j a_l b_j a_k = -b_k a_j b_j a_l - b_l a_j b_j = a_k b_k a_l + b_l a_k = -2\delta_{lk}$$

so that the quadratic elements  $b_k a_l$  generate a Clifford algebra.

Next one can consider several extensions of this algebra by starting from any collection of factorization relations of the form  $\partial_{a|x} = \partial_{b|x} = O_1, \partial_{b|x}\partial_{c|x} = O_2, \partial_{c|x}\partial_{d|x} = O_3, \dots$  whereby  $O_1, O_2, O_3, \dots$  are scalar second order operators. Obviously, there are many ways to define algebras starting from the idea of factorization.

**Example 2.5** (Generalizations of radial algebra). In both the previous cases one can weaken the assumption to the statement that  $\partial_{b|x}\partial_{a|x}$  (or also  $\partial_{a|x}\partial_{b|x}$ ) are commutative objects. This leads to commutation relations of the form

$$\begin{aligned} [a_j, b_k a_l + b_l a_k] &= 0, & [b_j, b_k a_l + b_l a_k] &= 0 \\ [a_j, a_k b_l + a_l b_k] &= 0, & [b_j, a_k b_l + a_l b_k] &= 0 \end{aligned}$$

while also the evaluation of  $\partial_{b|x}\partial_{a|x}\partial_{b|x}$  leads to the identification  $b_k a_l + b_l a_k = a_k b_l + a_l b_k$ .

**Remark 2.6.**

**1.** Among the above examples only the Clifford algebra lead to large scale investigation; it is simply the best algebra for calculus. The other algebras are interesting to be considered from an axiomatic point of view, but already in [12] it was pointed out that the best function theoretic properties more or less require the consideration of Clifford algebras. This is even more clear in Example 2.4 where it turns out that the universal algebra contains a Clifford algebra.

**2.** Also the consideration of radial algebras in [26] has already lead to interesting applications (see [21, 27]). This is due to the fact that this algebra is closely related to the algebra of Clifford polynomials and that it allows a completely coordinate independent calculus.

**3.** The main reason why the other algebras never became more than curiosities may have to do with the fact that they are infinite dimensional but it is also due to the absence of intertwining relations for the different families of generators  $a_j, b_j, c_j, \dots$

Next let us turn to systems. We have to consider now a collection of operators of the form

$$\partial_{a_1|x_1}, \dots, \partial_{a_n|x_n}$$

whereby  $a_1 = (a_{1;1}, \dots, a_{1;m}), \dots, a_n = (a_{n;1}, \dots, a_{n;m})$  are  $m$ -tuples of elements  $a_{j;k}$  belonging to some associative algebra and  $x_1 = (x_{11}, \dots, x_{1m}), \dots, x_n = (x_{n1}, \dots, x_{nm})$  are  $m$ -tuples of scalar coordinates  $x_{jk}$  which may be repeated several times. There are two equivalent reformulations

**Form 1:** Without loss of generality one may assume that there is only one  $m$ -tuple of coordinates  $(x_1, \dots, x_m)$ , so that we are faced with the analysis of the operators  $\partial_{a_1|x}, \dots, \partial_{a_n|x}$ .

**Form 2:** One may also assume that all coordinates  $x_{jk}$  are different and add the identifications of coordinates as extra equations. In any case, the situation whereby all  $x_{jk}$  are independent coordinates has special extra possibilities which motivates its independent treatment.

Next we need to have good axioms which are in fact inspired by the above algebras coming from factorization. We consider

1. *The Clifford axiom applied to Form 1.* We assume that (applying standard Form 1)

$$\partial_{a_j|x} \partial_{a_k|x} + \partial_{a_k|x} \partial_{a_j|x} = -2 \sum G_{jk;il} \partial_{x_i} \partial_{x_l}$$

for some given quadratic form  $G_{jk;il} = G_{kj;il} = G_{jk;li}$ . This assumption is readily seen to lead to relations of the form

$$a_{j;i} a_{k;l} + a_{j;l} a_{k;i} + a_{k;i} a_{j;l} + a_{j;i} a_{k;l} = -4G_{jk;il}.$$

Unfortunately perhaps, these relations are too weak to be able to prove that the elements  $a_{k;l}$  belong to the space of vectors of a certain Clifford algebra; one may hence have to consider systems that cannot be expressed in the Clifford algebra language and require more general algebras. Indeed, let us consider the simplest case of two operators  $\partial_{a|x}$  and  $\partial_{b|x}$ . Then the above defining relations may be reformulated as

$$\left. \begin{aligned} a_j a_k + a_k a_j &= -2g_{jk} \\ b_j b_k + b_k b_j &= -2h_{jk} \\ a_j b_k + a_k b_j + b_j a_k + b_k a_j &= -4f_{jk} \end{aligned} \right\}$$

and, taking as special case  $g_{jk} = h_{jk} = \delta_{jk}$ , and  $f_{jk} = 0$ , the first two sets of relations define Clifford algebras  $Alg\{a_j\}$  and  $Alg\{b_j\}$ ; the remaining relations only allow us to introduce new objects

$$A_{jk} = a_j b_k + b_k a_j$$

satisfying  $A_{jk} + A_{kj} = 0$ , i.e.  $A_{kj} = -A_{jk}$ . These objects  $A_{jk}$  satisfy extra algebra relations such as

$$a_l A_{jk} + a_j A_{lk} = b_k(a_j a_l + a_l a_j) + a_l b_k a_j + a_j b_k a_l = A_{jk} a_l + A_{lk} a_j,$$

and similar with respect to  $b_j$ . At this moment it is hard to predict how more general than a Clifford algebra this new algebra could be; a deeper analysis is needed.

2. *The Clifford axiom applied to Form 2.* We are now faced with relations of the form

$$\partial_{a_j|x_j} \partial_{a_k|x_k} + \partial_{a_k|x_k} \partial_{a_j|x_j} = -2 \sum G_{jk;il} \partial_{x_{ji}} \partial_{x_{kl}}$$

which in the case all variables  $x_{jk}$  are independent does actually lead to Clifford algebra defining relations of the form

$$a_{j;i} a_{k;l} + a_{k;l} a_{j;i} = -2G_{jk;il}.$$

It is hence possible to find a single Clifford algebra with respect to some quadratic form in which all the elements  $a_{j;l}$  are Clifford vectors. But of course this is no longer true in the more general case where it is possible that certain variables  $x_{jl}$  coincide.

3. *Radial axioms for Forms 1 and 2.* Also in the case of systems it is possible to weaken the Clifford algebra axioms to the so called radial axioms. In the case of systems of Form 1 these axioms state that the expressions  $\partial_{a_j|x} \partial_{a_k|x} + \partial_{a_k|x} \partial_{a_j|x}$  are scalar objects, i.e. one has commutation relations

$$[a_{s;t}, a_{j;i} a_{k;l} + a_{j;l} a_{k;i} + a_{k;i} a_{j;l} + a_{k;l} a_{j;i}] = 0$$

and like in the Clifford case these relations lead to non-trivial generalizations of radial algebras. In the case of systems of Form 2 whereby all coordinates  $x_{jl}$  are different we obtain, as expected from the radial axioms, the fact that the set of vectors  $a_{j;l}$  forms a radial algebra, i.e. all anti-commutators are scalar.

In this paper we will further restrict ourselves to the case where all the elements  $a_{j;l}$  of a given system are vector-valued objects belonging to some over-all Clifford algebra of the standard type generated by basis elements of the form  $e_1, \dots, e_M$  satisfying the classical defining relations

$$e_i e_j + e_j e_i = -2\delta_{ij}.$$

In the true several variable case this is no serious restriction but one has to be aware of the possible existence of natural and interesting systems that cannot be defined within these linguistic bounds. A more detailed algebraic investigation is at stake and part of the future scope. Hence from now on we will consider systems in the following

**General form.** Let  $x_1, \dots, x_m$  be scalar variables and let  $e_1, \dots, e_M$  be a given Clifford basis. Then we select vectors

$$\underline{u}_j = \sum u_{jl} e_l, \quad \underline{v}_j = \sum v_{jl} e_l, \dots$$

to arrive at operators of the form

$$\partial_{u|x} = \sum \underline{u}_j \partial_{x_j}, \quad \partial_{v|x} = \sum \underline{v}_j \partial_{x_j}, \dots \quad (1)$$

Important hereby is to note that apart from the choice of the parameters  $u_{jl}, v_{jl}, \dots$  the inhomogeneous system

$$\partial_{u|x} = g_u, \quad \partial_{v|x} = g_v, \quad \dots$$

depends on three natural numbers, namely

1. the total number  $m$  of scalar variables
2. the total dimension  $M$  of the over-all space of vectors
3. the number  $n$  of equations.

In fact one may now study systems in many ways, e.g. one can consider the parameters  $u_{jl}, v_{jl}, \dots$  as variables and then study resolutions and try to figure out for which special values there are exceptional syzygies and resolutions. This however seems beyond the capacities of any computer system. All one can do in general in fact is to select random values for parameters  $u_{jl}, v_{jl}, \dots$  and one may expect that the behaviour of the system will (probabilistically speaking) not depend on this choice. In this way one hence obtains information about the general behaviour. To find out which systems will (or can be expected to) display singular behaviour, the brain is a much better tool than the computer. This requires long investigation which is beyond the scope of this paper. But we somehow expect that singular behaviour is more likely to happen if the parameters belong to a very special set; we hereby think in the first place to take the parameters  $v_{jl}, u_{jl}, \dots$  to belong to the set  $\{0, 1, -1\}$ .

**Seiffen type.** Among the systems with entries in the set  $\{0, 1, -1\}$  there is a canonical subclass which may be written in terms of the special operators  $\partial_{\underline{x}.j} = \sum \partial_{x_l} e_{l.j}$  we discussed in our paper [23] and which also play a role in the theory of higher spin monogenicity (see also work by V. Souček and others). To formulate the general form of a Seiffen-type system we need the following ingredients:

1. We need a matrix of variables  $x = (x_{jk})$  ( $j = 1, \dots, s; k = 1, \dots, m$ ) which we think of as a collection of  $s$  different  $m$ -tuples  $x_j = (x_{j1}, \dots, x_{jm})$ .
2. We need an over-all Clifford frame  $e_{k.l}$  ( $k = 1, \dots, m; l = 1, \dots, p$ ).



3. We now produce the collection of vector derivatives  $\partial_{\underline{x}_j.l} = \sum \partial_{x_{jk}} e_{k.l}$  ( $j = 1, \dots, s; l = 1, \dots, p$ ).
4. We finally need structural constants (lambda structure)  $\Lambda_{i|j.l} \in \mathbb{C}$  ( $i = 1, \dots, n; j = 1, \dots, s; l = 1, \dots, p$ ) with which we may now produce the system of operators  $\partial_i = \partial_{\Lambda_{i|x}} = \sum_{j,l} \Lambda_{i|j.l} \partial_{\underline{x}_j.l}$ .

Here we are especially interested in the cases where the structure constants  $\Lambda_{i|j.l}$  belong to special subsets of the complex numbers; we hereby think in particular of the set  $\{0, 1, -1\}$  but also of the set  $\{0, 1, -1, i, -i\}$ . Note that the systems considered in almost all papers of the references belong to this class.

**Remark 2.7.** In the above definition of Seiffen type we have not really defined a true subclass of what we called the "general form" because also single operators like  $\underline{u}_j \partial_{x_j} = \sum e_k u_{jk} \partial_{x_j}$  can be written in terms of the operators  $\partial_{x_j} e_k$  which are interpretable as special cases of  $\partial_{\underline{x}_j.k}$ . However, due to the fact that the operators  $\partial_{\underline{x}.k}$  and  $\partial_{\underline{u}.k}$  are in fact labels for all operators  $\sum e_{j.k} \partial_{x_j}$  and  $\sum e_{j.k} \partial_{u_j}$  with variable dimension  $m$ , the Seiffen type operators are classes of operators rather than single operators; they are operators acting on an extended radial algebra defined as follows. Take a set of objects of the form  $x.l, l \in \{1, 2, \dots, p\}$ . Then the "higher spin radial algebra" is the algebra generated by these objects together with the axioms

1.  $x.l u.j = -u.j x.l$  for  $j \neq l$
2.  $x.l u.l + u.l x.l = x.u$  is a commutative object independent of  $l$ .

One can now redefine  $\partial_{x.k}$  as endomorphisms on this algebra similar to what we did in [26], thus giving an abstract meaning to the Seiffen-type operators.

There are still extensions of this class in which a certain idea is used to still generalize the class of operators  $\partial_{x.l}$  which is at the basis of the construction. We hereby mention:

**Matrix derivatives.** Instead of the vector derivative  $\partial_{\underline{x}}$  one may consider the matrix derivative

$$\partial_A = \sum e_a^1 e_b^2 \partial_{A_{ab}}$$

whereby we need two Clifford frames  $e_a^1$  and  $e_b^2$  which together form a standard Clifford basis. One can now construct the extended class of operators

$$\partial_{A_{j.l}} = \sum e_{a.l}^1 e_{b.l}^2 \partial_{A_{j,ab}}$$

More in general one may consider tensor derivatives

$$\partial_{A_{j.l}} = \sum e_{a_1.l}^1 \cdots e_{a_k.l}^k \partial_{A_{j,a_1 \dots a_k}}$$

**Multivector derivatives.** Instead of the vector derivative  $\partial_x$  one may also consider derivatives with respect to bivectors  $\partial_b = \sum e_{jk} \partial_{b_{jk}}$  or general  $k$ -vectors  $\partial_X = \sum e_A \partial_{x_A}$  or more general multivector derivatives (see also [13]) and produce more complicated systems. Moreover, one can also investigate  $k$ -vector matrices  $\partial_X = \sum e_A^1 e_B^2 \partial_{X_{AB}}$  or  $k$ -vector tensors, etc.

Next we will describe two special techniques which may be applied for quite general operator systems.

**Synthesis operators.** In general, when one has a system of the form (e.g., written in Form 2)

$$\partial_{a_j|x_j} f = g_j \quad (j = 1, \dots, n)$$

one may consider also the single equation

$$\sum \partial_{a_j|x_j} f = \sum g_j = g.$$

Hereby one should note that there can be many ways to perform a synthesis of a system because the initial system is equivalent to

$$b_j \partial_{a_j|x_j} f = b_j g_j,$$

leading to another synthesis equation

$$\sum b_j \partial_{a_j|x_j} f = g.$$

**Example 2.8.** Consider simply the "gradient equation" in scalar coordinates

$$\partial_{x_j} f = g_j.$$

Then using the canonical Clifford basis  $b_j = e_j$  ( $j = 1, \dots, m$ ) we perform the synthesis equation

$$\partial_{\underline{x}} f = \sum e_j \partial_{x_j} f = \sum e_j g_j = g$$

which is another way to write the initial system, with the advantage however that it becomes embedded in the theory of the inhomogeneous Dirac equation  $\partial_{\underline{x}} f = g$ , for which surjectivity theorems are much easier to obtain.

**Example 2.9.** Consider the "Dirac system" investigated in [21]

$$\partial_{\underline{x}_j} f = g_j$$

whereby  $\partial_{\underline{x}_j} = \sum e_k \partial_{x_{jk}}$  are Dirac operators acting on Clifford algebra-valued functions in the Clifford algebra generated by  $e_1, \dots, e_m$ . Next, denote  $\overset{1}{e}_j =$

$e_j$  and select another Clifford basis  $\overset{2}{e}_j$  which together with the first basis generates the over-all Clifford algebra  $\mathcal{C}_{2m}$ . One can then make the synthesis

$$\sum \overset{2}{e}_j \partial_{\underline{x}_j} f = \sum \overset{2}{e}_j g_j = g$$

which again is equivalent to the initial system but which again embeds the problem into a wider class of problems for a single matrix Dirac equation.

**Cross systems.** Let us start with the synthesis equation

$$\sum_j \left( \sum_k a_{j;k} \partial_{x_{jk}} \right) f = g = \sum g_j$$

of the system  $\partial_{a_j|x_j} f = \sum_k a_{j;k} \partial_{x_{jk}} f = g_j$ . Then one may also consider the transposed system

$$\sum_j a_{j;k} \partial_{x_{jk}} f = G_k$$

which one may consider on top of the initial system. The simplest example of such a "cross system" is obtained from the matrix Dirac equation

$$\sum_j \sum_k \overset{1}{e}_j \overset{2}{e}_k \partial_{A_{jk}} f = g$$

which synthesises both systems

$$\sum_k \overset{2}{e}_k \partial_{A_{jk}} f = g_j, \quad \sum_j \overset{1}{e}_j \partial_{A_{jk}} f = G_k.$$

This also deserves algebraic analysis treatment.

Finally, we introduce a class of systems of Dirac type operators coming from finite geometry and combinatorics (see also [10]).

**Systems of combinatorial type (Turkish systems).** We hereby have in mind systems that are constructed as follows:

1. We have a total set of scalar coordinates  $x_1, \dots, x_m$ .
2. We also have a total set of Clifford algebra generators  $e_1, \dots, e_M$ .
3. We now may produce the  $2mM$  operators  $\pm e_k \partial_{x_j}$ .

Now we are able to produce operator systems according to the following axioms.

- (A1) Each operator is an addition of basic operators of the above type 3.

- (A2) Every partial derivative  $\partial_{x_j}$  occurs at most once in a given operator (within a term  $\pm e_k \partial_{x_j}$ ).
- (A3) Also, every basis element  $e_k$  occurs at most once in a given equation.
- (A4) Every term  $e_k \partial_{x_j}$  occurs at most once in the whole system, either preceded by plus or minus sign.

There is now the possibility to combine systems of the above type with incidence structures "finite geometries" that consist of a set of points or "tops"  $\{p_1, \dots, p_m\}$  and a collection of lines or "blocks"  $\{b_1, \dots, b_n\}$  whereby every block  $b_j$  is in fact a subset of  $\{p_1, \dots, p_m\}$ . If now we let correspond to each point  $p_j$  the partial derivative  $\partial_{x_j}$ , then to each block corresponds a set of partial derivatives with which one may form an operator of the above type by attaching to each  $\partial_{x_j}$  a well chosen basis element  $e_k$  and a signature and take the sum over all  $j$ -indices in the block. In this way one arrives at several possible systems consisting of  $n$  operators. From a combinatorial point of view, to be able to produce such a system, we have in fact to assign for each given block an element  $e_k$  as well as a signature to each top of that block (i.e. partial derivative of the system) and sum up the obtained terms. One may always do this if one is free to choose the number  $M$  of basis elements  $e_k$  high enough. Question is: which is the minimal number of elements  $e_k$  needed for this? So that we may assume:

- (A5) The number  $M$  of basis elements  $e_k$  is minimal.

This problem may be translated into a problem of coloring the edges of a certain bipartite graph which is obtained as follows:

The points of the graph consist of two disjoint sets, namely the set of tops of the finite geometry and the set of blocks of that geometry. The lines of the graph connect a point in the set of tops to a point in the set of blocks if that top belongs to the block. The set of elements  $\{e_1, \dots, e_M\}$  may be seen as set of colors with which we have to color the edges of the graph such that all edges issuing from a given point in the graph have different color and the total number of colors is minimal. This total number is the edge chromatic number for which König proved that it equals the highest number of edges per point. The answer to the previous question is hence a classical problem. Classical are also many other combinatorial problems such as the following ones:

- (P1) In how many ways can the basis elements  $e_k$  be chosen such that all conditions are met?
- (P2) What is the finite group of permutations of the tops leaving the set of blocks invariant that, if combined with a permutation of the elements  $e_k$ , leave the set of operators invariant?

Less classical is

- (P3) To extend the finite group invariance to a continuous group one whereby the continuous extensions of the permutations of the sets  $\{\partial_{x_1}, \dots, \partial_{x_m}\}$  and  $\{e_1, \dots, e_M\}$  are subgroups of  $SO(m)$  and  $SO(M)$ , respectively?
- (P4) To what extent is the resolution of the system dependent on the chosen coloring or the choice of the signatures of the terms?
- (P5) For which finite geometries is the resolution only dependent on the geometry and not on the coloring or choice of signature?
- (P6) To study the resolutions in the particular cases of finite projective planes or affine planes or  $t$ -designs etc.

It is clear that we have here a brand new field of research. Moreover, there is also the "dual alternative" of the above construction whereby one starts from a finite geometry of incidence structure with tops  $\{p_1, \dots, p_M\}$  and blocks  $\{b_1, \dots, b_n\}$  whereby this time to each top we assign a basis element  $p_k \rightarrow e_k$ . Then for each fixed block we have to choose now to each top  $e_k$  of the block a certain "color"  $\partial_{x_j}$  and signature to produce a system of operators. The problem is now to make sure:

- (A5') The number  $m$  of partial derivatives is minimal.

Again there are a number of natural questions to be considered. We will call the systems produced in this way *super-dual systems*, in order to distinguish them from the dual systems obtained by interchanging the words *point* and *line*. Finally, instead of imposing an axiom like (A5) or (A5'), there is the possibility to simply choose two sets of tops, namely  $\{\partial_{x_1}, \dots, \partial_{x_m}\}$  and  $\{e_1, \dots, e_M\}$  and a single set of blocks  $\{b_1, \dots, b_n\}$  so that to each block  $b_j$  corresponds a subset of the two sets of tops with the same cardinality and one may make operators with this. It is clear that there are many possibilities and one can formulate several problems.

### 3. Special systems and resolutions

We start this section with a short overview of the algebraic treatment of systems of partial differential equations. For more details we refer the reader to the fundamental books [9, 15] and, for the applications to Clifford analysis, to [1, 2, 3, 20, 21].

Let  $\vec{f} = (f_1, \dots, f_r)$  be an  $r$ -tuple of real differentiable functions on an open set  $U \subseteq \mathbb{R}^n$  and let

$$\sum_{j=1}^r P_{ij}(D)f_j = g_i \quad (i = 1, \dots, q) \tag{2}$$

be a  $q \times r$  system of linear partial differential equations with constant coefficients. Let  $P = [P_{ij}]$  be a  $q \times r$  matrix of complex polynomials in  $\mathbb{C}^n$  and  $D = (-i\partial_{x_1}, \dots, -i\partial_{x_n})$ . The polynomial matrix  $P$ , that is the symbol of the system, can be obtained from  $P(D) = [P_{ij}(D)]$  by replacing (formally)  $\partial_{x_k}$  by the complex variable  $z_k$  for every  $k = 1, \dots, n$ . This procedure, that is equivalent to take the Fourier transform of  $P(D)$ , can be applied when we have an equation or a system of the type treated in Section 2, since we can consider the real components of each equation to get a system of type (2). The transpose matrix  $P^t$  of  $P$  is an  $R$ -homomorphism  $R^q \rightarrow R^r$  whose cokernel is  $\mathcal{M} = R^r / P^t R^q = R^r / \langle P^t \rangle$ , where  $R = \mathbb{C}[z_1, \dots, z_n]$  and  $\langle P^t \rangle$  is the submodule of  $R^r$  generated by the columns of  $P^t$ . By the Hilbert syzygy theorem, there is a finite free resolution

$$0 \longrightarrow R^{a_s} \xrightarrow{P_{a_s}^t} R^{a_{s-1}} \longrightarrow \dots \xrightarrow{P_1^t} R^q \xrightarrow{P^t} R^r \longrightarrow \mathcal{M} \longrightarrow 0$$

that together with its transpose

$$0 \longrightarrow R^r \xrightarrow{P} R^q \xrightarrow{P_1} \dots \longrightarrow R^{a_{s-1}} \xrightarrow{P_{a_s}} R^{a_s} \longrightarrow 0$$

are key tools for the algebraic analysis of system (2). Even though there is a lot of information that arises from the resolutions above, we are mainly interested in the point of view of syzygies: every matrix  $P_{a_i}^t(D)$  gives the compatibility conditions for the system whose representative polynomial matrix is  $P_{a_{i-1}}^t$ . In particular, the matrix  $P_1(D)$  gives the compatibility conditions that a datum  $\vec{g}$  of an inhomogeneous system  $P(D)\vec{f} = \vec{g}$  must satisfy to have a solution  $\vec{f}$ . In [1, 3, 20, 21] we studied the Cauchy-Fueter and the Moisil-Theodorescu complexes in several quaternionic variables and the Dirac complexes in the Clifford algebra  $\mathcal{C}_m$ . In the first two cases the complex is quadratic at the first step and then linear, while in the last case the behaviour depends on the dimension  $m$  considered. In any case, at the first step the syzygies are of degree at least two. In what follows we show that various kind of behaviour are allowed when dealing with systems of type (2) and we produce examples of systems with linear first syzygies. The knowledge of the resolutions of a system is important because it provides a first global information on how interesting a system can be, before developing its analysis. We have computed the minimal free resolutions in this paper using CoCoA, version 3.7 on a Digital AlphaServer 4100/600, with 4 CPU and 3 GB RAM.

**3.1 General systems.** We begin our study by considering some systems in general form. In one case (see subsection 3.12), we will give the details of the construction of the resolution that we will consider as an example for the other resolutions computed in this paper.

**3.11** We begin our study with a system of  $n$  operators of the type

$$\left. \begin{aligned} D^1 f &= (\underline{u}_1^1 \partial_{x_{11}} + \underline{u}_2^1 \partial_{x_{12}} + \dots + \underline{u}_m^1 \partial_{x_{1m}}) f = g_1 \\ &\vdots \\ D^n f &= (\underline{u}_1^n \partial_{x_{n1}} + \underline{u}_2^n \partial_{x_{n2}} + \dots + \underline{u}_m^n \partial_{x_{nm}}) f = g_n \end{aligned} \right\} \quad (3)$$

where  $\underline{u}_j^i = \sum_{l=1}^M u_{jl}^i e_l$  ( $i = 1, \dots, n; j = 1, \dots, m$ ) and  $f : \mathbb{R}^{nm} \rightarrow \mathcal{C}_M$ . Note that system (3) is written in terms of  $nm$  scalar coordinates  $x_{11}, \dots, x_{1m}, \dots, x_{n1}, \dots, x_{nm}$  and  $M$  Clifford algebra generators  $e_1, \dots, e_M$ . In the following, we will say that a system of  $n$  differential operators in a Clifford algebra  $\mathcal{C}_M$  is *Dirac like* if it behaves as a system of  $n$  Dirac operators in  $\mathcal{C}_M$ .

Let us denote by  $T^i = [u_{jk}^i]$  the matrix of the coefficients  $u_{jk}^i$  for every fixed index  $i$ . We have the following

**Proposition 3.1.** *The resolution of system (3) is Dirac like when  $M \leq m$  and  $T^i$  is of maximal rank for every  $i = 1, \dots, n$ .*

**Proof.** For every fixed index  $i$ , let us consider the set of the  $m$  variables  $x_{i1}, \dots, x_{im}$  involved in the  $i$ -th equation of the system and let us rewrite the operator  $D^i$  as

$$D^i = \sum_{j=1}^m \underline{u}_j^i \partial_{x_{ij}} = \sum_{j=1}^m \sum_{k=1}^M e_k u_{jk}^i \partial_{x_{ij}} = \sum_{k=1}^M e_k \partial_{y_{ik}}$$

where we have set  $\partial_{y_{ik}} = \sum_{j=1}^m u_{jk}^i \partial_{x_{ij}}$ . The operators  $\partial_{y_{ik}}$  are a new set of partial derivatives in a linearly transformed space if and only if the  $M \times m$  matrix  $T^i = [u_{jk}^i]$  is of maximal rank. This assures that the system of operators  $D^i = \sum_{k=1}^M e_k \partial_{y_{ik}}$  behaves as a Dirac like system in  $M$  dimensions ■

**Remark 3.2.** In the case  $M > m$  we cannot say whether or not the system is Dirac like. With the use of CoCoA we have explicitly written the minimal resolutions in some particular cases (for the details on the procedure used, see the explicit description in the next subsection). For  $n = 3, 4$  and  $M = 3, 4$  the resolutions are Dirac like except the trivial case  $m = 1$ .

**3.12** Let us now consider the system

$$\left. \begin{aligned} D^1 f &= (\underline{u}_1^1 \partial_{x_1} + \underline{u}_2^1 \partial_{x_2} + \dots + \underline{u}_3^1 \partial_{x_m}) f = g_1 \\ &\vdots \\ D^n f &= (\underline{u}_1^n \partial_{x_1} + \underline{u}_2^n \partial_{x_2} + \dots + \underline{u}_3^n \partial_{x_m}) f = g_n \end{aligned} \right\} \quad (4)$$

where  $\underline{u}_j^i = \sum_{l=1}^M u_{jl}^i e_l$  ( $i = 1, \dots, n; j = 1, \dots, m$ ) and  $f : \mathbb{R}^m \rightarrow \mathcal{C}_M$ . In this case we have  $m$  scalar coordinates  $x_1, \dots, x_m$  and  $M$  Clifford generators  $e_1, \dots, e_M$ . The matrix  $T^i$  is defined as above.

We have the following

**Theorem 3.3.** *System (4) has a Dirac like resolution when  $m \geq nM$  and  $T = [T^1, \dots, T^n]^t$  is of maximal rank.*

**Proof.** For every  $i = 1, \dots, n$ , the operator  $D^i$  can be rewritten as

$$D^i = \sum_{j=1}^m \underline{u}_j^i = \partial_{x_j} = \sum_{j=1}^m \sum_{k=1}^M e_k u_{jk}^i \partial_{x_j} = \sum_{k=1}^M e_k \partial_{y_{ik}}$$

where we put  $\partial_{y_{ik}} = \sum_{j=1}^m u_{jk}^i \partial_{x_j}$ . So we have a set of  $nM$  new partial derivatives  $\partial_{y_{ik}}$  if and only if the matrix  $T$  is of maximal rank ■

As in the previous case, when the condition  $m \geq nM$  is not satisfied we cannot assure that the complex coming from system (4) is Dirac like. If we fix the integers  $n$  and  $M$ , there is only a finite number of resolutions to be checked, so we can decide whether the resolution is Dirac like or not by using CoCoA. To show that the behaviour Dirac like or non Dirac like are both possible, we have treated in detail the case  $M = 4$ ,  $n = 3$  so that  $m < 12$ . For this case, we will give some details of the construction of the complex. By taking the real components with respect to each unit of the equation  $D^i f = g_i$ , where  $f : \mathbb{R}^m \rightarrow \mathcal{C}_4$ , we obtain 16 real equations that can be written in the form  $V^i(D)\vec{f} = \vec{g}_i$ , where  $\vec{f}$  is a 16-tuple of real functions. The symbol of the previous system is the  $16 \times 16$  polynomial matrix

$$V^i = \begin{bmatrix} A^i & B^i \\ -(B^i)^t & C^i \end{bmatrix} \quad (5)$$

where  $A^i, B^i, C^i$  are the matrices  $A, B, C$  given in Appendix and  $P_r^i = \sum_{l=1}^m u_{li}^i x_l$  ( $r = 1, \dots, 4$ ). The matrix associated to the system is the column  $V = [V^1, V^2, V^3]$ . Note that we are always using the same variables  $x_i$  in passing from  $V^i(D)$  to  $V^i$ , but we hope that no confusion will arise.

We have the following

**Proposition 3.4.** *The resolution of system (4) for  $M = 4$ ,  $n = 3$ ,  $5 \leq m < 12$  is Dirac like, i.e.*

$$\begin{aligned} 0 &\longrightarrow R^{32}(-6) \longrightarrow R^{144}(-5) \longrightarrow R^{240}(-4) \\ &\longrightarrow R^{160}(-3) \longrightarrow R^{48}(-1) \longrightarrow R^{16} \longrightarrow \mathcal{M}_3 \longrightarrow 0. \end{aligned}$$

For  $m = 2, 3, 4$  the resolutions are

$$\begin{aligned} 0 &\longrightarrow R^{16}(-2) \longrightarrow R^{32}(-1) \longrightarrow R^{16} \longrightarrow \mathcal{M}_3 \longrightarrow 0 \\ 0 &\longrightarrow R^{16}(-3) \longrightarrow R^{48}(-2) \longrightarrow R^{48}(-1) \longrightarrow R^{16} \longrightarrow \mathcal{M}_3 \longrightarrow 0 \\ 0 &\longrightarrow R^{16}(-5) \longrightarrow R^{48}(-4) \longrightarrow R^{32}(-2) \oplus R^{32}(-3) \\ &\longrightarrow R^{48}(-1) \longrightarrow R^{16} \longrightarrow \mathcal{M}_3 \longrightarrow 0, \end{aligned}$$



respectively.

**Proof.** With the use of CoCoA, it suffices to consider the  $48 \times 16$  matrix  $V$  where  $P_r^i = \sum_{l=1}^m u_{jl}^i x_l$  and  $m = 2, \dots, 12$ . Since we want that the  $u_{jl}^i$  vary in  $\mathbb{R}$ , the elements  $u_{jl}^i$  are selected randomly by the system with the use of the function `Rand()`. The rows of  $P$  generate an  $R$ -module whose resolution can be calculated by CoCoA with the command `Res()` ■

**3.13.** Another type of operators we considered in the setting of general operators are the following, in which we consider sums of operators of the type  $\pm e_k \partial_{x_i}$ . Those systems are still of the form (4) and show different kind of resolutions. We do not add details on the procedure to obtain the resolutions, since all the matrices involved can be obtained from (5) when  $M = 4$  or by (21) in the Appendix when  $M = 3$ , via suitable formal substitutions.

First consider the systems

$$\left. \begin{array}{l} (e_1 \partial_{x_1} + e_2 \partial_{x_2})f = g_1 \\ (e_1 \partial_{x_3} + e_2 \partial_{x_4})f = g_2 \\ (e_3 \partial_{x_5} + e_4 \partial_{x_6})f = g_3 \end{array} \right\} \quad \left. \begin{array}{l} (e_1 \partial_{x_1} + e_2 \partial_{x_2})f = g_1 \\ (e_1 \partial_{x_3} + e_3 \partial_{x_4})f = g_2 \\ (e_2 \partial_{x_5} + e_3 \partial_{x_6})f = g_3 \end{array} \right\}$$

in which  $m = 6$ ,  $n = 3$  and  $M = 4$  (even though the unit  $e_4$  does not appear explicitly in the second system). The resolution of these systems (compare with Proposition 3.4) are both Dirac like.

The system

$$\left. \begin{array}{l} (e_1 \partial_{x_1} + e_2 \partial_{x_2} + e_3 \partial_{x_3})f = 0 \\ (e_2 \partial_{x_1} + e_3 \partial_{x_2} + e_4 \partial_{x_3})f = 0 \\ (e_3 \partial_{x_1} + e_4 \partial_{x_2} + e_1 \partial_{x_3})f = 0 \end{array} \right\}$$

where  $M = 4$ ,  $m = 3$ ,  $n = 3$  has the linear resolution

$$0 \longrightarrow R^{16}(-3) \longrightarrow R^{48}(-2) \longrightarrow R^{48}(-1) \longrightarrow R^{16} \longrightarrow \mathcal{M}_3 \longrightarrow 0. \quad (6)$$

Finally, the system

$$\left. \begin{array}{l} (e_1 \partial_{x_1} + e_2 \partial_{x_2} + e_3 \partial_{x_3})f = g_1 \\ (e_1 \partial_{x_2} + e_2 \partial_{x_3} + e_3 \partial_{x_4})f = g_2 \\ (e_1 \partial_{x_3} + e_2 \partial_{x_4} + e_3 \partial_{x_5})f = g_3 \end{array} \right\}$$

has  $M = 3$ ,  $m = 5$  and  $n = 3$ , so it does not satisfy the hypotheses of Theorem 3.3. Its resolution is neither linear nor Dirac like:

$$\begin{aligned} 0 \longrightarrow R^8(-5) \longrightarrow R^{24}(-4) \longrightarrow R^{16}(-2) \oplus R^{16}(-3) \\ \longrightarrow R^{24}(-1) \longrightarrow R^8 \longrightarrow \mathcal{M}_3 \longrightarrow 0. \end{aligned}$$

**3.14** Another system, that we treated in the case  $M = 6$  is

$$\left. \begin{aligned} (e_1\partial_{x_1} + e_2\partial_{x_2} + e_3\partial_{x_3} + e_4\partial_{x_4})f &= g_1 \\ (e_3\partial_{x_1} + e_4\partial_{x_2} + e_5\partial_{x_3} + e_6\partial_{x_4})f &= g_2 \\ (e_5\partial_{x_1} + e_6\partial_{x_2} + e_1\partial_{x_3} + e_2\partial_{x_4})f &= g_3 \end{aligned} \right\}$$

where  $n = 4$  and  $m = 4$ . Here we have used a kind of spinor formalism, so that the matrix associated to the each operator in the system can be obtained by making suitable formal substitutions in the  $8 \times 8$  matrix (7) given in [21], that is the matrix associated to the Dirac operator in  $\mathcal{C}_6$ . A computation with CoCoA provides the resolution

$$0 \longrightarrow R^4(-3) \longrightarrow R^{16}(-2) \longrightarrow R^{20}(-1) \longrightarrow R^8 \longrightarrow \mathcal{M}_3 \longrightarrow 0.$$

**Remark 3.5.** In this case, the fact that each operator contains two units less than the maximum allowed by the least over-all Clifford algebra, has a great influence on the resolution. Note that the module associated to the system has 24 generators but only 20 of them are independent, so that the minimal resolution starts with the module  $R^{20}$ . The minimality also implies that the number of rows in each matrix are less than the number one needs to form Clifford relations.

**3.2 Seiffen-type systems.** Now we turn our attention to systems of the type that we called "Seiffen type". The general setting in which we will work is the following (see Section 2): let  $\mathcal{C}_{m,p}$  be the Clifford algebra whose Clifford frame is given by

$$e_{1,l}, e_{2,l}, \dots, e_{m,l}, \quad (l = 1, \dots, p)$$

where for  $l$  fixed we have a basis of  $\mathcal{C}_m$ . Let us consider  $s$  different  $m$ -tuples and let  $f : \mathbb{R}^{sm} \rightarrow \mathcal{C}_{m,p}$ .

**3.21** Let first consider  $s = 2$  and  $p = 3$  and the system

$$\left. \begin{aligned} (\partial_{x.1} + \partial_{y.2})f &= g_1 \\ (\partial_{x.2} + \partial_{y.3})f &= g_2 \\ (\partial_{x.3} + \partial_{y.1})f &= g_3 \end{aligned} \right\}$$

where  $\partial_{x.i} = \sum_{k=1}^m e_{k,i}\partial_{x_k}$  and  $\partial_{y.i} = \sum_{k=1}^m e_{k,i}\partial_{y_k}$ . Note that every operator has a symbol matrix of size  $2^{3m} \times 2^{3m}$ . So, to reduce the size of the matrices

involved, we will restrict to the case  $m = 2$  and we rewrite the system using the spinor formalism as

$$\left. \begin{aligned} \partial_{x.1}f + \partial_{y.2}f &= g_1 \\ \partial_{x.2}f + if\partial_{y.1} &= g_2 \\ if\partial_{x.1} + \partial_{y.1}f &= g_3 \end{aligned} \right\}. \tag{7}$$

With those choices, we have that  $f : \mathbb{R}^4 \rightarrow \mathcal{C}_4 \oplus i\mathcal{C}_4$  and that the matrix symbol of the system is

$$\begin{bmatrix} V_1 & O \\ O & V_1 \\ V_2 & -W_2 \\ W_2 & V_2 \\ V_3 & -W_3 \\ W_3 & V_3 \end{bmatrix}$$

where each block  $V_i, W_i$  is a  $16 \times 16$  matrix. In particular,  $V_i$  ( $i = 1, 2, 3$ ) can be obtained (formally) from matrix (5) as follows: to get  $V_1$  put  $P_1 = x_1, P_2 = x_2, P_3 = y_1, P_4 = y_2$ ; to get  $V_2$  put  $P_j = 0$  for  $j = 1, 2$  and  $P_j = x_j$  for  $j = 3, 4$ ; finally,  $V_3$  is obtained by setting  $P_j = y_j$  for  $j = 1, 2$  and  $P_j = 0$  for  $j = 3, 4$ . The matrices of type  $W_i$  can be obtained by direct computation. The resolution is

$$0 \longrightarrow R^{16}(-3) \longrightarrow R^{64}(-2) \longrightarrow R^{80}(-1) \longrightarrow R^{32} \longrightarrow \mathcal{M} \longrightarrow 0.$$

**Remark 3.6.** Note that we have two linear syzygies at the first step and that the complex ends with one linear relation. This is what we already got for the system in subsection 3.14 since it is a particular case of systems in subsection 3.21 and can be obtained for  $m = 2$ .

**3.22** Another system, that is obtained by six Dirac operators compressed in three operators, is

$$\left. \begin{aligned} (\partial_{x.1} + \partial_{u.2})f &= g_1 \\ (\partial_{y.1} + \partial_{v.3})f &= g_2 \\ (\partial_{z.2} + \partial_{w.3})f &= g_3 \end{aligned} \right\}. \tag{8}$$

where  $s = 6$  and  $p = 3$ . In the case  $m = 2$  we have that  $f : \mathbb{R}^{12} \rightarrow \mathcal{C}_6$ , so it is convenient to rewrite the system using the spinor formalism i.e. in the form

$$\left. \begin{aligned} \partial_{x.1}f + \partial_{u.2}f &= g_1 \\ \partial_{y.1}f + if\partial_{v.1} &= g_2 \\ \partial_{z.2}f + if\partial_{w.1} &= g_3 \end{aligned} \right\}.$$

We have that  $f : \mathbb{R}^4 \rightarrow \mathcal{C}_4 \oplus i\mathcal{C}_4$  and the matrix symbol of the system can be obtained as in subsection 3.21, with suitable modifications. The resolution is

$$\begin{aligned} 0 \longrightarrow R^{32}(-9) \longrightarrow R^{96}(-8) \longrightarrow R^{256}(-6) \longrightarrow R^{192}(-4) \oplus R^{192}(-5) \\ \longrightarrow R^{256}(-3) \longrightarrow R^{96}(-1) \longrightarrow R^{32} \longrightarrow \mathcal{M} \longrightarrow 0. \end{aligned}$$

**Remark 3.7.** This resolution mimics the resolution we got in the case of three abstract vector variables (see [21]) since we have the same number of syzygies and the same degrees. This means that this system is a perfect model for the theory we have developed in [21].

**3.23** We next consider the case of  $s = 3$ ,  $p = 2$  and the system

$$\left. \begin{aligned} (\partial_{x.1} + \partial_{y.2})f &= g_1 \\ (\partial_{y.1} + \partial_{z.2})f &= g_2 \\ (\partial_{z.1} + \partial_{x.2})f &= g_3 \end{aligned} \right\}.$$

Since  $\mathcal{C}_{m.2} \cong \mathcal{C}_m \otimes \mathcal{C}_m$ , the operator matrix associated to the system is composed by three  $2^{2m} \times 2^{2m}$  matrices. We reduce their size by rewriting the system with the spinor formalism, so that it becomes

$$\left. \begin{aligned} \partial_x f + if\partial_y &= g_1 \\ \partial_y f + if\partial_z &= g_2 \\ \partial_z f + if\partial_x &= g_3 \end{aligned} \right\} \quad (9)$$

where  $f : \mathbb{R}^{3m} \rightarrow \mathcal{C}_m \otimes i\mathcal{C}_m$ . When  $m = 2$ , we have  $f : \mathbb{R}^6 \rightarrow \mathbb{H} \oplus i\mathbb{H}$  (where  $\mathbb{H}$  denotes the real algebra of quaternions),  $\partial_x = \sum_{i=1}^2 e_i \partial_{x_i}$ ,  $\partial_y = \sum_{i=1}^2 e_i \partial_{y_i}$ ,  $\partial_z = \sum_{i=1}^2 e_i \partial_{z_i}$  and the matrix symbol of the system is composed by three  $8 \times 8$  matrices of the type

$$\begin{bmatrix} A & B \\ -B & A \end{bmatrix} \quad (10)$$

where, for the first equation, we have

$$A = \begin{bmatrix} 0 & -x_1 & -x_2 & 0 \\ x_1 & 0 & 0 & x_2 \\ x_2 & 0 & 0 & -x_1 \\ 0 & -x_2 & x_1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & y_1 & y_2 & 0 \\ -y_1 & 0 & 0 & y_2 \\ -y_2 & 0 & 0 & -y_1 \\ 0 & -y_2 & y_1 & 0 \end{bmatrix}$$

while, in the other two cases, it suffices to substitute appropriately the monomials in  $A$  and  $B$ . In an analogous way, we can consider this system for  $m = 3$ . In this second case, every equation of the system is represented by a  $16 \times 16$  matrix of type (10) were  $A$  and  $B$  are suitable  $8 \times 8$  matrices that we do not write here for shortness.

This time we obtained two resolutions that are deeply different from the cases we have already treated:

**Proposition 3.8.** *System (9) has for  $m = 2$  the resolution*

$$\begin{aligned} 0 \longrightarrow R^8(-6) \longrightarrow R^{36}(-5) \longrightarrow R^4(-3) \oplus R^{60}(-4) \\ \longrightarrow R^{12}(-2) \oplus R^{40}(-3) \longrightarrow R^{24}(-1) \longrightarrow R^8 \longrightarrow \mathcal{M}_3 \longrightarrow 0 \end{aligned}$$

while for  $m = 3$  it has the resolution

$$\begin{aligned} 0 \longrightarrow R^{12}(-11) \longrightarrow R^8(-9) \oplus R^{84}(-10) \longrightarrow R^{96}(-8) \oplus R^{224}(-9) \\ \longrightarrow R^{16}(-6) \oplus R^{396}(-7) \oplus R^{276}(-8) \longrightarrow R^{72}(-5) \oplus R^{728}(-6) \\ \longrightarrow R^{156}(-7) \longrightarrow R^{120}(-4) \oplus R^{624}(-5) \oplus R^{36}(-6) \\ \longrightarrow R^{100}(-3) \oplus R^{204}(-4) \longrightarrow R^{48}(-1) \longrightarrow R^{16} \longrightarrow \mathcal{M}_3 \longrightarrow 0. \end{aligned}$$

**Remark 3.9.** Note that, in the two resolutions, the Betti numbers are not multiple of 8 or 16 as one can expect from a Clifford analysis point of view.

**3.24** Finally we consider the following system that is a Seiffen generalization of a combinatorial system, associated with a tetrahedron (see next Section):

$$\left. \begin{aligned} (\partial_{x.1} + \partial_{y.2} + \partial_{z.3})f &= g_1 \\ (\partial_{y.1} + \partial_{z.2} + \partial_{u.3})f &= g_2 \\ (\partial_{z.1} + \partial_{u.2} + \partial_{x.3})f &= g_3 \\ (\partial_{u.1} + \partial_{x.2} + \partial_{y.3})f &= g_4 \end{aligned} \right\} \quad (11)$$

where we have  $s = 4$ ,  $p = 3$  and  $f : \mathbb{R}^{4m} \rightarrow \mathcal{C}_{m.3}$ . Following the procedure already used in subsection 3.21, we rewrite the system using the spinor formalism and consider  $m = 2$  so that  $f : \mathbb{R}^8 \rightarrow \mathcal{C}_4 \oplus i\mathcal{C}_4$ . The matrix symbol of the system is formed by four  $32 \times 32$  matrices that can be obtained as in subsection 3.21. The resolution is

$$\begin{aligned} 0 \longrightarrow R^8(-10) \longrightarrow R^{80}(-9) \oplus R^{32}(-8) \longrightarrow R^{56}(-8) \oplus R^{480}(-7) \\ \longrightarrow R^{64}(-7) \oplus R^{1240}(-6) \longrightarrow R^{56}(-6) \oplus R^{1760}(-5) \\ \longrightarrow R^{32}(-5) \oplus R^{1408}(-4) \longrightarrow R^8(-4) \oplus R^{512}(-3) \oplus R^{72}(-2) \\ \longrightarrow R^{128}(-1) \longrightarrow R^{32} \longrightarrow \mathcal{M} \longrightarrow 0. \end{aligned}$$

**3.3 Combinatorial type.** In this subsection we will study some combinatorial type systems, according to the problem (P6) we put in the previous section. What we have shown is that there are some cases in which the resolution obtained has Betti numbers proportional to a suitable De Rham complex and same length. This phenomenon suggests that the synthesis operators of

those systems should behave like the Dirac operator. In other cases, we have proved that, by changing the signature of the system, i.e. by putting signs + or – in front of the operator  $\partial_{x_i} e_k$  appearing in each equation of the system, the resolution does not change. Another type of invariance is the invariance with respect to the coloring, that we expect for most of the systems we have considered, but possibly not for systems with more irregular underlying geometry. The final statement we have in mind is, so far, a conjecture: the resolution of a given system is related only to the geometry of the system and not to the coloring.

We start our discussion with the case in which the incidence structure is a design with a fixed number of points per line and a fixed number of lines per point; in the case every two lines have at most one point in common we obtain as examples the finite projective and affine planes over  $\mathbb{Z}_p$ ,  $p$  prime.

Let us begin with the most famous classical example of the Fano plane:

**3.31** The Fano plane is the smallest projective plane with 7 points, 7 lines, 3 points per line and 3 lines per point whereby every two lines have exactly one point in common. It is the projective plane over  $\mathbb{Z}_2$ . In this case, minimal combinatorial Dirac systems require only 3 Clifford basis elements and an example is given by

$$\left. \begin{aligned} (e_1 \partial_{x_1} + e_2 \partial_{x_6} + e_3 \partial_{x_2})f &= g_1 \\ (e_1 \partial_{x_2} + e_2 \partial_{x_4} + e_3 \partial_{x_3})f &= g_2 \\ (e_1 \partial_{x_5} + e_2 \partial_{x_3} + e_3 \partial_{x_1})f &= g_3 \\ (e_1 \partial_{x_7} + e_2 \partial_{x_2} + e_3 \partial_{x_5})f &= g_4 \\ (e_1 \partial_{x_3} + e_2 \partial_{x_7} + e_3 \partial_{x_6})f &= g_5 \\ (e_1 \partial_{x_4} + e_2 \partial_{x_1} + e_3 \partial_{x_7})f &= g_6 \\ (e_1 \partial_{x_6} + e_2 \partial_{x_5} + e_3 \partial_{x_4})f &= g_7 \end{aligned} \right\} . \quad (12)$$

Every equation appearing in the system can be associated to a  $8 \times 8$  matrix similar to (21) by substituting (formally) the  $P_i$  with the variables  $x_j$ . For example, the matrix symbol of the first operator can be obtained by setting  $P_1 = x_1$ ,  $P_2 = x_6$ ,  $P_3 = x_2$ . The matrix associated to the system is then a  $56 \times 8$  matrix in the monomials  $x_1, \dots, x_7$  and its resolution is contained in the following

**Proposition 3.10.** *The resolution of system (12) is*

$$\begin{aligned} 0 &\longrightarrow R^8(-7) \longrightarrow R^{56}(-6) \longrightarrow R^{168}(-5) \longrightarrow R^{280}(-4) \\ &\longrightarrow R^{280}(-3) \longrightarrow R^{168}(-2) \longrightarrow R^{56}(-1) \longrightarrow R^8 \longrightarrow \mathcal{M}_7 \longrightarrow 0. \end{aligned}$$

Moreover, the resolution is invariant with respect to changes of signature of the terms in (12).

**Proof.** The resolution was found by CoCoA and the invariance up to signature of the terms was proved by using coefficients chosen randomly in  $\{-1, +1\}$  ■

**Remark 3.11.** The resolution of system (12) has Betti numbers proportional to those ones appearing in the standard De Rham complex  $\partial_{x_1}, \dots, \partial_{x_7}$  and the same length.

**3.32** The affine geometry  $\mathbb{Z}_3 \times \mathbb{Z}_3 \setminus \{(0, 0)\}$  has 8 points and 8 lines in total and 3 points per line and 3 lines through each point such that any two lines have at most one point in common, while through every point outside a given line goes exactly one parallel line. Also here three Clifford basis elements are sufficient to produce the combinatorial system:

$$\left. \begin{aligned} (e_1\partial_{x_1} + e_2\partial_{x_2} + e_3\partial_{x_3})f &= g_1 \\ (e_1\partial_{x_4} + e_2\partial_{x_5} + e_3\partial_{x_6})f &= g_2 \\ (e_1\partial_{x_8} + e_2\partial_{x_1} + e_3\partial_{x_4})f &= g_3 \\ (e_1\partial_{x_6} + e_2\partial_{x_3} + e_3\partial_{x_7})f &= g_4 \\ (e_1\partial_{x_5} + e_2\partial_{x_7} + e_3\partial_{x_1})f &= g_5 \\ (e_1\partial_{x_3} + e_2\partial_{x_8} + e_3\partial_{x_5})f &= g_6 \\ (e_1\partial_{x_7} + e_2\partial_{x_4} + e_3\partial_{x_2})f &= g_7 \\ (e_1\partial_{x_2} + e_2\partial_{x_6} + e_3\partial_{x_8})f &= g_8 \end{aligned} \right\} . \tag{13}$$

Each operator appearing in system (13) has been applied to functions  $f : \mathbb{R}^8 \rightarrow \mathcal{C}_3$ , so that the associated matrix is obtained by (21) via suitable substitutions and the matrix of the system is of size  $64 \times 8$ . The resolution is contained is the following

**Proposition 3.12.** *The resolution of system (13) is*

$$\begin{aligned} 0 \longrightarrow R^8(-8) \longrightarrow R^{64}(-7) \longrightarrow R^{224}(-6) \longrightarrow R^{448}(-5) \longrightarrow R^{560}(-4) \\ \longrightarrow R^{448}(-3) \longrightarrow R^{224}(-2) \longrightarrow R^{64}(-1) \longrightarrow R^8 \longrightarrow \mathcal{M}_8 \longrightarrow 0. \end{aligned}$$

*It is De Rham like and invariant with respect to changes of signature of the terms.*

**3.33** Consider 10 points and 10 lines such that for every 3 points passes a line and every point is the intersection of 3 lines (Desargues configuration). A

system we obtained is

$$\left. \begin{aligned} (e_1\partial_{x_1} + e_2\partial_{x_2} + e_3\partial_{x_7})f &= g_1 \\ (e_1\partial_{x_9} + e_2\partial_{x_1} + e_3\partial_{x_3})f &= g_2 \\ (e_1\partial_{x_8} + e_2\partial_{x_3} + e_3\partial_{x_2})f &= g_3 \\ (e_1\partial_{x_4} + e_2\partial_{x_7} + e_3\partial_{x_5})f &= g_4 \\ (e_1\partial_{x_6} + e_2\partial_{x_9} + e_3\partial_{x_4})f &= g_5 \\ (e_1\partial_{x_5} + e_2\partial_{x_6} + e_3\partial_{x_8})f &= g_6 \\ (e_1\partial_{x_{10}} + e_2\partial_{x_4} + e_3\partial_{x_1})f &= g_7 \\ (e_1\partial_{x_2} + e_2\partial_{x_5} + e_3\partial_{x_{10}})f &= g_8 \\ (e_1\partial_{x_3} + e_2\partial_{x_{10}} + e_3\partial_{x_6})f &= g_9 \\ (e_1\partial_{x_7} + e_2\partial_{x_8} + e_3\partial_{x_9})f &= g_{10} \end{aligned} \right\}. \quad (14)$$

**Proposition 3.13.** *The resolution of system (14) is*

$$\begin{aligned} 0 &\longrightarrow R^8(-10) \longrightarrow R^{80}(-9) \longrightarrow R^{360}(-8) \longrightarrow R^{960}(-7) \\ &\longrightarrow R^{1680}(-6) \longrightarrow R^{2016}(-5) \longrightarrow R^{1680}(-4) \longrightarrow R^{960}(-3) \longrightarrow R^{360}(-2) \\ &\longrightarrow R^{80}(-1) \longrightarrow R^8 \longrightarrow \mathcal{M}_{10} \longrightarrow 0. \end{aligned}$$

*It is De Rham like and invariant with respect to changes of signature of the terms.*

**3.34** We now consider the system given by the design with 13 points, 13 lines, 4 lines per each point, 4 points on each line. This is the projective plane over  $\mathbb{Z}_3$ . A possible system one obtains is

$$\left. \begin{aligned} (e_1\partial_{x_1} + e_2\partial_{x_2} + e_3\partial_{x_3} + e_4\partial_{x_{12}})f &= g_1 \\ (e_1\partial_{x_5} + e_2\partial_{x_6} + e_3\partial_{x_{12}} + e_4\partial_{x_4})f &= g_2 \\ (e_1\partial_{x_9} + e_2\partial_{x_{12}} + e_3\partial_{x_7} + e_4\partial_{x_8})f &= g_3 \\ (e_1\partial_{x_7} + e_2\partial_{x_4} + e_3\partial_{x_1} + e_4\partial_{x_{10}})f &= g_4 \\ (e_1\partial_{x_{10}} + e_2\partial_{x_8} + e_3\partial_{x_2} + e_4\partial_{x_5})f &= g_5 \\ (e_1\partial_{x_3} + e_2\partial_{x_{10}} + e_3\partial_{x_6} + e_4\partial_{x_9})f &= g_6 \\ (e_1\partial_{x_{11}} + e_2\partial_{x_3} + e_3\partial_{x_5} + e_4\partial_{x_7})f &= g_7 \\ (e_1\partial_{x_2} + e_2\partial_{x_9} + e_3\partial_{x_4} + e_4\partial_{x_{11}})f &= g_8 \\ (e_1\partial_{x_8} + e_2\partial_{x_1} + e_3\partial_{x_{11}} + e_4\partial_{x_6})f &= g_9 \\ (e_1\partial_{x_6} + e_2\partial_{x_7} + e_3\partial_{x_{13}} + e_4\partial_{x_2})f &= g_{10} \\ (e_1\partial_{x_{13}} + e_2\partial_{x_5} + e_3\partial_{x_9} + e_4\partial_{x_1})f &= g_{11} \\ (e_1\partial_{x_4} + e_2\partial_{x_{13}} + e_3\partial_{x_8} + e_4\partial_{x_3})f &= g_{12} \\ (e_1\partial_{x_{12}} + e_2\partial_{x_{11}} + e_3\partial_{x_{10}} + e_4\partial_{x_{13}})f &= g_{13} \end{aligned} \right\}. \quad (15)$$



The matrix symbol of the system is composed by 13 blocks each obtained from (5) by suitable substitutions. The module  $M$  generated by the rows of this matrix has 208 generators and CoCoA could not compute the whole resolution. We then tried to obtain, at least partially, the resolution by computing the first  $N$  syzygies by giving the following commands: `GB.Start_Res(M)`; `GB.Steps(M,N)`; `GB.GetBettiNumbers(M)`: the system will display a table with the Betti numbers and the degree of the syzygies found; what one obtains is

**Proposition 3.14.** *System (15) has 78 linear syzygies at the first step, then 286 linear syzygies and at least 100 linear syzygies at the third step.*

**Remark 3.15.** Since the first two steps suggest that the resolution is De Rham like, we tried to compute the resolution for the complex  $\partial_{x_1}, \dots, \partial_{x_{13}}$ . Unfortunately, also in this case it was not possible to get the whole resolution but, with the same procedure illustrated above, we have computed at least some steps. The two complexes both have 78, then 286 linear syzygies and the second has at least 224 linear syzygies at the third step, so they coincide at least at the first two steps.

It is then natural to conjecture that the resolutions coincide entirely, so that we can make the following:

**Conjecture 3.16.** *The resolution of system (15) is De Rham like and coincide with*

$$\begin{aligned}
 0 \longrightarrow R^{16}(-13) &\longrightarrow R^{208}(-12) \longrightarrow R^{1248}(-11) \longrightarrow R^{4576}(-10) \\
 &\longrightarrow R^{12480}(-9) \longrightarrow 0 \longrightarrow R^{20592}(-8) \longrightarrow R^{27456}(-7) \\
 &\longrightarrow R^{27456}(-6) \longrightarrow R^{20592}(-5) \longrightarrow R^{12480}(-4) \longrightarrow R^{4576}(-3) \\
 &\longrightarrow R^{1248}(-2) \longrightarrow R^{208}(-1) \longrightarrow R^{16} \longrightarrow \mathcal{M}_8 \longrightarrow 0.
 \end{aligned}$$

**3.35** Another interesting system that can be obtained with 4 points and 6 lines is

$$\left. \begin{aligned}
 (e_1\partial_{x_1} + e_2\partial_{x_2})f &= g_1 \\
 (e_1\partial_{x_3} + e_2\partial_{x_4})f &= g_2 \\
 (e_2\partial_{x_1} + e_3\partial_{x_3})f &= g_3 \\
 (e_1\partial_{x_2} + e_3\partial_{x_4})f &= g_4 \\
 (e_1\partial_{x_4} + e_3\partial_{x_1})f &= g_5 \\
 (e_2\partial_{x_3} + e_3\partial_{x_2})f &= g_6
 \end{aligned} \right\} \tag{16}$$

where we have considered  $f : \mathbb{R}^4 \rightarrow \mathcal{C}_3$ . The matrix associated to each of the operators appearing in the system can be obtained by suitable substitutions in the matrix (21), so that the size of the matrix associated to the system

is  $48 \times 8$ . This system contains two equations that can be eliminated (for example the last two), as it appears clear from the resolution we have found that is contained in the following

**Proposition 3.17.** *System (16) has the De Rham like resolution*

$$0 \longrightarrow R^8(-4) \longrightarrow R^{32}(-3) \longrightarrow R^{48}(-2) \longrightarrow R^{32}(-1) \longrightarrow R^8 \longrightarrow \mathcal{M} \longrightarrow 0$$

that is invariant with respect to changes of signature of the terms.

The dual system of (16) (where dual means in the geometric sense) is

$$\left. \begin{aligned} (e_1 \partial_{x_1} + e_2 \partial_{x_6} + e_3 \partial_{x_2})f &= g_1 \\ (e_1 \partial_{x_3} + e_2 \partial_{x_2} + e_3 \partial_{x_4})f &= g_2 \\ (e_1 \partial_{x_5} + e_2 \partial_{x_1} + e_3 \partial_{x_3})f &= g_3 \\ (e_1 \partial_{x_4} + e_2 \partial_{x_5} + e_3 \partial_{x_6})f &= g_4 \end{aligned} \right\} \quad (17)$$

and, already at the beginning, does contain only four equations. Below we write its resolution that is quite different from the resolution we got above:

$$\begin{aligned} 0 \longrightarrow R^{16}(-7) \longrightarrow R^{88}(-6) \longrightarrow R^{192}(-5) \longrightarrow R^{200}(-4) \\ \longrightarrow R^{24}(-2) \oplus R^{80}(-3) \longrightarrow R^{32}(-1) \longrightarrow R^8 \longrightarrow \mathcal{M} \longrightarrow 0. \end{aligned}$$

For the finite projective planes (with 7 or 13 points) there are only little possibilities of choosing the labels of the Clifford generators  $e_i$  or the indices of the variables  $x_j$  in each equation while in this case there can be many different choices of the labels, possibly leading to different resolutions. So we tried to produce the same type of system by changing the labels of the indices of  $e_i$  and  $x_j$ . For example, we tried the following two systems (we are writing here, for shortness, only the indices  $j$  of the  $\partial_{x_j}$  that are multiplied by  $e_1, e_2, e_3$ , respectively): system (1, 4, 3), (3, 6, 2), (2, 1, 5), (4, 5, 6) and the system obtained from the previous one by substituting the first 3-tuple by (1, 3, 4). Both gave the same resolution as above, so it is natural to conjecture that the system is dependent only on the geometry and not on the other choices like colors and signature.

**3.36** We finally tried some cases of Platonic bodies. Consider first a tetrahedron: it has 4 vertices and 4 faces. Every vertex belongs to 3 faces and every face has 3 vertices. A system associated to it can be written as

$$\left. \begin{aligned} (e_1 \partial_{x_1} + e_2 \partial_{x_2} + e_3 \partial_{x_3})f &= g_1 \\ (e_1 \partial_{x_2} + e_2 \partial_{x_1} + e_3 \partial_{x_4})f &= g_2 \\ (e_1 \partial_{x_3} + e_2 \partial_{x_4} + e_3 \partial_{x_2})f &= g_3 \\ (e_1 \partial_{x_4} + e_2 \partial_{x_3} + e_3 \partial_{x_1})f &= g_4 \end{aligned} \right\} \quad (18)$$

Consider now a cube and its 8 vertices and 6 faces. Every vertex belongs to 3 faces and every 3 faces intersect at most in one point. A system describing it is the following

$$\left. \begin{aligned} (e_1\partial_{x_1} + e_2\partial_{x_2} + e_3\partial_{x_3} + e_4\partial_{x_4})f &= g_1 \\ (e_1\partial_{x_7} + e_2\partial_{x_8} + e_3\partial_{x_5} + e_4\partial_{x_6})f &= g_2 \\ (e_1\partial_{x_5} + e_2\partial_{x_6} + e_3\partial_{x_1} + e_4\partial_{x_2})f &= g_3 \\ (e_1\partial_{x_8} + e_2\partial_{x_7} + e_3\partial_{x_4} + e_4\partial_{x_3})f &= g_4 \\ (e_1\partial_{x_3} + e_2\partial_{x_5} + e_3\partial_{x_7} + e_4\partial_{x_1})f &= g_5 \\ (e_1\partial_{x_6} + e_2\partial_{x_4} + e_3\partial_{x_2} + e_4\partial_{x_8})f &= g_6 \end{aligned} \right\}. \quad (19)$$

Finally, we consider the octahedron that has 6 vertices and 8 faces. Every face has 3 vertices and every vertex belongs to 4 faces. A system associated to it is

$$\left. \begin{aligned} (e_1\partial_{x_1} + e_2\partial_{x_2} + e_3\partial_{x_5})f &= g_1 \\ (e_1\partial_{x_2} + e_2\partial_{x_1} + e_3\partial_{x_3})f &= g_2 \\ (e_1\partial_{x_4} + e_3\partial_{x_1} + e_4\partial_{x_3})f &= g_3 \\ (e_2\partial_{x_5} + e_3\partial_{x_4} + e_4\partial_{x_1})f &= g_4 \\ (e_1\partial_{x_6} + e_3\partial_{x_2} + e_4\partial_{x_5})f &= g_5 \\ (e_1\partial_{x_3} + e_2\partial_{x_6} + e_4\partial_{x_2})f &= g_6 \\ (e_2\partial_{x_3} + e_3\partial_{x_6} + e_4\partial_{x_4})f &= g_7 \\ (e_1\partial_{x_5} + e_2\partial_{x_4} + e_4\partial_{x_6})f &= g_8 \end{aligned} \right\}. \quad (20)$$

**Proposition 3.18.** *Systems (18) – (20) have the resolutions*

$$\begin{aligned} 0 &\longrightarrow R^8(-4) \longrightarrow R^{32}(-3) \longrightarrow R^{48}(-2) \longrightarrow R^{32}(-1) \longrightarrow R^8 \longrightarrow \mathcal{M} \longrightarrow 0 \\ 0 &\longrightarrow R^{32}(-9) \longrightarrow R^{240}(-8) \longrightarrow R^{768}(-7) \longrightarrow R^{1344}(-6) \\ &\longrightarrow R^{48}(-4) \oplus R^{1344}(-5) \longrightarrow R^{160}(-3) \oplus R^{720}(-4) \\ &\longrightarrow R^{192}(-2) \oplus R^{160}(-3) \longrightarrow R^{96}(-1) \longrightarrow R^{16} \longrightarrow \mathcal{M} \longrightarrow 0 \\ 0 &\longrightarrow R^{16}(-6) \longrightarrow R^{96}(-5) \longrightarrow R^{240}(-4) \longrightarrow R^{320}(-3) \\ &\longrightarrow R^{240}(-2) \longrightarrow R^{96}(-1) \longrightarrow R^{16} \longrightarrow \mathcal{M} \longrightarrow 0, \end{aligned}$$

*respectively.*

**Remark 3.19.** Systems (18) and (20) are De Rham like while system (19) associated to the cube has a more complicated structure.

**3.37** Note that, with the exception of systems in subsections 3.35 and 3.36, all the other are self-dual in the geometric sense. So it is not interesting to

produce the systems coming from this kind of duality. More significant are the systems that can be written by taking the super-dual of the systems considered. Following what we have written in Section 2, it suffices to rewrite the systems by changing, formally,  $\partial_{x_i}$  with  $e_i$ . For sake of shortness, we will not write the super-dual system obtained in this way. Some of the systems obtained all produce the same resolution, as described in the following propositions.

**Proposition 3.20.** *The super-dual of systems (12) (Fano plane), (13), (14) (Desargues) and (16) – (17) have the same De Rham like resolution with respect to three operators, so they have 3 linear first syzygies, then other 3 and 1 linear syzygies.*

**Proof.** Once that one has written the super-dual system in the various cases, one has to find the associated matrix. In the cases of subsections 3.31 and 3.32, the matrix is made respectively by 7 and 8 blocks of size  $16 \times 16$  coming from the symbol of the Dirac operator in  $\mathcal{C}_8$  via suitable substitutions. The resolution is

$$0 \longrightarrow R^{16}(-3) \longrightarrow R^{48}(-2) \longrightarrow R^{48}(-1) \longrightarrow R^{16} \longrightarrow \mathcal{M} \longrightarrow 0.$$

In the case of subsection 3.35 the matrix consists of 4 blocks again  $16 \times 16$ , coming from suitable substitutions in the matrix associated to the Dirac operator in  $\mathcal{C}_4$ ; the resolution obtained is the same as above. Finally, in the case of subsection 3.33 the matrix symbol of the system is made by 10 blocks of size  $32 \times 32$  that can be obtained applying the spinor formalism to the original equations that are  $\mathcal{C}_{10}$ -valued. After the reduction, each equation can be written in  $\mathcal{C}_4 \oplus i\mathcal{C}_4$  and once one has the matrix symbol one can produce with CoCoA the resolution that is

$$0 \longrightarrow R^{32}(-3) \longrightarrow R^{96}(-2) \longrightarrow R^{96}(-1) \longrightarrow R^{32} \longrightarrow \mathcal{M} \longrightarrow 0.$$

This finishes the proof ■

**Proposition 3.21** *The super-dual of systems (18) – (20) have the resolutions*

$$\begin{aligned} 0 &\longrightarrow R^8(-4) \longrightarrow R^{32}(-3) \longrightarrow R^{48}(-2) \longrightarrow R^{32}(-1) \longrightarrow R^{16} \longrightarrow \mathcal{M} \longrightarrow 0 \\ 0 &\longrightarrow R^{16}(-4) \longrightarrow R^{64}(-3) \longrightarrow R^{96}(-2) \longrightarrow R^{64}(-1) \longrightarrow R^{16} \longrightarrow \mathcal{M} \longrightarrow 0 \\ 0 &\longrightarrow R^{16}(-6) \longrightarrow R^{96}(-5) \longrightarrow R^{240}(-4) \longrightarrow R^{320}(-3) \\ &\longrightarrow R^{240}(-2) \longrightarrow R^{96}(-1) \longrightarrow R^8 \longrightarrow \mathcal{M} \longrightarrow 0, \end{aligned}$$

respectively. They are De Rham like.

**Proof.** The first resolution can be obtained by associating to the system a suitable matrix built with blocks of the type (5). The second system is associated to a matrix built with 6 blocks of size  $16 \times 16$  that can be obtained from the matrix associated to the Dirac operator in  $\mathcal{C}_8$  with suitable substitutions (see [21]). The matrix symbol of the symbol has 96 rows of which only 64 are independent. The last system is associated to a matrix obtained with blocks of size  $8 \times 8$  coming from the matrix representing the Dirac operator in  $\mathcal{C}_6$ . All the resolutions was obtained with CoCoA ■

**Open problems.** We conclude this section with a list of open problems and conjectures.

We have shown that there are cases in which the Betti numbers appearing in the resolutions cannot be divided by the dimension of the Clifford algebra considered. This happens when e.g. the rows of the matrix symbol of the system do not form a minimal set of generators of the module. Obviously, the minimal resolution calculated by CoCoA contains a number of relations that is not enough to express them in terms of Clifford algebra relations. The first question we address is the following:

- (Q1) When can the resolution of a Clifford complex be expressed in terms of operators with values in a Clifford algebra?

The next questions are the following:

- (Q2) How many inequivalent colorings are there per incidence geometry? (Inequivalent means, that they cannot be obtained from one another by permuting the colors.)
- (Q3) To what extent does the resolution depend on coloring or signature?

A certain subgroup of the permutation group of the points leaves the geometry invariant, so we wonder if there is a proper permutation of the colors such that the Turkish systems remain invariant. In other words, we ask

- (Q4) What is the group invariance of systems of combinatorial type?

At last, we ask:

- (Q5) When is the resolution proportional to a De Rham complex?

We conjecture that this happens in the case of all affine and projective geometries over  $\mathbb{Z}_p$ ,  $p$  prime, and also in the case of self-dual designs and for designs in which the number of lines is not less than the number of points. We also point out that from the idea of De Rham complex can arise a new type of finite geometries.

## 4. Appendix

In this Appendix we have listed the matrices  $A$ ,  $B$ ,  $C$ ,  $U^i$  appearing in Subsections 3.12, 3.13 and 3.31.

$$\begin{aligned}
 A &= \begin{bmatrix} 0 & -P_1 & -P_2 & -P_3 & -P_4 & 0 & 0 & 0 \\ P_1 & 0 & 0 & 0 & 0 & P_2 & P_3 & P_4 \\ P_2 & 0 & 0 & 0 & 0 & -P_1 & 0 & 0 \\ P_3 & 0 & 0 & 0 & 0 & 0 & -P_1 & 0 \\ P_4 & 0 & 0 & 0 & 0 & 0 & 0 & -P_1 \\ 0 & -P_2 & P_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -P_3 & 0 & P_1 & 0 & 0 & 0 & 0 \\ 0 & -P_4 & 0 & 0 & P_1 & 0 & 0 & 0 \end{bmatrix} \\
 B &= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ P_3 & P_4 & 0 & 0 & 0 & 0 & 0 & 0 \\ -P_2 & 0 & P_4 & 0 & 0 & 0 & 0 & 0 \\ 0 & -P_2 & -P_3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -P_3 & -P_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & P_2 & 0 & -P_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & P_2 & P_3 & 0 & 0 \end{bmatrix} \\
 C &= \begin{bmatrix} 0 & 0 & 0 & -P_1 & 0 & 0 & -P_4 & 0 \\ 0 & 0 & 0 & 0 & -P_1 & 0 & P_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & -P_1 & -P_2 & 0 \\ P_1 & 0 & 0 & 0 & 0 & 0 & 0 & P_4 \\ 0 & P_1 & 0 & 0 & 0 & 0 & 0 & -P_3 \\ 0 & 0 & P_1 & 0 & 0 & 0 & 0 & P_2 \\ P_4 & -P_3 & P_2 & 0 & 0 & 0 & 0 & -P_1 \\ 0 & 0 & 0 & -P_4 & P_3 & -P_2 & P_1 & 0 \end{bmatrix} \\
 U^i &= \begin{bmatrix} 0 & P_1^i & P_2^i & P_3^i & 0 & 0 & 0 & 0 \\ P_1^i & 0 & 0 & 0 & P_2^i & P_3^i & 0 & 0 \\ P_2^i & 0 & 0 & 0 & -P_1^i & 0 & P_3^i & 0 \\ P_3^i & 0 & 0 & 0 & 0 & -P_1^i & -P_2^i & 0 \\ 0 & -P_2^i & P_1^i & 0 & 0 & 0 & 0 & -P_3^i \\ 0 & -P_3^i & 0 & P_1^i & 0 & 0 & 0 & P_2^i \\ 0 & 0 & -P_3^i & P_2^i & 0 & 0 & 0 & -P_1^i \\ 0 & 0 & 0 & 0 & P_3^i & -P_2^i & P_1^i & 0 \end{bmatrix}
 \end{aligned} \tag{21}$$

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