

A Transmission Problem with a Fractal Interface

M. R. Lancia

Abstract. In this paper we study a transmission problem with a fractal interface K , where a second order transmission condition is imposed. We consider the case in which the interface K is the Koch curve and we prove existence and uniqueness of the weak solution of the problem in $V(\Omega, K)$, a suitable "energy space". The link between the variational formulation and the problem is possible once we recover a version of the Gauss-Green formula for fractal boundaries, hence a definition of "normal derivative".

Keywords: *Fractal boundaries, Dirichlet forms, transmission problems*

AMS subject classification: 31C25, 35J20, 78M30, 28A20

1. Introduction

In this paper we study some properties of the solution of a transmission problem with a fractal layer. In particular, we look for weak solutions in $V(\Omega, K)$, a suitable space to be defined, of the transmission problem formally stated as

$$\left. \begin{array}{ll} -\Delta u = g & \text{in } \Omega_i \ (i = 1, 2) \\ C\Delta_K u = \left[\frac{\partial u}{\partial n}\right]_K & \text{in } K \setminus \{A, B\} \\ u = 0 & \text{on } \partial\Omega, [u] = 0 \text{ across } K \end{array} \right\} \quad (1.1)$$

where Ω is a bounded open set in \mathbb{R}^2 with regular or Lipschitz boundary (for instance, we can think $\Omega = (0, 1) \times (-1, 1)$) and g is a given function in $L^2(\Omega)$. We assume the layer K to be a fractal curve and the set Ω to be divided into two subsets Ω_1 and Ω_2 such that $K = \partial\Omega_1 \cap \partial\Omega_2$, with the result that $\Omega = \Omega_1 \cup \Omega_2 \cup K$. By A and B we denote the points $(0, 0)$ and $(1, 0)$, respectively, where K intersects $\partial\Omega$. Just to fix the ideas, we choose K to be the Koch curve throughout the paper and we denote by D_f its fractal dimension.

M. R. Lancia: Univ. di Roma "La Sapienza", Dip. di Met. e Mod. Mat. per le Sci. Appl., Via A. Scarpa 10, 00161 Roma, Italia; lancia@dmmm.uniroma1.it

Further, Δ denotes the Laplace operator in \mathbb{R}^2 , Δ_K is the Laplace operator defined on the layer K (see Subsection 3.2), the functions u_i are the restriction of u to Ω_i ($i = 1, 2$), $u_i|_K$ denotes the "trace" of u_i to K according to Definition 2.1, and $[u] = u_1|_K - u_2|_K$. We denote by n_i the "outward normal vector" to Ω_i so that $\frac{\partial u_i}{\partial n_i}$ ($i = 1, 2$) (as it will be defined in (4.20) below) denotes the trace to K of "the normal derivative" of u_i . The term $[\frac{\partial u}{\partial n}]_K = (\frac{\partial u_1}{\partial n_1} + \frac{\partial u_2}{\partial n_2})$, classically, denotes the jump of the trace of the normal derivative across K (here it will be necessary to establish in which sense they must be intended), and $C > 0$ is a physical constant.

We point out that the transmission condition on the layer K is a second order condition. Namely, the operator which is involved in the transmission condition is a second order operator. Such a condition appears naturally in electrostatics or magnetostatics: in these cases the constant C represents the dielectric constant or the magnetic permeability, respectively. In the mathematical literature, there are many papers dealing with transmission problems, always assuming the interface K to be a regular curve or surface, and with different transmission conditions (for a complete list of references see [27]). The classical case (see [27]), the case of smooth interface, is the combination of two elliptic boundary value problems in a domain $\Omega \subset \mathbb{R}^3$ and in a domain $K \subset \mathbb{R}^2$. The two problems are coupled via the transmission condition.

Our purpose in this paper is to consider the case of K a fractal interface. The transmission condition puts in relation two different roles that a fractal set may have from the point of view of partial differential equations. Indeed, in problem (1.1), K occurs, on one side, as the boundary of the (Euclidean) domains Ω_1 and Ω_2 , and also, on the other side, as an intrinsic body supporting a suitable Laplace operator. This double role of K in problem (1.1) is indeed the main feature or interest of the present transmission problem.

In Section 2 we recall the definition and the properties of the Koch curve and of some relevant functional spaces which will be used.

In Section 3 we define the energy form and the Laplace operator on the Koch curve K . The construction of the energy form E and the related Laplace operator on the Koch curve K follows the by now standard constructions given in [10, 18, 19] for the Sierpinski gasket and the more general class of nested fractals. The form E turns out to be a non-trivial closed Dirichlet form which is regular and strongly local in the space $L^2(K, \mu)$. The Laplacian on K is the operator associated to the energy form E (see (3.6) and (3.7)).

In Section 4 we will consider the trace space on the Koch curve of $H^1(\Omega)$ -functions and we will obtain a Green formula for domains with a fractal boundary. These trace spaces are a particular case of some more general spaces which have been investigated by Jonsson and Wallin in [16] and by Triebel in [28]. Actually, their theory works for the class of the so-called D_f -sets (the Koch

curve is indeed a D_f set). A D_f -set, roughly speaking, is a set on which a doubling measure is supported. More precisely, for $u \in H^1(\Omega)$ the trace of u on the Koch curve K is in the Besov space $B_{\beta}^{2,2}(K)$, with β equal to $\frac{D_f}{2}$.

We then introduce the dual of the space $B_{\beta}^{2,2}(K)$, $\beta = \frac{D_f}{2}$, for the Koch curve which, as shown by Jonsson and Wallin [17], coincides with the space $B_{-\beta}^{2,2}(K)$ – a subspace of Schwartz distributions supported on K . Finally, we can give Green’s formula to deal with boundary value problems with fractal boundary. This will allow us, by duality arguments, to define the trace of the ”normal derivative” as an element of the dual of the Besov space $B_{\beta,0}^{2,2}(K)$.

In Section 5 we state a variational principle for problem (1.1) (see Theorem 5.2). We prove existence and uniqueness of the minimum in $V(\Omega, K)$ (a suitable Hilbert space which is a sort of ”energy space” defined both on Ω and on the layer K) of the energy functional

$$W_0[u] = \int_{\Omega} |Du|^2 dx_1 dx_2 + CE[u|_K] - 2 \int_{\Omega} gu dx_1 dx_2$$

where $V(\Omega, K) = \{u \in H_0^1(\Omega) | z = u|_K \in D_0\}$, D_0 is the ”energy space” of those functions vanishing on the boundary of K ” defined at the end of Section 3, and $u|_K$ denotes the trace of u to K . Above $L^2(\Omega)$ and $H_0^1(\Omega)$ denote the usual Lebesgue and Sobolev spaces on the open set Ω , respectively.

The link between the variational problem and problem (1.1) – i.e. a ”strong interpretation” of the trace of the normal derivative, hence of the transmission condition – is possible by the Green formula also for domains with fractal boundaries.

In section 6 we give a ”strong” interpretation of the problem by proving that the variational solution satisfies the transmission condition in the sense of the dual of $D_0(K)$ (see Theorem 6.2). If u were more regular, say $\Delta_K u \in L^2(K, \mu)$, i.e. $u \in D_{\Delta_K}$, then the transmission condition could be interpreted in the L^2 -sense. This problem, as far as we know, was still an open problem also in the case of the smooth layer considered in [27] and in the case of Lipschitz interface, such as the prefractal curve approximating the Koch curve, and it has been recently studied in [21].

As a final remark, we point out that Sections 2 and 3 and Subsection 4.1 contain many technical results – which we have recalled for completeness – obtained by adapting to the present problem more general results due respectively to Kusuoka, Fukushima, Mosco, Jonsson and Wallin. The principal result is the formulation of the transmission condition given in Theorem 6.2. This result requires the solution of some delicate problems, due to the presence of the fractal layer, which have been analyzed in Sections 4 - 6 (see, i.e., Proposition 4.8, Theorem 4.15 and Proposition 6.1).

2. Preliminaries

2.1 The Koch curve. Through the paper \mathbb{R}^D ($D \geq 2$) will denote the D -dimensional Euclidean space, $|x - y|$ the Euclidean distance, $B_e(x, r) = \{y \in \mathbb{R}^D : |x - y| < r\}$ ($x \in \mathbb{R}^D, r > 0$) are the Euclidean balls (denoted by B_r^e).

The Koch curve belongs to the class of so-called *nested fractals* introduced by Lindström [22], and it is obtained as follows [7]. Pose $A = (0, 0)$ and $B = (1, 0)$, and let $V_0 = \{A, B\}$. Consider the set of $N = 4$ contractive similitudes $\Psi = \{\psi_1, \dots, \psi_4\}$ with contraction factors $\alpha_i^{-1} = \alpha^{-1} = \frac{1}{3}$, i.e. $\psi_1 = \frac{z}{3}$, $\psi_2 = \frac{z}{3}e^{i\frac{\pi}{3}} + \frac{1}{3}$, $\psi_3 = \frac{z}{3}e^{-i\frac{\pi}{3}} + \frac{1}{2} + i\frac{\sqrt{3}}{6}$ and $\psi_4 = \frac{z+2}{3}$, where z denotes an element of \mathbb{C} . Set $I = [0, 1]$ and

$$\begin{aligned} K_1 &= \cup_{j=1}^4 \psi_j(I) \\ K_{h+1} &= \cup_{M \in K_h} \cup_{j=1}^4 \psi_j(M) \end{aligned} \quad (2.1)$$

where M denotes a segment of the " h -th" generation. The *Koch curve* K is the unique closed bounded set which is invariant under Ψ , that is $K = \psi(K) := \cup_{i=1}^4 \psi_i(K)$.

Further, $C_0(K)$ denotes the space of continuous functions with compact support on K . On the Koch curve K there exists an invariant measure μ [11], that is

$$\int_K \phi d\mu = \sum_{i=1}^4 \frac{1}{4} \int_K (\phi \circ \psi_i) d\mu \quad (\phi \in C_0(K))$$

which is given, after normalization, by the restriction to K of the D_f -dimensional Hausdorff measure of \mathbb{R}^2 , $H^{D_f} \llcorner K$ normalized:

$$\mu = (H^{D_f}(K))^{-1} H^{D_f} \llcorner K \quad (2.2)$$

where $D_f = \frac{\ln 4}{\ln 3}$. The measure μ has the property [7, 11] that there exists two constants $c_1 >$ and $c_2 > 0$ such that

$$c_1 r^{D_f} \leq \mu(B_e(x, r) \cap K) \leq c_2 r^{D_f} \quad (x \in K). \quad (2.3)$$

As μ is supported on K , it is not ambiguous to write in (2.3) $\mu(B_e(x, r))$. In the terminology of the following Section 4 we say that K is a D_f -set.

Let us go further, giving some more definitions which will help us later. For an arbitrary n -tuple of indices $i_1, \dots, i_n \in \{1, \dots, 4\}$ we define

$$\begin{aligned} \psi_{i_1 \dots i_n} &= \psi_{i_1} \circ \psi_{i_2} \circ \dots \circ \psi_{i_n} \\ K_{i_1 \dots i_n} &= \psi_{i_1 \dots i_n}(K) \end{aligned}$$

and we will call it an n -complex.

Figure 1: The curves K_h^l, K_h^r (dashed) and the prefractal curve K_h (solid)

Remark 2.1. $\Gamma = \{A, B\}$ is the boundary of K . We note that Γ coincides with the set F of essential fixed points of the given similitudes $\{\psi_1, \psi_2, \psi_3, \psi_4\}$ (see [22, 25]).

Remark 2.2. We note that K can be approximated also from above. The point $(\frac{1}{2}, \frac{\sqrt{3}}{6})$ divides K into two parts, the left K^l and the right K^r one, respectively. For each one of these one can consider the corresponding prefractal curves K_h^l and K_h^r – generated by the segments of endpoints $(0, 0), (\frac{1}{2}, \frac{\sqrt{3}}{6})$ and $(\frac{1}{2}, \frac{\sqrt{3}}{6}), (1, 0)$ – which approximate K^l and K^r from above, respectively.

2.2 Relevant functional spaces. Let Ω be an open set in \mathbb{R}^2 and K a compact subset such that $K \subset \bar{\Omega}$. We denote by $D(\mathbb{R}^2)$ the set of C^∞ -functions with compact support in \mathbb{R}^2 , by $D(\Omega)$ the set of C^∞ -functions with compact support in Ω , and by $D'(\mathbb{R}^2)$ and $D'(\Omega)$ the duals of $D(\mathbb{R}^2)$ and $D(\Omega)$, respectively. From now on $L^p(\Omega)$ denotes the usual Lebesgue space with respect to the two-dimensional Lebesgue measure. By $H^m(\Omega)$ ($m \in \mathbb{N}$) we denote the usual Sobolev space:

$$H^m(\Omega) = \left\{ u : \Omega \rightarrow \mathbb{R} \left| \begin{array}{l} u \in L^2(\Omega) \text{ and } D^\alpha u \in L^2(\Omega) \\ (|\alpha| \leq m) \text{ in the distributional sense} \end{array} \right. \right\}$$

equipped with the norm which we denote by $\|\cdot\|_m$:

$$\|u\|_m = \sum_{0 \leq |\alpha| \leq m} \|D^\alpha u\|_{L^2(\Omega)};$$

$H_0^1(\Omega)$ will denote the closure of $D(\Omega)$ with respect to the $\|u\|_1$ -norm [1]. By $Du = (u_{x_1}, u_{x_2})$ we denote the gradient of u , by $dy = dy_1 dy_2$ or $dx = dx_1 dx_2$ we denote the Lebesgue measure in \mathbb{R}^2 , and by $|\Omega|$ the Lebesgue measure of Ω . By $H_{loc}^2(\Omega)$ we denote the space of functions $u \in H^2(D)$ on every open set $D \subset\subset \Omega$.

Definition 2.1. Let Ω be an open set in \mathbb{R}^2 and let $f \in L^1(\Omega)$. We say that f can be *strictly defined* at $x \in \bar{\Omega}$ if the limit

$$\bar{f}(x) = \lim_{r \rightarrow 0} \frac{1}{|B_e(x, r) \cap \Omega|} \int_{B_e(x, r) \cap \Omega} f(y) dy \tag{2.4}$$

exists.

Remark 2.3. By the Lebesgue theorem any $f \in L^1(\Omega)$ can be strictly defined at every $x \in \bar{\Omega}$, except possibly a subset of two-dimensional Lebesgue measure zero, and $f = \bar{f}$ a.e. in Ω .

If we replace Ω by \mathbb{R}^2 in (2.4) and $f \in L_{loc}^1(\mathbb{R}^2)$, then f can be strictly defined at $x \in \mathbb{R}^2$ if the limit $\tilde{f}(x) = \lim_{r \rightarrow 0} \frac{1}{|B_e(x, r)|} \int_{B_e(x, r)} f(y) dy$ exists.

Theorem 2.2. *Let $f \in H^1(\mathbb{R}^2)$. Then f can be strictly defined at quasi-every $x \in \mathbb{R}^2$.*

The proof is based on the fact that every $f \in H^1(\mathbb{R}^2)$ has a (unique) quasi-continuous representative and can be strictly defined up to a set of Newtonian capacity null. Then by [1: Theorem 6.2.1] \tilde{f} exists quasi-everywhere, \tilde{f} is quasi-continuous and coincides with f quasi-everywhere.

Theorem 2.2 still holds if we replace \mathbb{R}^2 by Ω . More precisely, if $f \in H^1(\Omega)$, then f can be strictly defined at quasi-every $x \in \bar{\Omega}$.

Definition 2.3.

(i) For $f \in H^1(\mathbb{R}^2)$ we denote by $\gamma_{0,K,\mathbb{R}^2} f$ the trace of f on K : $\gamma_{0,K,\mathbb{R}^2} f := \blacksquare$
 \tilde{f} .

(ii) For $f \in H^1(\Omega)$ we denote by $\gamma_{0,K,\Omega} f$ the trace of f on K : $\gamma_{0,K,\Omega} f :=$
 \tilde{f} .

Thus \bar{f} and \tilde{f} are functions defined with respect to the usual Newtonian capacity.

3. The Laplacian on the Koch curve

3.1 The energy form. The construction of the energy form on the Koch curve K is based on suitable sequences of finite difference schemes. For every positive integer n we define

$$E^{(n)}(z, z) = \frac{1}{2}4^n \sum_{i_1, \dots, i_n=1}^4 \sum_{\substack{\xi, \eta \in F \\ \xi \neq \eta}} (z(\psi_{i_1 \dots i_n}(\xi)) - z(\psi_{i_1 \dots i_n}(\eta)))^2 \quad (3.1)$$

where the coefficient 4 is a renormalization factor. It is well known [19] that the limit of the right-hand side in (3.1) does exist, and the limit form

$$E(z, z) = \lim_{n \rightarrow \infty} E^{(n)}(z, z) \quad (3.2)$$

is non-trivial ($E \neq \infty$) for some class of z .

The form $E[z] = E(z, z)$ is a closed Dirichlet form which is regular and strongly local in $L^2(K, \mu)$, the Hilbert space of square summable functions on K with respect to the invariant measure μ , with dense domain D_E in $L^2(K, \mu)$,

$$D_E = \left\{ z : K \rightarrow \mathbb{R} \mid z \in L^2(K, \mu) \text{ and } E(z, z) < +\infty \right\}. \quad (3.3)$$

The regularity of the form $E(z, z)$ means that it possesses a core, a core being any subset C of $D_E \cap C_0(K)$, where $C_0(K) = \{z : K \rightarrow \mathbb{R} \mid z \in C(K) \text{ and } z = 0 \text{ on } \Gamma\}$, which is dense both in $C_0(K)$ with the uniform norm and in D_E with respect to the intrinsic norm

$$\|z\|_E = (E(z, z) + \|z\|_{L^2(K, \mu)}^2)^{\frac{1}{2}}. \quad (3.4)$$

This property implies that D_E is non-trivial (i.e. not only made by constant functions). Moreover, the functions in D_E are Hölder continuous on K :

Theorem 3.1. $D_E \subset C^{0, \beta}(K)$ where $\beta = \frac{\ln 4}{\ln 9}$.

In the following, let

$$C^{0, \beta}(K) = \left\{ z : K \rightarrow \mathbb{R} \mid |z(x) - z(y)| \leq M|x - y|^\beta \quad (x, y \in K) \right\}$$

denote the space of Hölder continuous functions on K . The proof of Theorem 3.1 is given in [20] as a consequence of the characterization of the functions in the domain D_E of the form, in terms of the so-called Lipschitz spaces. This characterization is analogous to that given in [15] for the Sierpinski gasket. A similar characterization, for the more general class of nested fractals (including the Koch curve) has been obtained independently in [34] and was further generalized in [32].

By Theorem 3.1, $z \in D_E$ implies $z \in C(K)$, thus we shall identify z with its continuous representative which will still be denoted by z . Thus the condition $z \in D_E$ and $z = 0$ on Γ (i.e. $z(A) = z(B) = 0$) has an obvious meaning. In the sequel we shall consider homogeneous Dirichlet conditions on the boundary Γ of K .

Definition 3.2. We introduce the subspace $D_0(K) = \{z \in D_E \mid z = 0 \text{ on } \Gamma\}$ and denote by $\|z\|_{D_0(K)} = (E(z, z))^{\frac{1}{2}}$ the norm in $D_0(K)$.

This is a closed subspace of D_E with respect to the intrinsic norm. It is non-empty because it contains the solution of the homogeneous Dirichlet problem for the Poisson equation on K [8]. The subspace $D_0(K)$ can be also characterized as the closure of the set $C_0(K \setminus \Gamma) \cap D_E$ with respect to the intrinsic norm. In fact, we have

Lemma 3.3. *The space $D_0(K)$ coincides with the closure of the set $C_0(K \setminus \Gamma) \cap D_E$ with respect to the intrinsic norm.*

In the following we shall use also the form $E(z, w)$ which is obtained from $E(z, z)$ by the polarization identity:

$$E(z, w) = \frac{1}{2} \left\{ E(z + w, z + w) - E(z, z) - E(w, w) \right\} \quad (z, w \in D_E). \quad (3.5)$$

3.2 The Laplacian on the Koch curve. We now define the Laplace operator on the fractal K with homogeneous Dirichlet boundary conditions. The form E with domain $D_0(K)$ is again a closed form in $L^2(K, \mu)$. Therefore, there exists a non-positive self-adjoint operator Δ_K in $L^2(K, \mu)$, with dense domain $D_{\Delta_K} \cap D_0(K)$ in $L^2(K, \mu)$, such that

$$E(z, w) = - \int_K (\Delta_K z) w \, d\mu \quad (3.6)$$

for $z \in D_{\Delta_K} \cap D_0(K)$ and for all $w \in D_0(K)$. Let $(D_0(K))'$ denote the dual of the space $D_0(K)$, i.e. the set of linear and continuous functionals on $D_0(K)$. We now introduce the Laplace operator on the fractal K as a variational operator from $D_0(K) \rightarrow (D_0(K))'$ by

$$E(z, w) = - \langle \Delta_K z, w \rangle_{(D_0(K))', D_0(K)} \quad (3.7)$$

for $z \in D_0(K)$ and for all $w \in D_0(K)$ where $\langle \cdot, \cdot \rangle_{(D_0(K))', D_0(K)}$ is the duality pairing between $(D_0(K))'$ and $D_0(K)$. We use the same symbol Δ_K to define the Laplace operator both as a selfadjoint operator in (3.6) and as a variational operator in (3.7). It will be clear from the context to which case we refer. We remark also that the two definitions given above have their analogous counterpart in the case of the Euclidean Laplacian. More precisely, one can define the Laplacian with homogeneous Dirichlet boundary conditions either as a self-adjoint operator with domain $H^2(\cdot) \cap H_0^1(\cdot)$ or as a variational operator from $H_0^1(\cdot)$ to $H^{-1}(\cdot)$ [4].

4. The layer K as a D_f -set, traces and the Green formula

According to [16], we give the following

Definition 4.1. Let $F \subset \mathbb{R}^D$ be a closed non-empty subset. It is a D_f -set ($0 < D_f \leq D$) if there exist a Borel measure m with $\text{supp } m = F$ such that for some constants $c_1 = c_1(F) > 0$ and $c_2 = c_2(F) > 0$

$$c_1 r^{D_f} \leq m(B_e(x, r)) \leq c_2 r^{D_f} \quad (x \in F, 0 < r \leq 1). \tag{4.1}$$

Such an m is called a D_f -measure on F .

If F is a D_f -set, then the D_f -dimensional Hausdorff measure H^{D_f} of \mathbb{R}^D restricted to F , defined by $H^{D_f} \llcorner F(E) = H^{D_f}(F \cap E)$ for Borel sets E , is a D_f -measure on F . Also, any D_f -measure m on F is equivalent to $H^{D_f} \llcorner F$ in the sense that, for some constants $c_3 > 0$ and $c_4 > 0$, $c_3 m \leq H^{D_f} \llcorner F \leq c_4 m$ (see [16: Chapter II]). To conclude, a D_f -measure on a D_f -set $F \subset \mathbb{R}^D$ is unique up to equivalence and it is given by the restriction to F of the D_f -dimensional Hausdorff measure in \mathbb{R}^D .

Examples of D_f -sets are $F = \mathbb{R}^2$ with m equal to the 2-dimensional Lebesgue measure and geometrically self-similar sets [30], in particular we have

Proposition 4.2. *The Koch curve is a D_f -set with $D_f = \frac{\log 4}{\log 3}$. The invariant measure $\mu = (H^{D_f})^{-1} H^{D_f} \llcorner K$ is a D_f -measure.*

From now on we assume Ω to be the open rectangle $(0, 1) \times (-1, 1)$ and K the unit Koch curve. Throughout the paper c will denote different constants.

We now come to the definition of the class of Besov spaces in those special cases which best fit our problem. We remind the reader that we shall only consider the case in which $D = 2$ (for a complete discussion see [16]). According to [16] we give the following

Definition 4.3. Let K denote the Koch curve, μ its invariant measure and $D_f = \frac{\ln 4}{\ln 3}$. By $B_\beta^{2,2}(K)$ with $0 < \beta < 1$ we denote the space of all functions ω such that

$$\|\omega\|_{B_\beta^{2,2}(K)} = \|\omega\|_{L^2(K, \mu)} + \left(\iint_{|x-y|<1} \frac{|\omega(x) - \omega(y)|^2}{|x - y|^{D_f+2\beta}} d\mu(x) d\mu(y) \right)^{\frac{1}{2}} < \infty. \tag{4.2}$$

Here β is also called the *smoothness index*.

Before stating the trace theorem in the case of interest for us, we note the following

Theorem 4.4. *If $f \in H^1(\Omega)$ and $\text{Ext}f$ is any function in $H^1(\mathbb{R}^2)$ with $\text{Ext}f = f$ a.e. in Ω , then $\gamma_{0,K,\Omega}f$ and $\gamma_{0,K,\mathbb{R}^2}\text{Ext}f$ exist and coincide μ -a.e. on K .*

Proof. This theorem is a particular case of [29: Theorem 1] which holds for the larger class of (ϵ, δ) domains (see [13] for the definition) ■

Theorem 4.5 (see [29]). *Let K be the Koch curve. Then the trace operator $\gamma_{0,K,\mathbb{R}^2} : f \rightarrow \tilde{f}$ is a bounded linear surjection $H^1(\mathbb{R}^2) \rightarrow B_\beta^{2,2}(K)$ with a bounded linear right inverse (the extension operator) $\mathcal{E}xt : B_\beta^{2,2}(K) \rightarrow H^1(\mathbb{R}^2)$ such that if $f \in B_\beta^{2,2}(K)$, then $\mathcal{E}xt f \in C^\infty(\mathbb{R}^2 \setminus K)$ and*

$$(i) \quad \|\mathcal{E}xt f\|_{H^1(\mathbb{R}^2)} \leq c \|f\|_{B_\beta^{2,2}(K)}$$

$$(ii) \quad \gamma_{0,K,\mathbb{R}^2}(\mathcal{E}xt f) = f \text{ } \mu\text{-a.e.}$$

Theorem 4.6. *Let Ω and K be as in problem (1.1) and let $f \in H^1(\Omega)$. Further let $\gamma_{0,K,\Omega}$ be the trace operator defined in Definition 2.3/(ii). Then the mapping*

$$\gamma_{0,K,\Omega} : H^1(\Omega) \rightarrow B_\beta^{2,2}(K) \tag{4.3}$$

is a bounded linear surjection, with a bounded right linear inverse, where $\beta = \frac{D_f}{2} > 0$.

Proof. As the domain Ω is the rectangle, the function f can be extended by reflexion (see [4]): there exists a bounded linear operator (extension operator) $\text{Ext} : H^1(\Omega) \rightarrow H^1(\mathbb{R}^2)$ such that $\text{Ext}f = f$ a.e. in Ω , for all $f \in H^1(\Omega)$, and $\|\text{Ext}f\|_{H^1(\mathbb{R}^2)} \leq c \|f\|_{H^1(\Omega)}$ where the constant $c > 0$ depends on Ω . From Theorem 4.5 and the extension theorem we have $\|\tilde{f}\|_{B_\beta^{2,2}(K)} \leq c \|\text{Ext}f\|_{H^1(\mathbb{R}^2)} \leq c' \|f\|_{H^1(\Omega)}$ and from Theorem 4.4 we have $\tilde{f} = \tilde{f}$ μ -a.e. on K ■

In the sequel, in force of Theorem 4.4, we denote by the same symbol $f|_K$ both $\gamma_{0,K,\Omega}f$ and $\gamma_{0,K,\mathbb{R}^2}f$.

Remark 4.1. Theorem 4.4 holds for a large class of domains, namely the (ϵ, δ) domains (see [13] for the definition and [29] for the proofs). If $K = \partial T$ for a (bounded) Lipschitz domain T , then $D_f = 1$ and the trace space for $H^1(T)$ is $B_\beta^{2,2}(\partial T) = H^{\frac{1}{2}}(\partial T)$ with $\beta = \frac{1}{2}$ (for the definition of the trace space $H^{\frac{1}{2}}(\partial T)$ see [26]), and Theorem 4.6 should become a particular case of [12: Theorem 3.1].

Theorem 4.7. *There exists a bounded linear extension operator $\mathcal{E} : D_0(K) \rightarrow H^1(\mathbb{R}^2)$ such that $\gamma_{0,K,\mathbb{R}^2}(\mathcal{E}z) = z$ μ -a.e. for every $z \in D_0(K)$.* ■

Proof. We recall that $D_0(K) \subset D_E(K)$. From [20: Theorem 3.1] we deduce that $D_0(K)$ is embedded into the space $\text{Lip}(D_f, 2, \infty)(K)$ (for the definition of this space see [20]). From [15: Corollary 1] there exists a bounded linear

extension operator \mathcal{E} from $\text{Lip}(D_f, 2, \infty)(K)$ to the Besov space $B_{D_f/2+1}^{2,\infty}(\mathbb{R}^2)$ (for the definition of this space see [16]). Then the embedding theorem (see [16: Chapter VIII/Proposition 5]) yields the thesis ■

Proposition 4.8. *The space $V(\Omega, K) = \{u \in H_0^1(\Omega) : u|_K \in D_0(K)\}$ is non-trivial.*

Proof. We shall prove that non-trivial functions in $D_0(K)$ have a suitable extension in $H_0^1(\Omega)$. To see this, let G denote a compact set such that $G \subset\subset K$. For instance, choose $G = K \cap \overline{B_\epsilon(\bar{x}, r)}$ with $r < \frac{1}{2}$ and $\bar{x} = (\frac{1}{2}, 0)$. Let ϕ be the capacity potential of G (for its existence and properties see [9: Theorem 2.1.5]). The function ϕ belongs to $D_0(K)$, its support is compact on K and, by Theorem 4.7, $\mathcal{E}\phi \in H^1(\mathbb{R}^2)$. Then $\eta\mathcal{E}\phi \in H_0^1(\Omega)$ where η is a suitable cut-off function ■

Remark 4.2. Actually, one can prove that the trace space of $V(\Omega, K)$ to K is $D_0(K)$ (see Section 6).

Proposition 4.9. *The space $D_0(K)$ is embedded into $B_{D_f/2}^{2,2}(K)$.*

Proof. From [20: Theorem 3.1] we deduce that $D_0(K)$ is embedded into the space $\text{Lip}(D_f, 2, \infty)(K)$ (for the definition of this space see [20]). On the other hand, the space $B_{D_f/2}^{2,2}(K)$ coincides with the space $\text{Lip}(D_f/2, 2, 2)(K)$ (see [16: pp. 114/Proposition 1]). The thesis follows from the embedding of $\text{Lip}(D_f, 2, \infty)(K)$ into $\text{Lip}(D_f/2, 2, 2)(K)$ (see [16]) ■

4.1 The dual of $B_\beta^{2,2}(K)$ on the Koch curve K . Let us now introduce the dual space of $B_\beta^{2,2}(K)$ where $\beta = \frac{D_f}{2}$. This space as shown in [17] coincides with $B_{-\beta}^{2,2}(K)$, a subspace of Schwartz distributions $D'(\mathbb{R}^2)$, which are supported in K . It is built by means of atomic decompositions. Actually, Johnsson and Wallin [17] proved this result in the general framework of D_f -sets.

Here we do not give a detailed description of the duals of Besov spaces on D_f -sets and we refer to [17] for a complete discussion. We will only recall the main features to deal with our case $D = 2$, K the Koch curve, and $\beta = \frac{D_f}{2}$.

As a preparation we introduce some notation. Let N denote a division of \mathbb{R}^2 into equally squares Q with side r , half-open of the form $\{x = (x_1, x_2) \in \mathbb{R}^2 : a_i < x_i \leq a_i+r (i = 1, 2)\}$, obtained intersecting \mathbb{R}^2 with lines orthogonal to the axes. We call such a division a *net N with mesh r* . By N_h we denote the net with mesh 2^{-h} such that the origin is a corner of some square in the net $N_h(K) = \{Q \in N_h : Q \cap K \neq \emptyset\}$. In the following definition we still denote by μ the measure (2.2) trivially extended to \mathbb{R}^2 , that is the measure that on every Borel set E of \mathbb{R}^2 takes the value $\mu(E \cap K)$.

According to [29] we give the following

Definition 4.10. Let K be the Koch curve, let $\beta = \frac{D_f}{2}$ and let Q with $Q \cap K \neq \emptyset$ be a square with edge length 2^{-h} where h is a non-negative integer. A function $a = a_Q \in L^2(\mathbb{R}^2, \mu)$ is a $(-\beta, 2)$ atom associated with Q if the conditions

- $\text{supp } a \subset 2Q$
- $\int_{\mathbb{R}^2} a(x) d\mu(x) = 0$ if $h > 0$
- $\|a\|_{L^2(\mathbb{R}^2, \mu)} \leq 2^{h\beta}$

are satisfied.

Let N_h ($h \in \mathbb{N}_0$) be a fixed net with mesh 2^{-h} , let $Q \in N_h(K)$, let a_Q be a $(-\beta, 2)$ atom associated with Q and let S_Q be numbers such that $S = \{S_h\} \in l^2$ where S_h is given by

$$S_h = \left(\sum_{Q \in N_h(K)} |S_Q|^2 \right)^{\frac{1}{2}}. \quad (4.4)$$

Then the function

$$g_h = \sum_{Q \in N_h(K)} S_Q a_Q \quad (4.5)$$

is in $L^1_{loc}(\mathbb{R}^2, \mu)$, since the sum defining g_h is a finite sum on any compact subset of \mathbb{R}^2 . We identify g_h with the distribution

$$\langle g_h, \phi \rangle = \sum_{Q \in N_h(K)} S_Q \int a_Q \phi d\mu \quad (\phi \in D(\mathbb{R}^2)). \quad (4.6)$$

Then $f_m = \sum_{h=0}^m g_h$ is the distribution given by

$$\langle f_m, \phi \rangle = \sum_{h=0}^m \sum_{Q \in N_h(K)} S_Q \int a_Q \phi d\mu \quad (\phi \in D(\mathbb{R}^2)). \quad (4.7)$$

We have $f_m \rightarrow f$ in the distributional sense, i.e.

$$\langle f_m, \phi \rangle \rightarrow \langle f, \phi \rangle \quad (\phi \in D(\mathbb{R}^2), m \rightarrow \infty) \quad (4.8)$$

where the distribution f is given by

$$\langle f, \phi \rangle = \sum_{h=0}^{\infty} \sum_{Q \in N_h(K)} S_Q \int a_Q \phi d\mu \quad (\phi \in D(\mathbb{R}^2)). \quad (4.9)$$

In fact, since $\phi \in D(\mathbb{R}^2)$, the trace theorem for Besov spaces [16: p. 141] in particular gives $\phi|_K \in B^{2,2}_{\beta}(K)$ and the claim follows from [17: Lemma 3.2].

When (4.9) holds, we write

$$f = \sum_{h=0}^{\infty} \sum_{Q \in N_h(K)} S_Q a_Q \quad (4.10)$$

and we refer to (4.10) as an *atomic decomposition* of f .

Remark 4.3. Note that the atomic decomposition (4.10) is not necessarily unique, i.e. different $(-\beta, 2)$ -atoms a_Q and numbers S_Q , $S = \{S_h\}_{h=0}^\infty \in l^2$ with S_h given by (4.4), may give the same distribution in (4.10).

Definition 4.11. Let K be the Koch curve and $\beta = \frac{D_f}{2}$. We define $B_{-\beta}^{2,2}(K)$ to consist of those $f \in D'(\mathbb{R}^2)$ which are given by (4.10) where we assume that a_Q are $(-\beta, 2)$ atoms and S_Q are numbers such that $S = \{S_h\} \in l^2$ and S_h is defined by (4.4). We define the norm of f by

$$\|f\|_{B_{-\beta}^{2,2}(K)} = \inf \|S\|_{l^2} \tag{4.11}$$

where the infimum is taken over all possible atomic decompositions (4.10).

In [17] it is proved that the dual of $B_{-\beta}^{2,2}(K)$ is $B_{\beta}^{2,2}(K)$. In fact, if $f \in B_{-\beta}^{2,2}(K)$ and $g \in B_{\beta}^{2,2}(K)$ is given by the atomic decomposition $g = \sum_{h=0}^\infty \sum_{Q \in N_h(K)} S_Q a_Q$, then the duality is given by

$$\langle g, f \rangle = \sum_{h=0}^\infty \sum_{Q \in N_h(K)} S_Q \int a_Q f \, d\mu. \tag{4.12}$$

Remark 4.4. As pointed out in [17], the double sum in (4.12) is independent of the particular atomic decomposition used for g .

In fact, the following duality result holds [17].

Proposition 4.12. Assume $\beta = \frac{D_f}{2}$.

(i) If

$$g = \sum_{h=0}^\infty \sum_{Q \in N_h(K)} S_Q a_Q \in B_{-\beta}^{2,2}(K) \tag{4.13}$$

and L is defined by

$$L(f) = \sum_{h=0}^\infty \sum_{Q \in N_h(K)} S_Q \int a_Q f \, d\mu \quad \text{for } f \in B_{\beta}^{2,2}(K), \tag{4.14}$$

then $L \in (B_{\beta}^{2,2}(K))'$ and

$$\|L\| \leq c \|g\|_{B_{-\beta}^{2,2}(K)} \tag{4.15}$$

where $c > 0$ is a constant depending only on K, μ and β .

(ii) If $L \in (B_{\beta}^{2,2}(K))'$, then there exists a unique g as in (4.13) such that (4.14) – (4.15) hold and

$$\|g\|_{B_{-\beta}^{2,2}(K)} \leq c \|L\|$$

where $c > 0$ is a constant depending only on K, μ and β .

4.2 The Green formulas. It is well known (see, for instance, Necas [26], and Dautray and Lions [6]) that if S denotes an appropriate interface, say of Lipschitz type, the normal vector to S is defined a.e. [26] so that a suitable Green formula to deal with this type of boundary value problems can be proved (see [2: Appendix 4] and [6: Chapter 6/Section 4]), and by duality arguments the normal derivative can be interpreted in the sense of the dual of the Sobolev-type space $H_{0,0}^{\frac{1}{2}}(S)$ (for the definition of this space see [23], but also [2, 6]). Following this philosophy, we shall prove that in the fractal case the normal derivative can be interpreted in the sense of the dual of the Besov space $B_{\beta,0}^{2,2}(K)$ (see (4.20) below).

We start by recalling the Green formula for Lipschitz domains, specialized to our case. For the sake of simplicity we assume $T \subset \mathbb{R}^2$ to be, as in problem (1.1), the open rectangle $(0, 1) \times (-1, 1)$. We assume that the layer S is of Lipschitz type and that it divides T into two subdomains T_1 and T_2 such that $S = \partial T_1 \cap \partial T_2$. Let $\Gamma = \{A, B\}$ denote the two points in which S intersects ∂T .

By $\tilde{V}(T_i)$ we denote the set of functions

$$\tilde{V}(T_i) = \{u \in H^1(T) \mid \Delta u_i \in L^2(T_i)\}$$

where $u_i = u|_{\Omega_i}$ ($i = 1, 2$), $\Delta u_i = u_{i x_1 x_1} + u_{i x_2 x_2}$ and the derivatives are intended in the distributional sense. We define the space $H_{0,0}^{\frac{1}{2}}(S)$ [6] as follows:

$$H_{0,0}^{\frac{1}{2}}(S) = \left\{ u \in L^2(S) \mid \text{There exists } w \in H_0^1(T) \text{ such that } w|_S = u \text{ on } S \right\}$$

equipped with the quotient norm

$$\|u\|_{H_{0,0}^{\frac{1}{2}}(S)} = \inf_{\substack{w \in H_0^1(T) \\ w|_S = u}} \|w\|_{H^1(T)}.$$

Here $L^2(S)$ denotes the usual Lebesgue space with respect to the one-dimensional Lebesgue measure and by $(H_{0,0}^{\frac{1}{2}}(S))'$ we denote the dual of $H_{0,0}^{\frac{1}{2}}(S)$. Then the following Green formula holds [2, 6]:

Theorem 4.13. *Let T_i ($i = 1, 2$) denote one of the two domains defined above. For $u \in \tilde{V}(T_i)$ we have*

$$\left\langle \frac{\partial u_i}{\partial n_i}, \theta|_S \right\rangle_{(H_{0,0}^{\frac{1}{2}}(S))', H_{0,0}^{\frac{1}{2}}(S)} = \int_{T_i} Du_i D\theta \, dx_1 dx_2 + \int_{T_i} \theta \Delta u_i \, dx_1 dx_2 \quad (4.16)$$

for every $\theta \in H_0^1(T)$ where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between $(H_{0,0}^{\frac{1}{2}}(S))'$ and $H_{0,0}^{\frac{1}{2}}(S)$.

From (4.16) it follows that for $u \in \tilde{V}(T_i)$ the normal derivative can be defined as a linear and continuous functional on $H_{0,0}^{\frac{1}{2}}(S)$.

We now come to the case in which the layer is the Koch curve K . Analogously to the Lipschitz case we need to define $\frac{\partial u_i}{\partial n_i}$ for boundary value problems with fractal boundaries. Following [23] we give the following

Definition 4.14. We define $B_{\beta,0}^{2,2}(K)$ with $\beta = \frac{D_f}{2}$ as

$$B_{\beta,0}^{2,2}(K) = \left\{ z \in L^2(K, \mu) \mid \text{There exists } w \in H_0^1(\Omega) \text{ such that } w|_K = z \text{ on } K \right\} \blacksquare$$

and equip it with the quotient norm

$$\|z\|_{B_{\beta,0}^{2,2}(K)} = \inf_{\substack{w \in H_0^1(\Omega) \\ w|_K = z}} \|w\|_{H^1(\Omega)}. \tag{4.17}$$

We observe that $B_{\beta,0}^{2,2}(K) \subset B_{\beta}^{2,2}(K)$. In the sequel by $(B_{\beta,0}^{2,2}(K))'$ we will denote the dual of $B_{\beta,0}^{2,2}(K)$. Further, we define the set of functions

$$\tilde{V}(\Omega_i) = \{u \in H^1(\Omega) \mid \Delta u_i \in L^2(\Omega_i)\}$$

where $u_i = u|_{\Omega_i}$, $\Delta u_i = u_{i_{x_1 x_1}} + u_{i_{x_2 x_2}}$ and the derivatives are intended in the distributional sense.

Theorem 4.15. *Let K and Ω_i as in problem (1.1) and $u \in \tilde{V}(\Omega_i)$. The normal derivative of u on K defined in (4.20) below is a linear and continuous functional on $B_{\beta,0}^{2,2}(K)$ with $\beta = \frac{D_f}{2}$.*

Proof. Let $u \in \tilde{V}(\Omega_i)$. We define

$$l_i(\theta) = \int_{\Omega_i} Du_i D\theta \, dx_1 dx_2 + \int_{\Omega_i} \theta \Delta u_i \, dx_1 dx_2 \quad (\theta \in H_0^1(\Omega)). \tag{4.18}$$

Let us show that $l_i(\theta)$ depends only on the trace of θ on K and is independent from the choice of the test function $\theta \in H_0^1(\Omega)$, i.e. if $\theta, \hat{\theta} \in H_0^1(\Omega)$ and $\theta|_K = \hat{\theta}|_K$, then $l_i(\theta) = l_i(\hat{\theta})$.

We consider the increasing sequence of domains Ω_h^1 (exhausting Ω_1) corresponding to the Lipschitz prefractal curve K_h (see (2.1)). Denote by $\chi_{\Omega_h^1}$ the

characteristic function of Ω_h^1 . From Theorem 4.13 specialized to our situation $T_i = \Omega_h^1, S = K_h$ and $T = \Omega$ we deduce that, for all $\theta \in H_0^1(\Omega)$,

$$\begin{aligned} & \left\langle \frac{\partial u_1}{\partial n_1}, \theta|_{K_h} \right\rangle_{(H_{0,0}^{\frac{1}{2}}(K_h))', H_{0,0}^{\frac{1}{2}}(K_h)} \\ &= \int_{\Omega_1} \chi_{\Omega_h^1} Du_1 D\theta \, dx_1 dx_2 + \int_{\Omega_1} \chi_{\Omega_h^1} \theta \Delta u_1 \, dx_1 dx_2. \end{aligned} \tag{4.16}_h$$

But for every $\theta \in H_0^1(\Omega)$,

$$\begin{aligned} l_1(\theta) &= \int_{\Omega_1} Du_1 D\theta \, dx_1 dx_2 + \int_{\Omega_1} \theta \Delta u_1 \, dx_1 dx_2 \\ &= \lim_{h \rightarrow \infty} \int_{\Omega_1} \chi_{\Omega_h^1} Du_1 D\theta \, dx_1 dx_2 + \int_{\Omega_1} \chi_{\Omega_h^1} \theta \Delta u_1 \, dx_1 dx_2 \\ &= \lim_{h \rightarrow \infty} \left\langle \frac{\partial u_1}{\partial n_1}, \theta|_{K_h} \right\rangle_{(H_{0,0}^{\frac{1}{2}}(K_h))', H_{0,0}^{\frac{1}{2}}(K_h)}. \end{aligned}$$

From Schwarz inequality we get

$$|l_1(\theta)| \leq (\|\Delta u_1\|_{L^2(\Omega_1)} + \|Du_1\|_{L^2(\Omega_1)}) \|\theta\|_{H^1(\Omega)}. \tag{4.19}$$

We prove that the distribution $l(\cdot)$ is indeed supported on K . Namely, we consider, for any ball $B = B_r^e, B \subset \Omega \setminus K$, a smooth function ϕ_B supported on B and we choose $\theta\phi_B$ as test function in (4.16)_h. Definitely $\phi_B|_{K_h} = 0$, hence $(\theta\phi_B)|_{K_h} = 0$ and $l_1(\theta\phi_B) = 0$. Analogously we proceed for Ω_2 , considering the sequence of increasing domains Ω_h^2 (exhausting Ω_2) corresponding to the two prefractal curves K_h^r and K_h^l (see Remark 2.2).

The previous considerations allow us to define for any $u \in \tilde{V}(\Omega_i)$ the "normal derivative" in the following way:

$$\left\langle \frac{\partial u_i}{\partial n_i}, \theta|_K \right\rangle_{(B_{\beta,0}^{2,2}(K))', B_{\beta,0}^{2,2}(K)} = \int_{\Omega_i} Du_i D\theta \, dx_1 dx_2 + \int_{\Omega_i} \theta \Delta u_i \, dx_1 dx_2 \tag{4.20}$$

for every $\theta \in H_0^1(\Omega)$. We show now that $\frac{\partial u_i}{\partial n_i}$ ($i = 1, 2$) is a linear and continuous functional on $B_{\beta,0}^{2,2}(K)$. From (4.17), we have that for every $z \in B_{\beta,0}^{2,2}(K)$ there exists a $w \in H_0^1(\Omega)$ such that

$$\|w\|_{H^1(\Omega)} \leq c \|z\|_{B_{\beta,0}^{2,2}(K)} \tag{4.21}$$

and $w|_K = z$ μ -a.e. By using (4.20) and the independence from extension we have

$$\left\langle \frac{\partial u_i}{\partial n_i}, z \right\rangle_{(B_{\beta,0}^{2,2}(K))', (B_{\beta,0}^{2,2}(K))} = l(w).$$

The thesis now follows from (4.19) and (4.21) ■

5. Variational formulation

5.1 Variational principle. In this section, we give the variational formulation of problem (1.1) formally stated in Section 1. We follow the approach used in [27] for the classical case where the layer K was a smooth curve.

From Propositions 4.8 and 4.9 it follows that the space of functions $u : \Omega \rightarrow \mathbb{R}$

$$V(\Omega, K) = \{u \in H_0^1(\Omega) : u|_K \in D_0(K)\} \tag{5.1}$$

is well defined.

Lemma 5.1. $V(\Omega, K)$ is an Hilbert space equipped with the scalar product

$$(u, v)_{V(\Omega, K)} = \int_{\Omega} DuDv \, dx_1 dx_2 + E(u|_K, v|_K) \tag{5.2}$$

where $E(u|_K, v|_K)$ is the Dirichlet form associated to the fractal Laplacian on the layer K (see (3.5) – (3.6)).

We denote by $\|u\|_{V(\Omega, K)}^2$ the corresponding "energy norm" in $V(\Omega, K)$.

Proof of Lemma 5.1. Let $u_n \in V(\Omega, K)$ be a Cauchy sequence. We want to prove that there exists a $u \in V(\Omega, K)$ such that

$$\|u_n - u\|_{V(\Omega, K)}^2 = \|D(u_n - u)\|_{L^2(\Omega)}^2 + E(u_n|_K - u|_K, u_n|_K - u|_K) \rightarrow 0.$$

We note the following:

(i) From the Poincarè inequality we deduce that $\{u_n\}$ is a Cauchy sequence in $H_0^1(\Omega)$, therefore there exists an $u \in H_0^1(\Omega)$ such that $\|u_n - u\|_{H^1(\Omega)} \rightarrow 0$ as $n \rightarrow \infty$.

(ii) $\{u_n|_K\}$ is a Cauchy sequence in $D_0(K)$, therefore there exists a $z \in D_0(K)$ such that $\|u_n|_K - z\|_{D_0(K)} \rightarrow 0$ as $n \rightarrow \infty$.

(iii) As consequence of Theorem 4.6 (see also Definition 4.3), $\|u_n|_K - u|_K\|_{L^2(K, \mu)} \leq c\|u_n - u\|_{H^1(\Omega)}$. Hence $z = u|_K$ ■

For further properties of this form, like its regularity in the case $\Omega = \mathbb{R}^D$, we refer to [31, 32] (see also [33]).

We now come to state our variational principle. We look for the weak solution u of problem (1.1). As the constant C is not relevant for our purposes we set it equal to one.

Theorem 5.2. Given $g \in L^2(\Omega)$, there exists a unique $u \in V(\Omega, K)$ such that

$$\int_{\Omega} DuD\theta \, dx_1 dx_2 + E(u|_K, \theta|_K) = \int_{\Omega} g\theta \, dx_1 dx_2 \tag{5.3}$$

for every $\theta \in V(\Omega, K)$. Moreover, u is obtained by

$$\min_{\theta \in V(\Omega, K)} \left\{ \int_{\Omega} |D\theta|^2 dx_1 dx_2 + E[\theta|_K] - 2 \int_{\Omega} g\theta dx_1 dx_2 \right\}.$$

Proof. The thesis follows by applying the LaxMilgram theorem (or Riesz-Frechét theorem) to the bilinear form

$$a(u, \theta) = \int_{\Omega} (u_{x_1} \theta_{x_1} + u_{x_2} \theta_{x_2}) dx_1 dx_2 + E(u|_K, \theta|_K)$$

defined in $V(\Omega, K) \times V(\Omega, K)$. This form is continuous and coercive in $V(\Omega, K)$ and the linear functional $\int_{\Omega} g\theta dx$ is bounded in $V(\Omega, K)$, with norm depending on $\|g\|_{L^2(\Omega)}$ and the Poincarè constant in Ω ■

5.2 "Regularity" of the weak solution. Let us now go back and interpret the solved problem. We recall that by u_i we denote the restriction to Ω_i of the solution $u \in V(\Omega, K)$ of (5.3). Let us choose in (5.3) $\theta = \phi_1 \in D(\Omega_1)$ and $\theta = \phi_2 \in D(\Omega_2)$, respectively, where ϕ_1 and ϕ_2 are arbitrary. From this we obtain

$$\int_{\Omega_i} Du_i D\phi_i dx_1 dx_2 = \int_{\Omega_i} g\phi_i dx_1 dx_2 \quad (5.4)$$

for every $\phi_i \in D(\Omega_i)$ ($i = 1, 2$). We have in the sense of distributions

$$-\Delta u_1 = g \quad \text{in } D'(\Omega_1) \quad (5.5)$$

$$-\Delta u_2 = g \quad \text{in } D'(\Omega_2). \quad (5.6)$$

From the density of $D(\Omega_i)$ in $L^2(\Omega_i)$ and from the fact that $g \in L^2(\Omega_i)$ we deduce that equations (5.5) - (5.6) hold also in $L^2(\Omega_i)$. This gives that $u \in \tilde{V}(\Omega_i) = \{u \in H^1(\Omega) \mid \Delta u_i \in L^2(\Omega_i)\}$ where the Laplacian is intended in the distributional sense. The classical theory on local regularity results (see [4]) gives also that $u_i \in H_{loc}^2(\Omega_i)$.

6. The transmission condition

The purpose of this section is to show that the weak solution u of problem (1.1) satisfies the transmission condition on the fractal layer K in a "suitable" sense.

We preliminary prove that the trace space of $V(\Omega, K)$ on K is the space $D_0(K)$. In fact, every function $u \in V(\Omega, K)$ by definition is such that $u|_K \in D_0(K)$. Every function $z \in D_0(K)$ has a suitable extension w in $H_0^1(\Omega)$ such that $w|_K = z$ μ -a.e. on K . This follows from the fact that $D_0(K) \subset B_{\beta,0}^{2,2}(K)$ (see Definition 4.14).

Proposition 6.1. *The space $D_0(K)$ is a subspace of the space $B_{\beta,0}^{2,2}(K)$.*

Proof. Let $z \in D_0(K)$. There exists a bounded linear extension operator \mathcal{E} from $D_0(K)$ to the Besov space $B_{1+D_f/2}^{2,\infty}(\mathbb{R}^2)$ which is embedded into $B_{1+D_f/2-\epsilon}^{2,2}(\mathbb{R}^2)$ where $\epsilon > 0$ (see the proof of Theorem 4.7); in particular, we deduce $\mathcal{E}z \in C^0(\mathbb{R}^2)$ and $\mathcal{E}z \in H^1(\mathbb{R}^2)$. Let us consider now the function $u^* = \eta\mathcal{E}z$ where η is a suitable smooth cut-off function. For instance, $\eta = 0$ in the set $\mathcal{R} = ([0, 1] \times [\frac{2}{3}, 1]) \cup ([0, 1] \times [-1, -\frac{2}{3}])$, $\eta = 1$ in the set \mathcal{S} where \mathcal{S} is the rectangle $[0, 1] \times [-h, h]$ with $\frac{\sqrt{3}}{6} < h < \frac{2}{3}$ and $0 \leq \eta \leq 1$ in the remaining part. The function u^* belongs to $H^1(\Omega)$, $u^* = 0$ in \mathcal{R} , and

$$\|u^*\|_{H^1(\Omega)} \leq c \|u^*\|_{B_{1+D_f/2-\epsilon}^{2,2}(\Omega)} \leq c \|\mathcal{E}z\|_{B_{1+D_f/2-\epsilon}^{2,2}(\mathbb{R}^2)} \leq c \|z\|_{D_0(K)}. \quad (6.1)$$

Consider now in the rectangle Ω the four open triangles T_1, T_2, T_3, T_4 where

$$\begin{aligned} T_1 & \text{ has vertexes } A = (0, 0), A_1 = (0, 1), A_2 = (\frac{1}{3}, 1) \\ T_2 & \text{ has vertexes } B = (0, 1), B_1 = (1, 1), B_2 = (\frac{2}{3}, 1) \end{aligned}$$

and T_3 and T_4 are symmetric triangles with respect to the x_1 -axis.

Let us now focus our attention on the triangle T_1 . We denote by L_2 the side AA_2 . The trace \tilde{z} of $\mathcal{E}z$ to the set L_2 belongs to the space $B_{1/2+D_f/2-\epsilon}^{2,2}(L_2)$, i.e. the Sobolev space $H^s(L_2)$ with $s = \frac{1}{2} + \frac{D_f}{2} - \epsilon$ (see, i.e., [23]). By Morrey type embedding \tilde{z} belongs to $C^0(\overline{L_2})$; recall that $\tilde{z}(A) = \tilde{z}(A_2) = 0$. Consider now the function $z^* = \tilde{z}$ on L_2 and $z^* = 0$ in $\partial T_1 \setminus L_2$. The function z^* still belongs to the space $H^s(\partial T_1)$ (see [5] and [26] for the definition of this space). Thus by the extension theorem there exists an extension $v_1 = \text{Ext } z^*$ in $H^{s+\frac{1}{2}}(T_1)$ (see [5: Theorem 2.7.1]). In particular, the function v_1 belongs to $H^1(T_1)$ and its trace to L_2 is continuous and we have

$$\begin{aligned} \|v_1\|_{H^1(T_1)} & \leq \|v_1\|_{H^{s+\frac{1}{2}}(T_1)} \\ & \leq c \|z^*\|_{H^s(\partial T_1)} \\ & \leq c \|\mathcal{E}z\|_{B_{1+D_f/2-\epsilon}^{2,2}(\mathbb{R}^2)} \\ & \leq c \|z\|_{D_0(K)}. \end{aligned} \quad (6.2)$$

Let us recall that the function u^* also belongs to $H^1(\Omega \setminus T_1)$ and its trace to L_2 is continuous.

Repeat the same argument for the other triangles and define the function w as

$$w = \begin{cases} v_i & \text{in } T_i \\ u^* & \text{in } \Omega \setminus \cup_{i=1}^4 T_i. \end{cases}$$

A straightforward computation shows that $w \in H_0^1(\Omega)$. From (6.1) - (6.2) we deduce

$$\|w\|_{H^1(\Omega)} \leq \sum_{i=1}^4 \|v_i\|_{H^1(T_i)} + \|u^*\|_{H^1(\Omega \setminus \cup_{i=1}^4 T_i)} \leq c \|z\|_{D_0(K)}. \quad (6.3)$$

Thus we have proved that, for each given function $z \in D_0(K)$, there exists a function $w \in H_0^1(\Omega)$ such that $w|_K = z$ on K , that is to say that $z \in B_{\beta,0}^{2,2}(K)$. From the definition of $B_{\beta,0}^{2,2}(K)$ and from (6.3) we have $\|z\|_{B_{\beta,0}^{2,2}(K)} \leq \|z\|_{D_0(K)}$ thus concluding the proof ■

Remark 6.1. A different proof of Proposition 6.1 can be also achieved by making use of a general extension theorem for Besov spaces defined on general closed sets which are not possibly D_f -sets (such as the set $\partial\Omega_1 \cup \partial\Omega_2$). More precisely, for any given $z \in B_{D_f-\epsilon}^{2,2}(K)$ with $z(A) = z(B) = 0$ consider the function z^* defined as $z^* = z$ on K and $z^* = 0$ in $\partial\Omega$. By [14: Theorem 1] there exists a function $w \in H_0^1(\Omega)$ which extends z^* , hence z .

We can finally study the transmission condition. We recall that the (weak) solution u of (5.3) belongs to $\tilde{V}(\Omega_i)$ (see Subsection 5.2). This allow us to make use of the Green formula (see Theorem 4.15 and (4.20)). In particular, for every $z \in D_0(K)$

$$-E(u|_K, z) - \left\langle \frac{\partial u_1}{\partial n_1}, z \right\rangle_{(B_{\beta,0}^{2,2}(K))', B_{\beta,0}^{2,2}(K)} - \left\langle \frac{\partial u_2}{\partial n_2}, z \right\rangle_{(B_{\beta,0}^{2,2}(K))', B_{\beta,0}^{2,2}(K)} = 0.$$

We set

$$\left[\frac{\partial u}{\partial n} \right]_K = \frac{\partial u_1}{\partial n_1} + \frac{\partial u_2}{\partial n_2}.$$

So we have, more concisely,

$$-E(u|_K, z) = \left\langle \left[\frac{\partial u}{\partial n} \right]_K, z \right\rangle_{(B_{\beta,0}^{2,2}(K))', B_{\beta,0}^{2,2}(K)}. \quad (6.4)$$

We now show that the transmission condition holds in the sense of $D_0(K)$. We recall that $[\frac{\partial u}{\partial n}]_K \in (B_{\beta,0}^{2,2}(K))'$, i.e. it is a linear and continuous functional on $B_{\beta,0}^{2,2}(K)$. We denote by the same symbol $[\frac{\partial u}{\partial n}]_K$ the restriction of this functional to $D_0(K)$ (see Proposition 6.1), which is still a linear and continuous functional on $D_0(K)$:

$$\left| \left\langle \left[\frac{\partial u}{\partial n} \right]_K, z \right\rangle_{(B_{\beta,0}^{2,2}(K))', B_{\beta,0}^{2,2}(K)} \right| \leq c \|z\|_{D_0(K)}$$

where c depends on u . Hence, as $\Delta_K(u|_K) \in (D_0(K))'$ (see (3.7)), equation (6.4) can be also written as

$$\langle \Delta_K(u|_K), z \rangle_{(D_0(K))', D_0(K)} = \left\langle \left[\frac{\partial u}{\partial n} \right]_K, z \right\rangle_{(D_0(K))', D_0(K)}. \tag{6.5}$$

Thus, from (6.5) we have in the sense of the duality defined by $D_0(K)$

$$\Delta_K(u|_K) = \left[\frac{\partial u}{\partial n} \right]_K \quad \text{on } K, \quad \text{in } (D_0(K))'. \tag{6.6}$$

Theorem 6.2. *Let $u \in V(\Omega, K)$ be the weak solution of problem (5.3). Then the transmission condition (6.6)*

$$\langle \Delta_K(u|_K), z \rangle_{(D_0(K))', D_0(K)} = \left\langle \left[\frac{\partial u}{\partial n} \right]_K, z \right\rangle_{(D_0(K))', D_0(K)}$$

holds where Δ_K is the variational operator from $D_0(K) \rightarrow (D_0(K))'$ defined in (3.7) and $\langle \cdot, \cdot \rangle_{(D_0(K))', D_0(K)}$ is the duality pairing between $(D_0(K))'$ and $D_0(K)$.

To interpret the transmission condition in the sense of $(B_{\beta,0}^{2,2}(K))'$ one could consider the extension of the operator $\Delta_K(u|_K)$ as a linear and continuous functional from $D_0(K)$ to $B_{\beta,0}^{2,2}(K)$. We do not know if this extension is unique because we do not know if $D_0(K)$ is dense in $B_{\beta,0}^{2,2}(K)$. To obtain a "stronger" formulation, u should be more regular, say $\Delta_K(u|_K) \in L^2(K, \mu)$, i.e. $u|_K \in D_{\Delta_K}$ so that the transmission condition could be interpreted in the L^2 -sense. This problem, as far as we know, was still an open problem also in the case of the smooth layer considered in [27] and in the case in which the Koch curve is replaced by the corresponding approximating prefractal (Lipschitz) curves. It has been recently studied in [21], where it is proved that the transmission condition can be interpreted in the L^2 -sense for both the case of the smooth layer and the prefractal curve. We hope to extend this result to the present case by limit arguments such as those in [24]. This would be interesting also from a numerical point of view in order to prove the convergence of approximating schemes for problem (5.4), as some preliminary computations seem to substantiate our conjecture.

It is also an open problem to establish if the normal derivative itself (which, as shown, is a distribution supported on K) is a measure.

We are now in position to summarize the properties of our solution u as "strong solution" of problem (1.1):

Conclusions. Let $u \in V(\Omega, K)$ be the weak solution of problem (5.3). Then $u \in \tilde{V}(\Omega_i) = \{u \in H^1(\Omega) \mid \Delta u_i \in L^2(\Omega_i)\}$ – where the Laplacian is intended in the distributional sense – and $u_i \in H_{loc}^2(\Omega_i)$. Its trace $u|_K \in C^0(K)$ (in particular, it is in $C^{0,\beta}(K)$ with $\beta = \frac{\ln 4}{\ln 9}$). The normal derivative $\frac{\partial u_i}{\partial n_i}$ is in the dual of the space $B_{\beta,0}^{2,2}(K)$. The transmission condition holds in the sense of the duality defined by $D_0(K)$.

Acknowledgements. The author wishes to thank Prof. Umberto Mosco for suggesting the problem and his support. The author is grateful to Prof. M. Agostina Vivaldi for her guidance and several stimulating discussions during the preparation of this paper.

References

- [1] Adams, D. R. and L. I. Hedberg: *Function Spaces and Potential Theory*. Berlin: Springer-Verlag 1966.
- [2] Baiocchi, C. and A. Capelo: *Variational and Quasivariational Inequalities: Applications to Free-Boundary Value Problems*. New York: Wiley 1984.
- [3] Beurling, A. and J. Deny: *Espaces de Dirichlet*. Acta Math. 99 (1958), 203 – 224.
- [4] Brezis, H.: *Analisi Funzionale*. Napoli (Italy): Liguori 1986.
- [5] Brezzi, F. and G. Gilardi: *Fundamentals of P.D.E. for numerical analysis*. In: Finite Element Handbook (ed.: H. Kardestuncer and D. H. Norrie). New York: McGraw-Hill Book Co. 1987.
- [6] Dautray, R. and J. L. Lions: *Mathematical Analysis and Numerical Methods for Science and Technology*, Vol. 2. Berlin: Springer-Verlag 1988.
- [7] Falconer, K.: *The Geometry of Fractal Sets*, 2nd ed. Cambridge: Univ. Press 1990.
- [8] Fukushima, M.: *Dirichlet forms, diffusion processes and spectral dimension for nested fractals*. In: Ideas and Methods in Mathematical Analysis, Stochastics and Applications (eds.: S. Albeverio et al.). Cambridge: Univ. Press 1992, pp. 151 – 161.
- [9] Fukushima, M., Oshima, Y. and M. Takeda: *Dirichlet Forms and Symmetric Markov Processes* (de Gruyter Studies in Mathematics: Vol. 19). Berlin: W. de Gruyter 1994.
- [10] Fukushima, M. and T. Shima: *On a spectral analysis for the Sierpinski gasket*. Pot. Anal. 1 (1992), 1 – 5.
- [11] Hutchinson, J. E.: *Fractals and selfsimilarity*. Indiana Univ. Math. J. 30 (1981), 713 – 747.
- [12] Jerison, D. and C. E. Kenig: *The inhomogeneous Dirichlet problem in Lipschitz domains*. J. Funct. Anal. 130 (1995), 161 – 219.

- [13] Jones, P. W.: *Quasiconformal mappings and extendability of functions in Sobolev spaces*. Acta Math. 147 (1981), 71 – 88.
- [14] Jonsson, A.: *Besov spaces on closed subsets of \mathbb{R}^n* . Trans. Amer. Math. Soc. 341 (1994), 355 – 370.
- [15] Jonsson, A.: *Brownian motion on fractals and function spaces*. Math. Z. 222 (1996), 495 – 504.
- [16] Jonsson, A. and H. Wallin: *Function Spaces on Subsets of \mathbb{R}^n* , Part 1 (Math. Reports: Vol. 2). London: Harwood Acad. Publ. 1984.
- [17] Jonsson, A. and H. Wallin H.: *The dual of Besov spaces on fractals*. Studia Math. 112 (1995), 285 – 300.
- [18] Kigami, J.: *Harmonic calculus on P.C.F. Self-similar sets*. Trans. Amer. Math. Soc. 335 (1993), 721 – 755.
- [19] Kusuoka, S.: *Diffusion Processes in Nested Fractals*. Lect. Notes Math. 1567 (1993).
- [20] Lancia, M. R. and M. A. Vivaldi: *Lipschitz spaces and Besov traces on self-similar fractals*. Rend. Accad. Naz. Sci. XL Met. Mat. Appl. (5) 23 (1999), 101 – 116.
- [21] Lancia, M. R. and M. A.: Vivaldi: *On the regularity of the solutions for transmission problems*. Adv. Math. Sci. Appl. (to appear).
- [22] Lindström, T.: *Brownian Motion on Nested Fractals*. Memoirs Amer. Math. Soc. 83 (1990), Nr. 420.
- [23] Lions, J.L. and E. Magenes: *Non-Homogeneous Boundary Value Problems and Applications*, Vol.1. Berlin: Springer-Verlag 1972.
- [24] Mosco, U.: *Composite media and asymptotic Dirichlet forms*. J. Funct. Anal. 123 (1994), 368 – 421.
- [25] Mosco, U.: *Variational fractals*. Ann. Scuola Norm. Sup. Pisa, Special Volume in Memory of E. De Giorgi, 25 (1997), 683 – 712.
- [26] Necăs, J.: *Les méthodes directes en théorie des équationes elliptiques*. Paris: Masson 1967.
- [27] Pham Huy, H. and E. Sanchez-Palencia E.: *Phénomènes des transmission à travers des couches minces de conductivité élevée*. J. Math. Anal. Appl. 47 (1974), 284 – 309.
- [28] Triebel, H.: *Fractals and Spectra. Related to Fourier Analysis and Function Spaces* (Monographs in Mathematics: Vol. 91). Basel: Birkhäuser Verlag 1997.
- [29] Wallin, H.: *The trace to the boundary of Sobolev spaces on a snowflake*. Manuscripta Math. 73 (1991), 117 – 125.
- [30] Wallin, H.: *Self-similarity, Markov's inequality and d -sets*. In: Constr. Theory of Functions. Varna - Sofia 1993, pp. 285 – 297.

Added in proof:

- [31] Jonsson, A.: *Dirichlet forms and Brownian motion penetrating fractals*. Potent. Anal. 13 (2000), 69 – 80.

- [32] Kumagai, T.: *Brownian motion penetrating fractals: an application of the trace theorem of Besov spaces*. J. Funct. Anal. 170 (2000), 69 - 92.
- [33] Lindstrøm, T.: *Brownian motion penetrating the Sierpinski gasket*. In: Asymptotic Problems in Probability Theory: Stochastic Models and Diffusions on Fractals (eds.: K. D. Elworthy and N. Ikeda). Longmann 1993, pp. 248 – 278.
- [34] Paluba, K. P.: *Some function spaces related to the Brownian motion on simple nested fractals*. Stoch. & Stoch. Rep. 67 (1999), 267 – 285.

Received 20.03.2001; in revised form 30.10.2001