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Solution Decompositions for Linear Convection-Diffusion Problems

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Abstract. We consider a singularly perturbed convection-diffusion problem. The existence of certain decompositions of the solution into a regular solution component and a layer component is studied. Such decompositions are useful for the convergence analysis of numerical methods. Our aim is to show that such decompositions exist under less restrictive assumptions on the data of the problem than those required in earlier publications.

Keywords: *Convection-diffusion, singular perturbation, solution decomposition*

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1. Introduction

We consider the singularly perturbed convection-diffusion problem

$$\left.\begin{aligned} \mathcal{L}u(x) := -\varepsilon u''(x) - a(x)u'(x) + b(x)u(x) = f(x) \quad &\text{for } x \in (0,1) \\ u(0) = \gamma_0 \quad & \\ u(1) = \gamma_1 \quad & \end{aligned}\right\} \tag{1}$$

where $0 < \varepsilon \ll 1$ is a small constant, $a(x) \geq \alpha > 0$, $b \geq 0$, $a, b, f \in C^k(0,1)$ with $k = 0$ or $k = 1$. Its solution u typically has an exponential boundary layer at $x = 0$.

A variety of special numerical methods for the approximate solution of problem (1) have been proposed and analysed in the literature. For a survey the reader is referred to [8]. One possible means of constructing robust methods, i.e. methods that perform equally well no matter how small the perturbation parameter ε, is the use of standard schemes on highly non-uniform

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meshes such as Bakhvalov-type meshes [1, 10] or Shishkin-type meshes [3, 7]. For the analysis of such methods decompositions of the solution of problem (1) into a regular solution component and a layer component turned out to be very useful.

In [3] it was proved that if $a, f \in C^3(0,1)$ and $b \equiv 0$, then u admits the representation $u = v + w$ where the regular solution component v satisfies

$$\left.\begin{aligned} &\mathcal{L}v(x) = f(x) \\ &|v^{(i)}(x)| \le C(1+\varepsilon^{2-i}) \quad (i=0,1,2,3) \end{aligned}\right\} \qquad (x \in (0,1)) \qquad (2)_a$$

while for the boundary layer component w we have

$$\left.\begin{aligned} &\mathcal{L}w(x) = 0 \\ &|w^{(i)}(x)| \le C\varepsilon^{-i} \exp\Big(-\frac{\alpha x}{\varepsilon}\Big) \quad (i=0,1,2,3) \end{aligned}\right\} \qquad (x \in (0,1)) \qquad (2)_b$$

where here and throughout the paper $C > 0$ – sometimes subscripted – denotes a generic constant that is independent of ε. This decomposition is used in [3] to prove that the simple upwind scheme

$$-\frac{2\varepsilon}{h_i + h_{i+1}} \left(\frac{U_{i+1} - U_i}{h_{i+1}} - \frac{U_i - U_{i-1}}{h_i} \right) - a_i \frac{U_{i+1} - U_i}{h_{i+1}} + b_i U_i = f_i$$

is almost first-order convergent, uniformly in the perturbation parameter ε, on a Shishkin mesh.

The purpose of the present paper is to derive (2) under less restrictive regularity assumptions on the data of the problem. Our construction requires a, b and f to lie only in $C^1(0,1)$ rather than in $C^3(0,1)$. The key to this improvement is that the decomposition into regular and boundary parts is not unique. Unlike [3] we define v not via solutions of first-order problems, but as the solution of a second-order problem with an appropriate boundary condition at the outflow boundary $x = 0$. This idea can also be used for two-dimensional problems, where one defines the regular solution component as the solution of an elliptic problem rather than by means of hyperbolic problems as done in [2, 3]. A first attempt in this direction can be found in [6]. However, in two dimensions regularity of the boundary and compatibility of the boundary conditions become an important issue too.

2. Construction of the decomposition

The decomposition will be constructed as follows. We define v and w to be the solution of the boundary-value problems

$$\left.\begin{aligned} \mathcal{L}v(x) &= f(x) \quad (x \in (0,1)) \\ (-av' + bv)(0) &= f(0) \\ v(1) &= \gamma_1 \end{aligned}\right\} \tag{3$_a$}$$

and

$$\left.\begin{aligned} \mathcal{L}w(x) &= 0 \quad (x \in (0,1)) \\ w(0) &= \gamma_0 - v(0) \\ w(1) &= 0 \end{aligned}\right\}. \tag{3$_b$}$$

First we study the regular solution component v. The operator $\mathcal{L}$ satisfies certain maximum and comparison principles [5]. For example, if two functions $\check{v}$ and $\hat{v}$ satisfy

$$\begin{aligned} \mathcal{L}\check{v}(x) &\le \mathcal{L}\hat{v}(x) \quad \text{in } (0,1) \\ (-a\check{v}' + b\check{v})(0) &\le (-a\hat{v}' + b\hat{v})(0) \\ \check{v}(1) &\le \hat{v}(1), \end{aligned}$$

then $\check{v}(x) \le \hat{v}(x)$ on $[0,1]$. Using this comparison principle with

$$v^{\pm} = \pm\left(\alpha^{-1}\|f\|_\infty(1-x) + |\gamma_1|\right)$$

we get

$$\|v\|_\infty \le \alpha^{-1}\|f\|_\infty + |\gamma_1| =: C_1.$$

To derive bounds on the derivatives of v, we set $h(x) = f(x) - b(x)u(x)$ and write v as

$$v(x) = \int_x^1 \vartheta(s)ds + \tfrac{h(0)}{a(0)} \int_x^1 \exp(-A(s))\, ds + \gamma_1$$

where

$$\begin{aligned} A(x) &= \tfrac{1}{\varepsilon} \int_0^x a(s)\, ds \\ \vartheta(x) &= \tfrac{1}{\varepsilon} \int_0^x h(s) \exp\left(A(s) - A(x)\right) ds. \end{aligned}$$

Differentiating once, we get

$$v'(x) = -\tfrac{1}{\varepsilon} \int_0^x h(s) \exp\left(A(s) - A(x)\right) ds - \tfrac{h(0)}{a(0)} \exp(-A(x))$$

which gives

$$\|v'\|_\infty \le 2\alpha^{-1}\big\{\|f\|_\infty + \|b\|_\infty C_1\big\} =: C_2$$

because

$$\begin{aligned}\tfrac{1}{\varepsilon}\int_0^x \exp\big(A(s)-A(x)\big)ds &\le \tfrac{1}{\varepsilon}\int_0^x \exp\Big(\frac{\alpha(s-x)}{\varepsilon}\Big)ds \\ &= \frac{1}{\alpha}\Big(1-\exp\Big(-\frac{\alpha x}{\varepsilon}\Big)\Big) \\ &\le \frac{1}{\alpha}.\end{aligned} \tag{4}$$

Invoking the differential equation we get

$$\|v''\|_\infty \le \frac{1}{\varepsilon}\{\|a\|_\infty C_2 + \|b\|_\infty C_1 + \|f\|_\infty\}.$$

However, if $a, f \in C^1(0,1)$, then we have

$$v''(x) = -\tfrac{a(x)}{\varepsilon}\int_0^x \Big(\frac{h}{a}\Big)'(s)\exp\big(A(s)-A(x)\big)ds$$

from which the sharper estimate $\|v''\|_\infty \le \frac{\|a\|_\infty}{\alpha}\big\|\big(\frac{h}{a}\big)'\big\|_\infty$ can be derived using (4). A bound for the third-order derivative $\|v'''\|_\infty \le C\varepsilon^{-1}$ is readily obtained from the differential equation and the bounds on v, v' and v''. This completes our analysis of the regular part of u.

Now let us consider the boundary-layer term w. The operator $\mathcal{L}$ satisfies another comparison principle: if two functions $\check{w}$ and $\hat{w}$ satisfy

$$\begin{aligned}|\mathcal{L}\check{w}(x)| &\le \mathcal{L}\hat{w}(x) \qquad &&\text{in } (0,1)\\ |\check{w}(x)| &\le \hat{w}(x) &&\text{for } x \in \{0,1\}\end{aligned}$$

then

$$|\check{w}(x)| \le \hat{w}(x) \qquad \text{on } [0,1]$$

(see [5]). Using this comparison principle with $w^\pm = \pm|\gamma_0 - v(0)|e^{-\frac{\alpha x}{\varepsilon}}$, we see that

$$|w(x)| \le (|\gamma_0| + C_1)\exp\Big(-\frac{\alpha x}{\varepsilon}\Big) =: C_3\exp\Big(-\frac{\alpha x}{\varepsilon}\Big) \qquad (x \in (0,1)). \tag{5}$$

To bound the derivatives of w we use the fact that

$$w(x) = \int_x^1 \vartheta_w(s)\,ds + \kappa\int_x^1 \exp(-A(s))\,ds$$

with

$$\vartheta_w(x) = -\tfrac{1}{\varepsilon}\int_0^x (bw)(s)\exp\big(A(s)-A(x)\big).$$

Estimates for ϑ_w are obtained using (5)

$$|\vartheta_w(x)| \le \tfrac{\|b\|_\infty C_3}{\varepsilon}\int_0^x \exp\Big(-\frac{\alpha s}{\varepsilon}\Big)\exp\big(A(s)-A(x)\big) \le \frac{C_4}{\varepsilon}\exp\Big(-\frac{\alpha x}{\varepsilon}\Big).$$

The coefficient κ is determined by the boundary condition for $w(0)$ as

$$\kappa = \Big(\gamma_0 - v(0) - \int_0^1 \vartheta_w(s)\,ds\Big)\Big/\beta$$

where

$$\beta = \int_0^1 \exp(-A(s))\,ds \ge \int_0^1 \exp\Big(-\frac{\|a\|_\infty s}{\varepsilon}\Big)ds \ge \frac{\varepsilon}{\|a\|_\infty}.$$

Thus

$$|\kappa| \le \frac{\|a\|_\infty(C_3+\alpha^{-1}C_4)}{\varepsilon}.$$

For w' we have

$$w'(x) = -\vartheta_w(x) - \kappa\exp(-A(x))$$

and therefore

$$|w'(x)| \le \frac{C_5}{\varepsilon}\exp\Big(-\frac{\alpha x}{\varepsilon}\Big) \qquad (x\in(0,1))$$

by the above bounds for κ and ϑ_w. Using the differential equation and our estimates for w and w', we get

$$|w''(x)| \le C\varepsilon^{-2}\exp\Big(-\frac{\alpha x}{\varepsilon}\Big) \qquad (x\in(0,1)).$$

If $a, b \in C^1(0,1)$, then we differentiate $(3)_b$ and apply our bounds for w, w' and w'' to get

$$|w'''(x)| \le C\varepsilon^{-3}\exp\Big(-\frac{\alpha x}{\varepsilon}\Big) \qquad (x\in(0,1)).$$

We summarize our results as follows.

Theorem 1. *Let $a, b, f \in C^k$ with $k \in \{0,1\}$. Then $u \in C^{k+2}$ can be decomposed as $u = v + w$ where the regular solution component v satisfies*

$$\left.\begin{aligned} &\mathcal{L}v(x) = f(x) \\ &|v^{(i)}(x)| \le C(1 + \varepsilon^{k+1-i}) \quad (i = 0, 1, \dots, k+2) \end{aligned}\right\} \qquad (x \in (0,1))$$

while the boundary layer component w satisfies

$$\left.\begin{aligned} &\mathcal{L}w(x) = 0 \\ &|w^{(i)}(x)| \le C\varepsilon^{-i} \exp\Big(-\frac{\alpha x}{\varepsilon}\Big) \quad (i = 0, 1, \dots, k+2) \end{aligned}\right\} \qquad (x \in (0,1)).$$

Some applications, e.g. the analysis of higher-order schemes [9] or of extrapolation schemes [4] require decompositions with bounds for derivatives of order greater than three. To derive them note that our boundary condition $(-av' + bv)(0) = f(0)$ imposed on v corresponds to $v''(0) = 0$. To prove Theorem 1 for $k = 2$ we would impose the boundary condition

$$\Big(-\big(a - \varepsilon(a' - b)\big)v' + (b - \varepsilon b')v\Big)(0) = (f - \varepsilon f')(0)$$

instead. This corresponds to setting $v'''(0) = 0$. The operator $\mathcal{L}$ with this boundary condition satisfies a comparison principle too, provided that ε is smaller than some threshold value ε_0. We use this principle to prove the boundedness of v first. Then we proceed as above to get bounds for the derivatives.

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