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Orienting Method for Obstacle Problems

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Abstract. This paper deals with obstacle problems of type

$$\begin{aligned}&\text{minimize } \textstyle\int_\Omega F(x,v,\nabla v)\,dx\\ &\text{subject to } v\in W^{1,p}(\Omega), v\geq r \text{ in } \Omega, v=g \text{ on } \partial\Omega\end{aligned}$$

where $\Omega\subset\mathbb{R}^n$ is a bounded open set and $r,g\in W^{1,p}(\Omega)$ $(1\leq p\leq\infty)$. To state some sufficient criteria for determining parts of the coincidence set $\mathcal{C}(u)=\{x\in\Omega:\ u(x)=r(x)\}$ and of the non-coincidence set $\mathcal{N}(u)=\{x\in\Omega:\ u(x)>r(x)\}$ of the optimal solution u to this obstacle problem, optimal solutions to some particular auxiliary problems without obstacle

$$\begin{aligned}&\text{minimize } \textstyle\int_{\widetilde\Omega} F(x,v,\nabla v)\,dx\\ &\text{subject to } v\in \mathcal{K}^{\widetilde\Omega,\tilde g}=\{v\in W^{1,p}(\widetilde\Omega):\ v=\tilde g \text{ on } \partial\widetilde\Omega\}\end{aligned}$$

are used as orienting tool. For this purpose, we do not assume any coercive assumption, but only the uniqueness of the optimal solution to auxiliary problems, which is ensured if e.g. the performance index is strictly convex in $\mathcal{K}^{\widetilde\Omega,\tilde g}$.

Keywords: *Obstacle problems, variation problems, variational inequality, coincidence and non-coincidence set, orienting method*

AMS subject classification: Primary 47J20, 49K20, secondary 35G30, 49S05

1. Introduction

Numerous mathematical and physical problems can be formulated as the minimum problem

$$\begin{aligned}&\text{minimize } \textstyle\int_\Omega F(x,v,\nabla v)\,dx\\ &\text{subject to } v\in\mathcal{K}\end{aligned}$$

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where Ω is some bounded open set in $\mathbb{R}^n$. For instance, the well known Dirichlet problem reads as

$$\begin{aligned}&\text{minimize } \int_\Omega |\nabla v|^2\,dx\\&\text{subject to } v = g \text{ on } \partial\Omega\end{aligned}$$

where $|\cdot|$ denotes the Euclidean norm. The corresponding Euler equation for its optimal solution u is the boundary value problem

$$\left.\begin{aligned}\triangle u = 0 &\quad \text{in } \Omega\\ u = g &\quad \text{on } \partial\Omega\end{aligned}\right\} \tag{1.1}$$

(see, e.g., [16]). If the state function v has to satisfy an additional restriction like $v \geq r$, there arises a so-called *obstacle problem*

$$\begin{aligned}&\text{minimize } \int_\Omega |\nabla v|^2 dx\\&\text{subject to } v \in \mathcal{K} = \big\{v \in W^{1,2}(\Omega) :\ v \geq r \text{ in } \Omega, v = g \text{ on } \partial\Omega\big\}.\end{aligned} \tag{1.2}$$

Because of the restriction $v \geq r$, this obstacle problem does not lead to boundary value problem (1.1), but to the variational inequality

$$u \in \mathcal{K} : \quad \int_\Omega \nabla u \cdot \nabla(v - u)\,dx \geq 0 \text{ for all } v \in \mathcal{K} \tag{1.3}$$

(see [7, 15]). Such a problem is more complicated than (1.1). In fact, the partial differential equation $\triangle u = 0$ is still valid in the so-called non-coincidence set

$$\mathcal{N}(u) = \big\{x \in \Omega :\ u(x) > r(x)\big\}.$$

Since u is determined in the coincidence set

$$\mathcal{C}(u) = \big\{x \in \Omega :\ u(x) = r(x)\big\} = \Omega \setminus \mathcal{N}(u),$$

under the continuity assumption of first derivatives, one still has to consider the remaining problem

$$\left.\begin{aligned}\triangle u = 0 &\qquad \text{in } \mathcal{N}(u)\\ u = g &\qquad \text{on } \partial\Omega \cap \partial\mathcal{N}(u)\\ u = r, \tfrac{\partial u}{\partial n} = \tfrac{\partial r}{\partial n} &\qquad \text{on } \Omega \cap \partial\mathcal{N}(u)\end{aligned}\right\}. \tag{1.4}$$

Since $\Omega \cap \partial\mathcal{N}(u)$ is not known *a priori*, it is called a *free boundary* (see [15: p. 5]). This notation may cause misunderstanding for strangers. In concrete examples, it is by nature not *free* at all, but already fixed by the given problem statement. The only problem is that one does not know something about it *a priori*.

Here, "*a priori*" normally means "before solving (1.2) or (1.3) or (1.4)". But in this paper we state some sufficient criteria for determining parts of coincidence and non-coincidence sets $\mathcal{C}(u)$ and $\mathcal{N}(u)$, without solving the original obstacle problem (1.2) or variational inequality (1.3) or its corresponding free boundary problem (1.4).

The idea originated from the so-called *Method of Orienting Curves* which was developed in [3, 4, 9, 12, 14] for solving optimal control problems with state constraints. Its application area consists of problems with ordinary differential equations (i.e. with one independent variable) having one state function. Although this area is rather narrow, we had successfully applied this method for solving some relevant problems, such as constrained Zermelo's navigation problem [10], Steiner's problem of finding an inpolygon of some given convex polygon with minimal circumference [11], inventory problem [12], optimal control of hydroelectric power plants [5], and robot motion along a prescribed trajectory [13]. By this method, following so-called orienting curves, optimal trajectories are constructed part by part.

In this paper, we investigate problems with several independent variables. Thus surfaces appear instead of curves. Therefore, the shortened name *"Orienting Method"* is more appropriate. It is understandable if we cannot obtain such a complete result as in the case with one independent variable. But in a similar way, barrier functions and bottle neck points can be used to locate some coincidence and non-coincidence points of optimal solutions.

In Section 2, after formulating the class of obstacle problems and their auxiliary problems without obstacle, we show some sufficient conditions for fulfilling the most important assumption $(\mathcal{A}_U)$, which requires that auxiliary problems have at most one optimal solution. The notions of "barriers" and "bottle-neck points" are introduced in Section 3 and then used to state some sufficient criteria for coincidence and non-coincidence points. Section 4 is devoted to examples of use. A special class of problems satisfying some invariance assumption $(\mathcal{A}_I)$ and a concrete numerical example are considered there.

2. The uniqueness of solution to problems without obstacle

Let $\Omega \subset \mathbb{R}^n$ be a bounded open set. For given $r, g \in W^{1,p}(\Omega)$ $(1 \le p \le \infty)$ satisfying $g \ge r$ on $\partial\Omega$ denote

$$\mathcal{K}_r^{\Omega,g} = \big\{v \in W^{1,p}(\Omega) : v \ge r \text{ in } \Omega,\ v = g \text{ on } \partial\Omega\big\}. \tag{2.1}$$

Consider the *obstacle problem*

$$\begin{aligned}&\text{minimize } \mathcal{F}^{\Omega}(v) = \textstyle\int_{\Omega} F(x, v, \nabla v)\, dx \\ &\text{subject to } v \in \mathcal{K}_r^{\Omega,g}\end{aligned} \qquad (\mathcal{P}_r^{\Omega,g})$$

where $F : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$. Our goal is to determine some parts of the *non-coincidence set*

$$\mathcal{N}(u_r^{\Omega,g}) = \big\{x \in \Omega : \, u_r^{\Omega,g}(x) > r(x)\big\} \tag{2.2}$$

and of the *coincidence set*

$$\mathcal{C}(u_r^{\Omega,g}) = \Omega \setminus \mathcal{N}(u_r^{\Omega,g}) = \big\{x \in \Omega : \, u_r^{\Omega,g}(x) = r(x)\big\} \tag{2.3}$$

of the optimal solution $u_r^{\Omega,g}$ to problem $(\mathcal{P}_r^{\Omega,g})$, whose elements are called non-coincidence or coincidence points, respectively. Note that the inequalities and the equality in (2.1) - (2.3) are in the sense of $W^{1,p}(\Omega)$ (compare, for instance, [7]).

To avoid difficulties caused by the obstacle $v \geq r$ we do not deal directly with obstacle problem $(\mathcal{P}_r^{\Omega,g})$, but investigate corresponding *auxiliary problems without obstacle*

$$\begin{aligned}&\text{minimize } \mathcal{F}^{\widetilde{\Omega}}(v) = \textstyle\int_{\widetilde{\Omega}} F(x, v, \nabla v)\, dx \\ &\text{subject to } v \in \mathcal{K}^{\widetilde{\Omega},\tilde{g}}\end{aligned} \qquad (\mathcal{P}^{\widetilde{\Omega},\tilde{g}})$$

where $\widetilde{\Omega}$ is some open subset of Ω, $\tilde{g} \in W^{1,p}(\widetilde{\Omega})$, and

$$\mathcal{K}^{\widetilde{\Omega},\tilde{g}} = \big\{v \in W^{1,p}(\widetilde{\Omega}) : \, v = \tilde{g} \text{ on } \partial\widetilde{\Omega}\big\}. \tag{2.4}$$

These problems are complicated enough, but they are easier than the original one.

The most essential assumption needed in this paper is concerned with the uniqueness of optimal solutions to problems without obstacle, namely:

$(\mathcal{A}_U)$ For each $\tilde{g} \in W^{1,p}(\widetilde{\Omega})$, the corresponding problem $(\mathcal{P}^{\widetilde{\Omega},\tilde{g}})$ admits *at most one optimal solution*. More precisely, if $u_1, u_2 \in \mathcal{K}^{\widetilde{\Omega},\tilde{g}}$ and $\mathcal{F}^{\widetilde{\Omega}}(u_1) = \mathcal{F}^{\widetilde{\Omega}}(u_2) = \inf_{v \in \mathcal{K}^{\widetilde{\Omega},\tilde{g}}} \mathcal{F}^{\widetilde{\Omega}}(v)$, then $u_1 = u_2$ in $W^{1,p}(\widetilde{\Omega})$.

The simplest sufficient condition for $(\mathcal{A}_U)$ is the following.

Proposition 2.1. *Suppose*

$$(v,w) \in \mathbb{R} \times \mathbb{R}^n \mapsto F(x,v,w) \in \mathbb{R} \text{ is strictly convex for a.a. } x \in \widetilde{\Omega}. \tag{2.5}$$

Then condition $(\mathcal{A}_U)$ holds true.

Proof. Let $u_1, u_2 \in \mathcal{K}^{\widetilde{\Omega},\tilde{g}}$. Then (2.5) implies for $\lambda_1, \lambda_2 > 0$ satisfying $\lambda_1 + \lambda_2 = 1$

$$\begin{aligned} \lambda_1 \mathcal{F}^{\widetilde{\Omega}}(u_1) + \lambda_2 \mathcal{F}^{\widetilde{\Omega}}(u_2) &= \int_{\widetilde{\Omega}} \big(\lambda_1 F(x,u_1,\nabla u_1) + \lambda_2 F(x,u_2,\nabla u_2)\big) dx \\ &\geq \int_{\widetilde{\Omega}} F\big(x, \lambda_1 u_1 + \lambda_2 u_2, \nabla(\lambda_1 u_1 + \lambda_2 u_2)\big) dx \\ &= \mathcal{F}^{\widetilde{\Omega}}(\lambda_1 u_1 + \lambda_2 u_2) \end{aligned}$$

where equality only holds true if

$$\lambda_1 F(x,u_1,\nabla u_1) + \lambda_2 F(x,u_2,\nabla u_2) = F\big(x, \lambda_1 u_1 + \lambda_2 u_2, \nabla(\lambda_1 u_1 + \lambda_2 u_2)\big)$$

almost everywhere in $\widetilde{\Omega}$, which yields $u_1 = u_2$ and $\nabla u_1 = \nabla u_2$ a.e. in $\widetilde{\Omega}$ and therefore $u_1 = u_2$ in $W^{1,p}(\widetilde{\Omega})$. Hence, $\mathcal{F}^{\widetilde{\Omega}}$ is strictly convex on $\mathcal{K}^{\widetilde{\Omega},\tilde{g}}$, which implies immediately condition $(\mathcal{A}_U)$ ∎

Condition (2.5) demands at least that F is strictly convex with respect to both v and w. This strong condition is not necessary to the strict convexity of $\mathcal{F}^{\widetilde{\Omega}}$, as the following proposition shows, whose proof is just the same as to [2: p. 53/Proposition 2.5].

Proposition 2.2. *Let $n = 1$, $\widetilde{\Omega} = (0,1)$, and $\tilde{g} \equiv 0$. Let $F_1, F_2 \in C^\infty(\mathbb{R})$,*

$$F(x,v,w) = F_1(v) + F_2(w) \quad \text{and} \quad \begin{cases} \bar{F}_1 = \inf\{F_1''(v) : v \in \mathbb{R}\} \\ \bar{F}_2 = \inf\{F_2''(w) : w \in \mathbb{R}\}. \end{cases}$$

Then $\mathcal{F}^{\widetilde{\Omega}}$ is strictly convex on $\mathcal{K}^{\widetilde{\Omega},\tilde{g}}$, which implies condition $(\mathcal{A}_U)$ if

$$\bar{F}_2 \geq 0 \quad \text{and} \quad \bar{F}_1 + \pi^2 \bar{F}_2 > 0. \tag{2.6}$$

Obviously, (2.6) is fulfilled if $\bar{F}_2 > 0 > \bar{F}_1 > -\pi^2 \bar{F}_2$, i.e. even if F is not convex with respect to v. For $n > 1$ the following problem class is still big enough which does not require F to be strictly convex with respect to v.

Proposition 2.3. *Let $\widetilde{\Omega} \subset \mathbb{R}^n$ be a bounded open set with Lipschitz boundary and $1 \le p < \infty$. Suppose*

$$\left.\begin{array}{l} F(x,v,w) = F_1(x,v) + F_2(x,w) \\ F_1(x,\cdot) \text{ convex and } F_2(x,\cdot) \text{ strictly convex for a.a. } x \in \widetilde{\Omega} \end{array}\right\}. \tag{2.7}$$

Then $\mathcal{F}^{\widetilde{\Omega}}$ is strictly convex on $\mathcal{K}^{\widetilde{\Omega},\tilde{g}}$ and therefore condition $(\mathcal{A}_U)$ is fulfilled.

Proof. Let $\mathcal{F}_1^{\widetilde{\Omega}}(v) = \int_{\widetilde{\Omega}} F_1(x,v)\,dx$ and $\mathcal{F}_2^{\widetilde{\Omega}}(v) = \int_{\widetilde{\Omega}} F_2(x,\nabla v)\,dx$. Then $\mathcal{F}^{\widetilde{\Omega}} = \mathcal{F}_1^{\widetilde{\Omega}} + \mathcal{F}_2^{\widetilde{\Omega}}$ and $\mathcal{F}_1^{\widetilde{\Omega}}$ is convex on $\mathcal{K}^{\widetilde{\Omega},\tilde{g}}$. Therefore, it suffices to show that $\mathcal{F}_2^{\widetilde{\Omega}}$ is strictly convex on $\mathcal{K}^{\widetilde{\Omega},\tilde{g}}$. For this, let $u_1, u_2 \in \mathcal{K}^{\widetilde{\Omega},\tilde{g}}$ and $\lambda_1, \lambda_2 > 0$ with $\lambda_1 + \lambda_2 = 1$. Since $F_2(x,\cdot)$ is strictly convex for almost all $x \in \widetilde{\Omega}$, we have

$$\begin{aligned} \lambda_1 \mathcal{F}_2^{\widetilde{\Omega}}(u_1) + \lambda_2 \mathcal{F}_2^{\widetilde{\Omega}}(u_2) &= \int_{\widetilde{\Omega}} \big(\lambda_1 F_2(x,\nabla u_1) + \lambda_2 F_2(x,\nabla u_2)\big)dx \\ &\ge \int_{\widetilde{\Omega}} F_2\big(x, \lambda_1 \nabla u_1 + \lambda_2 \nabla u_2\big)dx \\ &= \mathcal{F}_2^{\widetilde{\Omega}}(\lambda_1 u_1 + \lambda_2 u_2) \end{aligned}$$

where equality only holds true if

$$\lambda_1 F_2(x,\nabla u_1) + \lambda_2 F_2(x,\nabla u_2) = F_2(x, \lambda_1 \nabla u_1 + \lambda_2 \nabla u_2)$$

almost everywhere in $\widetilde{\Omega}$, which yields $\nabla u_1 = \nabla u_2$ a.e. in $\widetilde{\Omega}$ and therefore

$$\nabla u_1 = \nabla u_2 \qquad \text{in } L^p(\widetilde{\Omega}). \tag{2.8}$$

Since $u = v = \tilde{g}$ on $\partial\widetilde{\Omega}$, Poincaré inequality (see [2: p. 26]) implies $\|u - v\|_{L^p(\widetilde{\Omega})} \le K \, \|\nabla(u-v)\|_{L^p(\widetilde{\Omega})}$ for some $K > 0$. Therefore, (2.8) implies $\|u - v\|_{L^p(\Omega)} = 0$ and

$$\|u-v\|_{W^{1,p}(\widetilde{\Omega})} = \big(\|u-v\|_{L^p(\widetilde{\Omega})^p} + \|\nabla(u-v)\|^p_{L^p(\widetilde{\Omega})}\big)^{1/p} = 0.$$

This means that $\mathcal{F}_2^{\widetilde{\Omega}}$ is strictly convex on $\mathcal{K}^{\widetilde{\Omega},\tilde{g}}$ ∎

Numerous relevant obstacle problems fulfill (2.7) and therefore condition $(\mathcal{A}_U)$. For instance, (2.7) is satisfied by the Dirichlet problem where

$$\mathcal{F}^{\widetilde{\Omega}}(v) = \int_{\widetilde{\Omega}} \big(F_1(x,v) + F_2(x,\nabla v)\big)dx \quad \text{with } \begin{cases} F_1(x,v) = f(x)\,v \\ F_2(x,w) = |w|^2 \end{cases} \tag{2.9}$$

Another example is the problem of minimal surfaces (see, e.g., [1, 8, 15]) where

$$\mathcal{F}^{\widetilde{\Omega}}(v) = \int_{\widetilde{\Omega}} F_2(x, \nabla v)\, dx \quad \text{with } F_2(x,w) = \sqrt{1+|w|^2}. \tag{2.10}$$

Let I be the unit $(n \times n)$-matrix. Then the Hessian matrix

$$H(x,w) = \Big(\frac{\partial^2}{\partial w_i\, \partial w_j} F_2(x,w)\Big)_{1\le i,j\le n} = (1+|w|^2)^{-\frac{3}{2}}\big((1+|w|^2)I - (w_i\, w_j)_{1\le i,j\le n}\big)$$

is positively definite because it follows from Schwarz inequality (see [16: p. 8]) for $\eta \in \mathbb{R}^n \setminus \{0\}$ that

$$\eta^T H(x,w)\, \eta = (1+|w|^2)^{-\frac{3}{2}}\big(|\eta|^2 + |w|^2|\eta|^2 - (w\cdot\eta)^2\big) \ge \big(1+|w|^2\big)^{-\frac{3}{2}} |\eta|^2 > 0.$$

Hence, $F_2(x,\cdot)$ is strictly convex and (2.7) is satisfied.

For the problem of finding the equilibrium position of a deformation membrane we have with $\sigma > 0$

$$\mathcal{F}^{\widetilde{\Omega}}(v) = \int_{\widetilde{\Omega}} \big(F_1(x,v) + F_2(x,\nabla v)\big) dx \quad \text{with } \begin{cases} F_1(x,v) = f(x)\, v \\ F_2(x,w) = \sigma\sqrt{1+|w|^2} \end{cases} \tag{2.11}$$

(see [15: pp. 1 - 2]). Obviously, according to the previous example, (2.7) is also fulfilled here.

To complete this section, let us state a local optimal property of optimal solutions which will be used in Sections 3 - 4.

Proposition 2.4. *Suppose* $u^{\widetilde{\Omega},\tilde{g}} \in \mathcal{K}^{\widetilde{\Omega},\tilde{g}}$ *is optimal to problem* $(\mathcal{P}^{\widetilde{\Omega},\tilde{g}})$ *and* $\hat{\Omega} \subset \widetilde{\Omega}$ *is open. Let* $\hat{u}$ *and* $\hat{g}$ *be the restrictions of* $u^{\widetilde{\Omega},\tilde{g}}$ *on* $\hat{\Omega}$ *or* $\partial\hat{\Omega}$*, respectively. Then* $\hat{u}$ *is optimal to problem* $(\mathcal{P}^{\hat{\Omega},\hat{g}})$*. If condition* $(\mathcal{A}_U)$ *holds true (for the just mentioned* $\widetilde{\Omega}$*), then* $\hat{u}$ *is the unique optimal solution to problem* $(\mathcal{P}^{\hat{\Omega},\hat{g}})$*.*

Proof. Let u be arbitrary in $\mathcal{K}^{\hat{\Omega},\hat{g}} \setminus \{\hat{u}\}$. Then for the extension $u(x) = u^{\widetilde{\Omega},\tilde{g}}(x)$ for $x \in \widetilde{\Omega} \setminus \hat{\Omega}$) there holds

$$u \in \mathcal{K}^{\widetilde{\Omega},\tilde{g}} \setminus \{u^{\widetilde{\Omega},\tilde{g}}\}. \tag{2.12}$$

Since $u^{\widetilde{\Omega},\tilde{g}}$ is optimal to problem $(\mathcal{P}^{\widetilde{\Omega},\tilde{g}})$, we have

$$\begin{aligned} 0 &\le \int_{\widetilde{\Omega}} F(x,u,\nabla u)\, dx - \int_{\widetilde{\Omega}} F(x, u^{\widetilde{\Omega},\tilde{g}}, \nabla u^{\widetilde{\Omega},\tilde{g}})\, dx \\ &= \int_{\hat{\Omega}} F(x,u,\nabla u)\, dx - \int_{\hat{\Omega}} F(x,\hat{u},\nabla\hat{u})\, dx, \end{aligned} \tag{2.13}$$

which implies that $\hat{u} \in \mathcal{K}^{\hat{\Omega},\hat{g}}$ is optimal to problem $(\mathcal{P}^{\hat{\Omega},\hat{g}})$. If condition $(\mathcal{A}_U)$ holds true, then (2.12) - (2.13) imply $\int_{\hat{\Omega}} F(x,u,\nabla u)\, dx > \int_{\hat{\Omega}} F(x,\hat{u},\nabla\hat{u})\, dx$ (for arbitrary $u \in \mathcal{K}^{\hat{\Omega},\hat{g}} \setminus \{\hat{u}\}$), i.e. $\hat{u}$ is the unique optimal solution to problem $(\mathcal{P}^{\hat{\Omega},\hat{g}})$ ∎

3. Barriers and bottle-neck points

As mentioned in Section 2, we now consider some particular minimum problems without obstacle and use their solutions as orientation tool.

Definition 3.1. Let $\widetilde{\Omega}$ be an open subset of Ω and

$$\tilde{g}(x) \begin{cases} \in [r(x), g(x)] & \text{if } x \in \partial\widetilde{\Omega} \cap \partial\Omega \\ = r(x) & \text{if } x \in \partial\widetilde{\Omega} \cap \Omega. \end{cases} \tag{3.1}$$

Then $u^{\widetilde{\Omega},\tilde{g}} \in \mathcal{K}^{\widetilde{\Omega},\tilde{g}}$ is said to be a *lower barrier* if it is the unique optimal solution to problem $(\mathcal{P}^{\widetilde{\Omega},\tilde{g}})$.

The reason for calling such a function as a lower barrier is given in the following.

Proposition 3.1. *Let $u_r^{\Omega,g}$ be optimal to problem $(\mathcal{P}_r^{\Omega,g})$ and $u^{\widetilde{\Omega},\tilde{g}}$ be a lower barrier. Then*

$$u_r^{\Omega,g} \geq u^{\widetilde{\Omega},\tilde{g}} \qquad \text{in } \widetilde{\Omega}. \tag{3.2}$$

Proof. Define

$$u_0(x) = \begin{cases} u^{\widetilde{\Omega},\tilde{g}}(x) & \text{if } x \in \operatorname{cl}\widetilde{\Omega} \\ r(x) & \text{if } x \in \operatorname{cl}\Omega \setminus \operatorname{cl}\widetilde{\Omega}. \end{cases} \tag{3.3}$$

Then $u_0 \in W^{1,p}(\Omega)$ and $u_0 \leq g$ on $\partial\Omega$. For

$$\begin{aligned} u_1 &:= \min(u_r^{\Omega,g}, u_0) = u_0 - (u_0 - u_r^{\Omega,g})^+ \\ u_2 &:= \max(u_r^{\Omega,g}, u_0) = u_r^{\Omega,g} + (u_0 - u_r^{\Omega,g})^+ \end{aligned} \tag{3.4}$$

we have $u_1, u_2 \in W^{1,p}(\Omega)$ and

$$\begin{aligned} \nabla u_1 &= \begin{cases} \nabla u_r^{\Omega,g} & \text{a.e. in } \{x \in \Omega : u_0(x) > u_r^{\Omega,g}(x)\} \\ \nabla u_0 & \text{a.e. in } \{x \in \Omega : u_0(x) \leq u_r^{\Omega,g}(x)\} \end{cases} \\ \nabla u_2 &= \begin{cases} \nabla u_0 & \text{a.e. in } \{x \in \Omega : u_0(x) > u_r^{\Omega,g}(x)\} \\ \nabla u_r^{\Omega,g} & \text{a.e. in } \{x \in \Omega : u_0(x) \leq u_r^{\Omega,g}(x)\} \end{cases} \end{aligned} \tag{3.5}$$

(compare [7: p. 50] and [15: p. 65]). It follows from $u_0(x) = r(x) \leq u_r^{\Omega,g}(x)$ for $x \in \operatorname{cl}\Omega \setminus \operatorname{cl}\widetilde{\Omega}$ that

$$\begin{aligned} \big\{x \in \Omega : u_0(x) > u_r^{\Omega,g}(x)\big\} &= \big\{x \in \widetilde{\Omega} : u_0(x) > u_r^{\Omega,g}(x)\big\} \\ &= \big\{x \in \widetilde{\Omega} : u^{\widetilde{\Omega},\tilde{g}}(x) > u_r^{\Omega,g}(x)\big\} \\ &= \big\{x \in \widetilde{\Omega} : u^{\widetilde{\Omega},\tilde{g}}(x) > u_1(x)\big\} \end{aligned} \tag{3.6}$$

and, by (3.1),

$$u_1 = u_0 = u^{\widetilde{\Omega},\tilde{g}} = \tilde{g} \qquad \text{on } \partial\tilde{\Omega}. \tag{3.7}$$

Assume now the contrary that (3.2) does not hold true. Then the measure of the set $\{x \in \widetilde{\Omega} : u^{\widetilde{\Omega},\tilde{g}}(x) > u_r^{\Omega,g}(x)\}$ is positive. Since $u^{\widetilde{\Omega},\tilde{g}}$ is the unique optimal solution to problem $(\mathcal{P}^{\widetilde{\Omega},\tilde{g}})$, (3.3) and (3.7) imply

$$\begin{aligned} 0 &< \int_{\widetilde{\Omega}} F(x, u_1, \nabla u_1)\, dx - \int_{\widetilde{\Omega}} F(x, u^{\widetilde{\Omega},\tilde{g}}, \nabla u^{\tilde{\Omega},\tilde{g}})\, dx \\ &= \int_{\widetilde{\Omega}} \big(F(x, u_1, \nabla u_1) - F(x, u_0, \nabla u_0)\big)\, dx. \end{aligned}$$

Therefore, (3.4) - (3.6) yield

$$\begin{aligned} 0 &< \int_{\{x\in\widetilde{\Omega}: u_0(x) > u_r^{\Omega,g}(x)\}} \big(F(x, u_1, \nabla u_1) - F(x, u_0, \nabla u_0)\big)\, dx \\ &= \int_{\{x\in\widetilde{\Omega}: u_0(x) > u_r^{\Omega,g}(x)\}} \big(F(x, u_r^{\Omega,g}, \nabla u_r^{\Omega,g}) - F(x, u_2, \nabla u_2)\big)\, dx \\ &= \int_{\{x\in\Omega: u_0(x) > u_r^{\Omega,g}(x)\}} \big(F(x, u_r^{\Omega,g}, \nabla u_r^{\Omega,g}) - F(x, u_2, \nabla u_2)\big)\, dx \\ &= \int_{\Omega} \big(F(x, u_r^{\Omega,g}, \nabla u_r^{\Omega,g}) - F(x, u_2, \nabla u_2)\big)\, dx, \end{aligned}$$

that means

$$\int_{\Omega} F(x, u_r^{\Omega,g}, \nabla u_r^{\Omega,g})\, dx > \int_{\Omega} F(x, u_2, \nabla u_2)\, dx$$

when $u_2 \in \mathcal{K}_r^{\Omega,g}$, because $u_2 \in W^{1,p}(\Omega)$, $u_2 \geq r$ in Ω and $u_2 = g$ on $\partial\Omega$. This conflicts with the assumption that $u_r^{\Omega,g}$ is optimal to problem $(\mathcal{P}_r^{\Omega,g})$. Hence, (3.2) must be true ∎

The preceding result is useful for finding subsets of non-coincidence points. By denoting

$$\mathcal{L}^+(v, \widetilde{\Omega}) = \{x \in \widetilde{\Omega} : v(x) > 0\} \tag{3.8}$$

we have

Corollary 3.2. *Let $\widetilde{\Omega} \subset \Omega$ be open and satisfy condition $(\mathcal{A}_U)$. Suppose*

$$\tilde{g} = r \qquad \text{on } \partial\widetilde{\Omega} \tag{3.9}$$

and $u^{\widetilde{\Omega},\tilde{g}}$ *is optimal to problem* $(\mathcal{P}^{\widetilde{\Omega},\tilde{g}})$. *Then for the non-coincidence set of the optimal solution* $u_r^{\Omega,g}$ *to problem* $(\mathcal{P}_r^{\Omega,g})$

$$\mathcal{L}^+(u^{\widetilde{\Omega},\tilde{g}} - r, \widetilde{\Omega}) \subset \mathcal{N}(u_r^{\Omega,g}). \tag{3.10}$$

Proof. Since $g \geq r$ on $\partial\Omega$, (3.1) follows from (3.9). By definition and condition $(\mathcal{A}_U)$, $u^{\widetilde{\Omega},\tilde{g}}$ is the unique optimal solution to problem $(\mathcal{P}^{\widetilde{\Omega},\tilde{g}})$ and therefore it defines a lower barrier. Hence, Proposition 3.1 yields $u_r^{\Omega,g}(x) \geq u^{\widetilde{\Omega},\tilde{g}}(x) > r(x)$ for $x \in \mathcal{L}^+(u^{\widetilde{\Omega},\tilde{g}} - r, \widetilde{\Omega})$, that means (3.10) is fulfilled ∎

Corollary 3.3. *Suppose condition* $(\mathcal{A}_U)$ *is fulfilled for* $\widetilde{\Omega} = \Omega$ *and* $r \leq \tilde{g} \leq g$ *on* $\partial\Omega$. *Let* $u_r^{\Omega,g}$ *be optimal to problem* $(\mathcal{P}_r^{\Omega,g})$ *and* $u^{\Omega,\tilde{g}}$ *be optimal to problem* $(\mathcal{P}^{\Omega,\tilde{g}})$. *Then*

$$u_r^{\Omega,g} \geq u^{\Omega,\tilde{g}} \qquad \text{in } \Omega \tag{3.11}$$

and

$$\mathcal{L}^+(u^{\Omega,\tilde{g}} - r, \Omega) \subset \mathcal{N}(u_r^{\Omega,g}). \tag{3.12}$$

Proof. By condition $(\mathcal{A}_U)$, $u^{\Omega,\tilde{g}}$ is the unique optimal solution to problem $(\mathcal{P}^{\Omega,\tilde{g}})$. Therefore, by definition, $u^{\Omega,\tilde{g}}$ is a lower barrier. Hence (2.2), (3.2) and (3.8) yield immediately (3.11) - (3.12) ∎

Next, let us introduce another notion which is useful for locating some parts of the coincidence set.

Definition 3.2. Suppose

$$\tilde{g} \geq g \qquad \text{on } \partial\Omega. \tag{3.13}$$

Then $u^{\Omega,\tilde{g}}$ is said to be an *upper barrier* provided it is the unique optimal solution to problem $(\mathcal{P}^{\Omega,\tilde{g}})$ and satisfies

$$u^{\Omega,\tilde{g}} \geq r \qquad \text{in } \Omega. \tag{3.14}$$

Proposition 3.4. *Let* $u_r^{\Omega,g}$ *be optimal to problem* $(\mathcal{P}_r^{\Omega,g})$ *and* $u^{\Omega,\tilde{g}}$ *be an upper barrier. Then*

$$u_r^{\Omega,g} \leq u^{\Omega,\tilde{g}} \quad \text{in} \quad \Omega. \tag{3.15}$$

Proof. Define

$$\begin{aligned} u_1 &:= \min(u_r^{\Omega,g}, u^{\Omega,\tilde{g}}) = u_r^{\Omega,g} - (u_r^{\Omega,g} - u^{\Omega,\tilde{g}})^+ \\ u_2 &:= \max(u_r^{\Omega,g}, u^{\Omega,\tilde{g}}) = u^{\Omega,\tilde{g}} + (u_r^{\Omega,g} - u^{\Omega,\tilde{g}})^+. \end{aligned} \tag{3.16}$$

Relations (3.13) - (3.14) and (3.16) imply

$$\begin{aligned} &u_1 \in W^{1,p}(\Omega),\ u_1 \geq r \text{ in } \Omega,\ u_1 = g \text{ on } \partial\Omega, \quad \text{i.e. } u_1 \in \mathcal{K}_r^{\Omega,g} \\ &u_2 \in W^{1,p}(\Omega),\ u_2 = \tilde{g} \text{ on } \partial\Omega, \quad \text{i.e. } u_2 \in \mathcal{K}^{\Omega,\tilde{g}} \end{aligned}$$

and

$$\nabla u_1 = \begin{cases} \nabla u^{\Omega,\tilde{g}} & \text{a.e. in } \{x \in \Omega : u_r^{\Omega,g}(x) > u^{\Omega,\tilde{g}}(x)\} \\ \nabla u_r^{\Omega,g} & \text{a.e. in } \{x \in \Omega : u_r^{\Omega,g}(x) \le u^{\Omega,\tilde{g}}(x)\} \end{cases}$$
$$\nabla u_2 = \begin{cases} \nabla u_r^{\Omega,g} & \text{a.e. in } \{x \in \Omega : u_r^{\Omega,g}(x) > u^{\Omega,\tilde{g}}(x)\} \\ \nabla u^{\Omega,\tilde{g}} & \text{a.e. in } \{x \in \Omega : u_r^{\Omega,g}(x) \le u^{\Omega,\tilde{g}}(x)\} \end{cases} \tag{3.17}$$

(compare [7: p. 50] and [15: p. 65]).

Assume now the contrary that (3.15) does not hold true. Then the measure of the set $\{x \in \Omega : u_r^{\Omega,g}(x) > u^{\Omega,\tilde{g}}(x)\}$ must be positive. Since $u^{\Omega,\tilde{g}}$ is the unique optimal solution to problem $(\mathcal{P}^{\Omega,\tilde{g}})$, (3.16) - (3.17) yield

$$\begin{aligned} 0 &< \int_\Omega F(x, u_2, \nabla u_2)\, dx - \int_\Omega F(x, u^{\Omega,\tilde{g}}, \nabla u^{\Omega,\tilde{g}})\, dx \\ &= \int_{\{x\in\Omega: u_r^{\Omega,g}(x) > u^{\Omega,\tilde{g}}(x)\}} \big(F(x, u_2, \nabla u_2) - F(x, u^{\Omega,\tilde{g}}, \nabla u^{\Omega,\tilde{g}})\big) dx \\ &= \int_{\{x\in\Omega: u_r^{\Omega,g}(x) > u^{\Omega,\tilde{g}}(x)\}} \big(F(x, u_r^{\Omega,g}, \nabla u_r^{\Omega,g}) - F(x, u_1, \nabla u_1)\big) dx \\ &= \int_\Omega \big(F(x, u_r^{\Omega,g}, \nabla u_r^{\Omega,g}) - F(x, u_1, \nabla u_1)\big) dx. \end{aligned}$$

That means

$$\int_\Omega F(x, u_r^{\Omega,g}, \nabla u_r^{\Omega,g})\, dx > \int_\Omega F(x, u_1, \nabla u_1)\, dx$$

when $u_1 \in \mathcal{K}_r^{\Omega,g}$, which conflicts with the assumption that $u_r^{\Omega,g}$ is optimal to problem $(\mathcal{P}_r^{\Omega,g})$. Hence, (3.15) must be true ∎

Definition 3.3. $x^* \in \Omega$ is called a *bottle-neck point* provided there exists an upper barrier $u^{\Omega,\tilde{g}}$ satisfying $u^{\Omega,\tilde{g}}(x^*) = r(x^*)$.

Proposition 3.5. *Let $u_r^{\Omega,g}$ be optimal to problem $(\mathcal{P}_r^{\Omega,g})$ and x^* be a bottle-neck point. Then $x^* \in \mathcal{C}(u_r^{\Omega,g})$, i.e. $u_r^{\Omega,g}(x^*) = r(x^*)$.*

Proof. By definition, there exists an upper barrier $u^{\Omega,\tilde{g}}$ satisfying $r(x^*) = u^{\Omega,\tilde{g}}(x^*)$. Proposition 3.4 implies $u^{\Omega,\tilde{g}}(x^*) \ge u_r^{\Omega,g}(x^*)$. Since $u_r^{\Omega,g}(x^*) \ge r(x^*)$, it follows immediately $u_r^{\Omega,g}(x^*) = r(x^*)$ ∎

Proposition 3.6. *Let $u_r^{\Omega,g}$ be optimal to problem $(\mathcal{P}_r^{\Omega,g})$ and $u^{\Omega,\tilde{g}}$ be optimal to problem $(\mathcal{P}^{\Omega,\tilde{g}})$ where*

$$\tilde{g} \ge g \quad \text{on } \partial\Omega \qquad \text{and} \qquad u^{\Omega,\tilde{g}} \ge r \quad \text{in } \Omega. \tag{3.18}$$

Assume that condition $(\mathcal{A}_U)$ *holds true for* $\widetilde{\Omega} = \Omega$ *and there is an open subset* $\hat{\Omega} \subset \Omega$ *such that*

$$u^{\Omega,\tilde{g}} = \begin{cases} g & \text{on } \partial\hat{\Omega} \cap \partial\Omega \\ r & \text{on } \partial\hat{\Omega} \cap \Omega. \end{cases} \tag{3.19}$$

Then

$$u_r^{\Omega,g} = u^{\Omega,\tilde{g}} \qquad \text{in } \hat{\Omega}. \tag{3.20}$$

Proof. Let $\hat{g}$ and $u^{\hat{\Omega},\hat{g}}$ be the restrictions of $u^{\Omega,\tilde{g}}$ on $\partial\hat{\Omega}$ and $\hat{\Omega}$, respectively. Then, by Proposition 2.4, $u^{\hat{\Omega},\hat{g}}$ is the unique optimal solution to problem $(\mathcal{P}^{\hat{\Omega},\hat{g}})$. Consequently, by definition and (3.19), $u^{\hat{\Omega},\hat{g}}$ is a lower barrier. Therefore, Proposition 3.1 implies

$$u_r^{\Omega,g} \geq u^{\hat{\Omega},\hat{g}} = u^{\Omega,\tilde{g}} \qquad \text{in } \hat{\Omega}. \tag{3.21}$$

On the other hand, it follows from definition and (3.18) that $u^{\Omega,\tilde{g}}$ is an upper barrier. Thus Proposition 3.4 yields $u_r^{\Omega,g} \leq u^{\Omega,\tilde{g}}$ in Ω. Combining this with (3.21), we obtain (3.20) at once ∎

4. Examples of use

To illustrate the applicability of the result from the previous section, we now consider a special case, where the following invariance assumption is made:

$(\mathcal{A}_I)$ For $\widetilde{\Omega} \subset \Omega$ and every $m \in \mathbb{R}$,

$$\mathcal{F}^{\widetilde{\Omega}}(v+m) - \mathcal{F}^{\widetilde{\Omega}}(v) = \text{const} =: c^{\widetilde{\Omega}}(m) \qquad \text{for all } v \in W^{1,p}(\widetilde{\Omega}). \tag{4.1}$$

Of course, this is a strong restriction. But numerous relevant problems satisfy it. For instance, condition $(\mathcal{A}_I)$ holds for (2.10) because $\mathcal{F}^{\widetilde{\Omega}}(v+m) - \mathcal{F}^{\widetilde{\Omega}}(v) = 0$ and for (2.9) and (2.11) because $\mathcal{F}^{\widetilde{\Omega}}(v+m) - \mathcal{F}^{\widetilde{\Omega}}(v) = m \int_{\widetilde{\Omega}} f\, dx = \text{const}$ for all $v \in W^{1,p}(\widetilde{\Omega})$ and $m \in \mathbb{R}$. In general, if v does not appear explicitly or appears only affinely in $F(x, v, \nabla v)$, then condition $(\mathcal{A}_I)$ is fulfilled.

Actually, $(\mathcal{A}_I)$ belongs to such assumptions which can ensure the continuous dependence of the optimal solution $u^{\widetilde{\Omega},\tilde{g}}$ to problem $(\mathcal{P}^{\widetilde{\Omega},\tilde{g}})$ on the parameter $\tilde{g}$. Moreover, it allows vertical movement without changing the optimal shape, as the following says.

Proposition 4.1. *Let* $u^{\widetilde{\Omega},\tilde{g}}$ *be optimal to problem* $(\mathcal{P}^{\widetilde{\Omega},\tilde{g}})$*. Then, for arbitrary* $m \in \mathbb{R}$, $u^{\widetilde{\Omega},\tilde{g}} + m$ *is optimal to problem* $(\mathcal{P}^{\widetilde{\Omega},\tilde{g}+m})$.

Proof. Relation (4.1) implies

$$\mathcal{F}^{\widetilde{\Omega}}(u^{\widetilde{\Omega},\tilde{g}}+m)=\mathcal{F}^{\tilde{\Omega}}(u^{\tilde{\Omega},\tilde{g}})+c^{\widetilde{\Omega}}(m)\leq\mathcal{F}^{\widetilde{\Omega}}(v)+c^{\widetilde{\Omega}}(m)=\mathcal{F}^{\widetilde{\Omega}}(v+m)$$

for all $v\in\mathcal{K}^{\widetilde{\Omega},\tilde{g}}$. But by (2.4), $v\in\mathcal{K}^{\widetilde{\Omega},\tilde{g}}$ if and only if $v+m\in\mathcal{K}^{\tilde{\Omega},\tilde{g}+m}$. Therefore, $u^{\widetilde{\Omega},\tilde{g}}+m\in\mathcal{K}^{\widetilde{\Omega},\tilde{g}+m}$ and $\mathcal{F}^{\widetilde{\Omega}}(u^{\widetilde{\Omega},\tilde{g}}+m)\leq\mathcal{F}^{\widetilde{\Omega}}(v)$ for all $v\in\mathcal{K}^{\widetilde{\Omega},\tilde{g}+m}$, that means $u^{\widetilde{\Omega},\tilde{g}}+m$ is optimal to problem $(\mathcal{P}^{\widetilde{\Omega},\tilde{g}+m})$ ∎

The above property can be applied to determine some parts of the non-coincidence set.

Proposition 4.2. *Suppose conditions* $(\mathcal{A}_U)$ *and* $(\mathcal{A}_I)$ *hold for* $\widetilde{\Omega}\subset\Omega$, $\hat{\Omega}$ *is an open subset of* $\widetilde{\Omega}$, *and* $u^{\widetilde{\Omega},\tilde{g}}$ *is optimal to problem* $(\mathcal{P}^{\widetilde{\Omega},\tilde{g}})$ *and satisfies*

$$u^{\widetilde{\Omega},\tilde{g}}(x)-r(x)>m=u^{\widetilde{\Omega},\tilde{g}}(y)-r(y)\quad\forall x\in\hat{\Omega},y\in\partial\hat{\Omega}\text{ and some } m\in\mathbb{R}.\tag{4.2}$$

Then $\hat{\Omega}$ *is a subset of the non-coincidence set* $\mathcal{N}(u_r^{\Omega,g})$.

Proof. It follows from conditions $(\mathcal{A}_U)$, $(\mathcal{A}_I)$ and Proposition 4.1 that $u^{\widetilde{\Omega},\tilde{g}}-m$ is the unique optimal solution to problem $(\mathcal{P}^{\widetilde{\Omega},\tilde{g}-m})$. Therefore, by Proposition 2.4 and (4.2), the restriction $\hat{u}$ of $u^{\widetilde{\Omega},\tilde{g}}-m$ on $\hat{\Omega}$ is the unique optimal solution to problem $(\mathcal{P}^{\hat{\Omega},r})$. Consequently, Corollary 3.2 yields $\mathcal{L}^+(\hat{u}-r,\hat{\Omega})\subset\mathcal{N}(u_r^{\Omega,g})$. Since (3.8) and (4.2) imply $\hat{\Omega}=\mathcal{L}^+(u^{\widetilde{\Omega},\tilde{g}}-m-r,\hat{\Omega})=\mathcal{L}^+(\hat{u}-r,\hat{\Omega})$ we have $\hat{\Omega}\subset\mathcal{N}(u_r^{\Omega,g})$ ∎

We mention that $\hat{\Omega}$ given in Proposition 4.2 is an open subset. It is a bit more difficult to ensure a closed subset to be contained in the non-coincidence set. For this purpose, we need the following notion.

Definition 4.1. A closed subset $A\subset\Omega$ is said to be a *locally strictly maximal region* of v provided there exists an open subset $B\subset\Omega$ which contains A and satisfies

$$v(x)>v(y)\qquad\text{for all } x\in A, y\in B\setminus A.\tag{4.3}$$

It follows from (4.3) that $\inf_{x\in\partial A}v(x)\geq\inf_{x\in A}v(x)\geq\sup_{x\in B\setminus A}v(x)$. If v is continuous on B, we have in addition $\sup_{x\in B\setminus A}v(x)=\sup_{x\in(B\setminus A)\cup\partial A}v(x)\geq\sup_{x\in\partial A}v(x)$ which yields $v=\text{const}$ on ∂A. Therefore, in this case, (4.3) is equivalent to $v\equiv a$ on ∂A and $v(x)\geq a>v(y)$ for all $x\in A$ and $y\in B\setminus A$ (for some constant a).

Proposition 4.3. *Let* $u_r^{\Omega,g}$ *be optimal to problem* $(\mathcal{P}_r^{\Omega,g})$ *and* $u^{\widetilde{\Omega},\tilde{g}}$ *be optimal to problem* $(\mathcal{P}^{\widetilde{\Omega},\tilde{g}})$ *for some open* $\widetilde{\Omega} \subset \Omega$ *and some* $\tilde{g} \in W^{1,p}(\widetilde{\Omega})$. *Suppose conditions* $(\mathcal{A}_U)$ *and* $(\mathcal{A}_I)$, *and* $u^{\widetilde{\Omega},\tilde{g}}$ *and* r *are continuous on* $\operatorname{cl}\widetilde{\Omega}$. *If some closed subset* $A \subset \widetilde{\Omega}$ *is a locally strictly maximal region of* $u^{\widetilde{\Omega},\tilde{g}} - r$, *then* A *is a subset of the non-coincidence set* $\mathcal{N}(u_r^{\Omega,g})$.

Proof. By definition, there exists an open subset B satisfying $A \subset B \subset \widetilde{\Omega} \subset \Omega$ and $u^{\widetilde{\Omega},\tilde{g}}(x) - r(x) > u^{\widetilde{\Omega},\tilde{g}}(y) - r(y)$ for all $x \in A$ and $y \in B \setminus A$. Take an arbitrary open subset B' with

$$A \subset B' \subset \operatorname{cl} B' \subset B. \tag{4.4}$$

Since $\partial B'$ is compact and $u^{\widetilde{\Omega},\tilde{g}} - r$ is continuous on $\operatorname{cl}\widetilde{\Omega}$, there exists $x^* \in \partial B'$ such that

$$u^{\widetilde{\Omega},\tilde{g}}(x^*) - r(x^*) = m := \max_{x \in \partial B'} \big(u^{\widetilde{\Omega},\tilde{g}}(x) - r(x)\big).$$

Consider

$$\hat{\Omega} = \big\{x \in B' : u^{\widetilde{\Omega},\tilde{g}}(x) - r(x) > m\big\}.$$

Inclusions (4.4) yield $\partial B' \subset B \setminus A$. This implies by $x^* \in \partial B'$ that

$$u^{\widetilde{\Omega},\tilde{g}}(x) - r(x) > u^{\widetilde{\Omega},\tilde{g}}(x^*) - r(x^*) = m \qquad \text{for all } x \in A \subset B',$$

that means $A \subset \hat{\Omega}$. Moreover, by definition we have

$$u^{\widetilde{\Omega},\tilde{g}}(x) - r(x) > m \geq u^{\widetilde{\Omega},\tilde{g}}(y) - r(y) \qquad \text{for all } x \in \hat{\Omega}, y \in \operatorname{cl} B' \setminus \hat{\Omega}. \tag{4.5}$$

Since $\operatorname{cl} B' \subset B \subset \operatorname{cl}\widetilde{\Omega}$ and $u^{\widetilde{\Omega},\tilde{g}} - r$ is continuous on $\operatorname{cl}\widetilde{\Omega}$, it follows that $\hat{\Omega}$ is open and

$$u^{\widetilde{\Omega},\tilde{g}}(x) - r(x) = m \qquad \text{for } x \in \partial\hat{\Omega}. \tag{4.6}$$

Proposition 2.4 implies that the restriction $u^{\widetilde{\Omega},\tilde{g}}|_{\hat{\Omega}}$ is optimal to problem $(\mathcal{P}^{\hat{\Omega},\hat{g}})$ with $\hat{g} = u^{\widetilde{\Omega},\tilde{g}}|_{\partial\hat{\Omega}}$. Hence, by Proposition 4.1, the restriction of $u^{\widetilde{\Omega},\tilde{g}} - m$ on $\hat{\Omega}$ is optimal to problem $(\mathcal{P}^{\hat{\Omega},\hat{g}-m})$. By (4.6), we have $u^{\widetilde{\Omega},\tilde{g}} - m = r$ on $\partial\hat{\Omega}$. Therefore, the restriction of $u^{\widetilde{\Omega},\tilde{g}} - m$ on $\hat{\Omega}$ is a lower barrier, and Corollary 3.2 yields $\mathcal{L}^+(u^{\widetilde{\Omega},\tilde{g}} - m - r, \hat{\Omega}) \subset \mathcal{N}(u_r^{\Omega,g})$. Since $\hat{\Omega} = \mathcal{L}^+(u^{\widetilde{\Omega},\tilde{g}} - m - r, \hat{\Omega})$ follows from (4.5), we obtain finally $A \subset \hat{\Omega} \subset \mathcal{N}(u_r^{\Omega,g})$ ∎

In particular, if for the optimal solution $u^{\tilde{\Omega},\tilde{g}}$ to some problem $(\mathcal{P}^{\widetilde{\Omega},\tilde{g}})$ and for some neighborhood $\mathcal{U}(x^*)$ of x^*

$$u^{\widetilde{\Omega},\tilde{g}}(x^*) - r(x^*) > u^{\widetilde{\Omega},\tilde{g}}(x) - r(x) \qquad \text{for } x \in \mathcal{U}(x^*) \setminus \{x^*\} \subset \widetilde{\Omega},$$

then (4.3) is valid for $A = \{x^*\}$ and $v = u^{\widetilde{\Omega},\tilde{g}} - r$, i.e. x^* is a *locally strictly maximal point* of $v = u^{\widetilde{\Omega},\tilde{g}} - r$. In this case, Proposition 4.3 ensures $x^* \in \mathcal{N}(u_r^{\Omega,g})$, that means $u_r^{\Omega,g}(x^*) > r(x^*)$.

By the propositions stated above we can locate the non-coincidence set $\mathcal{N}(u_r^{\Omega,g})$. This can be considered as an outer approach to the coincidence set $\mathcal{C}(u_r^{\Omega,g})$.

Let us state a direct approach to the coincidence set now.

Proposition 4.4. *Let $u_r^{\Omega,g}$ be optimal to problem $(\mathcal{P}_r^{\Omega,g})$ and $u^{\Omega,\tilde{g}}$ be optimal to problem $(\mathcal{P}^{\Omega,\tilde{g}})$ for some $\tilde{g} \in W^{1,p}(\Omega)$ satisfying $\tilde{g} \geq g$. Suppose conditions $(\mathcal{A}_U)$ and $(\mathcal{A}_I)$ for $\widetilde{\Omega} = \Omega$. If some $x^* \in \Omega$ fulfils*

$$u^{\Omega,\tilde{g}}(x^*) - r(x^*) = \min_{x\in\Omega} \big(u^{\Omega,\tilde{g}}(x) - r(x)\big) \leq 0, \tag{4.7}$$

then $x^* \in \mathcal{C}(u_r^{\Omega,g})$, that means $u_r^{\Omega,g}(x^*) = r(x^*)$.

Proof. Denote $m = u^{\Omega,\tilde{g}}(x^*) - r(x^*) \leq 0$. By Proposition 4.1, $u^{\Omega,\tilde{g}} - m$ is optimal to problem $(\mathcal{P}^{\Omega,\tilde{g}-m})$. Relation (4.7) implies $\tilde{g} - m \geq \tilde{g} \geq g$ on $\partial\Omega$ and $u^{\Omega,\tilde{g}}(x) - m \geq r(x)$ in Ω, that means $u^{\Omega,\tilde{g}} - m$ is an upper barrier. Since $u^{\Omega,\tilde{g}}(x^*) - m = r(x^*)$, x^* is a bottle-neck point. Therefore, Proposition 3.5 yields $x^* \in \mathcal{C}(u_r^{\Omega,g})$ ∎

Example 4.1. To illustrate easily the conclusions of this paper, let us consider the simple problem

$$\left.\begin{array}{l} \text{minimize } \mathcal{F}^{(-2,3)}(v) = \int_{-2}^{3} (4v + |v'|^2)\, dx \\ \text{subject to } v \in W^{1,2}(-2,3), v \geq |x| \text{ on } (-2,3), v(-2) = 4, v(3) = 5 \end{array}\right\}. \tag{4.8}$$

Here we have $\Omega = (-2,3)$, $r(x) = |x|$, $g(-2) = 4$ and $g(3) = 5$. Due to Proposition 2.3 and the remark after (4.1), conditions $(\mathcal{A}_U)$ and $(\mathcal{A}_I)$ are fulfilled for every open subset $\widetilde{\Omega} \subset \Omega$. For $\widetilde{\Omega} \subset \mathbb{R}$, the embedding $W^{1,2}(\widetilde{\Omega}) \subset C(\mathrm{cl}\widetilde{\Omega})$ is compact (see [16: p. 1027]), therefore $u \in W^{1,2}(\widetilde{\Omega})$ may be considered as continuous on $\mathrm{cl}\,\widetilde{\Omega}$. Hence, we can apply all propositions or corollaries in this paper to (4.8) because all assumptions required are satisfied.

Consider the auxiliary problem

$$\begin{array}{l} \text{minimize } \mathcal{F}^{(t_a,t_b)}(v) = \int_{t_a}^{t_b} (4v + |v'|^2)\, dx \\ \text{subject to } v \in W^{1,2}(t_a,t_b), v(t_a) = a, v(t_b) = b \end{array} \tag{4.9}$$

where $\widetilde{\Omega} = (t_a,t_b) \subset (-2,3) \subset \mathbb{R}$. The corresponding Euler equation

$$2 - v'' = 0 \tag{4.10}$$

yields $v(x) = x^2 + cx + d$ for some $c, d \in \mathbb{R}$. Combining with the boundary condition

$$v = \tilde{g} \text{ on } \partial\widetilde{\Omega} \qquad \text{where } \tilde{g}(x) = \begin{cases} a & \text{for } x = t_a \\ b & \text{for } x = t_b \end{cases} \tag{4.11}$$

we obtain

$$u^{\widetilde{\Omega},\tilde{g}}(x) = x^2 + cx + d \qquad \text{where } \begin{cases} c = \frac{b-a}{t_b-t_a} - t_a - t_b \\ d = \frac{at_b - bt_a}{t_b - t_a} + t_a t_b \end{cases}$$

which satisfies Euler equation (4.10) and boundary condition (4.11). Moreover, 0 is the Gateaux derivative of the convex functional $\mathcal{F}^{(t_a,t_b)}$ at $u^{\widetilde{\Omega},\tilde{g}}$, therefore $0 \in \partial\mathcal{F}^{(t_a,t_b)}(u^{\widetilde{\Omega},\tilde{g}})$ (see [6: p. 22 and p. 46]), which is sufficient for $u^{\widetilde{\Omega},\tilde{g}}$ to be optimal to problem $(\mathcal{P}^{\widetilde{\Omega},\tilde{g}})$ given by (4.9) (see [6: p. 81]).

Let us now apply the conclusions stated above.

(a) Clearly, $u^{\Omega,g}(x) = x^2 - 0.8\,x - 1.6 \;\; (x \in \Omega)$ is optimal to problem $(\mathcal{P}^{\Omega,g})$. By Definition 3.1, $u^{\Omega,g}$ is a lower barrier. Therefore, Proposition 3.1 yields for the optimal solution $u_r^{\Omega,g}$ to problem $(\mathcal{P}_r^{\Omega,g})$ given by (4.8) that $u_r^{\Omega,g}(x) \geq x^2 - 0.8\,x - 1.6 \;\; (x \in \Omega)$. Since

$$\begin{aligned}\mathcal{L}^+(u^{\Omega,g} - r, \Omega) &= \big\{x \in (-2,3) : \, x^2 - 0.8\,x - 1.6 - |x| > 0\big\} \\ &= \big(-2, -0.1 - \sqrt{1.61}\big) \cup \big(0.9 + \sqrt{2.41}, 3\big),\end{aligned}$$

Corollary 3.2 implies $\big(-2, -0.1 - \sqrt{1.61}\big) \cup \big(0.9 + \sqrt{2.41}, 3\big) \subset \mathcal{N}(u_r^{\Omega,g})$.

(b) For $z \in [-2 + \sqrt{2}, -0.5], \tilde{g}(-2) = z^2 + 4z + 6, \tilde{g}(3) = z^2 - 6z + 6$ we can show that $u^{\Omega,\tilde{g}}(x) = x^2 - (1+2z)x + z^2$ is optimal to problem $(\mathcal{P}^{\Omega,\tilde{g}})$ and satisfies (3.13) - (3.14). Therefore, by Definition 3.2, $u^{\Omega,\tilde{g}}$ is an upper barrier, that yields by Proposition 3.4 $u_r^{\Omega,g}(x) \leq x^2 - (1+2z)x + z^2$. Moreover, since $u^{\Omega,\tilde{g}}(z) = -z = |z| = r(z)$, according to Definition 3.3, z is a bottle-neck point. Hence, Proposition 3.5 implies $u_r^{\Omega,g}(z) = r(z) = -z$ for $z \in [-2 + \sqrt{2}, -0.5]$, that means $[-2 + \sqrt{2}, -0.5] \subset \mathcal{C}(u_r^{\Omega,g})$.

(c) With $\tilde{g}(-2) = 4.25$ and $\tilde{g}(3) = 9.25$ we have

$$\begin{aligned} &u^{\Omega,\tilde{g}}(x) = x^2 + 0.25 \geq r(x) = |x| \;\; (-2 \leq x \leq 3) \\ &u^{\Omega,\tilde{g}}(\pm 0.5) = 0.5 = |\pm 0.5| = r(\pm 0.5) \end{aligned}$$

i.e. (3.18) - (3.19) hold for $\hat{\Omega} = (-0.5, 0.5)$. Consequently, Proposition 3.6 implies $u_r^{\Omega,g}(x) = x^2 + 0.25$ for $|x| \leq 0.5$.

(d) Assume now $\widetilde{\Omega} = \Omega = (-2,3), \tilde{g}(-2) = 4$ and $\tilde{g}(3) = 9$. Then $u^{\widetilde{\Omega},\tilde{g}}(x) = x^2$ is optimal to problem $(\mathcal{P}^{\widetilde{\Omega},\tilde{g}})$. Since

$$\frac{d}{dx}\big(u^{\widetilde{\Omega},\tilde{g}}(x) - r(x)\big) = \frac{d}{dx}(x^2 - |x|) \begin{cases} > 0 & \text{if } -0.5 < x < 0 \\ < 0 & \text{if } 0 < x < 0.5 \end{cases}$$

each closed interval $[-\alpha, \alpha]$ contained in $(-0.5, 0.5)$ is a locally strictly maximal region. Therefore, Proposition 4.3 yields $[-\alpha, \alpha] \subset \mathcal{N}(u_r^{\Omega,g})$ whenever $|\alpha| < 0.5$. Hence, $(-0.5, 0.5) \subset \mathcal{N}(u_r^{\Omega,g})$. Actually, we also obtain this inclusion from Proposition 4.2 because (4.2) is satisfied for $\hat{\Omega} = (-0.5, 0.5)$ and $m = -0.25$. Since $u^{\widetilde{\Omega},\tilde{g}} - r$ attains its global minimum at $x = \pm 0.5$ and $u^{\widetilde{\Omega},\tilde{g}}(\pm 0.5) - r(\pm 0.5) = -0.25 < 0$, Proposition 4.4 shows that $\pm 0.5 \in \mathcal{C}(u_r^{\Omega,g})$. Obviously, this result is appropriate to the one in (c).

(e) By choosing $\tilde{g}(3) = g(3) = 5$ and varying $\tilde{g}(-2) = 10z - 5$ for $z \in [0.9, 3 - \sqrt{2}]$ we obtain $u^{\Omega,\tilde{g}}(x) = x^2 + (1 - 2z)x + 6z - 7$ as optimal solution to problem $(\mathcal{P}^{\Omega,\tilde{g}})$, which satisfies $u^{\Omega,\tilde{g}}(-2) = \tilde{g}(-2) \geq 4 = g(-2)$ by $z \geq 0.9$. Moreover, for $z \in [0.9, 3 - \sqrt{2}]$, the function $u^{\Omega,\tilde{g}}(x) - r(x) = x^2 + (1 - 2z)x + 6z - 7 - |x| \quad (x \in \Omega = (-2, 3))$ attains its global minimum at $x = z$, and $u^{\Omega,\tilde{g}}(z) - r(z) = -(z - 3)^2 + 2 \leq 0$ by $z \leq 3 - \sqrt{2}$. Consequently, Proposition 4.4 implies $u_r^{\Omega,g}(z) = r(z) = z$ for $z \in [0.9, 3 - \sqrt{2}]$, that means $[0.9, 3 - \sqrt{2}] \subset \mathcal{C}(u_r^{\Omega,g})$.

(f) Similarly as in (e), by choosing $\tilde{g}(-2) = g(-2) = 4$ and varying $\tilde{g}(3) = 14 - 10z$ for $z \in [0.5, 0.9]$ we obtain $u^{\Omega,\tilde{g}}(x) = x^2 + (1 - 2z)x + 2(1 - 2z)$ as optimal solution to problem $(\mathcal{P}^{\Omega,\tilde{g}})$, which satisfies

$$\begin{aligned} &u^{\Omega,\tilde{g}}(3) = \tilde{g}(3) \geq 5 = g(3) \\ &u^{\Omega,\tilde{g}}(z) - r(z) = -z^2 - 4z + 2 \leq 0. \end{aligned}$$

Moreover, for $z \in [0.5, 0.9]$ the function $u^{\Omega,\tilde{g}}(x) - r(x) = x^2 + (1 - 2z)x + 2(1 - 2z) - |x| \quad (x \in \Omega = (-2, 3))$ attains its global minimum at $x = z$. Consequently, Proposition 4.4 implies $[0.5, 0.9] \subset \mathcal{C}(u_r^{\Omega,g})$.

Actually, the results of (e) - (f) can be obtained in a shorter way as it was done in (b). But we took this longer way on purpose to demonstrate how to use Proposition 4.4.

We have seen how the conclusions of this paper can be applied to investigate optimal solutions to obstacle problems. By choosing different $\widetilde{\Omega}$ and $\tilde{g}$, it is possible to locate noncoincidence and coincidence points of the optimal solution $u_r^{\Omega,g}$ to the obstacle problem $(\mathcal{P}_r^{\Omega,g})$. It was shown in such a way that

$$[-2 + \sqrt{2}, -0.5] \cup [0.5,\ 3 - \sqrt{2}] \subset \mathcal{C}(u_r^{\Omega,g}).$$

These are already all coincidence points of the optimal solution $u_r^{\Omega,g}$ to obstacle problem (4.8), which can be shown to equal

$$u_r^{\Omega,g}(x) = \begin{cases} x^2 + (3 - 2\sqrt{2})x + 6 - 4\sqrt{2} & \text{if } x \in [-2, -2 + \sqrt{2}] \\ x^2 + 0.25 & \text{if } |x| \leq 0.5 \\ x^2 - (5 - 2\sqrt{2})x + 11 - 6\sqrt{2} & \text{if } x \in [3 - \sqrt{2}, 3] \\ |x| & \text{if } x \in [-2+\sqrt{2}, -0.5] \cup [0.5, 3-\sqrt{2}]. \end{cases} \qquad \blacksquare$$

Concluding remarks. Using the notion of "supersolution", similar results as Proposition 3.4 were obtained for some problem classes which require some coercive assumption (see, for instance, [7, 15]). In this paper, we do not assume any coercive condition, but only the uniqueness of the optimal solution to auxiliary problems $(\mathcal{P}^{\widetilde{\Omega},\tilde{g}})$, which is ensured if e.g. the performance index is strictly convex in $\mathcal{K}^{\widetilde{\Omega},\tilde{g}}$. It is to mention that the existence question is not investigated here.

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