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A Modified and a Finite Index Weber Transforms

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Abstract. This paper introduces, by way of constructing, specific finite and infinite integral transforms with Bessel functions J_ν and Y_ν in their kernels. The infinite transform and its reciprocal look deceptively similar to the known Weber transform and its reciprocal, respectively, but fundamentally differ from them. The new transform enjoys an operational property that makes it useful for applications to some problems in differential equations with non-constant coefficients. The paper gives a characterization of the image of some spaces of square integrable functions with respect to some measure under the infinite and finite transforms.

Keywords: *Bessel functions, integral transforms, Weber transform, index transforms*

AMS subject classification: Primary 44A15, 34B24, secondary 33C10

1. Introduction

Solutions of the Bessel differential equation

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+(x^2-\nu^2)y=0$$

are known as Bessel functions. There is a great variety of linear integral transforms whose kernels are Bessel functions. The transforms may be made over either a bounded interval or over a half line, and involve various boundary conditions. An extensive table of integral transforms involving Bessel functions in their kernels can be found, for example, in [7]. The best known transform of this kind is the singular Hankel transform

$$(\mathcal{H}f)(\lambda)=\int_0^\infty xJ_\nu(\lambda x)f(x)\,dx$$

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that applies to functions f defined over the positive half of the real line $\mathbb{R}$. Here J_ν denotes the Bessel function of the first kind of order ν and the symbols $\mathbb{R}$ and $\mathbb{R}^+$ denote the set of all real numbers and the set of all positive real numbers, respectively. The transform $\mathcal{H}$ corresponding to a fixed parameter ν replaces the differential form

$$D = \frac{d^2}{dx^2} + \frac{1}{x}\frac{d}{dx} - \frac{\nu^2}{x^2}$$

on the half-line $x > 0$ by $-\lambda^2\mathcal{H}(y)$, that is

$$\mathcal{H}(Dy) = -\lambda^2\mathcal{H}(y). \tag{1}$$

This operational property is principle one for applications of the transform to problems in differential equations. Another linear transform whose kernel involves Bessel functions, and that applies to functions f defined on an interval of the form $[a, \infty)$ with $a > 0$ and that also satisfies operational property (1) is the so-called Weber transform

$$(\mathfrak{W}f)(s) = \int_a^\infty \sqrt{x}\big[Y_\nu(sa)J_\nu(sx) - J_\nu(sa)Y_\nu(sx)\big]f(x)\,dx \tag{2}$$

where Y_ν denotes the Bessel function of the second kind of order ν. The Weber transform can be applied, for example, to a heat process problem in a radial-symmetric region $a \le r < \infty$. Its solution uses the inverse transform of $\mathfrak{W}$ which has the form

$$f(x) = \int_0^\infty \sqrt{x}\frac{Y_\nu(sa)J_\nu(sx) - J_\nu(sa)Y_\nu(sx)}{J_\nu^2(sa) + Y_\nu^2(sa)}s\{(\mathfrak{W}f)(s)\}\,ds. \tag{3}$$

Throughout the paper we shall let

$$d\mu(s) = \frac{s}{J_s^2(sa) + Y_s^2(sa)}\,ds$$

and use this notation in all the proofs. However, we shall display $d\mu(s)$ explicitly in the statements of lemmas, theorems, and corollaries.

In this paper we shall first establish the existence of yet another linear integral transform with Bessel functions in its kernel that applies to functions on a half-line $[a, \infty)$ $(a > 1)$. More specifically, the transform we derive is

$$(\mathcal{W}f)(s) = \int_a^\infty \frac{x^2 - 1}{x}\big[Y_s(sa)J_s(sx) - J_s(sa)Y_s(sx)\big]f(x)\,dx. \tag{4}$$

We further show that its reciprocal has the form

$$f(x) = \int_0^\infty \frac{Y_s(sa)J_s(sx) - J_s(sa)Y_s(sx)}{J_s^2(sa) + Y_s^2(sa)}\, s\{(\mathcal{W}f)(s)\}\, ds. \tag{5}$$

The pair of transforms (4) - (5) fundamentally differs from the Weber transform and its inverse in that the orders of the Bessel functions in the kernels are not fixed, and in the reciprocal transform (5) the order is the variable of integration. Integral transforms of type (5) are called *index transforms*. Details about many other index transforms can be found in [11]. Thus we shall call $\mathcal{W}f$ the *modified Weber transform* of f, and the inverse transform the *index Weber transform*.

The modified Weber transform turns out to have an operational property similar to that of Hankel and Weber transforms. In fact, we shall show in Section 2 that it resolves the differential form

$$\frac{x}{x^2-1}\frac{d}{dx}x\frac{d}{dx} = \frac{x^2}{x^2-1}\frac{d^2}{dx^2} + \frac{x}{x^2-1}\frac{d}{dx},$$

that is

$$\mathcal{W}\Big(\Big\{\frac{x}{x^2-1}\frac{d}{dx}x\frac{d}{dx}\Big\}f\Big)(s) = -s^2(\mathcal{W}f)(s)$$

provided that $\lim_{x\to a^+} f(x) = 0$. The modified Weber transform $\mathcal{W}$ can be applied to solve some boundary value problems. For instance, consider the equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = k\Big(1 - \frac{1}{r^2}\Big)u + f(r) \qquad (r^2 = x^2 + y^2, k > 0)$$

in the exterior of the circle $x^2+y^2 = a^2$ with the boundary condition $u\big|_{r=a} = 0$. This problem can be reduced to the form

$$\left.\begin{aligned} &\frac{1}{r}\frac{d}{dr}\Big(r\frac{dv}{dr}\Big) = k\Big(1 - \frac{1}{r^2}\Big)v + f(r) \quad (a < r < \infty) \\ &\qquad v(a) = 0 \end{aligned}\right\} \quad (v(r) = u(x,y)). \tag{6}$$

Thus

$$\Big\{\frac{r}{r^2-1}\frac{d}{dr}r\frac{d}{dr}\Big\}v - kv = \frac{r^2}{r^2-1}f.$$

Since $\mathcal{W}(\frac{r}{r^2-1}\frac{d}{dr}r\frac{d}{dr}v) = -s^2\mathcal{W}v$ we obtain

$$\mathcal{W}v = -\frac{1}{s^2+k}\mathcal{W}\Big(\frac{r^2}{r^2-1}f\Big).$$

Applying the inverse transform (5) to both sides resolves v and solves the problem. Notice that the Weber transform (2) would not apply to equation (6) due to the non-constant coefficient $1 - \frac{1}{r^2}$ of v. Section 2 is principally devoted to detail the technical steps of deriving the modified and index Weber transforms and to establish the basic operational property we mentioned earlier.

In many instances one is interested to describe or to characterize the image of certain subspaces under a given transform. A unified approach had been developed in [9, 10] to handle a large class of integral transforms arising from singular Sturm-Liouville problems. In particular, the image under transform (3) of functions in $L_2\big(\mathbb{R}^+, d\mu(s)\big)$ (ν fixed) that have compact support and those that vanish in a neighborhood of a point $\lambda_0 \in [0, \infty)$ have been fully accounted for in [9, 10]. In Section 3 we turn our attention to the study of the finite index Weber transform that arises from restricting the index Weber transform (5) to functions F acting on an interval of the form $(0, A)$ with $A > 0$. We specifically give a description of the image of $L_2\big((0, A); d\mu(s)\big)$ under the finite index Weber transform. The reader is referred to ([2, 3, 9, 10] for characterization of the image of $L_2\big((0, A); d\rho\big)$ ($A > 0$) for some measure $d\rho$ under various integral transforms. It is worth remarking that, in general, it is much harder to describe the image of $L_2(d\rho)$ under a finite integral transform (see [6, 10]) than under an infinite integral transform (see [5, 8]).

2. The modified Weber transform

In this section we derive the modified Weber transform and its reciprocal, the index Weber transform, and make precise the space of functions on which they act on. This is the content of Theorem 1 below. Theorem 2 gives equivalent conditions under which a basic operational property will hold.

Theorem 1. *Fix $a > 1$ and set $I_0 = [a, \infty)$. Then the mapping $f \to \mathcal{W}f$ given by*

$$F(s) = (\mathcal{W}f)(s) = \int_a^\infty \frac{x^2 - 1}{x} \big[Y_s(sa)J_s(sx) - J_s(sa)Y_s(sx)\big] f(x)\, dx \qquad (7)$$

defines an isometric transform from $L_2(I_0, \frac{x^2-1}{x}dx)$ onto $L_2\big(\mathbb{R}^+, \frac{s}{J_s^2(sa)+Y_s^2(sa)}ds\big)$ and thus the Parseval identity

$$\int_a^\infty |f(x)|^2 \frac{x^2 - 1}{x}\, dx = \int_0^\infty \frac{s|F(s)|^2}{J_s^2(sa) + Y_s^2(sa)}\, ds$$

holds. Moreover, the inverse transform is given by

$$f(x) = \int_0^\infty \frac{Y_s(sa)J_s(sx) - J_s(sa)Y_s(sx)}{J_s^2(sa) + Y_s^2(sa)}\, sF(s)\, ds. \tag{8}$$

The integrals in (7) and (8) are understood to exist in the sense of mean convergence, that is

$$\lim_{N\to\infty} \int_0^\infty |F(s) - F_N(s)|^2 d\mu(s) = 0$$
$$\lim_{M\to\infty} \int_a^\infty |f(x) - f_M(x)|^2 \frac{x^2-1}{x}\, dx = 0$$

where

$$F_N(s) = \int_a^N \frac{x^2-1}{x}\big[Y_s(sa)J_s(sx) - J_s(sa)Y_s(sx)\big] f(x)\, dx$$
$$f_M(x) = \int_0^M \big[Y_s(sa)J_s(sx) - J_s(sa)Y_s(sx)\big]\, F(s)\, d\mu(s).$$

We shall expound upon the proof of the theorem before starting it. Consider a second order differential equation of the form

$$A(x)\frac{d^2Y}{dx^2} + B(x)\frac{dY}{dx} + \lambda Y = 0 \qquad (x_0 \le x < \infty)$$

where it is assumed that $A(x) > 0$ and twice differentiable, and $B(x)$ is differentiable. Set

$$r(x) = \sqrt[4]{A(x)} \exp\left(-\frac{1}{2}\int \frac{B(x)}{A(x)}\, dx\right).$$

The change of variables

$$t = \gamma(x) := \int \frac{dx}{\sqrt{A(x)}}$$

ensures that if $y = y(t)$ is twice differentiable, then one has

$$\left\{A(x)\frac{d^2}{dx^2} + B(x)\frac{d}{dx}\right\} r(x)y = r(x)\left\{\frac{d^2}{dt^2} - q(t)\right\} y \tag{9}$$

where

$$q(t) = \frac{1}{4}\beta^2(t) + \frac{1}{2}\frac{d\beta(t)}{dt} \quad \text{and} \quad \beta(t) = \frac{1}{\sqrt{A(x)}}\left\{B(x) - \frac{1}{2}\frac{dA(x)}{dx}\right\} \ (x = \gamma^{-1}(t)).$$

For shortness we shall let

$$L = \frac{d^2}{dt^2} - q(t) \qquad \text{and} \qquad \mathfrak{D} = A(x)\frac{d^2}{dx^2} + B(x)\frac{d}{dx}.$$

Then (9) will read as

$$\big(\mathfrak{D}\{r(x)y(\gamma(x))\}\big)(x) = r(x)\{Ly\}(\gamma(x)).$$

If f is twice differentiable with respect to x, then setting $y(t) = \frac{f(\gamma^{-1}(t))}{r(\gamma^{-1}(t))}$ in the above relation gives

$$(\mathfrak{D}f)(x) = r(x)\left\{L\left(\frac{f(\gamma^{-1}(t))}{r(\gamma^{-1}(t))}\right)\right\}(\gamma(x)).$$

More generally, for any $n \in \mathbb{N}_0$, $\mathfrak{D}^n f$ exists if and only if $L^n\big(\frac{f(\gamma^{-1}(t))}{r(\gamma^{-1}(t))}\big)$ exists. In this case

$$(\mathfrak{D}^n f)(x) = r(x)\left\{L^n\left(\frac{f(\gamma^{-1}(t))}{r(\gamma^{-1}(t))}\right)\right\}(\gamma(x)) \qquad (n \in \mathbb{N}_0). \tag{10}$$

If $Y = Y(x)$ is a solution for

$$(\mathfrak{D} + \lambda)Y = 0, \tag{11}$$

then

$$y(t) = \frac{Y(\gamma^{-1}(t))}{r(\gamma^{-1}(t))} \tag{12}$$

is a solution for

$$(L + \lambda)y = 0. \tag{13}$$

In the reverse direction, if $y = y(t)$ is a solution for equation (13), then $Y(x) = r(x)y(\gamma(x))$ is a solution for equation (11).

Specializing the above proceedings to the differential equation

$$\frac{x^2}{x^2-1}\frac{d^2y}{dx^2} + \frac{x}{x^2-1}\frac{dy}{dx} + \lambda y = 0 \qquad (a \le x < \infty) \tag{14}$$

gives

$$r(x) = \frac{1}{\sqrt[4]{x^2-1}}$$

$$t = \gamma(x) = \int \frac{\sqrt{x^2-1}}{x}\,dx = \sqrt{x^2-1} - \sec^{-1} x \quad (a \le x < \infty) \tag{15}$$

$$\mathfrak{D} = \frac{x^2}{x^2-1}\frac{d^2}{dx^2} + \frac{x}{x^2-1}\frac{d}{dx} = \frac{x}{x^2-1}\frac{d}{dx}x\frac{d}{dx} \tag{16}$$

and

$$L = \frac{d^2}{dt^2} - q(t) \tag{17}$$

where

$$q(t) = \frac{-x^2(x^2+4)}{4(x^2-1)^3} \qquad (x = \gamma^{-1}(t)). \tag{18}$$

Thus if $\mathfrak{D}^n f$ is well-defined, then (10) will read as

$$\{\mathfrak{D}^n f\}(x) = (x^2-1)^{-\frac{1}{4}}\big(L^n\{\sqrt[4]{[\gamma^{-1}(t)]^2-1}\, f(\gamma^{-1}(t))\}\big)(\gamma(x)). \tag{19}$$

Put $t_0 = \gamma(a)$. Since for any measurable function f on I_0

$$\int_a^\infty |f(x)|^2 \frac{x^2-1}{x}\, dx = \int_{t_0}^\infty \big|\big(\sqrt[4]{[\gamma^{-1}(t)]^2-1}\, f(\gamma^{-1}(t))\big)(t)\big|^2 dt$$

it follows that

$$f \in L_2(I_0, \tfrac{x^2-1}{x}\, dx) \quad \Longleftrightarrow \quad \sqrt[4]{[\gamma^{-1}(\cdot)]^2-1}\, f(\gamma^{-1}(\cdot)) \in L_2(t_0, \infty).$$

In particular, relation (19) ensures that if $\mathfrak{D}^n f$ exists for any $n \in \mathbb{N}_0$, then

$$\mathfrak{D}^n f \in L_2(I_0, \tfrac{x^2-1}{x}\, dx) \quad \Longleftrightarrow \quad L^n\{\sqrt[4]{[\gamma^{-1}(\cdot)]^2-1}\, f(\gamma^{-1}(\cdot))\} \in L_2(t_0, \infty)$$

and in this case

$$\int_a^\infty |\mathfrak{D}^n f|^2 \frac{x^2-1}{x}\, dx = \int_{t_0}^\infty \big|L^n\big(\sqrt[4]{[\gamma^{-1}(t)]^2-1}\, f(\gamma^{-1}(t))\big)\big|^2 dt.$$

Henceforth, and for the rest of this paper, the symbols $\mathfrak{D}$ and L will always denote the operators defined in (16) and (17), respectively, and q will be as in (18). Moreover, the variables t and x are understood to be related as in (15).

Let $\varphi = \varphi(t,\lambda)$ and $\theta = \theta(t,\lambda)$ be the solutions of

$$(L+\lambda)y = \frac{d^2y}{dt^2} + (\lambda - q(t))y = 0$$

that satisfy the initial conditions

$$\left.\begin{aligned} \varphi(t_0,\lambda) &= 0 \\ \varphi'(t_0,\lambda) &= -1 \end{aligned}\right\} \qquad \text{and} \qquad \left.\begin{aligned} \theta(t_0,\lambda) &= 1 \\ \theta'(t_0,\lambda) &= 0 \end{aligned}\right\}.$$

Then there always exists a function $m = m(\lambda)$, known as a *Titchmarsh-Weyl function* [8], which is analytic in the upper half plane $\operatorname{Im}\lambda > 0$ and such that

$$\Psi(t,\lambda) = \theta(t,\lambda) + m(\lambda)\varphi(t,\lambda)$$

is in $L_2(t_0,\infty)$ for each λ with $\operatorname{Im}\lambda > 0$. Moreover, the function

$$\rho(\lambda) = -\frac{1}{\pi}\lim_{\delta\to 0+}\int_0^{\lambda} \operatorname{Im} m(u+i\delta)\,du \qquad (\lambda\in\mathbb{R}) \tag{20}$$

is monotone increasing on $\mathbb{R}$, and thus defines a Lebesgue-Stieltjes measure on $\mathbb{R}$. Further, the mapping $g \to G$ given by

$$G(\lambda) = \int_{t_0}^{\infty} \varphi(t,\lambda)g(t)\,dt \qquad \big(g\in L_2(t_0,\infty)\big) \tag{21}$$

defines an isometric transform from $L_2(t_0,\infty)$ onto $L_2(\mathbb{R},d\rho)$, so that

$$\int_{t_0}^{\infty} |g(t)|^2 dt = \int_{-\infty}^{\infty} |G(\lambda)|^2 d\rho(\lambda), \tag{22}$$

and the inverse transform is given by

$$g(t) = \int_{-\infty}^{\infty} \varphi(t,\lambda)G(\lambda)\,d\rho(\lambda). \tag{23}$$

Because of relation (19) we shall be able to relate the theory concerning the operator L, and eluded to above, to the operator $\mathfrak{D}$.

The proof of Theorem 1 is not particularly difficult. It fundamentally revolves about the facts elucidated above; exhibits, somewhat, lengthy and manipulative computations, and contains a very small amount of trickery. To aid the reader in following the proof we list the steps involved in it.

1. Solving equation (14) and obtaining the general solution of the Sturm-Liouville equation associated with it.
2. Obtaining the function $\varphi = \varphi(t,\lambda)$.
3. Determining the Titchmarsh-Weyl function $m(\lambda)$.
4. Determining explicitly the measure $d\rho(\lambda)$ and prove that $d\rho(\lambda) = 0$ for $\lambda < 0$.
5. Obtaining the modified and index Weber transforms.

Proof of Theorem 1. Step 1. Put $\lambda = s^2$. Then the change of variables $z = sx$ gives $\frac{dy}{dx} = s\frac{dy}{dz}$ and $\frac{d^2y}{dx^2} = s^2\frac{d^2y}{dz^2}$ and transforms equation (14) into

$$z^2\frac{d^2y}{dz^2} + z\frac{dy}{dz} + (z^2 - s^2)y = 0.$$

Two linearly independent solutions to the last equation are the Bessel functions $J_s(z)$ and $Y_s(z)$. Therefore solutions to equation (14) are $J_s(sx)$ and $Y_s(sx)$ with Wronskian [1]

$$W\big(J_s(\cdot), Y_s(\cdot)\big)(u) = \frac{2}{\pi u}. \tag{24}$$

Moreover, because of relation (12) $\sqrt[4]{x^2-1}J_s(sx)$ and $\sqrt[4]{x^2-1}Y_s(sx)$ $(x = \gamma^{-1}(t))$ are two linearly independent solutions of the equation

$$(L+\lambda)y = \frac{d^2y}{dt^2} + (\lambda - q(t))y = 0 \qquad (t_0 \le t < \infty). \tag{25}$$

Thus its general solution is given by

$$y(t,\lambda) = \sqrt[4]{x^2-1}\big[A_\lambda J_s(sx) + B_\lambda Y_s(sx)\big] \qquad (x = \gamma^{-1}(t)) \tag{26}$$

where A_λ and B_λ are arbitrary constants that may depend on λ.

Step 2. Since

$$\begin{aligned}
\frac{d}{dt}\sqrt[4]{x^2-1} &= \frac{x}{2}(x^2-1)^{-\frac{3}{4}}\frac{dx}{dt} = \frac{x^2}{2(x^2-1)^{\frac{5}{4}}}\\
\frac{d}{dt}J_s(sx) &= \frac{sx}{\sqrt{x^2-1}}J_s'(sx)\\
\frac{d}{dt}Y_s(sx) &= \frac{sx}{\sqrt{x^2-1}}Y_s'(sx)
\end{aligned}$$

where J_s' means $\frac{dJ_s}{dx}$ and similarly for Y_s', differentiating $y(t,\lambda)$ given in (26) with respect to t results in

$$\frac{d}{dt}y(t,\lambda) = \frac{x^2}{2(x^2-1)^{\frac{3}{2}}}y(t,\lambda) + \frac{sx}{\sqrt[4]{x^2-1}}\big[A_\lambda J_s'(sx) + B_\lambda Y_s'(sx)\big]. \tag{27}$$

Let $\varphi(t,\lambda)$ be the solution of equation (25) such that $\varphi(t_0,\lambda) = 0$ and $\varphi'(t_0,\lambda) = -1$. Thus from (26) - (27) we see that the boundary conditions on $\varphi(t,\lambda)$ give the system of equations

$$\left.\begin{aligned}
A_\lambda J_s(sa) + B_\lambda Y_s(sa) &= 0\\
A_\lambda J_s'(sa) + B_\lambda Y_s'(sa) &= -\frac{\sqrt[4]{a^2-1}}{sa}
\end{aligned}\right\}$$

which upon solving yields

$$\left.\begin{aligned}
A_\lambda &= \frac{\sqrt[4]{a^2-1}Y_s(sa)}{sa\,W(J_s(z),Y_s(z))(sa)} = \frac{\pi\sqrt[4]{a^2-1}}{2}Y_s(sa)\\
B_\lambda &= \frac{-\sqrt[4]{a^2-1}J_s(sa)}{sa\,W(J_s(z),Y_s(z))(sa)} = -\frac{\pi\sqrt[4]{a^2-1}}{2}J_s(sa)
\end{aligned}\right\}$$

where we used (24) for $W\big(J_s(z),Y_s(z)\big)$. Therefore,

$$\varphi(t,\lambda) = \frac{\pi\sqrt[4]{a^2-1}}{2}\sqrt[4]{x^2-1}\big[Y_s(sa)J_s(sx) - J_s(sa)Y_s(sx)\big]. \tag{28}$$

Step 3. Since for any measurable function k on $[t_0, \infty)$

$$\int_{t_0}^{\infty} |k(t)|^2 dt = \int_a^{\infty} |k(\gamma(x))|^2 \frac{\sqrt{x^2-1}}{x} dx$$

and since $0 < \sqrt{1-\frac{1}{a^2}} < \sqrt{1-\frac{1}{x^2}} < 1$ for $a < x$, it follows that

$$\begin{aligned}\int_a^{\infty} |k(\gamma(x))|^2 dx &\geq \int_{t_0}^{\infty} |k(t)|^2 dt \\ &= \int_a^{\infty} |k(\gamma(x))|^2 \frac{\sqrt{x^2-1}}{x}\, dx \\ &\geq \sqrt{1-\frac{1}{a^2}} \int_a^{\infty} |k(\gamma(x))|^2 dx.\end{aligned}$$

Hence $k \in L_2(t_0, \infty)$ if and only if $k(\gamma(\cdot)) \in L_2(a, \infty)$.

Let $\theta = \theta(t, \lambda)$ be the solution of equation (25) such that $\theta(t_0, \lambda) = 1$ and $\theta'(t_0, \lambda) = 0$ and let $m = m(\lambda)$ be the Titchmarsh-Weyl function so that, for each λ with $\mathrm{Im}\lambda > 0$, $\Psi(\cdot, \lambda) = \theta(\cdot, \lambda) + m(\lambda)\varphi(\cdot, \lambda) \in L_2(t_0, \infty)$. We could determine explicitly the form of the solution $\theta(\cdot, \lambda)$, as we have done for $\varphi(\cdot, \lambda)$, and then determine $\Psi(\cdot, \lambda)$ in terms of $m(\lambda)$ and then use it finally to obtain $m(\lambda)$. This laborious effort, however, can be avoided, for from the asymptotic formulas for the Bessel functions [1]

$$J_\nu(z) = \left(\tfrac{2}{\pi z}\right)^{\frac{1}{2}} \left[\cos\left(z - \tfrac{\pi}{2}\nu - \tfrac{\pi}{4}\right) + e^{|\mathrm{Im} z|} O(|z|^{-1})\right] \quad (|\arg z| < \pi) \tag{29}$$

$$Y_\nu(z) = \left(\tfrac{2}{\pi z}\right)^{\frac{1}{2}} \left[\sin\left(z - \tfrac{\pi}{2}\nu - \tfrac{\pi}{4}\right) + e^{|\mathrm{Im} z|} O(|z|^{-1})\right] \quad (|\arg z| < \pi) \tag{30}$$

$$H_\nu^{(1)}(z) \sim \left(\tfrac{2}{\pi z}\right)^{\frac{1}{2}} e^{i(z - \frac{\pi}{2}\nu - \frac{\pi}{4})} \qquad (-\pi < \arg z < 2\pi)$$

as $|z| \to \infty$ where $H_\nu^{(1)} = J_\nu + iY_\nu$ it immediately follows that, for λ with $\mathrm{Im}\,\lambda > 0$, the linearly independent solutions $\sqrt[4]{x^2-1} J_s(sx)$ and $\sqrt[4]{x^2-1} Y_s(sx)$ of equation (25) do not belong to $L_2(a, \infty)$, while the solution $y_0(t, \lambda) = \sqrt[4]{x^2-1} H_s^{(1)}(sx)$ does. Thus the solutions of equation (25), which are in $L_2(a, \infty)$ for λ with $\mathrm{Im}\lambda > 0$, are precisely the constant multiples of $y_0(\cdot, \lambda)$. Hence $\Psi(\cdot, \lambda)$ must be a multiple of $y_0(\cdot, \lambda)$. Since $\Psi(t_0, \lambda) = 1$ and $\frac{d\Psi}{dt}(t_0, \lambda) = -m(\lambda)$ we must have

$$m(\lambda) = -\frac{\frac{d\Psi}{dt}(t_0, \lambda)}{\Psi(t_0, \lambda)} = -\frac{\frac{dy_0}{dt}(t_0, \lambda)}{y_0(t_0, \lambda)}.$$

From (27)

$$\frac{dy_0}{dt}(t_0, \lambda) = \frac{a^2}{2(a^2-1)^{\frac{3}{2}}}\, y_0(t_0, \lambda) + \frac{sa}{\sqrt[4]{a^2-1}} \left[J_s'(sa) + iY_s'(sa)\right].$$

Hence

$$m(\lambda) = -\frac{sa}{\sqrt{a^2-1}}\left\{\frac{J_s'(sa)+iY_s'(sa)}{J_s(sa)+iY_s(sa)}\right\} - \frac{a^2}{2(a^2-1)^{\frac{3}{2}}} \qquad (\operatorname{Im}\lambda > 0). \quad (31)$$

Since $\Psi(\cdot,\lambda)$ is a multiple of $\sqrt[4]{x^2-1}H_s^{(1)}(sx)$ and $\Psi(t_0,\lambda)=1$, we have

$$\Psi(t,\lambda) = \frac{1}{\sqrt[4]{a^2-1}H_s^{(1)}(sa)}\sqrt[4]{x^2-1}H_s^{(1)}(sx) \qquad (\operatorname{Im}\lambda > 0). \quad (32)$$

Step 4. Now as $m = m(\lambda)$ had been determined, the next step is to determine the measure $d\rho(\lambda)$. Formula (31) makes it clear that $\lim_{\delta\to 0^+} m(\lambda + i\delta) = m(\lambda)$ for all $0 \neq \lambda \in \mathbb{R}$ so that $m = m(\lambda)$ is continuously extendable from the upper half-plane to the real λ-axis ($\lambda \neq 0$). Thus (20) is reduced to

$$d\rho(\lambda) = -\tfrac{1}{\pi}\{\operatorname{Im} m(\lambda)\}d\lambda \qquad (0 \neq \lambda \in \mathbb{R}).$$

Suppose that $\lambda > 0$, that is s is real. Then $J_s(sa), Y_s(sa), J_s'(sa), Y_s'(sa)$ are real quantities and

$$\begin{aligned}
&\left[J_s'(sa)+iY_s'(sa)\right]\left[J_s(sa)-iY_s(sa)\right] \\
&\quad = J_s'(sa)J_s(sa) + Y_s(sa)Y_s'(sa) + iW(J_s,Y_s)(sa) \\
&\quad = J_s'(sa)J_s(sa) + Y_s(sa)Y_s'(sa) + i\tfrac{2}{\pi sa}.
\end{aligned}$$

Hence

$$\begin{aligned}
\operatorname{Im} m(\lambda) &= \frac{-sa}{\sqrt{a^2-1}}\operatorname{Im}\left\{\frac{J_s'(sa)+iY_s'(sa)}{J_s(sa)+iY_s(sa)}\right\} \\
&= \frac{-2}{\pi\sqrt{a^2-1}\left[J_s^2(sa)+Y_s^2(sa)\right]}
\end{aligned} \qquad (\lambda > 0). \quad (33)$$

Suppose that $\lambda < 0$ or, equivalently, s is purely imaginary, say $s = iu$ with u real. Then $\bar{s} = -s$, and with the aid of the relations

$$H_\nu^{(1)}(iz) = \tfrac{2}{\pi i}e^{-\frac{1}{2}\nu\pi i}K_\nu(z), \quad \overline{K_\nu(z)} = K_{\bar{\nu}}(\bar{z}), \quad K_{-\nu}(z) = K_\nu(z)$$

where K_ν is the Macdonald function we obtain

$$\begin{aligned}
s\left\{\frac{J_s'(sa)+iY_s'(sa)}{J_s(sa)+iY_s(sa)}\right\} &= s\,\frac{H_s^{(1)'}(sa)}{H_s^{(1)}(sa)} \\
&= iu\,\frac{H_s^{(1)'}(iua)}{H_s^{(1)}(iua)} \\
&= iu\,\frac{\frac{1}{i}\left\{\frac{2}{\pi i}e^{-\frac{1}{2}s\pi i}K_s'(ua)\right\}}{\frac{2}{\pi i}e^{-\frac{1}{2}s\pi i}K_s(ua)} \\
&= u\,\frac{K_s'(ua)}{K_s(ua)}.
\end{aligned}$$

Since $\overline{\left\{\frac{K_s'(ua)}{K_s(ua)}\right\}} = \frac{K_{-s}'(ua)}{K_{-s}(ua)} = \frac{K_s'(ua)}{K_s(ua)}$, $\frac{K_s'(ua)}{K_s(ua)}$ is real. Hence

$$\operatorname{Im} m(\lambda) = \frac{-a}{\sqrt{a^2-1}} \operatorname{Im} s\left\{\frac{J_s'(sa)+iY_s'(sa)}{J_s(sa)+iY_s(sa)}\right\} = 0 \qquad (\lambda < 0). \tag{34}$$

Putting together (33) and (34) yields

$$d\rho(\lambda) = -\tfrac{1}{\pi}\{\operatorname{Im} m(\lambda)\}d\lambda = \begin{cases} \frac{4}{\pi^2\sqrt{a^2-1}}\frac{sds}{J_s^2(sa)+Y_s^2(sa)} & \text{for } \lambda > 0 \\ 0 & \text{for } \lambda < 0. \end{cases} \tag{35}$$

Step 5. With $\varphi(t,\lambda)$ as given in (28) and $d\rho(\lambda)$ as given in (35), formula (21) will define a transform $g \to G$ from $L_2(t_0,\infty)$ onto $L_2(\mathbb{R}^+, d\rho(\lambda))$, whose reciprocal is given by formula (23), and the Parseval identity (22) holds. Armed with this information, we are now ready to deliver the statement of the theorem. Let $f \in L_2(I_a, \frac{x^2-1}{x}dx)$ and put

$$g(t) = \frac{2}{\pi\sqrt[4]{a^2-1}}\sqrt[4]{(\gamma^{-1}(t))^2-1}\,f(\gamma^{-1}(t)). \tag{36}$$

Then g belongs to $L_2(t_0,\infty)$ and

$$g(\gamma(x)) = \frac{2}{\pi\sqrt[4]{a^2-1}}\sqrt[4]{x^2-1}\,f(x).$$

In fact, the assignment $f \to g$ defines a homeomorphism from $L_2(I_a, \frac{x^2-1}{x}dx)$ onto $L_2\big(t_0,\infty\big)$. We define

$$(\mathcal{W}f)(s) = G(s^2)$$

where G is the transform of g, as given in (36), under transform (21). It is plain that the transform $\mathcal{W}$ is onto $L_2\big(\mathbb{R}^+, d\mu(s)\big)$. The change of variables $t = \gamma(x)$, the form of the solution $\varphi(t,\lambda)$ as given by (28), formula (35) for

$d\rho(\lambda)$ together with (36) give

$$\begin{aligned}
(\mathcal{W}f)(s) &= \int_{t_0}^{\infty} g(t)\varphi(t,s^2)dt \\
&= \int_{a}^{\infty} g(\gamma(x))\varphi(\gamma(x),s^2)\frac{\sqrt{x^2-1}}{x}\,dx \\
&= \int_{a}^{\infty} \frac{x^2-1}{x}\big[Y_s(sa)J_s(sx)-J_s(sa)Y_s(sx)\big]f(x)\,dx \\
f(x) &= \frac{\pi\sqrt[4]{a^2-1}}{2\sqrt[4]{x^2-1}}\,g(\gamma(x)) \\
&= \frac{\pi\sqrt[4]{a^2-1}}{2\sqrt[4]{x^2-1}}\int_0^{\infty} G(\lambda)\varphi(\gamma(x),\lambda)\,d\rho(\lambda) \\
&= \int_0^{\infty} G(s^2)\big[Y_s(sa)J_s(sx)-J_s(sa)Y_s(sx)\big]d\mu(s) \\
&= \int_0^{\infty} \{(\mathcal{W}f)(s)\}\big[Y_s(sa)J_s(sx)-J_s(sa)Y_s(sx)\big]d\mu(s)
\end{aligned}$$

and, finally,

$$\begin{aligned}
\int_a^{\infty} |f(x)|^2\frac{x^2-1}{x}\,dx &= \int_{t_0}^{\infty} \big|\sqrt[4]{(\gamma^{-1}(t))^2-1}f(\gamma^{-1}(t))\big|^2 dt \\
&= \frac{\pi^2\sqrt{a^2-1}}{4}\int_{t_0}^{\infty}|g(t)|^2dt \\
&= \frac{\pi^2\sqrt{a^2-1}}{4}\int_{0}^{\infty}|G(\lambda)|^2d\rho(\lambda) \\
&= \int_0^{\infty}|(\mathcal{W}f)(s)|^2d\mu(s).
\end{aligned}$$

This ends the proof of Theorem 1 ∎

For $f \in L_2(I_0, \frac{x^2-1}{x}dx)$ we call $\mathcal{W}f$ as defined in (7) the *modified Weber transform* of f. We call a function $f \in L_2(I_0, \frac{x^2-1}{x}dx)$ the *index Weber transform* of a function $F \in L_2\big(\mathbb{R}^+, d\mu(s)\big)$ if $f = \mathcal{W}^{-1}F$, that is if relation (8) holds between f and F.

To facilitate many of the proofs in this section, the interplay between the operators L and $\mathfrak{D}$ is emphasized. The following remark is of this nature.

Remark 1. It is clear from the proof of Theorem 1 that if $f \in L_2(I_0, \frac{x^2-1}{x}dx)$ and

$$g(t) = \frac{2}{\pi\sqrt[4]{a^2-1}}\sqrt[4]{(\gamma^{-1}(t))^2-1}f(\gamma^{-1}(t)),$$

then $g \in L_2(t_0, \infty)$,

$$\left.\begin{aligned} (\mathcal{W}f)(s) &= \int_{t_0}^{\infty} \varphi(t, s^2) g(t)\, dt \\ g(t) &= \int_0^{\infty} G(\lambda)\varphi(t,\lambda)\, d\rho(\lambda) \end{aligned}\right\} \quad \text{where } G(\lambda) = (\mathcal{W}f)(\sqrt{\lambda}).$$

Moreover, $G \in L_2(\mathbb{R}^+, d\rho(\lambda))$. In particular, if $\mathfrak{D}f$ is well-defined and belongs to $L_2(I_0, \frac{x^2-1}{x}dx)$, then

$$\{\mathcal{W}(\mathfrak{D}f)\}(s) = \int_{t_0}^{\infty} \varphi(t, s^2)(Lg)(t)\, dt$$

since because of (19) applied with $n = 1$ we have

$$\begin{aligned} \sqrt[4]{(\gamma^{-1}(t))^2 - 1}\big\{\{\mathfrak{D}f\}(\gamma^{-1}(t))\big\} &= \big(L\big\{\sqrt[4]{[\gamma^{-1}(t)]^2 - 1} f(\gamma^{-1}(t))\big\}\big)(t) \\ &= \tfrac{\pi\sqrt[4]{a^2-1}}{2}(Lg)(t). \end{aligned}$$

Lemma 1. *If g is any function such that $g, Lg \in L_2(t_0, \infty)$, then*

$$\begin{aligned} \lim_{t\to\infty} \varphi(t,\lambda)g(t) &= 0 = \lim_{t\to\infty} \varphi(t,\lambda)g'(t) \\ \lim_{t\to\infty} \varphi'(t,\lambda)g(t) &= 0 = \lim_{t\to\infty} \varphi'(t,\lambda)g'(t). \end{aligned}$$

Proof. From relation (15) between t and x we get $\sqrt{x^2-1} - \frac{\pi}{2} \le t = \gamma(x) \le \sqrt{x^2-1} < x$ and thus $t = \gamma(x) \to \infty$ if and only if $x \to \infty$. Hence the definition of q in (18) implies

$$\lim_{t\to\infty} q(t) = \lim_{x\to\infty} \frac{-x^2(x^2+4)}{4(x^2-1)^3} = 0.$$

We now turn our attention to the behavior of g and g' at ∞. The function q is bounded on (t_0, ∞) and therefore $g'' = qg + Lg \in L_2(t_0, \infty)$. We have, by a Hardy-Littlewood inequality [4: p. 187/Formula 259]

$$\int_{t_0}^{\infty} |g'(t)|^2 dt \le 2\Big(\int_{t_0}^{\infty} |g(t)|^2 dt\Big)^{\frac{1}{2}} \Big(\int_{t_0}^{\infty} |g''(t)|^2 dt\Big)^{\frac{1}{2}}.$$

Thus $g' \in L_2(t_0, \infty)$ since $g, g'' \in L_2(t_0, \infty)$, and therefore $gg' \in L_1(t_0, \infty)$. Now

$$2\int_{t_0}^{t} g(u)g'(u)\, du = \int_{t_0}^{t} \frac{d}{du} g^2(u)\, du = g^2(t) - g^2(t_0)$$

and the limit of the most left side exists as $t \to \infty$. Consequently, $\lim_{t\to\infty} g^2(t)$ exists. But $g^2 \in L_1(t_0, \infty)$, so this limit must be zero. Hence $\lim_{t\to\infty} g(t) = 0$. Similarly, the relation

$$2\int_{t_0}^{t} g'(u)g''(u)\,du = \int_{t_0}^{t} \frac{d}{du}[g'(u)]^2 du = [g'(t)]^2 - [g'(t_0)]^2$$

and the fact that $g'g'', g'^2 \in L_1(t_0, \infty)$ show that $\lim_{t\to\infty} g'(t) = 0$. From asymptotic expansion formulas (29) - (30) and formulas [1: p. 364/Formulas 9.2.11 - 9.2.12]

$$\left.\begin{aligned} J'_\nu(z) &= \left(\tfrac{2}{\pi z}\right)^{\frac{1}{2}}\left[-\sin(z - \tfrac{\pi}{2}\nu - \tfrac{\pi}{4}) + O(|z|^{-1})\right] \\ Y'_\nu(z) &= \left(\tfrac{2}{\pi z}\right)^{\frac{1}{2}}\left[\cos(z - \tfrac{\pi}{2}\nu - \tfrac{\pi}{4}) + O(|z|^{-1})\right] \end{aligned}\right\} \qquad (z \to \infty)$$

we see that

$$\sqrt[4]{x^2-1}\,J_s(sx) = \left(\tfrac{2}{\pi s}\right)^{\frac{1}{2}} \sqrt[4]{1 - \tfrac{1}{x^2}}\left[\cos(xa - \tfrac{\pi}{2}s - \tfrac{\pi}{4}) + O(|x|^{-1})\right]$$

$$\sqrt[4]{x^2-1}\,Y_s(sx) = \left(\tfrac{2}{\pi s}\right)^{\frac{1}{2}} \sqrt[4]{1 - \tfrac{1}{x^2}}\left[\sin(xa - \tfrac{\pi}{2}s - \tfrac{\pi}{4}) + O(|x|^{-1})\right]$$

$$\frac{x}{\sqrt[4]{x^2-1}}\frac{d}{dx}J_s(sx) = \left(\tfrac{2}{\pi s}\right)^{\frac{1}{2}} \frac{1}{\sqrt[4]{1-\frac{1}{x^2}}}\left[-\sin(sx - \tfrac{\pi}{2}s - \tfrac{\pi}{4}) + O(|x|^{-1})\right]$$

$$\frac{x}{\sqrt[4]{x^2-1}}\frac{d}{dx}Y_s(sx) = \left(\tfrac{2}{\pi s}\right)^{\frac{1}{2}} \frac{1}{\sqrt[4]{1-\frac{1}{x^2}}}\left[\cos(sx - \tfrac{\pi}{2}s - \tfrac{\pi}{4}) + O(|x|^{-1})\right].$$

From the first two expansions it follows that $\varphi(t,\lambda)$ is bounded as $t \to \infty$, and the last two expansions imply that $\varphi'(t,\lambda)$ is bounded as $t \to \infty$ since

$$\begin{aligned} \varphi'(t,\lambda) = {} & \frac{x^2}{2(x^2-1)^{\frac{3}{2}}}\varphi(t,\lambda) \\ & + \frac{\pi\sqrt[4]{a^2-1}}{2}\frac{sx}{\sqrt[4]{x^2-1}}\left[Y_s(sa)J'_s(sx) - J_s(sa)Y'_s(sx)\right]. \end{aligned}$$

Therefore the assertions hold ∎

Associated with the operator L given in (17) is the Green function $G(t,u,\sigma)$ which is defined for a non-real complex number σ with $\operatorname{Im}\sigma > 0$ as

$$G(t,u,\sigma) = \begin{cases} \Psi(t,\sigma)\varphi(u,\sigma) & \text{if } t_0 \le u \le t \\ \Psi(u,\sigma)\varphi(t,\sigma) & \text{if } u > t \ge t_0 \end{cases}$$

where φ and Ψ are as in (28) and (32), respectively. The resolvent of g is the function $R_\sigma g$ given by

$$\begin{aligned}(R_\sigma g)(t) &= \int_{t_0}^{\infty} G(t,u,\sigma)g(u)\,du \\ &= \Psi(t,\sigma)\int_{t_0}^{t}\varphi(u,\sigma)g(u)\,du + \varphi(t,\sigma)\int_{t}^{\infty}\Psi(u,\sigma)g(u)\,du.\end{aligned}$$

It is well-known [5, 8] that if $g \in L_2(t_0,\infty)$, then $R_\sigma g \in L_2(t_0,\infty)$, $R_\sigma g$ is twice differentiable and

$$(L+\sigma)R_\sigma g = g. \tag{37}$$

The resolvent function also has the integral representation [7: p. 52/Formula 3.7]

$$(R_\sigma g)(t) = \int_0^\infty \varphi(t,\lambda)\frac{G(\lambda)}{\sigma-\lambda}\,d\rho(\lambda). \tag{38}$$

Theorem 2. *Let $f \in L_2\big(I_0, \frac{x^2-1}{x}dx\big)$. Then the following assertions are equivalent:*

(i) $(\cdot)^2(\mathcal{W}f)(\cdot) \in L_2\big(\mathbb{R}^+, \frac{s}{J_s^2(sa)+Y_s^2(sa)}ds\big)$.

(ii) $\lim_{x\to a^+} f(x) = 0$, *$\mathfrak{D}f$ is well-defined and $\mathfrak{D}f \in L_2\big(I_0, \frac{x^2-1}{x}dx\big)$.*

(iii) *$\mathfrak{D}f$ is well-defined, $\mathfrak{D}f \in L_2\big(I_0, \frac{x^2-1}{x}dx\big)$ and $\{\mathcal{W}(\mathfrak{D}f)\}(s) = -s^2(\mathcal{W}f)(s)$.*

Proof. The implication (iii) $\Rightarrow$ (i) is easy. Since if $f, \mathfrak{D}f \in L_2\big(I_0, \frac{x^2-1}{x}dx\big)$, then $\mathcal{W}f$ and $\mathcal{W}(\mathfrak{D}f)$ are well-defined and belong to $L_2\big(\mathbb{R}^+, d\mu(s)\big)$. But then $\{\mathcal{W}(\mathfrak{D}f)\}(s) = -s^2(\mathcal{W}f)(s)$ implies assertion (i). Thus we only need to show that (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii).

(i) $\Rightarrow$ (ii): Suppose that $(\cdot)^2(\mathcal{W}f)(\cdot) \in L_2\big(\mathbb{R}^+, d\mu(s)\big)$. Then the function f_1 defined by

$$\begin{aligned}f_1(x) &= \big(\mathcal{W}^{-1}\{s^2\mathcal{W}f\}\big)(x) \\ &= \int_0^\infty s^2(\mathcal{W}f)(s)\big[Y_s(sa)J_s(sx) - J_s(sa)Y_s(sx)\big]\,d\mu(s)\end{aligned}$$

is well-defined and belongs to $L_2\big(I_0, \frac{x^2-1}{x}dx\big)$. Put

$$g(t) = \frac{2}{\pi\sqrt[4]{a^2-1}}\sqrt[4]{(\gamma^{-1}(t))^2-1}\,f(\gamma^{-1}(t)) \tag{39}$$

$$g_1(t) = \frac{2}{\pi\sqrt[4]{a^2-1}}\sqrt[4]{(\gamma^{-1}(t))^2-1}\,f_1(\gamma^{-1}(t))$$

and $G(\lambda) = (\mathcal{W}f)(\sqrt{\lambda})$. Then (see Remark 1) $g, g_1 \in L_2(t_0, \infty)$ and $G, (\cdot)G(\cdot) \in L_2(\mathbb{R}^+, d\rho(\lambda))$. Moreover,

$$g(t) = \int_0^\infty \varphi(t,\lambda)G(\lambda)\,d\rho(\lambda)$$
$$g_1(t) = \int_0^\infty \lambda G(\lambda)\varphi(t,\lambda)\,d\rho(\lambda).$$

Since

$$\lim_{t\to t_0} g(t) = \lim_{t\to t_0} \int_0^\infty \varphi(t,\lambda)G(\lambda)\,d\rho(\lambda)$$
$$\varphi(t_0,\lambda) = 0,$$

it is immediate that $\lim_{t\to t_0} g(t) = 0$ and consequently $\lim_{x\to a^+} f(x) = 0$. Fix a non-real complex number σ with $\operatorname{Im}\sigma > 0$. Then integral representation (38) yields

$$\begin{aligned} g(t) &= \int_0^\infty \varphi(t,\lambda)G(\lambda)\,d\rho(\lambda) \\ &= \int_0^\infty (\sigma-\lambda)\varphi(t,\lambda)\frac{G(\lambda)}{\sigma-\lambda}\,d\rho(\lambda) \\ &= \sigma\int_0^\infty \varphi(t,\lambda)\frac{G(\lambda)}{\sigma-\lambda}\,d\rho(\lambda) - \int_0^\infty \varphi(t,\lambda)\frac{\lambda G(\lambda)}{\sigma-\lambda}\,d\rho(\lambda) \\ &= \sigma(R_\sigma g)(t) - (R_\sigma g_1)(t) \\ &= R_\sigma(\sigma g - g_1)(t). \end{aligned}$$

Because $\sigma g - g_1 \in L_2(t_0,\infty)$, it follows that $g = R_\sigma(\sigma g - g_1)$ is twice differentiable. Moreover, formula (37) gives $(L+\sigma)g = (L+\sigma)R_\sigma(\sigma g - g_1) = \sigma g - g_1$. Thus $Lg = -g_1$ and therefore $Lg \in L_2(t_0,\infty)$. Consequently, $\mathfrak{D}f \in L_2\big(I_0, \frac{x^2-1}{x}dx\big)$.

(ii) $\Rightarrow$ (iii): Assume that $f \in L_2\big(I_0, \frac{x^2-1}{x}dx\big)$, $\lim_{x\to a^+} f(x) = 0$ and $\mathfrak{D}f$ is well-defined and belongs to $L_2\big(I_0, \frac{x^2-1}{x}dx\big)$. The functions $F = \mathcal{W}f$ and $H = \mathcal{W}(\mathfrak{D}f)$ are well-defined and both belong to $L_2\big(\mathbb{R}, d\mu(s)\big)$. Let g be as in (39). Then $\lim_{t\to t_0} g(t) = 0$ and (see Remark 1)

$$F(s) = \int_{t_0}^\infty \varphi(t,s^2)g(t)\,dt \tag{40}$$
$$H(s) = \int_{t_0}^\infty \varphi(t,s^2)(Lg)(t)\,dt. \tag{41}$$

We show that $H(s) = -s^2F(s)$. Choose a real sequences $\{t_n\}$ such that $t_0 < t_n < \infty$, $t_n \to \infty$ as $n \to \infty$ and

$$F(s) = \lim_{n\to\infty} \int_{t_0}^{t_n} \varphi(t, s^2)g(t)\,dt \tag{42}$$

$$H(s) = \lim_{n\to\infty} \int_{t_0}^{t_n} \varphi(t, s^2)(Lg)(t)\,dt \tag{43}$$

almost everywhere. This is possible since the integrals in (40) - (41) converge in $L_2(\mathbb{R}^+, d\rho)$-norm, and this guarantees the existence of a sequence $\{t_n\}$ for which (42) - (43) hold almost everywhere. Since $(Lg)(t) = g''(t) - q(t)g(t)$,

$$\int_{t_0}^{t_n} \varphi(t, s^2)(Lg)(t)\,dt = \int_{t_0}^{t_n} \varphi(t, s^2)g''(t)\,dt - \int_{t_0}^{t_n} q(t)\varphi(t, s^2)g(t)\,dt.$$

Apply integration by parts to obtain

$$\begin{aligned}
\int_{t_0}^{t_n} \varphi(t, s^2)g''(t)\,dt &= \int_{t_0}^{t_n} \varphi(t, s^2)(g'(t))'dt \\
&= \varphi(t, s^2)g'(t)\big|_{t_0}^{t_n} - \int_{t_0}^{t_n} \varphi'(t, s^2)g'(t)\,dt \\
&= \varphi(t_n, s^2)g'(t_n) - \left\{\varphi'(t, s^2)g(t)\big|_{t_0}^{t_n} - \int_{t_0}^{t_n} \varphi''(t, s^2)g(t)\,dt\right\} \\
&= \varphi(t_n, s^2)g'(t_n) - \varphi'(t_n, s^2)g(t_n) + \int_{t_0}^{t_n} \varphi''(t, s^2)g(t)\,dt
\end{aligned}$$

where we used in the above computation $\varphi(t_0, s^2) = 0$ and $\lim_{t\to t_0} g(t) = 0$ in the third and last line, respectively. Hence

$$\begin{aligned}
&\int_{t_0}^{t_n} \varphi(t, s^2)(Lg)(t)\,dt \\
&\quad = \varphi(t_n, s^2)g'(t_n) - \varphi'(t_n, s^2)g(t_n) + \int_{t_0}^{t_n} \{\varphi''(t, s^2) - q(t)\varphi(t, s^2)\}g(t)\,dt.
\end{aligned}$$

Since $\varphi''(t, s^2) - q(t)\varphi(t, s^2) = -s^2\varphi(t, s^2)$ we finally arrive at

$$\begin{aligned}
&\int_{t_0}^{t_n} \varphi(t, s^2)(Lg)(t)\,dt \\
&\quad = \varphi(t_n, s^2)g'(t_n) - \varphi'(t_n, s^2)g(t_n) - s^2 \int_{t_0}^{t_n} \varphi(t, s^2)g(t)\,dt.
\end{aligned} \tag{44}$$

By Lemma 1,

$$\lim_{n\to\infty} \varphi(t_n, s^2)g'(t_n) = 0 = \lim_{n\to\infty} \varphi'(t_n, s^2)g(t_n),$$

and because of (42) - (43), letting $n \to \infty$ in (44) yields

$$\big\{(\mathcal{W}(\mathfrak{D}f)\big\}(s) = H(s) = -s^2F(s) = -s^2(\mathcal{W}f)(s).$$

Theorem 2 is proved ∎

The following corollary is a "hyper" version of Theorem 2 and will serve a crucial role in Section 3.

Corollary 1. *Let $f \in L_2\big(I_0, \frac{x^2-1}{x}dx\big)$. Then the following assertions are equivalent:*

(i) $(\cdot)^n(\mathcal{W}f)(\cdot) \in L_2\big(\mathbb{R}^+, \frac{s}{J_s^2(sa)+Y_s^2(sa)}ds\big)$ *for any* $n \in \mathbb{N}_0$.

(ii) $\mathfrak{D}^n f$ *is well-defined,* $\mathfrak{D}^n f \in L_2\big(I_0, \frac{x^2-1}{x}dx\big)$ *and* $\lim_{x\to a^+}(\mathfrak{D}^n f)(x) = 0$ *for any* $n \in \mathbb{N}_0$.

(iii) $\mathfrak{D}^n f$ *is well-defined,* $\mathfrak{D}^n f \in L_2\big(I_0, \frac{x^2-1}{x}dx\big)$ *and* $\big\{\mathcal{W}(\mathfrak{D}^n f)\big\}(s) = (-s^2)^n(\mathcal{W}\ f)(s)$ *for any* $n \in \mathbb{N}_0$.

Proof. (i) $\Rightarrow$ (ii): Suppose that $(\cdot)^n(\mathcal{W}f)(\cdot) \in L_2(\mathbb{R}^+, d\mu(s))$ for any $n \in \mathbb{N}_0$. Let $F = \mathcal{W}f$ and put $F_n(s) = (-s^2)^n F(s)$. Then $F_n \in L_2(\mathbb{R}^+, d\mu(s))$ and $f_n = \mathcal{W}^{-1}F_n$ is well-defined and belongs to $L_2\big(I_0, \frac{x^2-1}{x}dx\big)$. Since

$$s^2(\mathcal{W}f_n)(s) = -(-s^2)^{n+1}F(s) = -F_{n+1}(s)$$

belongs to $L_2(\mathbb{R}^+, d\mu(s))$, Theorem 2 would apply for f_n. Therefore $\lim_{x\to a^+} f_n(x) =$ 0, $\mathfrak{D}f_n$ is well-defined and belongs to $L_2\big(I_0, \frac{x^2-1}{x}dx\big)$, and

$$\big\{\mathcal{W}(\mathfrak{D}f_n)\big\}(s) = -s^2(\mathcal{W}f_n)(s) = F_{n+1}(s)$$

or, equivalently,

$$\mathfrak{D}f_n = f_{n+1}. \tag{45}$$

By iterating (45) and recognizing that $f = f_0$ and $\mathfrak{D}^0 f = f$ we obtain $\mathfrak{D}^n f = f_n$. Thus $\mathfrak{D}^n f$ belongs to $L_2\big(I_0, \frac{x^2-1}{x}dx\big)$ since f_n does, and $\lim_{x\to a^+}(\mathfrak{D}^n f)(x) = \lim_{x\to a^+} f_n(x) = 0$.

(ii) $\Rightarrow$ (iii): Assume for any $n \in \mathbb{N}_0$, $\mathfrak{D}^n f$ is well-defined, $\mathfrak{D}^n f \in L_2(I_0, \frac{x^2-1}{x}dx)$ and $\lim_{x\to a^+}(\mathfrak{D}^n f)(x) = 0$. Then Theorem 2 would apply in relation to the function $\mathfrak{D}^n f$. Therefore $(\cdot)^2\{\mathcal{W}\mathfrak{D}^n f)\}(\cdot) \in L_2(\mathbb{R}^+, d\mu(s))$ and, moreover,

$$\{\mathcal{W}\mathfrak{D}^{n+1}f)\}(s) = \{\mathcal{W}(\mathfrak{D}(\mathfrak{D}^n f))\}(s) = -s^2\{\mathcal{W}(\mathfrak{D}^n f)\}(s) \qquad (n \in \mathbb{N}_0). \tag{46}$$

Iterating (46) we get $\{\mathcal{W}\mathfrak{D}^n f)\}(s) = (-s^2)^n(\mathcal{W}f)(s) \quad (n \in \mathbb{N}_0)$

(iii) $\Rightarrow$ (i): If assertion (iii) holds, then clearly $(\cdot)^{2n}(\mathcal{W}f)(\cdot) \in L_2(\mathbb{R}^+, d\mu(s))$ for any $n \in \mathbb{N}_0$. But then the function defined by $[s^{2n} + s^{2(n+1)}](\mathcal{W}f)(s)$ belongs to $L_2(\mathbb{R}^+, d\mu(s))$, and since

$$[s^{2n} + s^{2(n+1)}]\,|(\mathcal{W}f)(s)| = s^{2n}(1+s^2)|(\mathcal{W}f)(s)| \geq 2s^{2n+1}|(\mathcal{W}f)(s)|,$$

it follows that $(\cdot)^{2n+1}(\mathcal{W}f)(\cdot) \in L_2(\mathbb{R}^+, d\mu(s))$ for any $n \in \mathbb{N}_0$. This completes the proof ∎

3. The finite index Weber transform

Let A be a fixed positive real number, but otherwise arbitrary. Associate with each function $F \in L_2((0,A), d\mu(s))$ the function f_A defined by

$$f_A(x) = \int_0^A sF(s)\,\frac{Y_s(sa)J_s(sx) - J_s(sa)Y_s(sx)}{J_s^2(sa) + Y_s^2(sa)}\,ds \qquad (a \leq x < \infty). \quad (47)$$

We will call f_A the finite index Weber transform of F. Clearly, $f_A \in L_2\big(I_0, \frac{x^2-1}{x}\,dx\big)$ and

$$\int_a^\infty |f_A(x)|^2 \frac{x^2-1}{x}\,dx = \int_0^A |F(s)|^2\,d\mu(s).$$

This section is solely devoted to the description of the image of $L_2((0,A), d\mu(s))$ under transform (47). For this purpose we need the following

Lemma 2. *Let $(\cdot)^n F(\cdot) \in L_2\big(\mathbb{R}^+, \frac{s}{J_s^2(sa)+Y_s^2(sa)}\,ds\big)$ for any $n \in \mathbb{N}_0$. Then*

$$\lim_{n\to\infty}\left\{\int_0^\infty s^{4n}|F(s)|^2 \frac{s}{J_s^2(sa)+Y_s^2(sa)}\,ds\right\}^{\frac{1}{4n}} = \sup_{s\in \mathrm{supp}F} s.$$

Proof. The lemma is trivial if $F = 0$. Thus, first suppose that $F \neq 0$ has compact support and let $\sup_{s\in\mathrm{supp}F} s = A > 0$. Then

$$\int_0^\infty s^{4n}|F(s)|^2 d\mu(s) = \int_0^A s^{4n}|F(s)|^2 d\mu(s) \leq A^{4n}\int_0^A |F(s)|^2 d\mu(s).$$

Hence

$$\limsup_{n\to\infty}\left\{\int_0^\infty s^{4n}|F(s)|^2 d\mu(s)\right\}^{\frac{1}{4n}} \leq A \limsup_{n\to\infty}\left\{\int_0^A |F(s)|^2 d\mu(s)\right\}^{\frac{1}{4n}} = A.$$

On the other hand, if $0 < \varepsilon < A = \sup_{s\in \mathrm{supp} F} s$, then $\int_{A-\varepsilon}^{A} |F(s)|^2 d\mu(s) > 0$. Thus

$$\begin{aligned}\liminf_{n\to\infty}\left\{\int_0^\infty s^{4n}|F(s)|^2 d\mu(s)\right\}^{\frac{1}{4n}} &\geq \liminf_{n\to\infty}\left\{\int_{A-\varepsilon}^{A} s^{4n}|F(s)|^2 d\mu(s)\right\}^{\frac{1}{4n}}\\ &\geq \liminf_{n\to\infty}\left\{(A-\varepsilon)^{4n}\int_{A-\varepsilon}^{A}|F(s)|^2 d\mu(s)\right\}^{\frac{1}{4n}}\\ &\geq A-\varepsilon.\end{aligned}$$

Because $\varepsilon > 0$ is arbitrary, $\lim_{n\to\infty}\left\{\int_0^\infty s^{4n}|F(s)|^2 d\mu(s)\right\}^{1/4n} = A$.

Suppose now that $F \neq 0$ has unbounded support. Then, for any N large enough, $\int_N^\infty |F(s)|^2 d\mu(s) > 0$. Consequently,

$$\begin{aligned}\liminf_{n\to\infty}\left\{\int_0^\infty s^{4n}|F(s)|^2 d\mu(s)\right\}^{\frac{1}{4n}} &\geq \liminf_{n\to\infty}\left\{\int_N^\infty s^{4n}|F(s)|^2 d\mu(s)\right\}^{\frac{1}{4n}}\\ &\geq \liminf_{n\to\infty}\left\{N^{4n}\int_N^\infty |F(s)|^2 d\mu(s)\right\}^{\frac{1}{4n}}\\ &= N.\end{aligned}$$

Letting $N \to \infty$ we obtain $\lim_{n\to\infty}\left\{\int_0^\infty s^{4n}|F(s)|^2 d\mu(s)\right\}^{1/4n} = \infty$. This completes the proof ∎

The next theorem is the main result of this section. It describes the image of the space $L_2\big((0,A), d\mu(s)\big)$ under the finite index Weber transform.

Theorem 3. *A function f_A is the finite index Weber transform of a function $F \in L_2\big((0,A), \frac{s}{J_s^2(sa)+Y_s^2(sa)}ds\big)$ if and only if the following is true:*

(i) *For any $n \in \mathbb{N}_0$, $\mathfrak{D}^n f_A$ is well-defined, $\mathfrak{D}^n f_A \in L_2\big(I_0, \frac{x^2-1}{x}dx\big)$ and $\lim_{x\to a^+}(\mathfrak{D}^n f_A)(x) = 0$.*

(ii) $\lim_{n\to\infty}\|\mathfrak{D}^n f_A\|^{\frac{1}{2n}}_{L_2(I_0,\frac{x^2-1}{x}dx)} \leq A$.

Proof. We start with proving the "only if" part. Let $F \in L_2\big((0,A), d\mu(s)\big)$. Then its extension by 0 on (A,∞) yields a function $\tilde{F} \in L_2(\mathbb{R}^+, d\mu(s))$ such that $(\cdot)^n\tilde{F}(\cdot) \in L_2(\mathbb{R}^+, d\mu(s))$ for all n, $\mathrm{supp}\tilde{F} \subset (0,A)$ and the index Weber transform $\mathcal{W}^{-1}\tilde{F}$ is again f_A. Thus Corollary 1 applies to f_A. In particular, assertion (ii) of Corollary 1 holds which is precisely item (i) of the lemma, and for every n, assertion (iii) of Corollary 1 holds, namely, $\mathfrak{D}^n f_A$ is well-defined, belongs to $L_2\big(I_0, \frac{x^2-1}{x}\,dx\big)$ and $\{\mathcal{W}(\mathfrak{D}^n f_A)\}(s) = (-s^2)^n(\mathcal{W}f_A)(s)$.

Since $\mathcal{W}f_A = \tilde{F}$, we have $\{\mathcal{W}(\mathfrak{D}^n f_A)\}(s) = (-s^2)^n \tilde{F}(s)$ and by the Parseval identity

$$\|\mathfrak{D}^n f_A\|_{L_2(I_0, \frac{x^2-1}{x}\, dx)} = \|\mathcal{W}(\mathfrak{D}^n f_A)\|_{L_2(\mathbb{R}^+, d\mu(s))}.$$

Hence

$$\|\mathfrak{D}^n f_A\|_{L_2(I_0, \frac{x^2-1}{x}\, dx)} = \left\{ \int_0^\infty s^{4n} |\tilde{F}(s)|^2 d\mu(s) \right\}^{\frac{1}{2}}.$$

Therefore, Lemma 2 gives

$$\lim_{n\to\infty} \|\mathfrak{D}^n f_A\|^{\frac{1}{2n}}_{L_2(I_0, \frac{x^2-1}{x} dx)} = \lim_{n\to\infty} \left\{ \int_0^\infty s^{4n} |\tilde{F}(s)|^2 d\mu(s) \right\}^{\frac{1}{4n}} = \sup_{s\in \mathrm{supp}\tilde{F}} s \le A$$

which is condition (ii). This completes the proof of the "only if" part.

We prove now the "if" part of the statement. So assume that conditions (i) and (ii) hold. Then because of (i) Corollary 1 applies and therefore, for all $n \in \mathbb{N}_0$, $(\cdot)^n (\mathcal{W}f_A)(\cdot) \in L_2(\mathbb{R}^+, d\mu(s))$ and $\{\mathcal{W}(\mathfrak{D}^n f_A)\}(s) = (-s^2)^n (\mathcal{W}f_A)(s)$. Put $\tilde{F} = \mathcal{W}f_A$. Then the above relation gives

$$\left\{ \int_0^\infty s^{4n} |\tilde{F}(s)|^2 d\mu(s) \right\}^{\frac{1}{2}} = \|\mathcal{W}(\mathfrak{D}^n f_A)\|_{L_2(\mathbb{R}^+, d\mu(s))} = \|\mathfrak{D}^n f_A\|_{L_2(I_0, \frac{x^2-1}{x} dx)}$$

where the last equality follows since $\mathcal{W}$ is an isometry. Thus one can apply Lemma 2 to obtain

$$\sup_{s\in \mathrm{supp}\tilde{F}} s = \lim_{n\to\infty} \left\{ \int_0^\infty s^{4n} |\tilde{F}(s)|^2 d\mu(s) \right\}^{\frac{1}{4n}} = \lim_{n\to\infty} \|\mathfrak{D}^n f_A\|^{\frac{1}{2n}}_{L_2(I_0, \frac{x^2-1}{x} dx)} \le A.$$

Thus $\mathrm{supp}\tilde{F} \subset (0, A)$ and the index Weber transform (8) turns out to be the finite index Weber transform (47) of $F = \tilde{F}|_{[0,A]}$, the restriction of $\tilde{F}$ to $[0, A]$. The theorem is proved ∎

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