

On the Solutions of a Quadratic Integral and an Integral-Differential Equation

L. von Wolfersdorf

Dedicated to the memory of Prof. Dr. Gerhard Grüß

Abstract. An integral equation and a related integral-differential equation of first order over \mathbb{R}_+ with a quadratic integral term representing the so-called autocorrelation of the unknown function is dealt with. For both equations the general solution is constructed and estimated in the L^2 -norm. Further, the asymptotic behaviour and the stability of the solution are investigated.

Keywords: *Nonlinear integral equations, nonlinear integro-ordinary differential equations, boundary value problems for holomorphic functions* ■

AMS subject classification: Primary 45J05, 45G10, secondary 30E25

1. Introduction

In stochastics and related applications quadratic integral equations occur which contain the so-called autocorrelation

$$(Ap)(t) = \int_0^\infty p(s)p(s+t) ds \quad (t > 0) \quad (1.1)$$

of the unknown function p (which is often a probability density) (cf. [5]). In our recent paper [6] (see also [7]) a procedure for solving the corresponding integral equation of the second kind and a related integral-differential equation is developed and special solution classes for the equations are treated in some detail.

L. von Wolfersdorf: Fak. Math. & Inf. der TU Bergakad. Freiberg, D-09596
Freiberg
wolfersd@math.tu-freiberg.de

In the present paper the (properly defined) general solutions of these equations are dealt with and investigated in a more complete manner. Under different assumptions on the right-hand side and its Fourier cosine transform the asymptotic behaviour of the solutions at infinity is determined and the L^2 - and L -norm of the Fourier cosine transform of the solution (and hence the L^2 -norm and the supremum norm of the solution itself) are estimated. Further, the stability of the solutions in the L^2 -norm is shown.

Solutions of the considered integral equation of the second kind can be used for a regularization of the autocorrelation equation $Ap = r$ [5].

2. Integral equation

We deal with the integral equation

$$p(t) + \int_0^\infty p(s)p(s+t) ds = \frac{g(t)}{2} \quad (t > 0) \quad (2.1)$$

under the assumption $g \in L(\mathbb{R}_+) \cap L^2(\mathbb{R}_+)$ looking for solutions $p \in L^2(\mathbb{R}_+)$ with $Ap \in L^2(\mathbb{R}_+)$. In the following, g and p are always real-valued functions.

We introduce the Fourier cosine and sine transform of p [4]

$$P(x) = \int_0^\infty p(t) \cos xt dt \equiv \mathcal{F}_c p$$

$$Q(x) = \int_0^\infty p(t) \sin xt dt \equiv \mathcal{F}_s p.$$

By $p \in L^2(\mathbb{R}_+)$ we have $P, Q \in L^2(\mathbb{R}_+)$. Further, we extend P as an even and Q as an odd function to the real axis \mathbb{R} so that $P, Q \in L^2(\mathbb{R})$. Applying the Fourier cosine transformation to (2.1) we obtain the condition

$$|\hat{F}(x)|^2 = \hat{G}(x) \quad (x \in \mathbb{R}) \quad (2.2)$$

(cf. [5, 6]) where $\hat{F}(z) = 1 + P(z) + iQ(z)$ are the boundary values of the holomorphic function in the upper half-plane $\text{Im } z > 0$

$$\hat{F}(z) = 1 + \int_0^\infty p(t)e^{itz} dt \quad (z = x + iy) \quad (2.3)$$

$$\hat{G}(x) = 1 + G(x), \quad G(x) = (\mathcal{F}_c g)(x). \quad (2.4)$$

Problem (2.2) with solutions \hat{F} of form (2.3) where $p \in L^2(\mathbb{R}_+)$ is equivalent to integral equation (2.1).

By the assumption $g \in L(\mathbb{R}_+)$ the function G is continuous on \mathbb{R} and $G(x) \rightarrow 0$ for $x \rightarrow \pm\infty$ so that $\widehat{G}(x) \rightarrow 1$ for $x \rightarrow \pm\infty$. From (2.2) the necessary solvability condition $\widehat{G}(x) \geq 0$ on \mathbb{R} follows. Somewhat stronger, we assume that $\widehat{G}(x) > 0$ on \mathbb{R} .

We are looking for solutions p of equation (2.1) for which the function \widehat{F} is continuous in $\text{Im } z \geq 0$, has the limit 1 for $z \rightarrow \infty$ uniformly in $\text{Im } z \geq 0$ and $\widehat{F}(x) \neq 0$ on \mathbb{R} . Then \widehat{F} has at most finitely many zeros in $\text{Im } z > 0$ which for real-valued function p are of the form

$$\begin{aligned} z_k^0 &= i y_k^0 \quad (k = 1, \dots, K; y_k^0 > 0) \\ z_j^1 &= x_j + i y_j, \quad z_j^2 = -x_j + i y_j \quad (j = 1, \dots, L; x_j, y_j > 0). \end{aligned}$$

Here every zero is counted corresponding to its multiplicity. These zeros can be chosen arbitrarily in the following.

By taking the logarithm in (2.2) and applying the theory of Cauchy integrals [1 - 3] the following solution of boundary value problem (2.2) for \widehat{F} is obtained (cf. [6]):

$$\widehat{F}(z) = q_0(z)q(z) \exp \Phi(z) \tag{2.5}$$

where

$$q_0(z) = \prod_{k=1}^K \frac{z - z_k^0}{z + z_k^0}, \quad q(z) = \prod_{j=1}^L \frac{(z - z_j^1)(z - z_j^2)}{(z - \overline{z_j^1})(z - \overline{z_j^2})} \tag{2.6}$$

$$\Phi(z) = \frac{2z}{\pi i} \int_0^\infty \frac{H(\xi)}{\xi^2 - z^2} d\xi, \quad H(x) = \frac{1}{2} \ln \widehat{G}(x). \tag{2.7}$$

In view of the Plemelj-Sochozky formula from (2.5) the solution $p = \frac{2}{\pi} \mathcal{F}_c P$ of equation (2.1) with

$$P(x) = -1 + \widehat{G}(x)^{1/2} \left[\text{Re}(q_0(x)q(x)) \cos I(x) + \text{Im}(q_0(x)q(x)) \sin I(x) \right] \tag{2.8}$$

follows where

$$I(x) = \frac{x}{\pi} \int_0^\infty \frac{\ln \widehat{G}(\xi)}{\xi^2 - x^2} d\xi. \tag{2.9}$$

We define this solution as the *general solution* of equation (2.1). The corresponding expression to (2.8) for Q is given by

$$Q(x) = \widehat{G}(x)^{1/2} \left[\text{Im}(q_0(x)q(x)) \cos I(x) - \text{Re}(q_0(x)q(x)) \sin I(x) \right]. \tag{2.10}$$

3. Asymptotic behaviour of the solution

Let us first have a look to the behaviour of the solution (2.8) for $x \rightarrow \infty$. In addition to $\widehat{G}(x) > 0$ on \mathbb{R}_+ , we assume that G is a Hölder continuous function on \mathbb{R}_+ with $G(x) = O(x^{-\delta})$ ($\delta > 1$) as $x \rightarrow \infty$. Then also H is Hölder continuous with

$$H(x) = \frac{1}{2} \ln[1 + G(x)] = O(x^{-\delta}) \tag{3.1}$$

$$\widehat{G}(x)^{1/2} = \sqrt{1 + G(x)} = 1 + O(x^{-\delta}). \tag{3.2}$$

Further,

$$\begin{aligned} \frac{2x}{\pi} \int_0^\infty \frac{H(\xi)}{\xi^2 - x^2} d\xi &= -\frac{2}{\pi} \frac{1}{x} \int_0^\infty H(\xi) d\xi + \frac{2}{\pi} \frac{1}{x} \int_0^\infty \frac{\xi^2 H(\xi)}{\xi^2 - x^2} d\xi \\ \int_0^\infty \frac{\xi^2 H(\xi)}{\xi^2 - x^2} d\xi &= \frac{1}{2} \int_0^\infty \frac{f(\eta)}{\eta - y} d\eta = o(1) \quad (x \rightarrow \infty) \end{aligned}$$

where $y = x^2, \eta = \xi^2$ and $f(\eta) = \sqrt{\eta} H(\sqrt{\eta}) = O(\eta^{-\gamma})$ with $\gamma = \frac{\delta-1}{2} > 0$. Therefore,

$$I(x) = -H_0 x^{-1} + o(x^{-1}), \quad H_0 = \frac{2}{\pi} \int_0^\infty H(\xi) d\xi. \tag{3.3}$$

Finally,

$$q_0(x)q(x) = \prod_{k=1}^K \frac{(x - i y_k^0)^2}{x^2 + (y_k^0)^2} \prod_{j=1}^L \frac{(x - x_j - i y_j)^2 (x + x_j - i y_j)^2}{[(x - x_j)^2 + y_j^2][(x + x_j)^2 + y_j^2]}$$

implying

$$\operatorname{Re}(q_0(x)q(x)) = 1 - Zx^{-2} + O(x^{-4}) = 1 + O(x^{-2}) \tag{3.4}$$

$$\operatorname{Im}(q_0(x)q(x)) = -Yx^{-1} + O(x^{-3}) \tag{3.5}$$

where

$$\begin{aligned} Y &= 2 \sum_{k=1}^K y_k^0 + 4 \sum_{j=1}^L y_j. \\ Z &= 2 \sum_{k=1}^K (y_k^0)^2 + 8 \sum_{j=1}^L y_j^2 - 8 \sum_{k=1}^K y_k^0 \sum_{j=1}^L y_j + 4 \sum_{k=1}^{K-1} \sum_{l=k+1}^K y_k^0 y_l^0 + 16 \sum_{j=1}^{L-1} \sum_{m=j+1}^L y_j y_m. \end{aligned}$$

In view of (3.2) - (3.5), from (2.8) with (2.9) we obtain

$$P(x) = -1 + [1 + O(x^{-\delta})][1 + O(x^{-2})] = O(x^{-\lambda}) \tag{3.6}$$

with $\lambda = \min\{\delta, 2\} > 1$. Hence $P \in L(\mathbb{R}_+) \cap L^2(\mathbb{R}_+)$.

We remark that from (2.10) in analogous manner the asymptotic behaviour of Q

$$Q(x) \sim (H_0 - Y)x^{-1} \quad (x \rightarrow \infty) \tag{3.7}$$

follows so that $Q \in L^2(\mathbb{R}_+)$ but $Q \notin L(\mathbb{R}_+)$ if $H_0 \neq Y$. Further, the function \widehat{F} is bounded on $\text{Im } z \geq 0$ with $|\widehat{F}(x) - 1| = |P(x) + iQ(x)| \in L^2(\mathbb{R})$ implying representation (2.3) for \widehat{F} by a well-known theorem of Paley and Wiener. Finally, for the more specific asymptotic behaviour

$$G(x) \sim g_0x^{-2} \quad (x \rightarrow \infty) \tag{3.8}$$

we get

$$P(x) \sim p_0x^{-2} \quad (x \rightarrow \infty) \tag{3.9}$$

with $p_0 = \frac{g_0}{2} - Z - \frac{H_0^2}{2} + H_0Y$.

Theorem 3.1. *For any $g \in L(\mathbb{R}_+) \cap L^2(\mathbb{R}_+)$ with Hölder-continuous Fourier cosine transform $G = \mathcal{F}_c g \in L^2(\mathbb{R}_+)$ satisfying $G(x) + 1 > 0$ on \mathbb{R}_+ and $G(x) = O(x^{-\delta})$ ($\delta > 1$) for $x \rightarrow \infty$ equation (2.1) has the solution $p = \frac{2}{\pi} \mathcal{F}_c P$ with $P \in L(\mathbb{R}_+) \cap L^2(\mathbb{R}_+)$ given by (2.8) and satisfying (3.6). If G fulfills (3.8), then P satisfies (3.9).*

From $P \in L(\mathbb{R}_+) \cap L^2(\mathbb{R}_+)$ it follows that $p \in L^2(\mathbb{R}_+)$ and p is a continuous function on \mathbb{R}_+ with limit $p(\infty) = 0$.

4. Estimation of the solution

We estimate the solution (2.8) under the additional assumption $G(x) \geq 0$ on \mathbb{R}_+ . By [4: Theorem 124] there holds $G \in L(\mathbb{R}_+)$ with $G(x) > 0$ on \mathbb{R}_+ if g is bounded, convex downwards and steadily decreasing to zero as $x \rightarrow \infty$. The solution is decomposed in the form

$$P(x) = J_1 + J_2 + J_3 + J_4$$

where

$$\begin{aligned} J_1 &= -[1 - \cos I(x)] \\ J_2 &= (\widehat{G}(x)^{1/2} - 1) \cos I(x) \\ J_3 &= \widehat{G}(x)^{1/2} [\text{Re}(q_0(x)q(x)) - 1] \cos I(x) \\ J_4 &= \widehat{G}(x)^{1/2} \text{Im}(q_0(x)q(x)) \sin I(x). \end{aligned}$$

As in [5], J_1 and J_2 can be estimated using the Riesz theorem and the inequalities $\ln(1 + u) \leq u$ and $\sqrt{1 + u} - 1 \leq \frac{u}{2}$ for $u \geq 0$. We have

$$\begin{aligned} \|J_1\|_{L^2}^2 &= \int_0^\infty [1 - \cos I(x)]^2 dx \leq \int_0^\infty I^2(x) dx \\ &= \frac{1}{4} \int_0^\infty \ln^2(1 + G(x)) dx \leq \frac{1}{4} \int_0^\infty G^2(x) dx = \frac{1}{4} \|G\|_{L^2}^2 \end{aligned} \quad (4.2)$$

$$\|J_1\|_L = \int_0^\infty [1 - \cos I(x)] dx \leq \frac{1}{2} \int_0^\infty I^2(x) dx \leq \frac{1}{8} \|G\|_{L^2}^2 \quad (4.3)$$

and

$$\|J_2\|_{L^2}^2 = \int_0^\infty [(G(x) + 1)^{1/2} - 1]^2 dx \leq \frac{1}{4} \int_0^\infty G^2(x) dx = \frac{1}{4} \|G\|_{L^2}^2 \quad (4.4)$$

$$\|J_2\|_L = \int_0^\infty [(G(x) + 1)^{1/2} - 1] dx \leq \frac{1}{2} \int_0^\infty G(x) dx = \frac{1}{2} \|G\|_L. \quad (4.5)$$

For estimating J_3 and J_4 we use the inequalities

$$\begin{aligned} |\operatorname{Re}(q_0(x)q(x)) - 1| &\leq \frac{C_1}{1 + x^2} \\ |\operatorname{Im}(q_0(x)q(x))| &\leq \frac{C_2 x}{1 + x^2} \end{aligned} \quad (4.6)$$

where

$$\begin{aligned} C_1 &= \sup_{x \in \mathbb{R}_+} [1 + x^2] |\operatorname{Re}(q_0(x)q(x)) - 1| \\ C_2 &= \sup_{x \in \mathbb{R}_+} \frac{1 + x^2}{x} |\operatorname{Im}(q_0(x)q(x))| \end{aligned}$$

are finite by (3.4) - (3.5) and $\operatorname{Im}(q_0(x)q(x)) = O(x)$ for $x \rightarrow 0$. Further,

$$\widehat{G}(x)^{1/2} \leq M = \sup_{x \in \mathbb{R}_+} (G(x) + 1)^{1/2} < \infty \quad (4.7)$$

due to the continuity of G on \mathbb{R}_+ and the limit $G(\infty) = 0$. Hence we obtain the estimates

$$\|J_3\|_{L^2}^2 \leq M^2 C_1^2 \int_0^\infty \frac{dx}{(1 + x^2)^2} = \frac{\pi}{4} M^2 C_1^2 \quad (4.8)$$

$$\|J_3\|_L \leq M C_1 \int_0^\infty \frac{dx}{1 + x^2} = \frac{\pi}{2} M C_1 \quad (4.9)$$

and

$$\|J_4\|_{L^2}^2 \leq M^2 C_2^2 \int_0^\infty \frac{x^2}{(1+x^2)^2} dx = \frac{\pi}{4} M^2 C_2^2 \tag{4.10}$$

$$\begin{aligned} \|J_4\|_L &\leq M C_2 \int_0^\infty \frac{x}{1+x^2} |I(x)| dx \\ &\leq M C_2 \left(\int_0^\infty \frac{x^2}{(1+x^2)^2} dx \right)^{\frac{1}{2}} \left(\int_0^\infty I^2(x) dx \right)^{\frac{1}{2}} \\ &\leq M C_2 \frac{\sqrt{\pi}}{2} \left(\frac{1}{4} \int_0^\infty G^2(x) dx \right)^{\frac{1}{2}} = \sqrt{\pi} \frac{M}{4} C_2 \|G\|_{L^2}. \end{aligned} \tag{4.11}$$

From (4.1) and (4.2), (4.4), (4.8), (4.10), resp. (4.3), (4.5), (4.9), (4.11) the estimations

$$\|P\|_{L^2} \leq \|G\|_{L^2} + \frac{\sqrt{\pi}}{2} M(C_1 + C_2) \tag{4.12}$$

$$\|P\|_L \leq \frac{1}{8} \|G\|_{L^2}^2 + \frac{1}{2} \|G\|_L + M \left(\frac{\pi}{2} C_1 + \frac{\sqrt{\pi}}{4} C_2 \|G\|_{L^2} \right) \tag{4.13}$$

follow. Also the Paley-Wiener conditions for representation (2.3) of \widehat{F} are fulfilled.

Theorem 4.1. *For any $g \in L(\mathbb{R}_+) \cap L^2(\mathbb{R}_+)$ with non-negative $G = \mathcal{F}_c g \in L^2(\mathbb{R}_+)$ equation (2.1) has the solution $p = \frac{2}{\pi} \mathcal{F}_c P \in L^2(\mathbb{R}_+)$ with $P \in L^2(\mathbb{R}_+)$ given by (2.8) satisfying estimation (4.12).*

If, in addition, $G \in L(\mathbb{R}_+)$, then also $P \in L(\mathbb{R}_+)$ satisfying estimation (4.13), and p is continuous on \mathbb{R}_+ with limit $p(\infty) = 0$.

Theorem 4.1 generalizes [6: Theorem 1].

In avoiding the assumption $G(x) \geq 0$ on \mathbb{R}_+ we further consider functions G obeying an inequality of the form

$$-\frac{C_0}{a_0^2 + x^2} \leq G(x) \leq \frac{C}{a^2 + x^2} \quad \text{on } \mathbb{R}_+ \tag{4.14}$$

where $a, a_0 > 0$ and $C, C_0 > 0$ with $C_0 < a_0^2$ ensuring that $G(x) + 1 > 0$ on \mathbb{R}_+ . Then

$$\int_0^\infty \ln^2(1 + G(x)) dx \leq A^2$$

where

$$A^2 = \max \left\{ \int_0^\infty \ln^2 \frac{(C + a^2) + x^2}{a^2 + x^2} dx, \int_0^\infty \ln^2 \frac{(a_0^2 - C_0) + x^2}{a_0^2 + x^2} dx \right\}$$

implying the estimations for J_1

$$\|J_1\|_{L^2}^2 \leq \frac{1}{4}A^2, \quad \|J_1\|_L \leq \frac{1}{8}A^2. \tag{4.15}$$

Further,

$$|(G(x) + 1)^{1/2} - 1| \leq N(x)$$

where

$$N(x) = \max \left\{ \frac{\sqrt{(C + a^2) + x^2} - \sqrt{a^2 + x^2}}{\sqrt{a^2 + x^2}}, \frac{\sqrt{a_0^2 + x^2} - \sqrt{(a_0^2 - C_0) + x^2}}{\sqrt{a_0^2 + x^2}} \right\}$$

with $N(x) = O(x^{-2})$ for $x \rightarrow +\infty$. Hence for J_2 the estimates

$$\|J_2\|_{L^2}^2 \leq B^2, \quad \|J_2\|_L \leq D \tag{4.16}$$

with the integrals $B^2 = \int_0^\infty N^2(x) dx$ and $D = \int_0^\infty N(x) dx$ hold.

The estimates for J_3 and J_4 are independent of the assumption $G(x) \geq 0$. Hence, for functions G fulfilling (4.14) we obtain the estimations

$$\|P\|_{L^2} \leq \frac{1}{2}A + B + \frac{\sqrt{\pi}}{2}M(C_1 + C_2) \tag{4.17}$$

$$\|P\|_L \leq \frac{1}{8}A^2 + D + M\left(\frac{\pi}{2}C_1 + \frac{\sqrt{\pi}}{4}C_2\|G\|_{L^2}\right) \tag{4.18}$$

with the constants M, C_1, C_2 from above and A, B, D from (4.15) - (4.16).

Theorem 4.2. *For any $g \in L(\mathbb{R}_+) \cap L^2(\mathbb{R}_+)$ with $G = \mathcal{F}_c g$ obeying inequality (4.14) equation (2.1) has the solution $p = \frac{2}{\pi}\mathcal{F}_c P$ with $P \in L(\mathbb{R}_+) \cap L^2(\mathbb{R}_+)$ given by (2.8) satisfying estimations (4.17) - (4.18). The solution p is continuous on \mathbb{R}_+ with limit $p(\infty) = 0$.*

We remark that analogously to (4.14) functions G satisfying $-\frac{C_0}{(a_0^2+x^2)^{\gamma_0}} \leq G(x) \leq \frac{C}{(a^2+x^2)^\gamma}$ where $\gamma, \gamma_0 > \frac{1}{2}$ can be dealt with, too.

5. Stability theorem

We investigate the stability of the solution p in $L^2(\mathbb{R}_+)$ for fixed zeros of \widehat{F} . Let $g_j \in L(\mathbb{R}_+) \cap L^2(\mathbb{R}_+)$ ($j = 1, 2$) with $G_j = \mathcal{F}_c g_j$ satisfying the assumptions

$$\widehat{G}_j(x) = 1 + G_j(x) \geq \delta_j^2 > 0 \quad \text{on } \mathbb{R}_+. \tag{5.1}$$

The difference of the corresponding Fourier cosine transforms $P_j = \mathcal{F}_c p_j$ of the solutions p_j ($j = 1, 2$) of (2.8) is given by

$$P_1(x) - P_2(x) = K_1(x) + K_2(x) \tag{5.2}$$

where

$$\begin{aligned}
 K_1(x) &= [\widehat{G}_1(x)^{1/2} - \widehat{G}_2(x)^{1/2}] \\
 &\quad \times \left[\operatorname{Re}(q_0(x)q(x)) \cos I_1(x) + \operatorname{Im}(q_0(x)q(x)) \sin I_1(x) \right] \\
 K_2(x) &= \widehat{G}_2(x)^{1/2} \left\{ \operatorname{Re}(q_0(x)q(x)) [\cos I_1(x) - \cos I_2(x)] \right. \\
 &\quad \left. + \operatorname{Im}(q_0(x)q(x)) [\sin I_1(x) - \sin I_2(x)] \right\}
 \end{aligned}$$

with

$$I_j(x) = \frac{x}{\pi} \int_0^\infty \frac{\ln \widehat{G}_j(\xi)}{\xi^2 - x^2} d\xi \quad (j = 1, 2).$$

There holds

$$\widehat{G}_1(x)^{1/2} - \widehat{G}_2(x)^{1/2} = \frac{\widehat{G}_1(x) - \widehat{G}_2(x)}{\widehat{G}_1(x)^{1/2} + \widehat{G}_2(x)^{1/2}}$$

implying

$$|\widehat{G}_1(x)^{1/2} - \widehat{G}_2(x)^{1/2}| \leq \frac{1}{\delta} |G_1(x) - G_2(x)|$$

with $\delta = \delta_1 + \delta_2 > 0$. Further,

$$|\operatorname{Re}(q_0q) \cos I_1 + \operatorname{Im}(q_0q) \sin I_1| = |\operatorname{Re}[\exp(-iI_1)q_0q]| \leq |q_0(x)q(x)| = 1.$$

Therefore, $|K_1(x)| \leq \frac{1}{\delta} |G_1(x) - G_2(x)|$ on \mathbb{R}_+ and

$$\|K_1\|_{L^2} \leq \frac{1}{\delta} \|G_1 - G_2\|_{L^2}. \tag{5.3}$$

From

$$\begin{aligned}
 &|\operatorname{Re}(q_0q)[\cos I_1 - \cos I_2] + \operatorname{Im}(q_0q)[\sin I_1 - \sin I_2]| \\
 &= 2|\operatorname{Im}(\exp(-\frac{i}{2}[I_1 + I_2])q_0q)| \cdot \left| \sin \frac{I_1 - I_2}{2} \right| \\
 &\leq 2|q_0q| \left| \sin \frac{I_1 - I_2}{2} \right| \\
 &\leq |I_1(x) - I_2(x)|
 \end{aligned}$$

we obtain

$$|K_2(x)| \leq M_2 |I_1(x) - I_2(x)| \tag{5.4}$$

where $M_2 = \sup_{x \in \mathbb{R}_+} \widehat{G}_2(x)^{1/2}$. Further, by the Riesz theorem

$$\|I_1 - I_2\|_{L^2} = \frac{1}{2} \left\| \ln \frac{\widehat{G}_1}{\widehat{G}_2} \right\|_{L^2} \tag{5.5}$$

and

$$\left| \ln \frac{\widehat{G}_1(x)}{\widehat{G}_2(x)} \right| \leq \ln \left(1 + \frac{|G_1(x) - G_2(x)|}{\gamma_0} \right) \leq \frac{1}{\gamma_0} |G_1(x) - G_2(x)| \quad (5.6)$$

where $\gamma_0 = \delta_0^2 = \min\{\delta_1^2, \delta_2^2\} > 0$. In view of (5.4) - (5.6) we get

$$\|K_2\|_{L^2} \leq \frac{1}{\gamma_0} \frac{M_2}{2} \|G_1 - G_2\|_{L^2}. \quad (5.7)$$

From (5.2) and (5.3), (5.7) the inequality

$$\|P_1 - P_2\|_{L^2} \leq E \|G_1 - G_2\|_{L^2}, \quad E = \frac{1}{\delta} + \frac{M_2}{2\gamma_0}$$

follows which is equivalent to the *stability estimation* for the solution p_j ($j = 1, 2$) of equation (2.1)

$$\|p_1 - p_2\|_{L^2} \leq E \|g_1 - g_2\|_{L^2}. \quad (5.8)$$

Theorem 5.1. *Let $g_j \in L(\mathbb{R}_+) \cap L^2(\mathbb{R}_+)$ ($j = 1, 2$) with $G_j = \mathcal{F}_c g_j$ satisfying assumptions (5.1). Then for the corresponding solutions $p_j \in L^2(\mathbb{R}_+)$ of equation (2.1) stability estimation (5.8) holds.*

Theorem 5.1 generalizes [5: Theorem 3] (see there also for a relaxation of the assumptions $g_j \in L(\mathbb{R}_+)$).

6. Integral-differential equation

In the following we deal with the integral-differential equation of first order

$$p'(t) + \mu p(t) + \int_0^\infty p(s)p(s+t) ds = \frac{g(t)}{2} \quad (t > 0) \quad (6.1)$$

where $\mu \in \mathbb{R}$ and we again assume $g \in L(\mathbb{R}_+) \cap L^2(\mathbb{R}_+)$ looking for solutions $p \in L^2(\mathbb{R}_+)$ which are continuous on \mathbb{R}_+ with $p(\infty) = 0$ having the derivative $p' \in L^2(\mathbb{R}_+)$.

Applying the Fourier cosine transformation to equation (6.1) we obtain the condition

$$|\widehat{F}(x)|^2 = \widehat{G}(x) + 2\alpha \quad (x \in \mathbb{R}) \quad (6.2)$$

(cf. [6]) where

$$\widehat{F}(x) = \mu + ix + P(x) + iQ(x)$$

are the boundary values of the holomorphic function in the upper half-plane $\text{Im } z > 0$

$$\widehat{F}(z) = \mu + iz + \int_0^\infty p(t)e^{itz} dt \quad (z = x + iy) \tag{6.3}$$

and $\alpha = p(0)$,

$$\widehat{G}(x) = \mu^2 + x^2 + G(x) \quad (x \in \mathbb{R}). \tag{6.4}$$

Again $P = \mathcal{F}_c p, Q = \mathcal{F}_s p$ and $G = \mathcal{F}_c g$. Assuming $P \in L(\mathbb{R}_+)$, we have

$$\alpha = \frac{2}{\pi} \int_0^\infty P(x) dx. \tag{6.5}$$

Problem (6.2) with (6.5) and solutions \widehat{F} of form (6.3) where $p' \in L^2(\mathbb{R}_+)$ is equivalent to integral-differential equation (6.1). Further, stronger than the necessary solvability condition $\widehat{G}(x) + 2\alpha \geq 0$ on \mathbb{R} for condition (6.2) we assume $\widehat{G}(x) + 2\alpha > 0$ on \mathbb{R} .

We are looking for solutions p of equation (6.1) for which the function $\widehat{F}(z) - iz$ is continuous in $\text{Im } z \geq 0$ with the limit μ for $z \rightarrow \infty$ uniformly in $\text{Im } z \geq 0$ and $\widehat{F}(x) \neq 0$ on \mathbb{R} . Then \widehat{F} has at most finitely many zeros in $\text{Im } z > 0$ of the form as above:

$$\begin{aligned} z_k^0 &= i y_k^0 \quad (k = 1, \dots, K) \\ z_j^{1,2} &= \pm x_j + i y_j \quad (j = 1, \dots, L) \end{aligned}$$

where $y_k^0, x_j, y_j > 0$. Again these zeros of \widehat{F} can be chosen in arbitrary manner.

From (6.2) the following solution for \widehat{F} is obtained (cf. [6] again):

$$\widehat{F}(z) = (iz - b)q_0(z)q(z)\tilde{F}(z), \quad \tilde{F}(z) = \exp \tilde{\Phi}(z) \tag{6.6}$$

where $b > 0$ is an arbitrarily chosen number (for instance, $b = 1$), q_0 and q are given by (2.6) again and

$$\tilde{\Phi}(z) = \frac{2z}{\pi i} \int_0^\infty \frac{\tilde{H}(\xi)}{\xi^2 - z^2} d\xi, \quad \tilde{H}(x) = \frac{1}{2} \ln \tilde{G}(x) \tag{6.7}$$

with

$$\tilde{G}(x) = \frac{\widehat{G}(x) + 2\alpha}{x^2 + b^2} \rightarrow 1 \quad (x \rightarrow \pm\infty).$$

Therefore, the general solution of equation (6.1) is given by $p = \frac{2}{\pi} \mathcal{F}_c P$ with

$$\begin{aligned} P(x) &= -\mu + \tilde{G}(x)^{1/2} \\ &\times \left[\text{Re}((ix - b)q_0(x)q(x)) \cos I(x) + \text{Im}((ix - b)q_0(x)q(x)) \sin I(x) \right] \end{aligned} \tag{6.8}$$

where now

$$I(x) = \frac{x}{\pi} \int_0^\infty \frac{\ln \tilde{G}(\xi)}{\xi^2 - x^2} d\xi. \tag{6.9}$$

The parameter α has to fulfill equation (6.5). The integral in (6.5) can be calculated by

$$\int_0^\infty P(x) dx = \frac{1}{2} \lim_{R \rightarrow \infty} \int_{K_R} [\mu - \widehat{F}(z)] dz$$

where K_R denotes the upper semi-circle with centre 0 and radius R . There hold the asymptotic relations

$$\begin{aligned} \tilde{H}(x) &\sim h_1 x^{-2} \quad (x \rightarrow \pm\infty) & h_1 &= \alpha + \frac{1}{2}(\mu^2 - b^2) \\ \tilde{\Phi}(z) &\sim i H_1 z^{-1} + h_1 z^{-2} & H_1 &= \frac{2}{\pi} \int_0^\infty \tilde{H}(\xi) d\xi \end{aligned}$$

for $z \rightarrow \infty$, $\text{Im } z > 0$ (cf. [6]) implying

$$\begin{aligned} \tilde{F}(z) &\sim 1 + \tilde{\Phi}(z) + \tilde{\Phi}^2(z) \\ &\sim 1 + i H_1 z^{-1} + [h_1 - \frac{1}{2} H_1^2] z^{-2} \end{aligned} \tag{6.10}$$

for $z \rightarrow \infty$, $\text{Im } z > 0$. Further, for $z \rightarrow \infty$

$$\begin{aligned} q_0(z) &\sim 1 - 2i \sum_{k=1}^K y_k^0 \cdot z^{-1} - \left(2 \sum_{k=1}^K (y_k^0)^2 + 4 \sum_{k=1}^{K-1} \sum_{l=k+1}^K y_k^0 y_l^0 \right) z^{-2} \\ q(z) &\sim 1 - 4i \sum_{j=1}^L y_j \cdot z^{-1} - \left(8 \sum_{j=1}^L y_j^2 + 16 \sum_{j=1}^{L-1} \sum_{m=j+1}^L y_j y_m \right) z^{-2} \end{aligned}$$

yielding

$$\mu - \widehat{F}(z) \sim -i z + A + B i z^{-1} \tag{6.11}$$

for $z \rightarrow \infty$, $\text{Im } z > 0$ where

$$A = \mu + b + H_1 - Y, \quad Y = 2 \sum_{k=1}^K y_k^0 + 4 \sum_{j=1}^L y_j$$

and some longer expression for B . From (6.11) it follows that relation (6.5) holds if and only if $A = 0$ and $B = -\alpha$.

Some algebra shows that $B = -\alpha$ is a consequence of $A = 0$. Hence relation (6.5) is equivalent to the condition $A = 0$, i.e. $H_1 = Y - \mu - b$ or, explicitly,

$$\frac{1}{\pi} \int_0^\infty \ln \frac{\mu^2 + x^2 + 2\alpha + G(x)}{x^2 + b^2} dx = Y - \mu - b. \tag{6.12}$$

This condition for α is independent of $b(> 0)$ as it must be. Its left-hand side is a strictly increasing, concave function of α tending to $+\infty$ as $\alpha \rightarrow +\infty$. So there exists at most one root α_0 of this equation for which $\widehat{G}(x) + 2\alpha_0 > 0$ on \mathbb{R} .

7. Asymptotic behaviour of the solution

We again make the stronger assumption that G is a Hölder continuous function on \mathbb{R}_+ with $G(x) = O(x^{-\delta})$ ($\delta > 1$) as $x \rightarrow \infty$. Then also \tilde{G} and \tilde{H} are Hölder continuous with

$$\begin{aligned} \tilde{G}(x)^{1/2} &= 1 + h_1 x^{-2} + O(x^{-2-\lambda}) \\ \tilde{H}(x) &= h_1 x^{-2} + O(x^{-2-\lambda}) \end{aligned}$$

where $h_1 = \alpha + \frac{1}{2}(\mu^2 - b^2)$ as above and $\lambda = \min\{\delta, 2\} > 1$. Further (cf. [6]),

$$I(x) = -H_1 x^{-1} + O(x^{-3}), \quad H_1 = \frac{2}{\pi} \int_0^\infty \tilde{H}(\xi) d\xi$$

implying

$$\tilde{F}(x) = 1 + i H_1 x^{-1} + [h_1 - \frac{1}{2} H_1^2] x^{-2} + i O(x^{-3}) + O(x^{-2-\lambda}) \tag{7.1}$$

with real O -terms. Relation (7.1) is analogous to (6.10) above and a similar algebra as there leads from (7.1) to

$$\hat{F}(x) = i x + [\mu - A] - i B x^{-1} + O(x^{-2}) + i O(x^{-1-\lambda}) \tag{7.2}$$

with real O -terms which is analogous to (6.11) with the same constants A and B . Assuming the solvability condition $A = 0$ with $B = -\alpha$, we obtain the asymptotic relations

$$\left. \begin{aligned} P(x) &= O(x^{-2}) \\ xQ(x) - \alpha &= O(x^{-\lambda}) \quad (\lambda > 1) \end{aligned} \right\} \quad (x \rightarrow \infty) \tag{7.3}$$

which yield $P, xQ - \alpha \in L(\mathbb{R}_+) \cap L^2(\mathbb{R}_+)$. Therefore, the solution $p = \frac{2}{\pi} \mathcal{F}_c P \in L^2(\mathbb{R}_+)$ and its derivative $p' = \frac{2}{\pi} \mathcal{F}_c (xQ - \alpha) \in L^2(\mathbb{R}_+)$ are continuous functions on \mathbb{R}_+ with limits $p(\infty) = p'(\infty) = 0$.

Finally, the function $\hat{F} - \mu - iz$ fulfills the Paley-Wiener condition on $\text{Im } z \geq 0$ with $|\hat{F}(x) - \mu - ix| = |P(x) + iQ(x)| \in L^2(\mathbb{R})$ implying representation (6.3) for \hat{F} .

Theorem 7.1. *For any $g \in L(\mathbb{R}_+) \cap L^2(\mathbb{R}_+)$ with Hölder continuous Fourier cosine transform $G = \mathcal{F}_c g \in L^2(\mathbb{R}_+)$ satisfying $G(x) = O(x^{-\delta})$ with $\delta > 1$ for $x \rightarrow \infty$ and a real parameter α satisfying relation (6.12) for $b > 0$ where $G(x) + \mu^2 + x^2 + 2\alpha > 0$ on \mathbb{R}_+ , equation (6.1) has the solution $p = \frac{2}{\pi} \mathcal{F}_c P$ with $P = \mathcal{F}_c p \in L(\mathbb{R}_+) \cap L^2(\mathbb{R}_+)$ given by (6.8) and satisfying together with $Q = \mathcal{F}_s p$ the asymptotic relations (7.3) for $x \rightarrow \infty$. The solution p and its derivative p' are in $L^2(\mathbb{R}_+)$ and are continuous functions on \mathbb{R}_+ with limits $p(\infty) = p'(\infty) = 0$.*

Theorem 7.1 generalizes [6: Theorem 2].

8. Estimation of the solution

In the following we estimate the L^2 -norm of the solution under the assumptions $g \in L(\mathbb{R}_+) \cap L^2(\mathbb{R}_+)$ and

$$G(x) + \mu^2 + 2\alpha \geq \delta^2 > 0 \quad \text{on } \mathbb{R}_+ \quad (8.1)$$

with some $\delta > 0$ where α satisfies equation (6.12). Further, without loss of generality, we choose b with $0 < b \leq \delta$ so that

$$G_0(x) \equiv G(x) + \mu^2 + 2\alpha - b^2 \geq 0 \quad \text{on } \mathbb{R}_+. \quad (8.2)$$

In this and the next section we write $\|\cdot\|$ instead of $\|\cdot\|_{L^2}$, for simplicity. Substituting $\mu = Y - b - H_1$ from (6.12) into (6.8), the function P can be decomposed in the form

$$P(x) = \sum_{k=1}^7 J_k \quad (8.3)$$

where

$$\begin{aligned} J_1 &= (b - Y)[1 - \cos I(x)] \\ J_2 &= \{b - Y + \operatorname{Re}[(ix - b)q_0(x)q(x)]\} \cos I(x) \\ J_3 &= [\tilde{G}(x)^{1/2} - 1] \operatorname{Re}[(ix - b)q_0(x)q(x)] \cos I(x) \\ J_4 &= H_1 + xI(x) \\ J_5 &= x[\sin I(x) - I(x)] \\ J_6 &= \{\operatorname{Im}[(ix - b)q_0(x)q(x)] - x\} \sin I(x) \\ J_7 &= [\tilde{G}(x)^{1/2} - 1] \operatorname{Im}[(ix - b)q_0(x)q(x)] \sin I(x) \end{aligned}$$

and I is given by (6.9) with

$$\tilde{G}(x) = 1 + \frac{G_0(x)}{x^2 + b^2}. \quad (8.4)$$

At first we estimate the expressions J_1, J_4 and J_5 . We have

$$\begin{aligned}
 \|J_1\|^2 &= \int_0^\infty J_1^2 dx \\
 &= (b - Y)^2 \int_0^\infty [1 - \cos I(x)]^2 dx \\
 &\leq (b - Y)^2 \int_0^\infty I^2(x) dx \\
 &= \frac{(b-Y)^2}{4} \int_0^\infty \ln^2 \tilde{G}(x) dx \\
 &\leq \frac{(b-Y)^2}{4} \int_0^\infty \frac{G_0^2(x)}{(x^2 + b^2)^2} dx \\
 &\leq \frac{(b-Y)^2}{2} \left[(\mu^2 + 2\alpha - b^2)^2 \int_0^\infty \frac{dx}{(x^2 + b^2)^2} + \int_0^\infty \frac{G^2(x)}{(x^2 + b^2)^2} dx \right] \\
 &\leq \frac{(b-Y)^2}{2} \left[\frac{\pi}{4b^3} (\mu^2 + 2\alpha - b^2)^2 + \frac{1}{b^4} \|G\|^2 \right]
 \end{aligned}$$

and hence

$$\|J_1\| \leq \frac{|b-Y|}{\sqrt{2b}} \left[\frac{1}{2} \left(\frac{\pi}{b}\right)^{1/2} |\mu^2 + 2\alpha - b^2| + \frac{1}{b} \|G\| \right]. \tag{8.5}$$

Further, we use the integral inequality of Hardy and Littlewood (cf. [2: Chapter II/§ 3])

$$I_{\beta,p} \equiv \int_0^\infty x^\beta \left| \frac{x}{\pi} \int_0^\infty \frac{f(\xi)}{\xi^2 - x^2} d\xi \right|^p dx \leq C_{\beta,p}^p \int_0^\infty x^\beta |f(x)|^p dx$$

holding for $-p - 1 < \beta < p - 1$ ($p > 1$) with $C_{\beta,p} = \frac{1}{2} M_{\beta,p}$ where $M_{\beta,p}$ is the norm of the Hilbert transformation in the corresponding weighted Lebesgue space. So, for the integral

$$J_4 = \frac{1}{\pi} \int_0^\infty \frac{\xi^2 \ln \tilde{G}(\xi)}{\xi^2 - x^2} d\xi$$

there holds

$$\begin{aligned}
 \|J_4\|^2 &\leq A_2^2 \int_0^\infty x^2 \ln^2 \tilde{G}(x) dx \\
 &\leq A_2^2 \int_0^\infty \frac{x^2 G_0^2(x)}{(x^2 + b^2)^2} dx \\
 &\leq 2A_2^2 \int_0^\infty \frac{1}{x^2 + b^2} [(\mu^2 + 2\alpha - b^2)^2 + G^2(x)] dx \\
 &\leq 2A_2^2 \left[\frac{1}{b} (\mu^2 + 2\alpha - b^2)^2 + \frac{1}{b^2} \|G\|^2 \right]
 \end{aligned}$$

where $A_2 = C_{-2,2}$ and

$$\|J_4\| \leq \sqrt{2}A_2 \left[\frac{1}{\sqrt{b}} |\mu^2 + 2\alpha - b^2| + \frac{1}{b} \|G\| \right]. \tag{8.6}$$

Moreover,

$$\begin{aligned} \|J_5\|^2 &= \int_0^\infty x^2 [I(x) - \sin I(x)]^2 dx \\ &\leq \frac{1}{36} \int_0^\infty x^2 I^6(x) dx \\ &\leq \frac{A_3^2}{36} \int_0^\infty x^2 \ln^6 \tilde{G}(x) dx \\ &\leq \frac{16}{9} A_3^2 \left[|\mu^2 + 2\alpha - b^2|^6 \int_0^\infty \frac{x^2}{(x^2 + b^2)^6} dx + \int_0^\infty \frac{x^2 G^6(x)}{(x^2 + b^2)^6} dx \right] \\ &\leq \frac{16}{9} A_3^2 \left[|\mu^2 + 2\alpha - b^2|^6 \int_0^\infty \frac{dx}{(x^2 + b^2)^5} + \frac{1}{b^{10}} \int_0^\infty G^6(x) dx \right] \\ &\leq \frac{16}{9} A_3^2 \left[|\mu^2 + 2\alpha - b^2|^6 \frac{\pi}{2} \frac{1}{b^9} + \frac{1}{b^{10}} M_1^2 \|G\|^2 \right] \end{aligned}$$

where $A_3 = C_{2,6}^3$ and $M_1 = \sup_{x \in \mathbb{R}_+} G^2(x)$, yielding

$$\|J_5\| \leq \frac{4}{3} \frac{1}{b^5} A_3 \left[\left(\frac{\pi b}{2}\right)^{1/2} |\mu^2 + 2\alpha - b^2|^3 + M_1 \|G\| \right]. \tag{8.7}$$

Secondly, we estimate J_2 and J_6 . We get

$$\begin{aligned} |J_2| &\leq |b - Y + \operatorname{Re}[(ix - b)q_0(x)q(x)]| \\ |J_6| &\leq |\operatorname{Im}[(ix - b)q_0(x)q(x)] - x|. \end{aligned}$$

The asymptotic relation

$$q_0(x)q(x) = 1 - iYx^{-1} + O(x^{-2}) + iO(x^{-3}) \quad (x \rightarrow \pm\infty)$$

with real O -terms (cp. (3.4) - (3.5)) implies

$$\begin{aligned} b - Y + \operatorname{Re}[(ix - b)q_0(x)q(x)] &= O(x^{-2}) \\ \operatorname{Im}[(ix - b)q_0(x)q(x)] - x &= O(x^{-1}). \end{aligned}$$

Therefore the inequalities

$$\begin{aligned} |J_2| &\leq \frac{D_1}{1 + x^2} \\ |J_6| &\leq \frac{D_2 x}{1 + x^2} \end{aligned} \quad (x \in \mathbb{R}_+)$$

hold where

$$D_1 = \sup_{x \in \mathbb{R}_+} [1 + x^2] |b - Y + \operatorname{Re}[(i x - b)q_0(x)q(x)]| < \infty$$

$$D_2 = \sup_{x \in \mathbb{R}_+} [1 + x^2] \frac{1}{x} |\operatorname{Im}[(i x - b)q_0(x)q(x)]| < \infty$$

(cp. (4.6)). This gives the estimates

$$\begin{aligned} \|J_2\| &\leq \frac{D_1}{2} \sqrt{\pi} \\ \|J_6\| &\leq \frac{D_2}{2} \sqrt{\pi}. \end{aligned} \tag{8.8}$$

Finally, we have to estimate J_3 and J_7 . It is

$$\begin{aligned} |J_3| &\leq A(x) |\operatorname{Re}[(i x - b)q_0(x)q(x)]| \\ |J_7| &\leq A(x) |\operatorname{Im}[(i x - b)q_0(x)q(x)]| \end{aligned}$$

where $A(x) = \tilde{G}^{1/2}(x) - 1$. Now by (8.4)

$$A(x) \leq \frac{G_0(x)}{x^2 + b^2} \leq \frac{M_0}{x^2 + b^2}, \quad M_0 = \sup_{x \in \mathbb{R}_+} G_0(x)$$

and, in view of $|q_0(x)q(x)| = 1$,

$$\left. \begin{aligned} &|\operatorname{Re}[(i x - b)q_0(x)q(x)]| \\ &|\operatorname{Im}[(i x - b)q_0(x)q(x)]| \end{aligned} \right\} \leq x + b \quad \text{on } \mathbb{R}_+.$$

Therefore,

$$\|J_3\|^2, \|J_7\|^2 \leq M_0^2 \int_0^\infty \frac{(x + b)^2}{(x^2 + b^2)^2} dx \leq 2M_0^2 \int_0^\infty \frac{dx}{x^2 + b^2} = \frac{M_0^2 \pi}{b}$$

and

$$\|J_3\|, \|J_7\| \leq \left(\frac{\pi}{b}\right)^{1/2} M_0. \tag{8.9}$$

From (8.3) and (8.2) - (8.9) we obtain the estimation $\|P\| \leq E_0 + E_1 \|G\|$ where E_0 and E_1 are determined by the coefficients in estimates (8.2) - (8.9). This yields the estimation for the L^2 -norm of the solution p

$$\|p\| \leq \frac{2}{\pi} E_0 + E_1 \|g\|. \tag{8.10}$$

Theorem 8.1. *For any $g \in L(\mathbb{R}_+) \cap L^2(\mathbb{R}_+)$ satisfying (8.1) and choosing $0 < b \leq \delta$ the norm of the solution $p \in L^2(\mathbb{R}_+)$ fulfills estimation (8.10).*

Remark. For obtaining an analogous estimation for the norm of p' one can use equation (6.1) where in estimating the Fourier cosine transform of the integral term the norm of P^2 can be dealt with in a similar way as the norm of P and it should be observed that Q is the Hilbert transform of P .

9. Stability theorem

Let $g_j \in L(\mathbb{R}_+) \cap L^2(\mathbb{R}_+)$ ($j = 1, 2$) with $G_j = \mathcal{F}_c g_j$ satisfying the inequalities

$$G_j(x) + \mu^2 + 2\alpha_j \geq \delta_j^2 > 0 \quad \text{on } \mathbb{R}_+ \tag{9.1}$$

with some $\delta_j > 0$ where α_j obeys the corresponding equation (6.12) with some joint $b > 0$. By (6.8) the difference $P_1 - P_2$ of the corresponding solutions with the same fixed zeros of \widehat{F}_j can be decomposed in the form

$$P_1 - P_2 = \sum_{m=1}^4 K_m \tag{9.2}$$

where

$$\begin{aligned} K_1 &= [\widetilde{G}_1^{1/2}(x) - \widetilde{G}_2^{1/2}(x)] \operatorname{Re}[(ix - b)q_0(x)q(x)] \cos I_1(x) \\ K_2 &= [\widetilde{G}_1^{1/2}(x) - \widetilde{G}_2^{1/2}(x)] \operatorname{Im}[(ix - b)q_0(x)q(x)] \sin I_1(x) \\ K_3 &= \widetilde{G}_2^{1/2}(x) \operatorname{Re}[(ix - b)q_0(x)q(x)] (\cos I_1(x) - \cos I_2(x)) \\ K_4 &= \widetilde{G}_2^{1/2}(x) \operatorname{Im}[(ix - b)q_0(x)q(x)] (\sin I_1(x) - \sin I_2(x)) \end{aligned}$$

with

$$\widetilde{G}_j(x) = \frac{\widehat{G}_j(x) + 2\alpha_j}{x^2 + b^2}, \quad \widehat{G}_j(x) = \mu^2 + x^2 + G_j(x), \quad I_j(x) = \frac{x}{\pi} \int_0^\infty \frac{\ln \widetilde{G}_j(\xi)}{\xi^2 - x^2} d\xi.$$

At first we estimate K_1 and K_2 . We have

$$\begin{aligned} |K_1| &\leq A_0(x) |\operatorname{Re}[(ix - b)q_0(x)q(x)]| \\ |K_2| &\leq A_0(x) |\operatorname{Im}[(ix - b)q_0(x)q(x)]| \end{aligned}$$

where $A_0(x) = |\widetilde{G}_1^{1/2}(x) - \widetilde{G}_2^{1/2}(x)|$. Now

$$A_0(x) \leq \frac{(x^2 + b^2)^{-1/2}}{(x^2 + \delta_1^2)^{1/2} + (x^2 + \delta_2^2)^{1/2}} [|G_1(x) - G_2(x)| + 2|\alpha_1 - \alpha_2|]$$

and again

$$\left. \begin{aligned} &|\operatorname{Re}[(ix - b)q_0(x)q(x)]| \\ &|\operatorname{Im}[(ix - b)q_0(x)q(x)]| \end{aligned} \right\} \leq x + b \quad \text{on } \mathbb{R}_+$$

implying

$$|K_1|, |K_2| \leq \frac{1}{\sqrt{2(x^2 + \delta_0^2)^{1/2}}} [|G_1(x) - G_2(x)| + 2|\alpha_1 - \alpha_2|]$$

where $\delta_0 = \min\{\delta_1, \delta_2\} > 0$. Therefore

$$\begin{aligned} \|K_1\|^2, \|K_2\|^2 &\leq \frac{1}{2} \int_0^\infty \frac{|G_1(x) - G_2(x)|^2}{x^2 + \delta_0^2} dx + 2(\alpha_1 - \alpha_2)^2 \int_0^\infty \frac{dx}{x^2 + \delta_0^2} \\ &\leq \frac{1}{2\delta_0^2} \|G_1 - G_2\|^2 + \frac{\pi}{\delta_0} (\alpha_1 - \alpha_2)^2 \end{aligned}$$

and

$$\|K_1\|, \|K_2\| \leq \frac{1}{\sqrt{2}\delta_0} \|G_1 - G_2\| + \left(\frac{\pi}{\delta_0}\right)^{1/2} |\alpha_1 - \alpha_2|. \tag{9.3}$$

It remains to estimate K_3 and K_4 . As above we have

$$|K_3|, |K_4| \leq D_2(x + b) |I_1(x) - I_2(x)|$$

where $D_2^2 = \sup_{x \in \mathbb{R}} \tilde{G}_2(x) < \infty$. Further, by the Riesz theorem,

$$\begin{aligned} \int_0^\infty |I_1(x) - I_2(x)|^2 dx &= \frac{1}{4} \left\| \ln \frac{\tilde{G}_1}{\tilde{G}_2} \right\|^2 \\ &\leq \frac{1}{4} \int_0^\infty \ln^2 \left(1 + \frac{|G_1(x) - G_2(x)| + 2|\alpha_1 - \alpha_2|}{x^2 + \delta_0^2} \right) dx \\ &\leq \frac{1}{4} \int_0^\infty \left(\frac{|G_1(x) - G_2(x)| + 2|\alpha_1 - \alpha_2|}{x^2 + \delta_0^2} \right)^2 dx \\ &\leq \frac{1}{2} \left[\frac{1}{\delta_0^4} \|G_1 - G_2\|^2 + \frac{\pi}{\delta_0^3} (\alpha_1 - \alpha_2)^2 \right]. \end{aligned}$$

Finally, in view of relation (6.12), $\int_0^\infty \ln \frac{\tilde{G}_1(x)}{\tilde{G}_2(x)} dx = 0$ and therefore

$$I_1(x) - I_2(x) = \frac{x^{-1}}{\pi} \int_0^\infty \frac{\xi^2 \ln \frac{\tilde{G}_1(\xi)}{\tilde{G}_2(\xi)}}{\xi^2 - x^2} d\xi.$$

Applying the Hardy-Littlewood inequality from above, we obtain

$$\begin{aligned} &\int_0^\infty x^2 |I_1(x) - I_2(x)|^2 dx \\ &\leq A_2^2 \int_0^\infty x^2 \ln^2 \frac{\tilde{G}_1(x)}{\tilde{G}_2(x)} dx \\ &\leq 2A_2^2 \left[\int_0^\infty \frac{x^2}{(x^2 + \delta_0^2)^2} |G_1(x) - G_2(x)|^2 dx + 4(\alpha_1 - \alpha_2)^2 \int_0^\infty \frac{x^2}{(x^2 + \delta_0^2)^2} dx \right] \\ &\leq 2A_2^2 \left[\frac{1}{\delta_0^2} \|G_1 - G_2\|^2 + \frac{2\pi}{\delta_0} (\alpha_1 - \alpha_2)^2 \right] \end{aligned}$$

where $A_2 = C_{-2,2}$. Hence

$$\|K_3\|^2, \|K_4\|^2 \leq D_2^2 \left\{ \left(\frac{b^2}{\delta_0^4} + \frac{4}{\delta_0^2} A_2^2 \right) \|G_1 - G_2\|^2 + \left(\frac{2\pi}{\delta_0^3} + \frac{8\pi}{\delta_0} A_2^2 \right) (\alpha_1 - \alpha_2)^2 \right\}$$

and

$$\left. \begin{array}{l} \|K_3\| \\ \|K_4\| \end{array} \right\} \leq D_2 \left\{ \frac{1}{\delta_0^2} (b^2 + 4\delta_0^2 A_2^2)^{\frac{1}{2}} \|G_1 - G_2\| + \left(\frac{2\pi}{\delta_0^3} \right)^{\frac{1}{2}} (1 + 4\delta_0^2 A_2^2)^{\frac{1}{2}} |\alpha_1 - \alpha_2| \right\}. \quad (9.4)$$

From (9.2) - (9.4) the estimation

$$\|P_1 - P_2\| \leq B_0 |\alpha_1 - \alpha_2| + B_1 \|G_1 - G_2\|$$

follows where B_0 and B_1 are determined by the coefficients in estimates (9.3) - (9.4). Then for the solutions p_1 and p_2 we get

$$\|p_1 - p_2\| \leq \frac{2}{\pi} B_0 |\alpha_1 - \alpha_2| + B_1 \|g_1 - g_2\|. \quad (9.5)$$

Theorem 9.1. *For any $g_j \in L(\mathbb{R}_+) \cap L^2(\mathbb{R}_+)$ ($j = 1, 2$) satisfying (9.1) the norm of the difference of the solutions $p_j \in L^2(\mathbb{R}_+)$ ($j = 1, 2$) fulfills estimation (9.5).*

Acknowledgements. The author thanks the referees for several hints which improve the understanding of the paper.

References

- [1] Koosis, P.: *Introduction to H_p Spaces*. Cambridge: Cambridge Univ. Press 1980.
- [2] Michlin, S. G. and S. Prößdorf: *Singuläre Integraloperatoren*. Berlin: Akademie-Verlag 1980.
- [3] Muskhelishvili, N. I.: *Singular Integral Equations*. Groningen: Noordhoff 1953.
- [4] Titchmarsh, E. C.: *Introduction to the Theory of Fourier Integrals*. Oxford: Clarendon Press 1948.
- [5] von Wolfersdorf, L.: *A regularization procedure for the auto-correlation equation*. Math. Meth. Appl. Sci. 24 (2001), 1073 – 1088.
- [6] von Wolfersdorf, L.: *A class of quadratic integral-differential equations*. Complex Variables (to appear).
- [7] von Wolfersdorf, L.: *Einige Klassen quadratischer Integralgleichungen*. Sitzber. Sächs. Akad. Wiss. Leipzig, Math.-nat. Klasse 128 (2000), Heft 2.

Received 30.07.2001; in revised form 13.12.2001