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Inequalities for the Tail of the Exponential Series

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Abstract. Let

$$I_n(x) = e^{-x} - \sum_{k=0}^{n} (-1)^k \frac{x^k}{k!} = \sum_{k=n+1}^{\infty} (-1)^k \frac{x^k}{k!}.$$

We prove: if $\alpha, \beta > 0$ are real numbers and $n \geq 1$ is an integer, then the inequalities

$$\frac{n+1}{n+2} \frac{\left(1+\frac{x}{n+\alpha}\right)^2}{\left(1+\frac{x}{n-1+\alpha}\right)\left(1+\frac{x}{n+1+\alpha}\right)} < \frac{I_{n-1}(x) I_{n+1}(x)}{\left(I_n(x)\right)^2} < \frac{n+1}{n+2} \frac{\left(1+\frac{x}{n+\beta}\right)^2}{\left(1+\frac{x}{n-1+\beta}\right)\left(1+\frac{x}{n+1+\beta}\right)}$$

hold for all real numbers $x > 0$ if and only if $\alpha \leq 1$ and $\beta \geq 2$. Our result improves inequalities published by M. Merkle in 1997.

Keywords: *Exponential function, infinite series, integral means, inequalities, rational approximation*

AMS subject classification: 26D15, 33B10

1. Introduction

In 1943, P. Kesava Menon [7] proved the inequality

$$\frac{1}{2} < \frac{J_{n-1}(x) J_{n+1}(x)}{(J_n(x))^2} \qquad (x > 0;\, n \in \mathbb{N}) \tag{1.1}$$

where

$$J_n(x) = e^x - \sum_{k=0}^{n} \frac{x^k}{k!} = \sum_{k=n+1}^{\infty} \frac{x^k}{k!}$$

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denotes the tail of the Maclaurin series of the exponential function. Inequality (1.1) can be refined and complemented as

$$\frac{n+1}{n+2} < \frac{J_{n-1}(x)J_{n+1}(x)}{(J_n(x))^2} < 1 \qquad (x > 0;\ n \in \mathbb{N}). \tag{1.2}$$

Both bounds are sharp (see [2, 6, 8]).

In the recent past, several mathematicians continued the research of inequalities (1.1) and (1.2) and provided different extensions of these results (see [3 - 5, 8 - 11]). Of special interest is a paper of Merkle [10] published in 1997. He presented remarkable properties of $J_n(x)$, where x is a negative real number, that is, he investigated

$$I_n(x) = e^{-x} - \sum_{k=0}^{n}(-1)^k \frac{x^k}{k!} = \sum_{k=n+1}^{\infty}(-1)^k \frac{x^k}{k!} \qquad (x > 0;\ n \in \mathbb{N}_0).$$

His main result is the following striking companion of (1.2).

Proposition. *Let $n \geq 1$ be an integer. Then, for all real numbers $x > 0$,*

$$\frac{n}{n+1} \leq \frac{I_{n-1}(x)I_{n+1}(x)}{(I_n(x))^2} \leq \frac{n+1}{n+2}. \tag{1.3}$$

Both bounds are best possible.

Moreover, Merkle established the representation

$$(-1)^{n+1} I_n(x) = \frac{x^{n+1}}{(n+1)!\big[1 + \frac{x}{n+\theta(n,x)}\big]} \tag{1.4}$$

where $\theta(n,x) \in (1,2)$ with $\lim_{x\to\infty} \theta(n,x) = 1$ and

$$\lim_{x\to 0^+} \theta(n,x) = 2. \tag{1.5}$$

An application of (1.4) leads to an additive counterpart of (1.3). If $n \geq 1$ is an integer, then for all $x > 0$

$$0 < x^{-2(n+1)}\big[\big(I_n(x)\big)^2 - I_{n-1}(x)I_{n+1}(x)\big] < \frac{1}{(n+1)!\,(n+2)!}$$

where both bounds are sharp.

It is not difficult to show that the ratio $\frac{I_{n+1}(x)}{I_n(x)}$ can be approximated by linear functions. Indeed, for all integers $n \geq 0$ and real numbers $x > 0$ we have

$$a_n x < \frac{I_{n+1}(x)}{I_n(x)} < b_n x \tag{1.6}$$

where the best possible factors (which depend only on n) are given by $a_n = -\frac{1}{n+1}$ and $b_n = -\frac{1}{n+2}$. In view of (1.6) it is natural to look for simple rational functions r_1 and r_2 such that the double-inequality

$$r_1(x) \le \frac{I_{n-1}(x)I_{n+1}(x)}{(I_n(x))^2} \le r_2(x) \tag{1.7}$$

is valid for all $x > 0$ and $n \ge 1$. It is the aim of this paper to show that in fact there exist four quadratic polynomials p_1, p_2 and q_1, q_2 such that (1.7) holds with $r_1 = \frac{p_1}{q_1}$ and $r_2 = \frac{p_2}{q_2}$. It turns out that our upper and lower bounds for $\frac{I_{n-1}(x)I_{n+1}(x)}{(I_n(x))^2}$ improve those given in (1.3).

2. Main result

The following rational approximation to $\frac{I_{n-1}(x)I_{n+1}(x)}{(I_n(x))^2}$ is valid.

Theorem. *Let $\alpha, \beta > 0$ be real numbers and let $n \ge 1$ be an integer. The inequalities*

$$\begin{aligned} &\frac{n+1}{n+2}\frac{\left(1+\frac{x}{n+\alpha}\right)^2}{\left(1+\frac{x}{n-1+\alpha}\right)\left(1+\frac{x}{n+1+\alpha}\right)} \\ &\qquad < \frac{I_{n-1}(x)I_{n+1}(x)}{(I_n(x))^2} < \frac{n+1}{n+2}\frac{\left(1+\frac{x}{n+\beta}\right)^2}{\left(1+\frac{x}{n-1+\beta}\right)\left(1+\frac{x}{n+1+\beta}\right)} \end{aligned} \tag{2.1}$$

hold for all real numbers $x > 0$ if and only if $\alpha \le 1$ and $\beta \ge 2$.

Proof. First, we prove: if $0 < \alpha \le 1$ and $\beta \ge 2$, then (2.1) is valid for all $n \ge 1$ and $x > 0$. We define for $t > 0$

$$\delta(t,n,x) = \frac{(1+\frac{x}{n+t})^2}{(1+\frac{x}{n-1+t})(1+\frac{x}{n+1+t})}$$

and set $z = n + t > 1$. Then we obtain

$$\frac{\partial \delta(t,n,x)}{\partial t} = 2x\,\delta(t,n,x)\frac{x^2+3zx+3z^2-1}{z(z^2-1)(x+z)\big((x+z)^2-1\big)} > 0$$

which implies that $t \mapsto \delta(t,n,x)$ is strictly increasing on $(0,\infty)$. Thus, it suffices to establish (2.1) for $\alpha = 1$ and $\beta = 2$.

Taylor's formula yields the integral representations

$$n!\,(-1)^{n+1}I_n(x) = \int_0^x (x-t)^n e^{-t}dt = x^{n+1}e^{-x}\int_0^1 t^n e^{xt}dt. \tag{2.2}$$

From (2.2) we conclude that the right-hand side of (2.1) with $\beta = 2$ is equivalent to

$$\big[f(n-1,x)f(n+1,x)\big]^{1/2} < f(n,x) \tag{2.3}$$

where

$$f(n,x) = \frac{(n+1)(n+2+x)}{n+2}\int_0^1 t^n e^{xt}dt.$$

Inequality (2.3) is a consequence of the stronger inequality

$$\frac{1}{2}\big[f(n-1,x)+f(n+1,x)\big] < f(n,x). \tag{2.4}$$

We prove (2.4) for real numbers $n \geq 1$ and $x > 0$. Using

$$f(n+1,x) = \frac{(n+2)(n+3+x)}{(n+3)x}e^x - \frac{(n+2)^2(n+3+x)}{(n+3)(n+2+x)x}f(n,x)$$
$$f(n-1,x) = \frac{n+1+x}{n+1}e^x - \frac{(n+2)(n+1+x)x}{(n+1)^2(n+2+x)}f(n,x)$$

and

$$\int_0^1 t^n e^{xt}dt = \frac{1}{x^{n+1}}\int_0^x s^n e^s ds \tag{2.5}$$

we obtain

$$\begin{aligned} &2f(n,x) - f(n-1,x) - f(n+1,x) \\ &\quad = x^{-n-2}u(n,x)\left[\int_0^x s^n e^s ds - \frac{v(n,x)}{w(n,x)}x^{n+1}e^x\right] \end{aligned} \tag{2.6}$$

where

$$u(n,x) = \frac{1}{n+1}x^3 + \frac{3n+4}{n+2}x^2 + \frac{(n+1)(3n+8)}{n+3}x + (n+1)(n+2)$$
$$v(n,x) = x^2 + \frac{(n+1)(2n+5)}{n+3}x + (n+1)(n+2)$$
$$w(n,x) = x^3 + \frac{(n+1)(3n+4)}{n+2}x^2 + \frac{(n+1)^2(3n+8)}{n+3}x + (n+1)^2(n+2).$$

Let

$$g(n,x) = \int_0^x s^n e^s ds - \frac{v(n,x)}{w(n,x)}x^{n+1}e^x. \tag{2.7}$$

Partial differentiation leads to

$$\frac{\partial g(n,x)}{\partial x} = 2\frac{A(n,x)}{\big(B(n,x)\big)^2}x^{n+3}e^x$$

where

$$\begin{aligned} A(n,x) &= [n^2+5n+6]x^2 \\ &\quad + [4n^3+22n^2+36n+18]x \\ &\quad + 6n^4+44n^3+116n^2+130n+52 \\ B(n,x) &= [n^2+5n+6]x^3 \\ &\quad + [3n^3+16n^2+25n+12]x^2 \\ &\quad + [3n^4+20n^3+47n^2+46n+16]x \\ &\quad + n^5+9n^4+31n^3+51n^2+40n+12. \end{aligned}$$

Thus, $x \mapsto g(n,x)$ is strictly increasing on $[0,\infty)$. Hence,

$$g(n,x) > g(n,0) = 0 \tag{2.8}$$

so that (2.6) - (2.8) imply the validity of inequality (2.4).

Next, we consider the left-hand inequality of (2.1). Let

$$h(n,x) = (n+1+x)\int_0^1 t^n e^{xt}dt.$$

Applying (2.2) we obtain that the first inequality of (2.1) with $\alpha = 1$ is equivalent to

$$(h(n,x))^2 < h(n-1,x)h(n+1,x). \tag{2.9}$$

We establish (2.9) for real numbers $n \geq 1$ and $x > 0$. Using

$$\begin{aligned} h(n+1,x) &= \frac{n+2+x}{x}e^x - \frac{(n+1)(n+2+x)}{(n+1+x)x}h(n,x) \\ h(n-1,x) &= \frac{n+x}{n}e^x - \frac{(n+x)x}{n(n+1+x)}h(n,x) \end{aligned}$$

and (2.5) we obtain

$$h(n-1,x)h(n+1,x) - \big(h(n,x)\big)^2 = \frac{1}{n}x^{-2n-2}\lambda(n,x)\mu(n,x) \tag{2.10}$$

where

$$\begin{aligned} \lambda(n,x) &= x^2+(2n+2)x+n^2+n \\ \mu(n,x) &= e^{2x}p(n,x) - e^x q(n,x)\int_0^x s^n e^s ds + \left(\int_0^x s^n e^s ds\right)^2 \\ p(n,x) &= x^{2n+1}\frac{(n+x)(n+2+x)}{\lambda(n,x)} \\ q(n,x) &= x^n\frac{(n+x)(n+1+x)(n+2+x)}{\lambda(n,x)}. \end{aligned}$$

Differentiation gives

$$\frac{\partial \mu(n,x)}{\partial x} = \frac{b(n,x)}{(\lambda(n,x))^2} x^{n-1} e^x \left[\frac{a(n,x)}{b(n,x)} x^{n+1} e^x - \int_0^x s^n e^s ds \right] \tag{2.11}$$

where

$$\begin{aligned} a(n,x) = {} & x^5 + [5n+4]x^4 \\ & + [10n^2+15n+4]x^3 \\ & + [10n^3+21n^2+8n]x^2 \\ & + [5n^4+13n^3+6n^2-2n]x \\ & + n^5+3n^4+2n^3 \\ b(n,x) = {} & x^6 + [6n+4]x^5 \\ & + [15n^2+20n+4]x^4 \\ & + [20n^3+40n^2+18n]x^3 \\ & + [15n^4+40n^3+29n^2+4n]x^2 \\ & + [6n^5+20n^4+20n^3+4n^2-2n]x \\ & + n^6+4n^5+5n^4+2n^3. \end{aligned}$$

Let

$$\phi(n,x) = \frac{a(n,x)}{b(n,x)} x^{n+1} e^x - \int_0^x s^n e^s ds. \tag{2.12}$$

Then

$$\frac{\partial \phi(n,x)}{\partial x} = 2n \frac{c(n,x)}{(b(n,x))^2} x^{n+2} e^x$$

where

$$\begin{aligned} c(n,x) = {} & 6x^6 + [36n+29]x^5 \\ & + [90n^2+137n+43]x^4 \\ & + [120n^3+258n^2+147n+12]x^3 \\ & + [90n^4+242n^3+189n^2+25n-12]x^2 \\ & + [36n^5+113n^4+109n^3+26n^2-6n]x \\ & + 6n^6+21n^5+24n^4+9n^3. \end{aligned}$$

This implies that $x \mapsto \phi(n,x)$ is strictly increasing on $[0,\infty)$. Hence,

$$\phi(n,x) > \phi(n,0) = 0. \tag{2.13}$$

From (2.11) - (2.13) we conclude

$$\mu(n,x) > \mu(n,0) = 0 \tag{2.14}$$

so that (2.10) and (2.14) lead to inequality (2.9).

It remains to prove that in (2.1) the parameters $\alpha = 1$ and $\beta = 2$ are best possible. We assume that there exist numbers $\alpha, \beta > 0$ and $n \geq 1$ such that (2.1) is valid for all $x > 0$. Since $\lim_{x\to\infty} \frac{I_{n+1}(x)}{xI_n(x)} = -\frac{1}{n+1}$, we obtain from the left-hand side of (2.1) if $x \to \infty$

$$\frac{n+1}{n+2}\frac{(n-1+\alpha)(n+1+\alpha)}{(n+\alpha)^2} \leq \frac{n}{n+1}$$

which is equivalent to $\alpha \leq 1$. Applying (1.4) we conclude that the right-hand side of (2.1) is equivalent to

$$0 < \frac{\left(1+\frac{x}{n+\beta}\right)^2}{\left(1+\frac{x}{n-1+\beta}\right)\left(1+\frac{x}{n+1+\beta}\right)} - \frac{\left(1+\frac{x}{n+\theta(n,x)}\right)^2}{\left(1+\frac{x}{n-1+\theta(n-1,x)}\right)\left(1+\frac{x}{n+1+\theta(n+1,x)}\right)}. \tag{2.15}$$

Denote herein the right part by $\sigma(n, x, \beta)$. A short computation gives that (1.5) and (2.15) lead to

$$0 \leq \lim_{x\to 0^+} \frac{\sigma(n,x,\beta)}{x} = \omega(n,\beta) - \omega(n,2) \tag{2.16}$$

where

$$\omega(n,y) = \frac{2}{n+y} - \frac{1}{n-1+y} - \frac{1}{n+1+y}.$$

Since

$$\frac{\partial\omega(n,y)}{\partial y} = \frac{2[3(n+y)^2-1]}{(n+y)^2(n-1+y)^2(n+1+y)^2} > 0 \qquad (y > 0)$$

we conclude that $y \mapsto \omega(n, y)$ is strictly increasing on $(0, \infty)$. Thus, we get from (2.16) that $\beta \geq 2$. This completes the proof of the Theorem ∎

Remarks.

(1) A simple calculation shows that the inequalities

$$\begin{aligned} \frac{n}{n+1} &< \frac{n+1}{n+2}\frac{\left(1+\frac{x}{n+1}\right)^2}{\left(1+\frac{x}{n}\right)\left(1+\frac{x}{n+2}\right)} \\ \frac{n+1}{n+2}\frac{\left(1+\frac{x}{n+2}\right)^2}{\left(1+\frac{x}{n+1}\right)\left(1+\frac{x}{n+3}\right)} &< \frac{n+1}{n+2} \end{aligned} \tag{2.17}$$

hold for all $n \geq 1$ and $x > 0$. Hence, (2.1) (with $\alpha = 1$ and $\beta = 2$) improves the bounds given in (1.3). Moreover, from (2.1) and (2.17) we conclude that inequalities (1.3) are strict.

(2) Let

$$\Delta_n(a,x) = (n+1)!\frac{n+a+x}{n+a}|I_n(x)| \qquad (n \in \mathbb{N}_0).$$

The Theorem yields: if $0 < \alpha \le 1$ and $x > 0$, then $n \mapsto \Delta_n(\alpha, x)$ is strictly log-convex, whereas, if $\beta \ge 2$ and $x > 0$, then $n \mapsto \Delta_n(\beta, x)$ is strictly log-concave.

(3) Applying (2.2) we obtain an identity, which connects the functions I_n and J_n with the integral

$$\beta_n(x) = \int_{-1}^{1} t^n e^{-xt} dt.$$

We have

$$\beta_n(x) = \frac{n!}{x^{n+1}}[e^{-x}J_n(x) - e^x I_n(x)] \qquad (x > 0;\, n \in \mathbb{N}_0).$$

This formula and further properties of β_n are given in [1: Chapter 5].

Let $A_n(x)$ be the arithmetic mean of the function $t \mapsto \exp(xt)$ $(x > 0)$ on $[0,1]$ with the weight function $t \mapsto t^n$ $(n \in \mathbb{N}_0)$, that is,

$$A_n(x) = \frac{\int_0^1 t^n e^{xt} dt}{\int_0^1 t^n dt} = (n+1)\int_0^1 t^n e^{xt} dt.$$

The Theorem and (2.2) imply the following integral inequalities.

Corollary. *Let $\alpha, \beta > 0$ be real numbers and let $n \ge 1$ be an integer. The double-inequality*

$$\frac{\left(1+\frac{x}{n+\alpha}\right)^2}{\left(1+\frac{x}{n-1+\alpha}\right)\left(1+\frac{x}{n+1+\alpha}\right)} < \frac{A_{n-1}(x)A_{n+1}(x)}{(A_n(x))^2} < \frac{\left(1+\frac{x}{n+\beta}\right)^2}{\left(1+\frac{x}{n-1+\beta}\right)\left(1+\frac{x}{n+1+\beta}\right)}$$

is valid for all real numbers $x > 0$ if and only if $\alpha \le 1$ and $\beta \ge 2$.

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