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Weighted Hölder Continuity of Hyperbolic Harmonic Bloch functions

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Abstract. Characterizations of weighted Hölder continuity and weighted Lipschitz continuity are obtained for the hyperbolic Bloch functions on the unit ball of $\mathbb{R}^n$. Similar results are extended to hyperbolic little Bloch and Besov spaces.

Keywords: *Hyperbolic Bloch space, hyperbolic Besov spaces, invariant Laplacian*

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1. Introduction

Let $\mathbb{B}$ be the unit ball in $\mathbb{R}^n$ with $n \geq 2$, $d\nu$ the normalized measure on $\mathbb{B}$ and $d\sigma$ the normalized surface measure on the unit sphere $S = \partial\mathbb{B}$. We shall consider the Poincaré metric in $\mathbb{B}$

$$ds^2 = \frac{|dx|^2}{(1-|x|^2)^2}.$$

The corresponding Laplace-Beltrami operator and gradient are given by

$$\begin{aligned} \widetilde{\triangle} f(x) &= (1-|x|^2)^2 \left(\triangle f(x) + \frac{2(n-2)}{1-|x|^2} \sum_{i=1}^{n} x_i \frac{\partial f}{\partial x_i}(x) \right), \\ \widetilde{\nabla} f(x) &= (1-|x|^2) \nabla f(x), \end{aligned}$$

where $\triangle$ and ∇ denote the usual Laplacian and gradient, respectively. They are invariant in the sense

$$\begin{aligned} \widetilde{\triangle} f(x) &= \triangle (f \circ \varphi_x)(0), \\ \widetilde{\nabla} f(x) &= \nabla (f \circ \varphi_x)(0), \end{aligned}$$

where the Möbius transformation $\varphi_x \in \mathrm{Aut}(B)$, $x \in \mathbb{B}$, is an involutionary automorphism of $\mathbb{B}$ with $\varphi_x(0) = x$. Notice that for any $f \in C^2(\mathbb{B})$

$$|\widetilde{\nabla} f(x)| = (1-|x|^2)|\nabla f(x)|.$$

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A function $f \in C^2(\mathbb{B})$ is called *hyperbolic harmonic* or simply *$\mathcal{H}$-harmonic* if it is annihilated by the invariant Laplacian on $\mathbb{B}$. The

- *$\mathcal{H}$-harmonic Bloch space* $\mathcal{B}$ is the space of all $\mathcal{H}$-harmonic functions on $\mathbb{B}$ for which $\sup_{x\in\mathbb{B}} |\widetilde{\nabla} f(x)| < \infty$
- *$\mathcal{H}$-harmonic little Bloch space* $\mathcal{B}_0$ consists of all functions $f \in \mathcal{B}$ such that $\lim_{|x|\to 1} |\widetilde{\nabla} f(x)| = 0$
- *$\mathcal{H}$-harmonic Besov space* $\mathcal{B}_p$ is the space of all -harmonic functions on B for which $\int_{\mathbb{B}} |\widetilde{\nabla} f(x)|^p d\tau(x) < \infty$ where $d\tau(x) = (1-|x|^2)^{-n} d\nu(x)$ is the invariant measure on $\mathbb{B}$.

Let $\alpha, \beta \geq 0$ and $0 < \lambda < 1$, and let f be a continuous function in $\mathbb{B}$. If there exist a constant C such that

$$(1-|x|^2)^\alpha (1-|y|^2)^\beta |f(x)-f(y)| \leq C|x-y| \tag{1.1}$$

for any $x, y \in \mathbb{B}$, then we say that f satisfies a *weighted Lipschitz condition* of indices (α, β). If there exist a constant C such that

$$(1-|x|^2)^\alpha (1-|y|^2)^\beta |f(x)-f(y)| \leq C|x-y|^\lambda \tag{1.2}$$

for any $x, y \in \mathbb{B}$, then we say that f satisfies a *weighted Hölder condition* of indices (α, β, λ).

The main purpose of this paper is to give some characterizations of $\mathcal{B}$, $\mathcal{B}_0$ and $\mathcal{B}_p$ in terms of weighted Hölder or Lipschitz conditions. We refer to [3, 4, 7, 8] for corresponding results in the complex unit ball for holomorphic or $\mathcal{M}$-harmonic functions. See [6, 9, 12, 13, 15, 16] for various characterization of the Bloch, little Bloch, and Besov spaces in the unit ball of $\mathbb{C}^n$.

Our main results are the following three theorems.

Theorem 1.1. *Let f be a hyperbolic harmonic function on $\mathbb{B}$. Then the following statements are equivalent:*

(i) *$f \in \mathcal{B}$.*

(ii) *f satisfies a weighted Lipschitz condition of indices (α, β) with $\alpha + \beta = 1$, $\alpha, \beta > 0$.*

(iii) *f satisfies a weighted Hölder condition of indices (α, β, λ) with $\alpha + \beta = \lambda$, $\alpha, \beta > 0$ and $0 < \lambda < 1$.*

Theorem 1.2. *Let $0 < \lambda < 1$ and $\alpha, \beta > 0$ with $\alpha + \beta = \lambda$. For any hyperbolic harmonic function f on $\mathbb{B}$, $f \in \mathcal{B}_0$ if and only if*

$$\lim_{|x|\to 1^-} \sup \left\{ (1-|x|^2)^\alpha (1-|y|^2)^\beta \frac{|f(x)-f(y)|}{|x-y|^\lambda} : y \in \mathbb{B}, y \neq x \right\} = 0.$$

Theorem 1.3. *Let $p \in (2(n-1), \infty)$. For any hyperbolic harmonic function f on $\mathbb{B}$, $f \in \mathcal{B}_p$ if and only if*

$$\int_{\mathbb{B}} \int_{\mathbb{B}} (1-|x|^2)^{\frac{p}{2}} (1-|y|^2)^{\frac{p}{2}} \Big(\frac{|f(x)-f(y)|}{|x-y|} \Big)^p d\tau(x) d\tau(y) < \infty.$$

2. Preliminaries

We shall be using the following notation: for $x, y \in \mathbb{R}^n$ we write in polar coordinates $x = |x|x'$ and $y = |y|y'$. For any $y, w \in \mathbb{R}^n$ the symmetric lemma (see [2: p. 10]) shows

$$\big||y|w - y'\big| = \big||w|y - w'\big|. \tag{2.1}$$

The same deduction yields

$$\big||y|w - (1 - |w|^2)y'\big| = \big||w|y - (1 - |w|^2)w'\big|$$

so that

$$\big||y|^2 w - (1 - |w|^2)y\big| = |y|\,\big||w|y - (1 - |w|^2)w'\big|. \tag{2.2}$$

For any $a \in \mathbb{B}$ we denote the Möbius transformation in $\mathbb{B}$ by φ_a. It is an involutionary automorphism of $\mathbb{B}$ such that $\varphi_a(0) = a$ and $\varphi_a(a) = 0$, which is of the form (see [1: p.25])

$$\varphi_a(x) = \frac{|x - a|^2 a - (1 - |a|^2)(x - a)}{\big||x|a - x'\big|^2} \qquad (a, x \in \mathbb{B}). \tag{2.3}$$

From (2.2) with $w = a$ and $y = x - a$ we have

$$|\varphi_a(x)| = \frac{|x - a|}{\big||a|x - a'\big|} \tag{2.4}$$

such that

$$1 - |\varphi_a(x)|^2 = \frac{(1 - |x|^2)(1 - |a|^2)}{\big||a|x - a'\big|^2}. \tag{2.5}$$

For any $a \in \mathbb{B}$ and $\delta \in (0, 1)$ we denote

$$E(a, \delta) = \big\{x \in \mathbb{B} : |\varphi_a(x)| < \delta\big\},$$
$$B(a, \delta) = \big\{x \in \mathbb{B} : |x - a| < \delta\big\}.$$

Clearly, $E(a, \delta) = \varphi_a(B(0, \delta))$.

Lemma 2.1. *Let $x, w \in \mathbb{B}$ and $y \in E(w, \delta)$. Then*

$$\frac{1 - \delta}{1 + \delta}\big||x|w - x'\big| \le \big||x|y - x'\big| \le \frac{1 + \delta}{1 - \delta}\big||x|w - x'\big|.$$

Proof. From (2.4) and (2.1) we have $|\varphi_y(w)| = |\varphi_w(y)|$, so that $y \in E(w, \delta)$ is equivalent to $w \in E(y, \delta)$. By symmetry, we need only to prove the right inequality. Since

$$\big||x|y - x'\big| \le \big||x|(y - w)\big| + \big||x|w - x'\big|$$

it is enough to show

$$|y - w| \le \frac{2\delta}{1 - \delta}\big||x|w - x'\big|$$

for any $y \in E(w, \delta)$. Denoting $\eta = \varphi_w(y)$ we have $y = \varphi_w(\eta)$ and $|\eta| < \delta$. From (2.3), a direct computation yields

$$|\varphi_w(\eta) - w| = \frac{|\eta|}{\big||w|\eta - w'\big|}(1 - |w|^2).$$

Therefore, by the simple inequality $1 - |w| \le \big||x|w - x'\big|$ we get

$$|y - w| = |\varphi_w(\eta) - w| \le \frac{\delta}{1 - \delta}(1 - |w|^2) \le \frac{\delta}{1 - \delta}2\big||x|w - x'\big|$$

as desired ∎

As a direct corollary, we have

$$1-|x|^2 \simeq 1-|y|^2 \qquad (x \in E(y,\delta)). \tag{2.6}$$

In fact, taking $w = x$ in Lemma 2.1 we get $\big||x|y - x'\big| \simeq 1 - |y|^2$. The assertion now follows from (2.1).

Let F be the hypergeometric function (see [5, 10])

$$F(a,b;c;s) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{k!(c)_k} s^k$$

for $a, b, c \in \mathbb{R}$ and c neither zero nor a negative integer, where $(a)_k$ denotes the Pochhammer symbol with $(a)_0 = 1$ and $(a)_k = a(a+1)\cdots(a+k-1)$, $k \in \mathbb{N}$. These functions have some well-known properties:

(i) Bateman's integral formula

$$F(a,b;c+\mu;s) = \frac{\Gamma(c+\mu)}{\Gamma(c)\Gamma(\mu)} \int_0^1 t^{c-1}(1-t)^{\mu-1} F(a,b;c;ts)\,dt \tag{2.7}$$

with $c, \mu > 0$ and $s \in (-1,1)$.

(ii) For any integer m [12: p. 69]

$$\begin{aligned} F(-m,b;c;1) &= \frac{(c-b)_m}{(c)_m} \\ F(-m,a+m;c;1) &= \frac{(-1)^m(1+a-c)_m}{(c)_m}. \end{aligned} \tag{2.8}$$

The following identity furnishes the hypergeometric function with an integral representation.

Lemma 2.2. *Let $t > 1$, $\lambda \in \mathbb{R}$ and $r \in (-1,1)$. Then*

$$\int_{-1}^{1} \frac{(1-u^2)^{(t-3)/2}}{(1-2ru+r^2)^{\lambda}}\,du = \frac{\Gamma(\frac{t-1}{2})\Gamma(\frac{1}{2})}{\Gamma(\frac{t}{2})} F\big(\lambda, \lambda+1-\tfrac{t}{2}; \tfrac{t}{2}; r^2\big). \tag{2.9}$$

Proof. Let C_m^λ be the Gegenbauer polynomials. These polynomials can be defined by the generating function

$$(1-2ru+r^2)^{-\lambda} = \sum_{m=0}^{\infty} C_m^\lambda(u) r^m \tag{2.10}$$

where

$$\begin{aligned} C_{2m}^\lambda(u) &= (-1)^m \frac{(\lambda)_m}{m!} F\big(-m, m+\lambda; \tfrac{1}{2}; u^2\big) \\ C_{2m+1}^\lambda(u) &= (-1)^m \frac{(\lambda)_m}{m!} 2uF\big(-m, m+\lambda+1; \tfrac{3}{2}; u^2\big). \end{aligned} \tag{2.11}$$

To calculate the integral in (2.9), we apply (2.10) and (2.11) and can deduce that it is only left to evaluate the integral

$$\int_{-1}^{1} (1-u^2)^{(t-3)/2} F\big(-m, m+\lambda; \tfrac{1}{2}; u^2\big)\,du$$

or rather an integral over the interval $(0,1)$ by the simple change of variables $t = u^2$. For this integral, we first use Bateman's integral formula (2.7) with $s = 1$, then we apply (2.8) so that it can be represented by Pochhammer symbols. The calculation of integral (2.9) leads to a series which by definition is the desired hypergeometric function ∎

Lemma 2.3. *Let $\alpha > -1$ and $\beta \in \mathbb{R}$. Then for any $x \in \mathbb{B}$*

$$\int_{\mathbb{B}} \frac{(1-|y|^2)^\alpha}{\big||x|y - x'\big|^{n+\alpha+\beta}} d\nu(y) \approx \begin{cases} (1-|x|^2)^{-\beta} & \text{if } \beta > 0 \\ \log \frac{1}{1-|x|^2} & \text{if } \beta = 0 \\ 1 & \text{if } \beta < 0 \end{cases}$$

where $a(x) \approx b(x)$ means the ratio $\frac{a(x)}{b(x)}$ has a positive finite limit as $|x| \to 1$.

Proof. Denote the above integral by $J_{\alpha,\beta}(x)$. From Stirling's formula we need only to show

$$J_{\alpha,\beta}(x) = \frac{\Gamma(\frac{n}{2}+1)\Gamma(\alpha+1)}{\Gamma(\alpha+\frac{n}{2}+1)} F\big(\tfrac{n+\alpha+\beta}{2}, \tfrac{2+\alpha+\beta}{2}; \alpha + \tfrac{n}{2} + 1; |x|^2\big).$$

For any continuous function f of one variable and any $\eta \in \partial\mathbb{B}$, we have the formula (see [2: p. 216])

$$\int_{\partial\mathbb{B}} f(\langle\zeta,\eta\rangle)\, d\sigma(\zeta) = \frac{\Gamma(\frac{n}{2})}{\Gamma(\frac{n-1}{2})\Gamma(\frac{1}{2})} \int_{-1}^{1} (1-u^2)^{\frac{n-3}{2}} f(u)\, du$$

where $\langle\zeta,\eta\rangle$ stands for the inner product in $\mathbb{R}^n$. Taking

$$f(u) = \big(1 - 2ru + r^2\big)^{-\frac{n+\alpha+\beta}{2}} \qquad \big(r \in (0,1) \text{ fixed}\big)$$

and combining it with Lemma 2.2 we get

$$\begin{aligned} \int_{\partial\mathbb{B}} &\big(1 - 2r\langle\zeta,\eta\rangle + r^2\big)^{-\frac{n+\alpha+\beta}{2}} d\sigma(\zeta) \\ &= \frac{\Gamma(\frac{n}{2})}{\Gamma(\frac{n-1}{2})\Gamma(\frac{1}{2})} \int_{-1}^{1} \frac{(1-u^2)^{\frac{n-3}{2}}}{\big(1-2ru+r^2\big)^{\frac{n+\alpha+\beta}{2}}}\, du \\ &= F\big(\tfrac{n+\alpha+\beta}{2}, \tfrac{2+\alpha+\beta}{2}; \tfrac{n}{2}; r^2\big). \end{aligned}$$

Consequently, from the polar coordinates formula we get

$$\begin{aligned} J_{\alpha,\beta}(x) &= n \int_0^1 r^{n-1}(1-r^2)^\alpha dr \int_S \big(1 - 2r|x|\langle x',\zeta\rangle + r^2|x|^2\big)^{-\frac{n+\alpha+\beta}{2}} d\sigma(\zeta) \\ &= C \int_0^1 r^{n-1}(1-r^2)^\alpha F\big(\tfrac{n+\alpha+\beta}{2}, \tfrac{2+\alpha+\beta}{2}; \tfrac{n}{2}; r^2|x|^2\big) dr. \end{aligned}$$

The assertion now follows from Bateman's integral formula (2.7) ∎

3. Bloch space

In this section we give the proof of Theorems 1.1 and 1.2. Theorem 1.1 can be rephrased as the following

Theorem 3.1. *Let $0 < \alpha < \lambda \leq 1$. For any hyperbolic harmonic function f on $\mathbb{B}$, $f \in \mathcal{B}$ if and only if*

$$\sup\left\{(1-|x|^2)^{\alpha}(1-|y|^2)^{\lambda-\alpha}\frac{|f(x)-f(y)|}{|x-y|^{\lambda}} : x, y \in \mathbb{B}, x \neq y\right\} < \infty. \tag{3.1}$$

Proof. We may assume $\alpha \leq \frac{\lambda}{2}$, since one of the indices α and $\lambda - \alpha$ is no greater than $\frac{\lambda}{2}$.

First, let us assume that $f \in \mathcal{B}$. For any $a \in \mathbb{B}$ we have

$$f(a) - f(0) = \int_0^1 \frac{df}{dt}(ta)\,dt = \sum_{k=1}^{n} a_k \int_0^1 \frac{\partial f}{\partial x_k}(ta)\,dt$$

so that

$$|f(a) - f(0)| \leq n\|f\|_{\mathcal{B}} \int_0^1 \frac{|a|}{1-t^2|a|^2}dt = \frac{n}{2}\|f\|_{\mathcal{B}} \log\frac{1+|a|}{1-|a|}.$$

Now, replacing f by $f \circ \varphi_y$ and substituting $x = \varphi_y(a)$ we get

$$|f(x) - f(y)| \leq \frac{n}{2}\|f\|_{\mathcal{B}} \log\frac{1+|\varphi_y(x)|}{1-|\varphi_y(x)|}.$$

To estimate the last factor, we can apply the fact that

$$\log\frac{1+|a|}{1-|a|} = 2|a|\sum_{n=0}^{\infty}\frac{|a|^{2n}}{2n+1} \leq C|a|\sum_{n=0}^{\infty}\frac{\Gamma(n+\alpha)}{n!\Gamma(\alpha)}|a|^{2\alpha} = C\frac{|a|}{(1-|a|^2)^{\alpha}}$$

for any $0 < \alpha < 1$ and $a \in \mathbb{B}$. Now, from identities (2.4) - (2.5), we get

$$\begin{aligned}\log\frac{1+|\varphi_y(x)|}{1-|\varphi_y(x)} &\leq C\frac{|\varphi_y(x)|}{(1-|\varphi_y(x)|^2)^{\alpha}}\\ &\leq C\frac{|\varphi_y(x)|^{\lambda}}{(1-|\varphi_y(x)|^2)^{\alpha}}\\ &= C\frac{|x-y|^{\lambda}}{(1-|x|^2)^{\alpha}(1-|y|^2)^{\lambda-\alpha}}\left(\frac{1-|y|^2}{\|x|y-x'|}\right)^{\lambda-2\alpha}\\ &\leq C\frac{2^{\lambda-2\alpha}|x-y|^{\lambda}}{(1-|x|^2)^{\alpha}(1-|y|^2)^{\lambda-\alpha}}.\end{aligned}$$

Here we used the assumption $\alpha \leq \frac{\lambda}{2}$ and the inequality $1 - |y| \leq \big||x|y - x'\big|$ for any $x, y \in \mathbb{B}$. Notice that $2^{\lambda-2\alpha} \leq 2^{\lambda} \leq 2$, which combined with the above results yields

$$|f(x) - f(y)| \leq nC\frac{|x-y|^{\lambda}}{(1-|x|^2)^{\alpha}(1-|y|^2)^{\lambda-\alpha}}\|f\|_{\mathcal{B}}.$$

This proves the necessity.

Conversely, suppose that f is hyperbolic harmonic and (3.1) is satisfied. We will show that $f \in \mathcal{B}$. For any fixed $\delta \in (0,1)$, it is known that

$$|\widetilde{\nabla} f(0)| \leq C \int_{\delta B} |f(a)| d\tau(a).$$

Now, replacing f by $f \circ \varphi_x - f(x)$ and taking $y = \varphi_x(a)$ we get

$$|\widetilde{\nabla} f(x)| \leq C \int_{E(x,\delta)} |f(x) - f(y)|\, d\tau(y). \tag{3.2}$$

Therefore,

$$|\widetilde{\nabla} f(x)| \leq C \sup \Big\{ |f(x) - f(y)| : y \in E(x,\delta), x \in B \Big\}.$$

Note that, for any $y \in E(x,\delta)$, $|\varphi_y(x)| \leq \delta$ and $1 - |x|^2 \simeq 1 - |y|^2$, so that

$$\begin{aligned}\frac{(1-|x|^2)^\alpha (1-|y|^2)^{\lambda-\alpha}}{|x-y|^\lambda} &\simeq \frac{(1-|x|^2)^{\lambda/2}(1-|y|^2)^{\lambda/2}}{|x-y|^\lambda} \\ &= \Big(\frac{\sqrt{1-|\varphi_y(x)|^2}}{|\varphi_y(x)|} \Big)^\lambda \\ &\geq C.\end{aligned} \tag{3.3}$$

Consequently,

$$\begin{aligned}|\widetilde{\nabla} f(x)| &\leq C \sup \left\{ \frac{(1-|x|^2)^\alpha (1-|y|^2)^{\lambda-\alpha}}{|x-y|^\lambda} |f(x) - f(y)| : y \in E(x,\delta), x \in B \right\} \\ &\leq C \sup \left\{ \frac{(1-|x|^2)^\alpha (1-|y|^2)^{\lambda-\alpha}}{|x-y|^\lambda} |f(x) - f(y)| : x, y \in B \right\}.\end{aligned}$$

This completes the proof of Theorem 3.1 ∎

Theorem 3.2. *Let $0 < \alpha < \lambda \leq 1$. For any hyperbolic harmonic function f on $\mathbb{B}$, $f \in \mathcal{B}_0$ if and only if*

$$\lim_{|x| \to 1^-} \sup \left\{ (1-|x|^2)^\alpha (1-|y|^2)^{\lambda-\alpha} \frac{|f(x) - f(y)|}{|x-y|^\lambda} : y \in \mathbb{B}, y \neq x \right\} = 0. \tag{3.4}$$

Proof. Assume that $f \in \mathcal{B}_0$ and let $f_t(x) = f(tx)$ $(t \in (0,1))$. By (3.1), we have

$$(1-|x|^2)^\alpha (1-|y|^2)^{\lambda-\alpha} \frac{\big|(f - f_t)(x) - (f - f_t)(y)\big|}{|x-y|^\lambda} \leq C \|f - f_t\|_{\mathcal{B}}$$

and

$$\begin{aligned}&(1-|x|^2)^\alpha (1-|y|^2)^{\lambda-\alpha} \frac{|f_t(x) - f_t(y)|}{|x-y|^\lambda} \\ &= t^\lambda \frac{(1-|x|^2)^\alpha (1-|y|^2)^{\lambda-\alpha}}{(1-|tx|^2)^\alpha (1-|ty|^2)^{\lambda-\lambda}} (1-|tx|^2)^\alpha (1-|ty|^2)^{\lambda-\alpha} \frac{|f(tx) - f(ty)|}{|tx-ty|^\lambda} \\ &\leq C \frac{t^\lambda}{(1-t^2)^\lambda} (1-|x|^2)^\alpha \|f\|_{\mathcal{B}}.\end{aligned}$$

By the triangle inequality we obtain

$$\sup\Big\{(1-|x|^2)^\alpha(1-|y|^2)^{\lambda-\alpha}\frac{|f(x)-f(y)|}{|x-y|^\lambda}\,:\,y\in\mathbb{B},y\neq x\Big\}$$
$$\leq C\frac{t^\lambda}{(1-t^2)^\lambda}(1-|x|^2)^\alpha\|f\|_{\mathcal{B}}+\|f-f_t\|_{\mathcal{B}}.$$

In the above inequality, by first letting $|x|\to 1^-$, the first term on the right side converges to 0, and then letting $t\to 1^-$, the second term on the right side also converges to 0.

Now suppose that f is hyperbolic harmonic and (3.3) is satisfied. We will show that $f\in\mathcal{B}_0$. Fix $r\in(0,1)$. From (3.2) - (3.3) we have

$$|\widetilde{\nabla}f(x)|\leq C(n,r)\int_{E(x,r)}(1-|x|^2)^\alpha(1-|y|^2)^{\lambda-\alpha}\frac{|f(x)-f(y)|}{|x-y|^\lambda}\,d\tau(y).$$

By assumption (3.4), for any given $\varepsilon>0$ there exists $\delta\in(0,1)$ such that

$$\sup\Big\{(1-|x|^2)^\alpha(1-|y|^2)^{\lambda-\alpha}\frac{|f(x)-f(y)|}{|x-y|^\lambda}\,:\,y\in\mathbb{B},y\neq x\Big\}<\varepsilon$$

whenever $|x|>\delta$. Since

$$\int_{E(x,r)}d\tau=\tau(E(a,r))=\tau(B(0,r))=n\int_0^r t^{n-1}(1-t^2)^{-n}dt$$

we have $|\widetilde{\nabla}f(x)|<C\varepsilon$ for any $|x|>\delta$, which means $|\widetilde{\nabla}f(x)|\to 0$ as $|x|\to 1^-$. This completes the proof ∎

4. $\mathcal{H}$-Besov spaces

In this section, we give the Holland-Walsh characterization for $\mathcal{H}$-Besov spaces. When $p\to\infty$, it also reveals the weighted Lipschitz characterization of Bloch spaces.

Theorem 4.1. *Let $p\in(2(n-1),\infty)$ and f be hyperbolic harmonic on $\mathbb{B}$. Then $f\in\mathcal{B}_p$ if and only if*

$$\int_{\mathbb{B}}\int_{\mathbb{B}}(1-|x|^2)^{\frac{p}{2}}(1-|y|^2)^{\frac{p}{2}}\Big(\frac{|f(x)-f(y)|}{|x-y|}\Big)^p d\tau(x)d\tau(y)<\infty. \tag{4.1}$$

To prove this theorem, we need the following

Lemma 4.2. *Let $p\geq 1$ and $\alpha>-1$. If f is hyperbolic harmonic on $\mathbb{B}$, then*

$$\int_{\mathbb{B}}\Big(\int_0^1\frac{|\widetilde{\nabla}f(ta)|}{1-t|a|}dt\Big)^p d\nu_\alpha(a)\leq C\int_{\mathbb{B}}|\widetilde{\nabla}f(a)|^p d\nu_\alpha(a). \tag{4.2}$$

Proof. Fix $\varepsilon \in (0,1)$. Observe that for any $t \in [0,1]$ and $a \in \mathbb{B}$ if at least one of t and $|a|$ is less than ε, then $|ta| = t|a| < \varepsilon$, such that $\frac{1}{1-t|a|} \le \frac{1}{1-\varepsilon}$. Thus the left side in (4.2) can be controlled by

$$\int_{\mathbb{B}-\varepsilon\mathbb{B}} \left(\int_\varepsilon^1 \frac{|\widetilde{\nabla} f(ta)|}{1-t|a|} dt \right)^p d\nu_\alpha(a) + C \sup_{x\in\varepsilon\mathbb{B}} |\widetilde{\nabla} f(x)|^p.$$

Denote the first summand above by I. From the polar coordinate integral formula and Minkowski's inequality we get

$$\begin{aligned} I &= n \int_\varepsilon^1 \int_{\partial\mathbb{B}} \left(\int_\varepsilon^1 \frac{|\widetilde{\nabla} f(ts\zeta)|}{1-ts} dt \right)^p d\sigma(\zeta) s^{n-1}(1-s^2)^\alpha ds \\ &\le C \int_\varepsilon^1 \left(\int_\varepsilon^1 \frac{M_p(ts, |\widetilde{\nabla} f|)}{1-ts} dt \right)^p s^{n-1}(1-s^2)^\alpha ds \\ &\le C \int_\varepsilon^1 \left(\int_{\varepsilon^2}^s h(\rho) d\rho \right)^p (1-s^2)^\alpha ds \end{aligned}$$

where

$$h(\rho) = \frac{\rho^{(n-1)/p} M_p(\rho, |\widetilde{\nabla} f|)}{1-\rho}.$$

From Hölder's inequality and Fubini's theorem, we can get the following Hardy's inequality:

$$\begin{aligned} \int_0^1 \left(\int_0^s h(\rho) d\rho \right)^p (1-s)^\alpha ds &\le \int_0^1 \int_0^s h^p(\rho) d\rho (1-s)^\alpha ds \\ &\le \int_0^1 \int_\rho^1 (1-s)^\alpha ds h^p(\rho)\, d\rho \\ &\le C \int_0^1 h^p(t)(1-t)^{\alpha+1} dt \end{aligned}$$

for any $p \ge 1$, $\alpha > -1$, and $h \ge 0$. As a result,

$$\begin{aligned} I &\le C \int_0^1 \left(\int_0^s h(\rho) d\rho \right)^p (1-s)^\alpha ds \\ &\le C \int_0^1 t^{n-1}(1-t)^\alpha M_p^p(t, |\widetilde{\nabla} f|)\, dt \\ &= C \int_{\mathbb{B}} |\widetilde{\nabla} f(a)|^p d\nu_\alpha(a). \end{aligned}$$

It remains to show that $\sup_{\varepsilon\mathbb{B}} |\widetilde{\nabla} f(x)|^p \le C \int_{\mathbb{B}} |\widetilde{\nabla} f(a)|^p d\nu_\alpha(a)$. For this it is sufficient to prove the inequality

$$|\widetilde{\nabla} f(x)|^p \le C \int_{E(x,\delta)} |\widetilde{\nabla} f|^p(a)\, d\tau(a). \tag{4.3}$$

Since f is hyperbolic harmonic, we have $f(0) = \int_{\partial\mathbb{B}} f(r\xi)\,d\sigma(\xi)$ for any $0 < r < 1$. Replacing f by $f \circ \varphi_x$, we see that $f(x) = \int_{\partial\mathbb{B}} f(\varphi_x(r\xi))d\sigma(\xi)$ for any $x \in \mathbb{B}$ and $0 < r < 1$. Now we take the gradient about x, evaluate at $x = 0$, and denote $\psi_a(x) = \varphi_x(a)$ to get $|\nabla f(0)| \leq C \int_{\partial\mathbb{B}} |\nabla(f \circ \psi_{r\xi})(0)\,d\sigma(\xi)$. Since

$$|\nabla(f \circ \psi_{r\xi})(0)| \leq C|\nabla f(\psi_{r\xi}(0))| \sup_{0<s<1} |\nabla\psi_{s\xi}(0)|$$

and $\psi_{r\xi}(0) = \varphi_0(r\xi) = r\xi$, it follows that $|\nabla f(0)| \leq C \int_{\partial\mathbb{B}} |\nabla f(r\xi)|d\sigma(\xi)$. Multiplying both sides by $nr^{n-1}(1-r^2)^{-n}dr$ and integrating from 0 to δ, we notice that $|\nabla f(r\xi)| \leq (1-\delta^2)^{-1}|\widetilde{\nabla} f(r\xi)|$ for any $r \in (0,\delta)$ and we conclude

$$|\nabla f(0)| \leq C(1-\delta^2)^{-1}\delta^{-n} \int_{\delta B} |\widetilde{\tau} f(w)|\,d\lambda(w).$$

If we replace f by $f \circ \varphi_x$, then assertion (4.3) follows. This finishes the proof ∎

Proof of Theorem 4.1 Assume that $f \in \mathcal{B}_p$. For any $a \in \mathbb{B}$ we have

$$\frac{|f(a) - f(0)|}{|a|} = \left|\int_0^1 \nabla f(ta)\frac{a}{|a|}\,dt\right| \leq \int_0^1 \frac{|\widetilde{\nabla} f(ta)|}{1 - t|a|}\,dt.$$

Therefore, Lemma 4.2 means

$$\int_B \frac{|f(a) - f(0)|^p}{|a|^p}\,d\nu_\alpha(a) \leq C \int_B |\widetilde{\nabla} f(a)|^p d\nu_\alpha(a).$$

Replacing f with $f \circ \varphi_x$, integrating with respect to $d\tau(x)$, taking $y = \varphi_x(a)$ and setting $\alpha = \frac{p}{2} - n$, we get

$$\begin{aligned}
&\int_B \int_B \frac{|f(y) - f(x)|^p}{|\varphi_x(y)|^p}\big(1 - |\varphi_x(y)|^2\big)^{\frac{p}{2}} d\tau(x)d\tau(y) \\
&\quad\leq C \int_B \int_B |\widetilde{\nabla} f(y)|^p\big(1 - |\varphi_x(y)|^2\big)^{\frac{p}{2}} d\tau(x)d\tau(y) \\
&\quad\leq C \int_B |\widetilde{\nabla} f(y)|^p d\tau(y) \int_B \big(1 - |\varphi_x(y)|^2\big)^{\frac{p}{2}} d\tau(x) \\
&\quad\leq C \int_B |\widetilde{\nabla} f(y)|^p d\tau(y).
\end{aligned}$$

In the last step, we used the estimate $\int_B \big(1 - |\varphi_x(y)|^2\big)^{\frac{p}{2}} d\tau(x) \leq C$ for $p > 2(n-1)$, which follows from (2.5) and the Forelli-Rudin estimate in Lemma 2.3. Since

$$\frac{(1 - |\varphi_x(y)|^2)^{\frac{p}{2}}}{|\varphi_x(y)|^p} = \frac{(1 - |x|^2)^{\frac{p}{2}}(1 - |y|^2)^{\frac{p}{2}}}{|x - y|^p},$$

we get (4.1).

Conversely, supposing that f is hyperbolic harmonic and satisfying (4.1), we will show that $f \in \mathcal{B}_p$. For any fixed $\delta \in (0,1)$,

$$|\widetilde{\nabla} f(x)| \leq C \int_{E(x,\delta)} |f(x) - f(y)|\, d\tau(y).$$

Then, by applying Hölder's inequality and (3.3) with $\lambda = p$ and $\alpha = \frac{p}{2}$,

$$\begin{aligned} |\widetilde{\nabla} f(x)|^p &\leq C \int_{E(x,\delta)} |f(x) - f(y)|^p d\tau(y) \\ &\leq \int_{E(x,\delta)} |f(x) - f(y)|^p \frac{(1-|x|^2)^{\frac{p}{2}}(1-|y|^2)^{\frac{p}{2}}}{|x-y|^p}\, d\tau(y). \end{aligned}$$

Thus, (4.1) implies $f \in \mathcal{B}_p$. This completes the proof ∎

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