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Index Transforms Associated with Bessel and Lommel Functions

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Abstract. In this paper we extend a variety of index integral transforms (i.e. integral transforms over an index as integration variable) with Bessel and Lommel functions as kernels by considering mapping properties of the related integral operators. This class of transforms includes, for instance, operators of Titchmarsh type. Useful integral representations of the considered kernels are deduced and boundedness properties, Parseval equalities, Plancherel type theorem and inversion formula are given.

Keywords: *Lommel, Bessel, Macdonald and hypergeometric functions; Mellin, Kontorovich-Lebedev, Fourier cosine and sine transforms; Parseval equality, Plancherel theorem*

AMS subject classification: 44A20, 44A15, 33C10

1. Introduction and preliminary results

The aim of this paper is to study boundedness and inversion properties in weighted Lebesgue spaces of the non-convolution integral transform

$$(\mathcal{T}_\mu f)(x) = 2^{1-\mu} \int_0^\infty \left[S_{\mu,i\tau}(x) - s_{\mu,i\tau}(x)\right] f(\tau) \frac{d\tau}{\left|\Gamma(\frac{\mu+i\tau+1}{2})\right|^2} \tag{1.1}$$

where $x > 0$ is a variable, $\mu \in \mathbb{R}$ is a parameter, $\Gamma(z)$ is Euler's Gamma function and $S_{\mu,i\tau}(x)$ as well as $s_{\mu,i\tau}(x)$ are Lommel's functions (cf. [1: Vol. II]). Such type of integral transforms, where the integration process is realized over the index (a subscript) of the kernel consisting of special functions had been studied intensively in [6]. Recently (see [7, 8]) similar Titchmarsh transforms of index type associated with Bessel functions as kernel have been considered. However, despite the kernel in (1.1) is related to the Titchmarsh kernel via the identity [1: Vol. II]

$$\begin{aligned} S_{\mu,\nu}(x) &= s_{\mu,\nu}(x) + 2^{\mu-1} \frac{\Gamma(\frac{\mu-\nu+1}{2})\Gamma(\frac{\mu+\nu+1}{2})}{\sin(\nu\pi)} \\ &\quad \times \left[\cos\frac{(\mu-\nu)\pi}{2} J_{-\nu}(x) - \cos\frac{(\mu+\nu)\pi}{2} J_\nu(x)\right], \end{aligned} \tag{1.2}$$

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transform (1.1) has completely different structure and should be studied independently from our consideration of Titchmarsh operators. Note that the kernel (1.2) can be simplified by letting there $\nu = i\tau \quad (\tau \in \mathbb{R}_+)$. Hence it is not difficult to obtain the representation

$$\begin{aligned} &S_{\mu,i\tau}(x) - s_{\mu,i\tau}(x) \\ &\quad = 2^{\mu-1}\left|\Gamma\left(\tfrac{\mu+i\tau+1}{2}\right)\right|^2 \left[\frac{\sin\frac{\pi\mu}{2}}{\cosh\frac{\pi\tau}{2}}\,\mathrm{Re}J_{i\tau}(x) - \frac{\cos\frac{\pi\mu}{2}}{\sinh\frac{\pi\tau}{2}}\,\mathrm{Im}J_{i\tau}(x)\right] \end{aligned} \tag{1.3}$$

where we denote as usual by $\mathrm{Re}J_{i\tau}(x)$ and $\mathrm{Im}J_{i\tau}(x)$ the real and imaginary parts of the Bessel function $J_{i\tau}(x)$. Furthermore, it is also worth mentioning that the Lommel functions $S_{\mu,i\tau}(x)$ and $s_{\mu,i\tau}(x)$ are particular solutions of the inhomogeneous Bessel differential equation $x^2\frac{d^2\omega}{dx^2} + x\frac{d\omega}{dx} + (x^2+\tau^2)\omega = x^{\mu+1}$.

Since all special functions considered here are of hypergeometric type, we will use their integral representations through the pair of the Mellin direct and inverse transforms [2, 4]

$$\left.\begin{aligned} f^{\mathcal{M}}(s) &= \int_0^\infty f(x)x^{s-1}dx \\ f(x) &= \tfrac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} f^{\mathcal{M}}(s)x^{-s}ds \end{aligned}\right\} \qquad (s = \gamma + it, x > 0) \tag{1.4}$$

as one of the essential tools of our investigation. The integrals in (1.4) are convergent, in particular, in the norm of the spaces $L_2(\gamma - i\infty, \gamma + i\infty)$ and $L_2(\mathbb{R}_+; x^{2\gamma-1})$, respectively. In addition, the Parseval equality

$$\int_0^\infty |f(x)|^2 x^{2\gamma-1}dx = \tfrac{1}{2\pi}\int_{-\infty}^\infty |f^{\mathcal{M}}(\gamma+it)|^2 dt \tag{1.5}$$

is true. For example, Mellin transforms (1.4) of the Lommel functions $S_{\mu,i\tau}(x)$ and $s_{\mu,i\tau}(x)$ can be obtained via [2: Vol. III/Relations (8.4.27.1) and (8.4.27.3)]. Precisely, we find

$$\frac{2^{1-\mu}}{\left|\Gamma\left(\frac{\mu+i\tau+1}{2}\right)\right|^2}\int_0^\infty s_{\mu,i\tau}(x)x^{s-1}dx = 2^{s-1}\frac{\Gamma\left(\frac{s+\mu+1}{2}\right)\Gamma\left(\frac{1-\mu-s}{2}\right)}{\Gamma\left(\frac{2+i\tau-s}{2}\right)\Gamma\left(\frac{2-i\tau-s}{2}\right)} \tag{1.6}$$

where $-1-\mu < \gamma < 1-\mu, \frac{3}{2}$ and

$$\frac{2^{1-\mu}}{\left|\Gamma\left(\frac{\mu+i\tau+1}{2}\right)\right|^2}s_{\mu,i\tau}(x) = \tfrac{1}{\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} 2^{s-2}\frac{\Gamma\left(\frac{s+\mu+1}{2}\right)\Gamma\left(\frac{1-\mu-s}{2}\right)}{\Gamma\left(\frac{2+i\tau-s}{2}\right)\Gamma\left(\frac{2-i\tau-s}{2}\right)}x^{-s}ds,$$

and

$$2^{1-\mu}\left|\Gamma\left(\tfrac{1-\mu+i\tau}{2}\right)\right|^2\int_0^\infty S_{\mu,i\tau}(x)x^{s-1}dx = 2^{s-1}\pi\frac{\Gamma\left(\frac{s+i\tau}{2}\right)\Gamma\left(\frac{s-i\tau}{2}\right)}{\cos\frac{\pi(\mu+s)}{2}} \tag{1.7}$$

where $-1-\mu, 0 < \gamma < 1-\mu$ and

$$2^{1-\mu}\left|\Gamma\left(\tfrac{1-\mu+i\tau}{2}\right)\right|^2 S_{\mu,i\tau}(x) = -i\int_{\gamma-i\infty}^{\gamma+i\infty} 2^{s-2}\,\frac{\Gamma\left(\frac{s+i\tau}{2}\right)\Gamma\left(\frac{s-i\tau}{2}\right)}{\cos\frac{\pi(\mu+s)}{2}}\,x^{-s}ds.$$

Further, in view of [2: Vol.II/Relation (2.16.3.15)] the Lommel function $S_{\mu,i\tau}(x)$ in turn can be represented through the known Widder transform [5] of the Macdonald function $K_{i\tau}(x)$. Namely, we have the formula

$$\left|\Gamma(\tfrac{1-\mu+i\tau}{2})\right|^2 S_{\mu,i\tau}(x) = (2x)^{1+\mu} \int_0^\infty \frac{y^{-\mu}}{x^2+y^2} K_{i\tau}(y)\, dy \qquad (|\mu|<1). \tag{1.8}$$

The Macdonald function $K_{i\tau}(x)$ just appeared is a very important kernel in index transforms theory and represents the kernel of the familiar Kontorovich-Lebedev transform [6]

$$(KLf)(x) = \int_0^\infty K_{i\tau}(x) f(\tau)\, d\tau. \tag{1.9}$$

The Mellin-Barnes type integral representation for the Macdonald function is due to [2: Vol. III/Relation (8.4.23.1)] and has the form

$$K_{i\tau}(x) = \tfrac{1}{\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} 2^{s-3}\Gamma(\tfrac{s+i\tau}{2})\Gamma(\tfrac{s-i\tau}{2}) x^{-s} ds \qquad (x,\gamma>0). \tag{1.10}$$

Moreover, we mention here its Fourier-type integral

$$K_{i\tau}(x) = \int_0^\infty e^{-x\cosh u}\cos(\tau u)\, du \qquad (x>0). \tag{1.11}$$

We put down here one more important integral, which gives the representation of the Gauss hypergeometric function in terms of the Laplace integral of the Macdonald function (1.10) (cf. [2: Vol. II/Relation (2.16.6.3)]). Precisely, we find

$$\begin{aligned} \frac{x^{i\tau-\alpha}}{2^\alpha}\sqrt{\pi}\,\frac{|\Gamma(\alpha+i\tau)|^2}{\Gamma(\alpha+\frac12)}\, {}_2F_1\big(\tfrac{\alpha-i\tau}{2}, \tfrac{\alpha-i\tau+1}{2}; \alpha+\tfrac12; 1-\tfrac{1}{x^2}\big) \\ = \int_0^\infty t^{\alpha-1}e^{-xt}K_{i\tau}(t)\, dt \end{aligned} \tag{1.12}$$

for $x,\alpha>0$. Concerning the kernel in (1.1) one can calculate its Mellin transform $(1.4)_1$ using relations (1.6) - (1.7). Thus we obtain

$$\begin{aligned} \frac{2^{1-\mu}}{\left|\Gamma(\frac{\mu+i\tau+1}{2})\right|^2} \int_0^\infty \big[S_{\mu,i\tau}(x) - s_{\mu,i\tau}(x)\big] x^{s-1} dx \\ = \frac{2^{s-1}}{\pi}\Gamma(\tfrac{s+i\tau}{2})\Gamma(\tfrac{s-i\tau}{2})\cos\big(\tfrac{\pi}{2}(\mu-s)\big) \end{aligned}$$

where $\operatorname{Re} s=\gamma$ and $0<\gamma<\frac32$. If we call now formula (1.10) and the integral [2: Vol. I/Relation (2.2.4.25)]

$$\tfrac{1}{\pi}\int_0^\infty \frac{x^{-\frac{1+\mu-s}{2}}}{1+x}\, dx = \frac{1}{\cos\big(\frac{\pi}{2}(\mu-s)\big)},$$

then appealing to the Mellin-Parseval formula (1.5) we deduce the very useful integral representation

$$K_{i\tau}(x) = \frac{(2x)^{1-\mu}}{\left|\Gamma(\frac{\mu+i\tau+1}{2})\right|^2} \int_0^\infty \frac{y^\mu [S_{\mu,i\tau}(y) - s_{\mu,i\tau}(y)]}{x^2+y^2}\, dy \qquad (|\mu|<1) \tag{1.13}$$

of the Macdonald function through the Widder transform of the kernel in (1.1).

2. Boundedness properties of the Lommel and Bessel function transforms

In this section we will study the behaviour in Lebesgue spaces of the integral operator given by (1.1). Namely, we will consider a space of summable functions where there exists the regular Fourier cosine transform given by

$$(F_c f)(\tau) = \sqrt{\tfrac{2}{\pi}} \int_0^\infty f(t) \cos(\tau t)\, dt. \tag{2.1}$$

For the time being we assume that $f \in C_0^\infty(\mathbb{R}_+)$, i.e. f belongs to the space of smooth functions with compact support, which is dense, for instance, in $L_p \quad (p \geq 1)$. Consequently, the integral in (1.1) converges absolutely for each $x > 0$ since its kernel is a continuous function of $\tau \in \mathbb{R}_+$.

Let us introduce auxiliary operators by the formulas

$$\begin{aligned} (\mathrm{Re}J[f])(x) &= \int_0^\infty \frac{f(\tau)}{\cosh \frac{\pi\tau}{2}} \mathrm{Re}J_{i\tau}(x)\, d\tau \\ (\mathrm{Im}J[f])(x) &= \int_0^\infty \frac{f(\tau)}{\sinh \frac{\pi\tau}{2}} \mathrm{Im}J_{i\tau}(x)\, d\tau. \end{aligned} \tag{2.2}$$

Taking into account identity (1.3) we obtain

$$(\mathcal{T}_\mu f)(x) = \sin \tfrac{\pi\mu}{2} (\mathrm{Re}J[f])(x) - \cos \tfrac{\pi\mu}{2} (\mathrm{Im}J[f])(x). \tag{2.3}$$

Furthermore, we will use below boundedness properties in the space $L_2(\mathbb{R}_+)$ of the singular operator or integral transform of Hilbert type (see [4: Chapter 8])

$$(\Phi f)(x) = \tfrac{2}{\pi} \lim_{N \to \infty} \int_{1/N}^N \frac{t f(t)}{t^2 - x^2}\, dt. \tag{2.4}$$

In order to prove the main results of this section we denote by $X(\mathbb{R}_+)$ the space of Lebesgue measurable functions, which are summable over the measure $x\, dx$ and let the indefinite integral

$$\psi_f(t) = \int_t^\infty f(y)\, dy \tag{2.5}$$

be such that $\lim_{t\to 0} \psi_f(t) = 0$. It is not difficult to show that $X(\mathbb{R}_+)$ is a closed subspace of $L_1(\mathbb{R}_+; x\, dx)$.

Theorem 1. *Let $f \in X(\mathbb{R}_+)$. Then the integral operators* $\mathrm{Re}J[f], \mathrm{Im}J[f] : X(\mathbb{R}_+) \to L_2(\mathbb{R}_+)$ *satisfy the isometric identity*

$$\|\mathrm{Re}J[f]\|_{L_2(\mathbb{R}_+)} = \|\mathrm{Im}J[f]\|_{L_2(\mathbb{R}_+)} \tag{2.6}$$

in $L_2(\mathbb{R}_+)$ and for $f, g \in X(\mathbb{R}_+)$ the Parseval-type relations in terms of the Fourier cosine operator (2.1)

$$\begin{aligned} \int_0^\infty (\mathrm{Re}J[f])(x)\, \overline{(\mathrm{Re}J[g])(x)}\, dx &= \int_0^\infty (F_c f)(\tau)\, \overline{(F_c g)(\tau)}\, \frac{d\tau}{\sinh \tau} \\ \int_0^\infty (\mathrm{Im}J[f])(x)\, \overline{(\mathrm{Im}J[g])(x)}\, dx &= \int_0^\infty (F_c f)(\tau)\, \overline{(F_c g)(\tau)}\, \frac{d\tau}{\sinh \tau} \end{aligned} \tag{2.7}$$

hold true. Furthermore, the reciprocal formulas in mean convergence sense through operator (2.4)

$$\begin{aligned}(\mathrm{Im}J[f])(x) &= \tfrac{2}{\pi}\lim_{N\to\infty}\int_{1/N}^{N}\frac{t(\mathrm{Re}J[f])(t)}{t^2-x^2}\,dt\\ (\mathrm{Re}J[f])(x) &= \tfrac{2}{\pi}\lim_{N\to\infty}\int_{1/N}^{N}\frac{x(\mathrm{Im}J[f])(t)}{x^2-t^2}\,dt\end{aligned} \tag{2.8}$$

take place. Finally, for almost all $x > 0$ operators (2.2) *are defined by the formulas*

$$\begin{aligned}(\mathrm{Re}J[f])(x) &= \frac{d}{dx}\int_0^\infty\int_0^x \frac{f(\tau)}{\cosh\frac{\pi\tau}{2}}\,\mathrm{Re}J_{i\tau}(y)\,dyd\tau\\ (\mathrm{Im}J[f])(x) &= \frac{d}{dx}\int_0^\infty\int_0^x \frac{f(\tau)}{\sinh\frac{\pi\tau}{2}}\,\mathrm{Im}J_{i\tau}(y)\,dyd\tau.\end{aligned} \tag{2.9}$$

Proof. First we observe that the indefinite integral (2.5) is an absolutely continuous function on (a,∞) $(a>0)$ tending to zero in the origin and belonging according to the conditions of the theorem to $L_1(t,\infty)$ for each $t>0$. Indeed, we have

$$|\psi_f(t)| \le \int_t^\infty |f(y)|\,dy \le t^{-1}\int_t^\infty y|f(y)|\,dy < \frac{C}{t}$$

where $C>0$ is an absolute constant. Hence taking the Fourier cosine transform (2.1) of $f\in X(\mathbb{R}_+)$, after integration by parts and elimination of the outintegrated terms we represent it as

$$(F_c f)(\tau) = \tau\sqrt{\tfrac{2}{\pi}}\int_0^\infty \psi_f(t)\sin(\tau t)\,dt = \tau\,(F_s\psi_f)(\tau) \tag{2.10}$$

where $F_s\psi_f$ is the Fourier sine transform of the function ψ_f. As a consequence of relations (2.7) we will prove the norm equality

$$\begin{aligned}\|\mathrm{Re}J[f]\|_{L_2(\mathbb{R}_+)} &= \left(\int_0^\infty |(\mathrm{Re}J[f])(x)|^2dx\right)^{\frac12}\\ &= \left(\int_0^\infty |(F_c f)(\tau)|^2\frac{d\tau}{\sinh\tau}\right)^{\frac12}.\end{aligned} \tag{2.11}$$

Making use of representation (2.10) it is majorized by the expression

$$\sqrt{\tfrac{2}{\pi}}\left(\int_0^\infty \frac{\tau^2 d\tau}{\sinh\tau}\left(\int_0^\infty |\psi_f(t)\sin(\tau t)|\,dt\right)^2\right)^{\frac12}. \tag{2.12}$$

However,

$$\begin{aligned}\int_0^\infty |\psi_f(t)\sin(\tau t)|\,dt &\le \int_0^\infty |\psi_f(t)|\,dt\\ &\le \int_0^\infty dt\int_t^\infty |f(y)|\,dy\\ &= \int_0^\infty |f(y)|\,dy\int_0^y dt\\ &= \int_0^\infty y|f(y)|\,dy.\end{aligned}$$

Therefore, combining with (2.12) it becomes

$$\left(\int_0^\infty |(F_c f)(\tau)|^2 \frac{d\tau}{\sinh\tau}\right)^{\frac{1}{2}} \le \sqrt{\tfrac{2}{\pi}}\left(\int_0^\infty \frac{\tau^2 d\tau}{\sinh\tau}\right)^{\frac{1}{2}} \|f\|_{L_1(\mathbb{R}_+;x\,dx)}$$
$$= \sqrt{\tfrac{7}{\pi}\,\zeta(3)}\,\|f\|_{X(\mathbb{R}_+)}$$

where $\zeta = \zeta(z)$ is Riemann's zeta function (see [1: Vol. I]).

Now, in order to deduce Parseval type equalities (2.7) we appeal to the integral representation

$$\frac{\pi}{2\cosh\frac{\pi\tau}{2}}\,\mathrm{Re}J_{i\tau}(x) = \int_0^\infty \sin(x\cosh t)\cos(\tau t)\,dt. \tag{2.13}$$

of the kernel in $(2.2)_1$ (see [8]). We substitute it into $(2.2)_1$, take into account the uniform convergence by $\tau \in \mathrm{supp} f$ and change the order of integration. As the result, for all $f \in C_0^\infty(\mathbb{R}_+)$ after the substitution $\cosh t = u$ we have

$$\begin{aligned}(\mathrm{Re}J[f])(x) &= \sqrt{\tfrac{2}{\pi}}\int_0^\infty \sin(x\cosh t)(F_c f)(t)\,dt\\ &= \sqrt{\tfrac{2}{\pi}}\int_1^\infty \sin(xu)\frac{(F_c f)\big(\log(u+\sqrt{u^2-1}\big)}{\sqrt{u^2-1}}\,du.\end{aligned} \tag{2.14}$$

Hence applying the Parseval equality for the Fourier sine and cosine transforms and returning to the original variable we obtain

$$\begin{aligned}\|\mathrm{Re}\,J[f]\|_{L_2(\mathbb{R}_+)} &= \left(\int_1^\infty \frac{\big|(F_c f)\big(\log(u+\sqrt{u^2-1})\big)\big|^2}{u^2-1}\,du\right)^{\frac{1}{2}}\\ &= \left(\int_0^\infty |(F_c f)(t)|^2 \frac{dt}{\sinh t}\right)^{\frac{1}{2}}\\ &\le \sqrt{\tfrac{7}{\pi}\,\zeta(3)}\,\|f\|_{X(\mathbb{R}_+)}\end{aligned} \tag{2.15}$$

which yields (2.11) for the dense set of functions of $C_0^\infty(\mathbb{R}_+)$. Consequently, equality $(2.7)_1$ follows immediately invoking the Parseval equality for the Fourier transform of functions $f, g \in C_0^\infty(\mathbb{R}_+)$. Hence, as it is easily to verify from (2.15), the desired equality $(2.7)_1$ holds true by continuity for all functions from the space $X(\mathbb{R}_+)$. Moreover, we have the boundedness of the operator given by $(2.2)_1$ and acting from $X(\mathbb{R}_+)$ into $L_2(\mathbb{R}_+)$.

In order to prove the isometric identity (2.6), which will immediately imply equality $(2.7)_2$, we use the Mellin transform formulas

$$\begin{aligned}\frac{1}{\cosh\frac{\pi\tau}{2}}\int_0^\infty \mathrm{Re}J_{i\tau}(x)x^{s-1}dx &= 2^{s-1}\frac{\Gamma(\frac{s+i\tau}{2})\Gamma(\frac{s-i\tau}{2})}{\Gamma(\frac{s}{2})\Gamma(1-\frac{s}{2})}\\ -\frac{1}{\sinh\frac{\pi\tau}{2}}\int_0^\infty \mathrm{Im}J_{i\tau}(x)x^{s-1}dx &= 2^{s-1}\frac{\Gamma(\frac{s+i\tau}{2})\Gamma(\frac{s-i\tau}{2})}{\Gamma(\frac{1+s}{2})\Gamma(\frac{1-s}{2})}\end{aligned} \tag{2.16}$$

(see [7]) for the kernels of the operators given by (2.2). Indeed, applying the Mellin transform (1.4) through operators (2.2) we employ formulas (2.16). Interchanging the order of integration for functions from the dense set $C_0^\infty(\mathbb{R}_+)$ we obtain

$$(\mathrm{Re}J[f])^{\mathcal{M}}(\tfrac{1}{2}+it)\cot\big(\tfrac{\pi}{2}(\tfrac{1}{2}+it)\big) = -(\mathrm{Im}J[f])^{\mathcal{M}}(\tfrac{1}{2}+it).$$

Since, however, $\left|\cot\big(\frac{\pi}{2}(\frac{1}{2}+it)\big)\right| = 1$, the isometric identity (2.6) immediately follows by virtue of the Mellin-Parseval equality (1.6) with $\gamma = \frac{1}{2}$. It can be extended continuously for all $f \in X(\mathbb{R}_+)$. Moreover, in view of [4: Theorem 129] it is not difficult to establish the reciprocal formulas (2.8) in terms of the singular operator (2.4).

Formulas (2.9) one can establish in view of representation (2.14) and Fubini's theorem, if we write (2.13) and the corresponding integral for the kernel in (2.3) (see [1: Vol. II]) as

$$\begin{aligned} \frac{\pi}{2\cosh\frac{\pi\tau}{2}}\,\mathrm{Re}J_{i\tau}(x) &= -\frac{d}{dx}\int_0^\infty \frac{\cos(x\cosh t)}{\cosh t}\cos(\tau t)\,dt \\ \frac{\pi}{2\sinh\frac{\pi\tau}{2}}\,\mathrm{Im}J_{i\tau}(x) &= -\frac{d}{dx}\int_0^\infty \frac{\sin(x\cosh t)}{\cosh t}\cos(\tau t)\,dt. \end{aligned} \tag{2.17}$$

Thus we completed the proof of Theorem 1 ∎

Invoking identity (2.3) we obtain

Corollary 1. *The integral operators given by* (1.1), (2.2), (2.3) *and the operator of Hilbert type given by* (2.4) *are connected each other by the relations*

$$\begin{aligned} (\mathcal{T}_\mu f)(x) &= \sin\frac{\pi\mu}{2}(\Phi\mathrm{Im}J[f])(x) - \cos\frac{\pi\mu}{2}(\mathrm{Re}J[f])(x) \\ (\mathcal{T}_\mu f)(x) &= \sin\frac{\pi\mu}{2}(\mathrm{Re}J[f])(x) - \cos\frac{\pi\mu}{2}(\Phi\mathrm{Re}J[f])(x). \end{aligned}$$

Note that the latter equalities are similar in some sense to relations connecting fractional integrals with singular operators (cf. [3: Chapter 3]).

Corollary 2. *The integral operator $\mathcal{T}_\mu f : X(\mathbb{R}_+) \to L_2(\mathbb{R}_+)$ $(\mu \in \mathbb{R})$ is bounded and we have*

$$\|\mathcal{T}_\mu f\|_2 \le 2\sqrt{\tfrac{7}{\pi}\,\zeta(3)}\,\|f\|_{X(\mathbb{R}_+)}.$$

Moreover, it can be defined for almost all $x > 0$ by the formula

$$(\mathcal{T}_\mu f)(x) = 2^{1-\mu}\frac{d}{dx}\int_0^\infty\int_0^x \big[S_{\mu,i\tau}(y) - s_{\mu,i\tau}(y)\big] f(\tau)\frac{dy\,d\tau}{\left|\Gamma\big(\frac{\mu+i\tau+1}{2}\big)\right|^2}. \tag{2.18}$$

3. Kontorovich-Lebedev type representation theorem and inversion of the Lommel function index transformation

In this final section we will establish a representation theorem for the Kontorovich-Lebedev operator (1.9). As a corollary we obtain an inversion formula for the Lommel function transformation $\mathcal{T}_\mu f$ defined by (2.18).

Let us consider on $\mathbb{R}_+$ the following Banach space of functions $f \in L^*(\mathbb{R}_+)$ or A-type space (see, for instance, in [5]) whose Fourier cosine transforms (2.1) lie in $L_1(\mathbb{R}_+)$. We can introduce a norm in L^* by setting $\|f\|_{L^*(\mathbb{R}_+)} = \int_0^\infty |(F_c f)(t)|\, dt$. As it is known, elements of this space are continuous, bounded functions vanishing at infinity.

The following result says about the boundedness of the Kontorovich-Lebedev operator (1.9) in $L^*(\mathbb{R}_+)$.

Lemma 1. *The Kontorovich-Lebedev operator $KLf : L^*(\mathbb{R}_+) \to L^*(\mathbb{R}_+)$ defined by* (1.9) *is bounded and* $\|KLf\|_{L^*(\mathbb{R}_+)} \le \frac{\pi}{2}\|f\|_{L^*(\mathbb{R}_+)}$.

Proof. According to the definition of $L^*(\mathbb{R}_+)$, we find

$$f(\tau) = \sqrt{\tfrac{2}{\pi}} \int_0^\infty (F_c f)(t) \cos(t\tau)\, dt. \tag{3.1}$$

Consequently, after substituting (3.1) into (1.9) one can change the order of integration via the Fubini theorem. Then using the reciprocal Fourier cosine formula (1.11) we obtain

$$(KLf)(x) = \sqrt{\tfrac{\pi}{2}} \int_0^\infty e^{-x\cosh u} (F_c f)(u)\, du. \tag{3.2}$$

Further, one can calculate the Fourier cosine transform (2.1) of KLf. Indeed, after substitution the latter expression we change the order of integration by the Fubini theorem. Evaluating an elementary integral we arrive at the representation

$$\sqrt{\tfrac{2}{\pi}} \int_0^\infty (KLf)(t) \cos(t\tau)\, dt = \int_0^\infty \frac{(F_c f)(u) \cosh u}{\cosh^2 u + \tau^2}\, du.$$

This result will immediately imply the norm estimate

$$\begin{aligned}\|KLf\|_{L^*(\mathbb{R}_+)} &\le \int_0^\infty |(F_c f)(u)| \cosh u\, du \int_0^\infty \frac{d\tau}{\cosh^2 u + \tau^2} \\ &= \int_0^\infty |(F_c f)(u)|\, du \int_0^\infty \frac{d\tau}{1+\tau^2} \\ &= \tfrac{\pi}{2}\|f\|_{L^*(\mathbb{R}_+)}\end{aligned}$$

and completes the proof of Lemma 1 ∎

The next theorem states representation properties of an arbitrary function $f \in L^*(\mathbb{R}_+)$ in terms of the Kontorovich-Lebedev operator (1.9).

Theorem 2. *For any $f \in L^*(\mathbb{R}_+)$ and for all $\tau \in \mathbb{R}_+$, the representation*

$$f(\tau) = \tfrac{2}{\pi^2}\tau\sinh(\pi\tau)\lim_{\varepsilon\to 0+}\int_0^\infty x^{\varepsilon-1}K_{i\tau}(x)(KLf)(x)\,dx \tag{3.3}$$

is valid.

Proof. By using formula (3.2) we substitute it in the latter integral and according to the absolute convergence of the iterated integral change the order of integration. Hence invoking with formula (1.12), the right-hand side of (3.3) denoted by $I(\tau,\varepsilon)$ can be represented as

$$\begin{aligned} I(\tau,\varepsilon) = \frac{2^{\frac{1}{2}-\varepsilon}}{\pi}\tau\sinh(\pi\tau)\frac{|\Gamma(\varepsilon+i\tau)|^2}{\Gamma(\varepsilon+\frac{1}{2})}&\int_0^\infty (\cosh u)^{i\tau-\varepsilon} \\ \times {}_2F_1\big(\tfrac{\varepsilon-i\tau}{2},\tfrac{\varepsilon+i\tau}{2};\varepsilon+\tfrac{1}{2};\tanh^2 u\big)&(F_cf)(u)\,du. \end{aligned} \tag{3.4}$$

In order to deduce (3.3) we only must motivate the passing to the limit under the integral sign in (3.4). Indeed, due to the Boltz formula for the Gauss hypergeometric function (see [1: Vol. I/Relation (2.1.4.22)]) we have the equality

$$(\cosh u)^{i\tau-\varepsilon}{}_2F_1\big(\tfrac{\varepsilon-i\tau}{2},\tfrac{\varepsilon-i\tau+1}{2};\varepsilon+\tfrac{1}{2};\tanh^2 u\big) = {}_2F_1\big(\tfrac{\varepsilon-i\tau}{2},\tfrac{\varepsilon+i\tau}{2};\varepsilon+\tfrac{1}{2};-\sinh^2 u\big).$$

Hence owing to well known formulas of the Gauss function and its analytic continuation and to the definition of the Pochammer symbol we deduce the relations of the latter hypergeometric function

$$\begin{aligned} &{}_2F_1\big(\tfrac{\varepsilon-i\tau}{2},\tfrac{\varepsilon+i\tau}{2};\varepsilon+\tfrac{1}{2};-\sinh^2 u\big) \\ &\quad = \frac{\Gamma(\varepsilon+\frac{1}{2})}{\big|\Gamma(\frac{\varepsilon+i\tau}{2})\big|^2}\sum_{n=0}^\infty \frac{\Gamma(\frac{\varepsilon-i\tau}{2}+n)\Gamma(\frac{\varepsilon+i\tau}{2}+n)}{\Gamma(\varepsilon+\frac{1}{2}+n)}\,\frac{(-1)^n\sinh^{2n}u}{n!} \end{aligned} \tag{3.5}$$

when $0 < \sinh u \le 1$ and

$$\begin{aligned} &{}_2F_1\big(\tfrac{\varepsilon-i\tau}{2},\tfrac{\varepsilon+i\tau}{2};\varepsilon+\tfrac{1}{2};-\sinh^2 u\big) \\ &\quad = \frac{\Gamma(\varepsilon+\frac{1}{2})\Gamma(i\tau)(\sinh u)^{i\tau-\varepsilon}}{|\Gamma(\frac{\varepsilon+i\tau}{2})|^2\Gamma(\frac{1+\varepsilon+i\tau}{2})}\,\frac{\Gamma(1-i\tau)}{\Gamma(\frac{1-\varepsilon-i\tau}{2})} \\ &\qquad \times\sum_{n=0}^\infty \frac{\Gamma(\frac{\varepsilon-i\tau}{2}+n)\Gamma(\frac{1-\varepsilon-i\tau}{2}+n)}{\Gamma(1-i\tau+n)\sinh^{2n}u}\,\frac{(-1)^n}{n!} \\ &\qquad + \frac{\Gamma(\varepsilon+\frac{1}{2})\Gamma(-i\tau)(\sinh u)^{-i\tau-\varepsilon}}{|\Gamma(\frac{\varepsilon+i\tau}{2})|^2\Gamma(\frac{1+\varepsilon-i\tau}{2})}\,\frac{\Gamma(1+i\tau)}{\Gamma(\frac{1-\varepsilon+i\tau}{2})} \\ &\qquad \times\sum_{n=0}^\infty \frac{\Gamma(\frac{\varepsilon+i\tau}{2}+n)\Gamma(\frac{1-\varepsilon+i\tau}{2}+n)}{\Gamma(1+i\tau+n)\sinh^{2n}u}\,\frac{(-1)^n}{n!} \end{aligned} \tag{3.6}$$

when $\sinh u > 1$. Therefore, via the Stirling asymptotic formula of the Gamma function (see [1: Vol. 1]), the elementary inequality $|\Gamma(z)| \le \Gamma(\mathrm{Re}z)$ $(\mathrm{Re}z > 0)$ and the convergence of the hypergeometric series one can majorize the functions in (3.5) - (3.6) uniformly for all $u, \tau \in \mathbb{R}_+$ and $0 \le \varepsilon \le \varepsilon_0 < \frac{1}{2}$ as

$$\begin{aligned}\left|{}_2F_1\left(\tfrac{\varepsilon-i\tau}{2}, \tfrac{\varepsilon+i\tau}{2}; \varepsilon+\tfrac{1}{2}; -\sinh^2 u\right)\right| &\\ \le 1 + \frac{\Gamma(\varepsilon_0+\frac{1}{2})}{|\Gamma(\frac{\varepsilon+i\tau}{2})|^2} \sum_{n=1}^{\infty} \frac{[\Gamma(\frac{\varepsilon_0}{2}+n)]^2}{\Gamma(\frac{1}{2}+n)} \frac{1}{n!} &\\ \le 1 + \frac{1}{|\Gamma(\frac{\varepsilon+i\tau}{2})|^2} O\left(\sum_{n=1}^{\infty} \frac{1}{n^{\frac{3}{2}-\varepsilon_0}}\right)&\end{aligned}$$

when $0 < \sinh u \le 1$ and as

$$\begin{aligned}&\left|{}_2F_1\left(\tfrac{\varepsilon-i\tau}{2}, \tfrac{\varepsilon+i\tau}{2}; \varepsilon+\tfrac{1}{2}; -\sinh^2 u\right)\right| \\ &\le \frac{\Gamma(\varepsilon_0+\frac{1}{2})|\Gamma(i\tau)|(\sinh u)^{-\varepsilon}}{|\Gamma(\frac{\varepsilon+i\tau}{2})\Gamma(\frac{1+\varepsilon+i\tau}{2})|} \\ &\quad \times \left[1 + \frac{1}{|\Gamma(\frac{\varepsilon-i\tau}{2})\Gamma(\frac{1-\varepsilon-i\tau}{2})|} \sum_{n=1}^{\infty} \frac{\Gamma(\frac{\varepsilon}{2}+n)\Gamma(\frac{1-\varepsilon}{2}+n)}{|\Gamma(1-i\tau+n)|\, n!}\right] \\ &\quad + \frac{\Gamma(\varepsilon_0+\frac{1}{2})|\Gamma(-i\tau)|(\sinh u)^{-\varepsilon}}{|\Gamma(\frac{\varepsilon-i\tau}{2})\Gamma(\frac{1+\varepsilon-i\tau}{2})|} \\ &\quad \times \left[1 + \frac{1}{|\Gamma(\frac{\varepsilon+i\tau}{2})\Gamma(\frac{1-\varepsilon+i\tau}{2})|} \sum_{n=1}^{\infty} \frac{\Gamma(\frac{\varepsilon}{2}+n)\Gamma(\frac{1-\varepsilon}{2}+n)}{|\Gamma(1+i\tau+n)|\, n!}\right] \\ &\le \frac{|\Gamma(i\tau)|}{|\Gamma(\frac{\varepsilon+i\tau}{2})\Gamma(\frac{1+\varepsilon+i\tau}{2})|}\left[1 + \frac{1}{|\Gamma(\frac{\varepsilon-i\tau}{2})\Gamma(\frac{1-\varepsilon-i\tau}{2})|} O\left(\sum_{n=1}^{\infty} \frac{1}{n^{\frac{3}{2}}}\right)\right] \\ &\quad + \frac{|\Gamma(-i\tau)|}{|\Gamma(\frac{\varepsilon-i\tau}{2})\Gamma(\frac{1+\varepsilon-i\tau}{2})|}\left[1 + \frac{1}{|\Gamma(\frac{\varepsilon+i\tau}{2})\Gamma(\frac{1-\varepsilon+i\tau}{2})|} O\left(\sum_{n=1}^{\infty} \frac{1}{n^{\frac{3}{2}}}\right)\right]\end{aligned}$$

when $\sinh u > 1$. The obtained estimates imply that the integrand in (3.4) is uniformly bounded over the interval $(0, \varepsilon_0)$ and that it is majorized by $C_\tau |(F_c f)(u)|$, with $C_\tau > 0$ being a constant depending on τ only. Furthermore, since (see [1: Vol. I/Relation (2.8.11)])

$$\lim_{\varepsilon \to 0+} {}_2F_1\left(\tfrac{\varepsilon-i\tau}{2}, \tfrac{\varepsilon+i\tau}{2}; \varepsilon+\tfrac{1}{2}; -\sinh^2 u\right) = \cos(\tau u),$$

one can appeal to the Lebesgue dominated convergence theorem. Passing to the limit in (3.4) when $\varepsilon \to 0+$, via (2.1) we obtain the desired representation (3.3) and Theorem 2 is proved ∎

Finally, Theorem 2 allows us to establish an inversion of the Lommel function index transform (2.18) in the space $L^*(\mathbb{R}_+)$:

Theorem 3. *If $f \in L^*(\mathbb{R}_+)$ and $\mathcal{T}_\mu f$ $(|\mu| < 1)$ is given by (2.18), then for all $\tau \in \mathbb{R}_+$ the inversion formula*

$$\begin{aligned}f(\tau) = \frac{\tau \sinh(\pi\tau)}{\pi^2} \lim_{\varepsilon \to 0+} 2^{\varepsilon-\mu} \left|\Gamma\left(\tfrac{1+\varepsilon-\mu+i\tau}{2}\right)\right|^2 &\\ \times \int_0^\infty x^{\varepsilon-1} S_{\mu-\varepsilon, i\tau}(x)(\mathcal{T}_\mu f)(x)\, dx &\end{aligned} \tag{3.7}$$

is valid.

Proof. First we observe that from identity (1.3) and integral representations (2.17) it is not difficult to derive the formula

$$\frac{2^{-\mu}}{\left|\Gamma\left(\frac{\mu+i\tau+1}{2}\right)\right|^2}\left[S_{\mu,i\tau}(x)-s_{\mu,i\tau}(x)\right]=-\tfrac{1}{\pi}\frac{d}{dx}\int_0^\infty \cos(\tau y)\frac{\sin(\frac{\pi\mu}{2}-x\cosh y)}{\cosh y}\,dy$$

$(x>0)$ for the kernel of the Lommel transform (1.1). Meanwhile, taking integral (1.13) which is absolutely convergent, we substitute it into the Kontorovich-Lebedev operator given by (1.9). Then, at least for $f\in C_0^\infty(\mathbb{R}_+)$, one can immediately invert the order of integration and arrive at the representation

$$(KLf)(x)=x^{1-\mu}\int_0^\infty \frac{y^\mu(\mathcal{T}_\mu f)(y)}{x^2+y^2}\,dy. \tag{3.8}$$

In similar manner one can find

$$(\mathcal{T}_\mu f)(x)=\sqrt{\tfrac{2}{\pi}}\int_0^\infty (F_cf)(y)\cos\left(\tfrac{\pi\mu}{2}-x\cosh y\right)dy$$

for each $f\in C_0^\infty(\mathbb{R}_+)$. The function $\mathcal{T}_\mu f$ is bounded and continuous on $\mathbb{R}_+$ and, moreover,

$$|(\mathcal{T}_\mu f)(x)|\le\sqrt{\tfrac{2}{\pi}}\int_0^\infty |(F_cf)(y)|dy=\sqrt{\tfrac{2}{\pi}}\|f\|_{L^*(\mathbb{R}_+)}. \tag{3.9}$$

Hence, via (3.8) - (3.9) we obtain

$$\begin{aligned}|(KLf)(x)|&\le x^{1-\mu}\int_0^\infty \frac{y^\mu|(\mathcal{T}_\mu f)(y)|}{x^2+y^2}\,dy\\&\le\sqrt{\tfrac{2}{\pi}}\|f\|_{L^*(\mathbb{R}_+)}x^{1-\mu}\int_0^\infty\frac{y^\mu}{x^2+y^2}\,dy\\&=\sqrt{\tfrac{2}{\pi}}\|f\|_{L^*(\mathbb{R}_+)}\int_0^\infty\frac{y^\mu}{1+y^2}\,dy\\&=\sqrt{\tfrac{\pi}{2}}\frac{1}{\cos\frac{\pi\mu}{2}}\|f\|_{L^*(\mathbb{R}_+)}.\end{aligned}$$

Consequently, the operator KLf presented by (3.8) is bounded as an operator from $L^*(\mathbb{R}_+)$ into the space of bounded continuous functions on $\mathbb{R}_+$. As already known from Lemma 1, the Kontorovich-Lebedev operator (1.9) is bounded in $L^*(\mathbb{R}_+)$. Moreover, elements of this space are bounded continuous functions vanishing at infinity. It is clear now that since (3.8) holds true for the dense set of C_0^∞-functions, then in view of the Banach theorem equality (3.8) is true for all $f\in L^*(\mathbb{R}_+)$. Hence one can substitute (3.8) into integral (3.3) and in view of the Fubini theorem change the order of integration. Then due to formula (1.8) we immediately arrive at the inversion formula (3.7) and complete the proof of Theorem 3 ∎

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