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A multilinear generalisation of the Hilbert transform and fractional integration

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Abstract. We study a multilinear analogue of the Hilbert transform. As can be expected, the finiteness of the form depends on cancellation properties in the kernel and care must be taken in the definition of the form. We show how to define the form in terms of distributions and prove L^p bounds for that form.

In the second part, we study an analogous form on the level of fractional integration. This has been studied in one form by Drury. We note the L^p bounds for it and find the optimal constant for this bound in the case with the most symmetries. We also determine all functions which are optimisers for this inequality.

Finally, we consider analogues of the fractional integration form in directions similar to those of Beckner's approach for multilinear Hardy–Littlewood–Sobolev inequalities.

1. Introduction

1.1. Overview

The Hilbert transform and the fractional integration operator acting on functions on $\mathbb{R}$ can be viewed as convolution operators with a kernel which involves calculating the volume of the simplex in $\mathbb{R}$ whose vertices are two points, x_1 and x_2, that is, calculating the difference $x_2 - x_1$.

One way of extending these operators to higher dimensions is to multilinearise them and consider n-linear forms, defined for functions on $\mathbb{R}^{n-1}$ whose kernel involves calculating the volume of the simplex in $\mathbb{R}^{n-1}$ whose vertices are the n points, $x_1, \dots, x_n$, that is, calculating $\det \left(\begin{smallmatrix} 1 & \cdots & 1 \\ x_1 & \cdots & x_n \end{smallmatrix}\right)$.

In this article we will consider two such forms. The first lives on the level of the Hilbert transform, so that the L^p–boundedness of the form relies on cancellation properties of the kernel and care must be taken to properly define the form. This form has not been considered before in the literature.

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The second form lives on the level of fractional integration. Such a form has been considered before by Christ [7], Drury [8] and Baernstein and Loss [1] because of its relation to questions regarding the k-plane transform and the restriction of the Fourier transform. In particular Drury proved L^p-boundedness for the form or rather a closely related analogue of it for functions defined on the $(n-1)$-sphere. For this form we address the question of finding the best constant in the inequality which the gives the L^p-boundedness of the form. Such questions have been answered for the fractional integration operator by Carlen and Loss [6], see also the book of Lieb and Loss [10], using symmetrisation and conformal invariance. We adapt these techniques for the form we wish to consider.

Furthermore, Beckner [2] has considered multilinearising fractional integration in a different way, and we remark that it is possible in some sense to combine these two methods of multilinearisation.

Finally, all of our techniques rely on some geometric invariance which among other things makes it possible to formulate all of our results on Euclidean space, spherical space or hyperbolic space.

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We now turn to the statement of our results.

1.2. The singular integral

The object we wish to study is the n-linear form given formally by

$$\Lambda(f_1,\dots,f_n) := \int_{(\mathbb{R}^{n-1})^n} \frac{f_1(x_1)\cdots f_n(x_n)}{\det\left(\begin{smallmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_n\end{smallmatrix}\right)}\, \mathrm{d}x_1\,\mathrm{d}x_2\dots\mathrm{d}x_n\,, \tag{1.1}$$

where $x_i \in \mathbb{R}^{n-1}$. In the determinant we interpret the variables x_i as column vectors, adjoin a top row containing only 1 and thus get a square matrix. For $n=2$ we have

$$\Lambda(f,g) = \int_{\mathbb{R}}\int_{\mathbb{R}} \frac{f(x)g(y)}{\det\left(\begin{smallmatrix} 1 & 1 \\ x & y\end{smallmatrix}\right)}\,\mathrm{d}x\,\mathrm{d}y = \int_{\mathbb{R}}\int_{\mathbb{R}} \frac{f(x)g(y)}{y-x}\,\mathrm{d}x\,\mathrm{d}y = \pi\langle Hf, g\rangle,$$

where Hf denotes the Hilbert transform of f, so in this case Λ is the bilinear form associated to the Hilbert transform. For $n \geq 3$ we can see Λ as an n-linear generalisation of the Hilbert transform.

There is a closely related form defined for n functions on the unit sphere S^{n-1} given by

$$\Lambda_S(f_1,\dots,f_n) := \int_{(S^{n-1})^n} \frac{f_1(\omega_1)\cdots f_n(\omega_n)}{\det(\omega_1,\dots,\omega_n)}\,\mathrm{d}\omega_1\,\mathrm{d}\omega_2\dots\mathrm{d}\omega_n. \tag{1.2}$$

The integrals (1.1) and (1.2) are not absolutely convergent so we replace them with

$$\Lambda(f_1,\dots,f_n) := \frac{1}{2}\int \frac{(f_1(x_1)-f_1(x_1^*))f_2(x_2)\cdots f_n(x_n)}{\det\left(\begin{smallmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_n\end{smallmatrix}\right)}\,\mathrm{d}x_1\,\mathrm{d}x_2\dots\mathrm{d}x_n\,, \tag{1.3}$$

where x_1^* is the reflection of x_1 in the hyperplane determined by the other variables and

$$\Lambda_S(f_1,\ldots,f_n) := \frac{1}{2}\int \frac{(f_1(\omega_1)-f_1(\omega_1^*))f_2(\omega_2)\cdots f_n(\omega_n)}{\det(\omega_1,\ldots,\omega_n)}\,\mathrm{d}\omega_1\,\mathrm{d}\omega_2\ldots\mathrm{d}\omega_n\,, \tag{1.4}$$

where ω_1^* is the reflection of ω_1 in the great hypercircle determined by the other variables.

As a purely formal exercise we can calculate

$$\begin{aligned}
\Lambda(f_1,\ldots,f_n) &= \frac{1}{2}\int \frac{(f_1(x_1)-f_1(x_1^*))f_2(x_2)\cdots f_n(x_n)}{\det\left(\begin{smallmatrix}1&1&\ldots&1\\ x_1&x_2&\ldots&x_n\end{smallmatrix}\right)}\,\mathrm{d}x_1\,\mathrm{d}x_2\ldots\mathrm{d}x_n\\
&= \frac{1}{2}\int \frac{f_1(x_1)f_2(x_2)\cdots f_n(x_n)}{\det\left(\begin{smallmatrix}1&1&\ldots&1\\ x_1&x_2&\ldots&x_n\end{smallmatrix}\right)}\,\mathrm{d}x_1\,\mathrm{d}x_2\ldots\mathrm{d}x_n\\
&\quad - \frac{1}{2}\int \frac{f_1(x_1^*)f_2(x_2)\cdots f_n(x_n)}{\det\left(\begin{smallmatrix}1&1&\ldots&1\\ x_1&x_2&\ldots&x_n\end{smallmatrix}\right)}\,\mathrm{d}x_1\,\mathrm{d}x_2\ldots\mathrm{d}x_n\\
&= \int \frac{f_1(x_1)f_2(x_2)\cdots f_n(x_n)}{\det\left(\begin{smallmatrix}1&1&\ldots&1\\ x_1&x_2&\ldots&x_n\end{smallmatrix}\right)}\,\mathrm{d}x_1\,\mathrm{d}x_2\ldots\mathrm{d}x_n
\end{aligned}$$

since the change of variables $x_1\mapsto x_1^*$ has Jacobian 1, $x_1^{**}=x_1$ and

$$\det\begin{pmatrix}1&1&\ldots&1\\ x_1&x_2&\ldots&x_n\end{pmatrix} = -\det\begin{pmatrix}1&1&\ldots&1\\ x_1^*&x_2&\ldots&x_n\end{pmatrix},$$

which follows by noting that the determinants are the signed volumes of the simplices whose vertices are $x_1,\ldots,x_n$ and $x_1^*,x_2,\ldots,x_n$ respectively and these simplices have the same unsigned volume but different orientations.

The following lemma establishes that (1.3) and (1.4) are sensible definitions.

Lemma 1.1. *Let $f_1,\ldots,f_n$ be functions in $C_c^\infty(\mathbb{R}^{n-1})$ or $C^\infty(S^{n-1})$. Then*

(1) *the integrals in* (1.3) *or* (1.4) *are absolutely convergent,*

(2) *the numerator $(f_1(x_1)-f_1(x_1^*))f_2(x_2)\cdots f_n(x_n)$ in* (1.3) *can be replaced by*

$$f_1(x_1)\cdots f_{i-1}(x_{i-1})(f_i(x_i)-f_i(x_i^*))f_{i+1}(x_{i+1})\cdots f_n(x_n)$$

for any $i=1,\ldots,n$ without affecting the value of the integral and

(3) *the numerator $(f_1(\omega_1)-f_1(\omega_1^*))f_2(\omega_2)\cdots f_n(\omega_n)$ in* (1.4) *can be replaced by*

$$f_1(\omega_1)\cdots f_{i-1}(\omega_{i-1})(f_i(\omega_i)-f_i(\omega_i^*))f_{i+1}(\omega_{i+1})\cdots f_n(\omega_n)$$

for any $i=1,\ldots,n$ without affecting the value of the integral.

The symbols x_i^* and ω_i^* have the obvious meaning analogous to x_1^* and ω_1^*.

What we are interested in are estimates such as

$$|\Lambda(f_1,\ldots,f_n)| \lesssim \|f_1\|_{L^{p_1}(\mathbb{R}^{n-1})}\cdots\|f_n\|_{L^{p_n}(\mathbb{R}^{n-1})} \tag{1.5}$$

and

$$|\Lambda_S(f_1,\dots,f_n)| \lesssim \|f_1\|_{L^{p_1}(S^{n-1})}\cdots\|f_n\|_{L^{p_n}(S^{n-1})} \tag{1.6}$$

where $A \lesssim B$ signifies that there is an absolute constant C, depending only on the dimension, such that $A \le CB$. We shall prove the following theorems.

Theorem 1.2. *Let $\mathcal{S}$ be the closed polytope in $\mathbb{R}^n$ whose vertices are the n permutations of the n-tuple $(\frac{n-2}{n-1},\dots,\frac{n-2}{n-1},1)$. Then (1.5) holds if and only if $(\frac{1}{p_1},\dots,\frac{1}{p_n})$ lies in the interior of $\mathcal{S}$, relative to the hyperplane that $\mathcal{S}$ lies in. For $n \ge 3$, the estimate holds on the boundary of $\mathcal{S}$ if each f_j with j for which $\frac{1}{p_j} = \frac{n-2}{n-1}$ is restricted to be a characteristic function of a set but the other f_j's may be unrestricted. The estimate fails if any f_j with j for which $\frac{1}{p_j} = \frac{n-2}{n-1}$ is taken unrestricted.*

Remark 1.3. Each point (q_j) in $\mathcal{S}$ lies in the hyperplane Π defined by the equation

$$\sum_{j=1}^{n} q_j = n-1.$$

When we speak about the exterior, the interior and the boundary of $\mathcal{S}$ we understand it to be taken relative to Π.

To state our result concerning (1.6), consider the set of points $(z_1,\dots,z_n) \in \mathbb{R}^n$ such that $z_i \ge 0$, $\sum_{i=1}^n z_i \le n-1$ and $\sum_{j=1}^k z_{i_j} < \frac{(n-2)k+1}{n-1}$ for any subset $A = \{i_1,\dots,i_k\}$ of $\{1,\dots,n\}$. Call this set $\tilde{\mathcal{S}}$.

Theorem 1.4. *Inequality (1.6) holds if $(\frac{1}{p_1},\dots,\frac{1}{p_n})$ lies in $\tilde{\mathcal{S}}$. It fails if $(\frac{1}{p_1},\dots,\frac{1}{p_n})$ lies in the exterior of $\tilde{\mathcal{S}}$. For $n \ge 3$, the estimate holds at a vertex of $\mathcal{S}$ if each f_j is restricted to be a characteristic function of a set but it fails if the f_j's are unrestricted.*

By specialising these theorems to the centre of $\mathcal{S}$ we get:

Corollary 1.5.

$$|\Lambda(f_1,\dots,f_n)| \lesssim \prod_{i=1}^{n} \|f_i\|_{L^{\frac{n}{n-1}}(\mathbb{R}^{n-1})}. \tag{1.7}$$

Corollary 1.6.

$$|\Lambda_S(f_1,\dots,f_n)| \lesssim \prod_{i=1}^{n} \|f_i\|_{L^{\frac{n}{n-1}}(S^{n-1})}. \tag{1.8}$$

To make the geometric picture complete, we note that the integrals (1.1) on Euclidean space and (1.2) on the sphere have a close relative on hyperbolic space. To formulate that, following Beckner [2], we let $\mathbb{H}^{n-1}$ denote the two-sheeted hyperboloid in $\mathbb{R}^n$ given by

$$\mathbb{H}^{n-1} = \{Q = (q_0,\bar{q}) \in \mathbb{R}\times\mathbb{R}^{n-1} : q_0^2 - |\bar{q}| = 1\}.$$

This set has a measure, $\mathrm{d}\nu$, which invariant under actions of the Lorenz group $O(1, n-1)$ and this set-up is a model for hyperbolic space.

We consider the form

$$\Lambda_{\mathbb{H}}(f_1, \dots, f_n) := \int_{(\mathbb{H}^{n-1})^n} \frac{f_1(q_1) \cdots f_n(q_n)}{\det(q_1, \dots, q_n)} \, \mathrm{d}\nu(q_1) \cdots \mathrm{d}\nu(q_n). \tag{1.9}$$

Note that when calculating the determinant, each q_i is viewed as a column vector in $\mathbb{R}^n$. This is a singular integral but a suitable variant of Lemma 1.1 holds so that the definition is sensible. We are interested in estimates of the form

$$|\Lambda_{\mathbb{H}}(f_1, \dots, f_n)| \lesssim \|f_1\|_{L^{p_1}(\mathrm{d}\nu)} \cdots \|f_n\|_{L^{p_n}(\mathrm{d}\nu)}. \tag{1.10}$$

We prove:

Theorem 1.7. *An identical statement to the one in Theorem* 1.4 *holds for inequality* (1.10)*.*

1.3. Best constants and optimisers

In Section 3 we look at fractional integral analogues of the multilinear forms above. Define, for $0 < \alpha < 1$,

$$\Lambda_\alpha(f_1, \dots, f_n) := \int \frac{f_1(x_1) \cdots f_n(x_n)}{\left|\det \left(\begin{smallmatrix} 1 & 1 & \dots & 1 \\ x_1 & x_2 & \dots & x_n \end{smallmatrix}\right)\right|^\alpha} \, \mathrm{d}x_1 \, \mathrm{d}x_2 \dots \mathrm{d}x_n \,, \tag{1.11}$$

where $x_i \in \mathbb{R}^{n-1}$. Also define

$$\Lambda_{S,\alpha}(f_1, \dots, f_n) := \int \frac{f_1(\omega_1) \cdots f_n(\omega_n)}{|\det(\omega_1, \dots, \omega_n)|^\alpha} \, \mathrm{d}\omega_1 \, \mathrm{d}\omega_2 \dots \mathrm{d}\omega_n \,, \tag{1.12}$$

where $\omega_i \in S^{n-1}$. Finally, define

$$\Lambda_{\mathbb{H},\alpha}(f_1, \dots, f_n) := \int_{(\mathbb{H}^{n-1})^n} \frac{f_1(q_1) \cdots f_n(q_n)}{|\det(q_1, \dots, q_n)|^\alpha} \, \mathrm{d}\nu(q_1) \cdots \mathrm{d}\nu(q_n). \tag{1.13}$$

As in the Hardy–Littlewood–Sobolev theorem concerning fractional integrals, the boundedness of these multilinear forms does not rely on cancellation properties of the kernel. Indeed, we have that

$$|\Lambda_\alpha(f_1, \dots, f_n)| \lesssim \|f_1\|_{p_{0,\alpha}} \dots \|f_{n-1}\|_{p_{0,\alpha}} \|f_n\|_1; \tag{1.14}$$

$$|\Lambda_{S,\alpha}(f_1, \dots, f_n)| \lesssim \|f_1\|_{p_{0,\alpha}} \dots \|f_{n-1}\|_{p_{0,\alpha}} \|f_n\|_1 \tag{1.15}$$

and

$$|\Lambda_{\mathbb{H},\alpha}(f_1, \dots, f_n)| \lesssim \|f_1\|_{p_{0,\alpha}} \dots \|f_{n-1}\|_{p_{0,\alpha}} \|f_n\|_1 \tag{1.16}$$

where $1/p_{0,\alpha} = 1 - \alpha/(n-1)$. As before, interpolation gives that

$$|\Lambda_\alpha(f_1, \dots, f_n)| \lesssim \|f_1\|_{p_\alpha} \dots \|f_n\|_{p_\alpha}; \tag{1.17}$$

$$|\Lambda_{S,\alpha}(f_1, \dots, f_n)| \lesssim \|f_1\|_{p_\alpha} \dots \|f_n\|_{p_\alpha} \tag{1.18}$$

and

$$|\Lambda_{\mathbb{H},\alpha}(f_1,\ldots,f_n)| \lesssim \|f_1\|_{p_\alpha}\cdots\|f_n\|_{p_\alpha} \tag{1.19}$$

where $1/p_\alpha = 1-\alpha/n$. These results can be proved with the same methods we used for the singular integral version and in fact this has already been done for $\Lambda_{S,\alpha}$ by Drury [8]. Because of the absolute convergence there is no question about how the forms are defined and this makes the proof slightly simpler.

There is an implied constant on the right hand side of inequalities (1.17), (1.18) and (1.19). We will give a minimum value for these constants and identify the functions that give equality with them.

To state the theorem, let us define

$$\mathcal{H}(f_1,\ldots,f_n) := \frac{|\Lambda_\alpha(f_1,\ldots,f_n)|}{\|f_1\|_p\cdots\|f_n\|_p},$$

where for the rest of this section we have fixed p as p_α. Also define

$$k(x) = \frac{1}{(1+|x|^2)^{\frac{n}{2p}}}. \tag{1.20}$$

We prove the following.

Theorem 1.8. *The n-tuple $(k,\ldots,k)$ is an optimiser for the operator Λ_α in the sense that*

$$\sup_{f_i\geq 0}\mathcal{H}(f_1,\ldots,f_n) = \mathcal{H}(k,\ldots,k).$$

Furthermore, if the tuple $(f_1,\ldots,f_n)$ of non-negative functions is an optimiser for Λ_α then there exists an $n\times n$ matrix A with determinant 1 and $c_i\geq 0$ for $1\leq i\leq n$ such that

$$f_i(x) = c_i\|A\left(\begin{smallmatrix}x\\1\end{smallmatrix}\right)\|^{-\frac{n}{p}} \text{ for each } 1\leq i\leq n \tag{1.21}$$

and conversely, all tuples of functions of this form are optimisers.

The analogous theorems for $\Lambda_{S,\alpha}$ and $\Lambda_{\mathbb{H},\alpha}$ are stated and proved at the end of Section 3.

1.4. Relations to the results of Beckner

In [2], Beckner considers multilinear analogues of the Hardy–Littlewood–Sobolev inequality which take the form

$$\int\prod_{k=1}^N f_k(x_k)\prod_{1\leq i<j\leq N}|x_i-x_j|^{-\gamma_{ij}}\,\mathrm{d}x_1\cdots\mathrm{d}x_N \leq C\prod_{k=1}^N\|f_k\|_{p_k}$$

for non-negative valued functions on $\mathbb{R}^n$. For certain ranges of the parameters γ_{ij} and p_k this inequality possesses a conformal invariance and he shows how it is possible to write down equivalent inequalities for the (Riemann) sphere S^n and for hyperbolic space $\mathbb{H}^n$. Furthermore, by playing this invariance against symmetrisation techniques as Carlen and Loss did for the original Hardy–Littlewood–Sobolev inequality, Beckner gives the optimal value of the constant C and also all the functions which furnish it. This idea is also a theme in our work but we would like to note that in the work of Beckner and Carlen and Loss the mappings between the underlying spaces are conformal whereas in ours they are not.

Furthermore, it is possible to extend our results in the spirit of Beckner to multilinear forms of the type

$$I_\gamma(f_1,\ldots,f_N) = \int \prod_{k=1}^{N} f_k(x_k) \prod_{P\in\mathcal{P}} V(x_i|i\in P)^{-\gamma_P}\, \mathrm{d}x_1\cdots\mathrm{d}x_N, \tag{1.22}$$

with $x_i \in \mathbb{R}^{n-1}$ for $i = 1,\ldots,N$ and each P in the collection $\mathcal{P}$ is a set of n indices from $\{1,\ldots,N\}$ so that $\{x_i|i\in P\}$ is the vertex set of a simplex in $\mathbb{R}^{n-1}$ whose (unsigned) volume is denoted $V(x_i|i\in P)$. In this case we are interested in inequalities of the form

$$I_\gamma(f_1,\ldots,f_N) \le C\prod_{k=1}^{N}\|f_k\|_{p_k}. \tag{1.23}$$

The condition

$$\frac{n}{p_k} + \sum_{\substack{P\in\mathcal{P}\\ k\in P}} \gamma_P = n \tag{1.24}$$

must hold for $k = 1,\ldots,N$ in order for the geometric invariance which we want to exploit to exist. Additionally, the kernel must be locally integrable. Clearly, a sufficient condition for this is that

$$\sum_{P\in\mathcal{P}} \gamma_P < 1,$$

but in general this is not necessary.

We will not attempt to locate a set of sufficient and necessary conditions for integrability, but we remark that in the simplest new case, when the integration is over x, y, z and w which are elements of $\mathbb{R}^2$ and the integral takes the form

$$\int_{(\mathbb{R}^2)^4} \frac{f_1(x)f_2(y)f_3(z)f_4(w)}{\left|\det\left(\begin{smallmatrix}1&1&1\\x&y&z\end{smallmatrix}\right)\right|^{\alpha}\left|\det\left(\begin{smallmatrix}1&1&1\\x&y&w\end{smallmatrix}\right)\right|^{\beta}\left|\det\left(\begin{smallmatrix}1&1&1\\x&z&w\end{smallmatrix}\right)\right|^{\gamma}\left|\det\left(\begin{smallmatrix}1&1&1\\y&z&w\end{smallmatrix}\right)\right|^{\delta}}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,\mathrm{d}w \tag{1.25}$$

then the sufficient and necessary conditions are

$$0\le\alpha,\beta,\gamma,\delta<1 \quad\text{and}\quad \alpha+\beta+\gamma+\delta<2. \tag{1.26}$$

If the integrability condition and (1.24) hold and if I_γ cannot be factorised into the product of two integrals then more or less the same arguments as for the case of a single determinant give that the optimisers for (1.23) are exactly those of the form (1.21) with the power p in the exponent replaced by p_k as appropriate.

We discuss these issues in Section 3.1.

Remark 1.9. If we take N functions on $\mathbb{R}^{n-1}$ then the multilinear fractional integration kernel of Beckner which is a function of N points is formed by taking pairs of these points and for each pair considering the convex set determined by the points in the pair and taking the suitably defined volume of this convex set. Similarly, the form (1.22) involves a kernel which again is a function of N points and is formed by taking subsets containing n of these, considering the convex set determined by them and taking the volume of this convex set. Curiously, taking subsets of k points, for $2 < k < n$, and considering the volume of the convex set they form and forming a kernel as a product of those gives forms which do not seem to possess a geometric invariance which is suitable for the type of analysis which we do here.

2. The singular integral

For $n \geq 3$ the positive results of Theorem 1.2 follow from the following estimate.

Theorem 2.1. *Let $n \geq 3$ and $\chi_{E_1}, \ldots, \chi_{E_{n-1}}$ be characteristic functions of $n-1$ measurable sets in $\mathbb{R}^{n-1}$ and f_n be a measurable function on $\mathbb{R}^{n-1}$. Then*

$$|\Lambda(\chi_{E_1}, \ldots, \chi_{E_{n-1}}, f_n)| \lesssim \|\chi_{E_1}\|_{\frac{n-1}{n-2}} \cdots \|\chi_{E_{n-1}}\|_{\frac{n-1}{n-2}} \|f_n\|_1. \tag{2.1}$$

Let us note how we can use multilinear interpolation to pass from this estimate to the general result of the theorem. Firstly note that convexity gives directly that (1.5) holds for tuples $\left(\frac{1}{p_1}, \ldots, \frac{1}{p_n}\right)$ in $\mathcal{S}$ if each f_j is restricted to be a characteristic function f_j. Now take an element $\bar{p} \in \mathcal{S}$ and assume that $\frac{1}{p_j} = \frac{n-2}{n-1}$ if and only if $j \leq k$ where $k \leq n$. Let us fix sets E_j for $1 \leq j \leq k$ and note that we have

$$|\Lambda(\chi_{E_1}, \ldots, \chi_{E_n})| \lesssim \prod_{j \leq k} \|\chi_{E_j}\|_{\frac{n-1}{n-2}} \prod_{j > k} \|\chi_{E_j}\|_{q_j} \tag{2.2}$$

if each q_j is sufficiently close to p_j and

$$\sum_{j=k+1}^{n} \frac{1}{q_j} = 1 + \frac{n-2}{n-1}(n-k-1).$$

This shows that we can use Marcinkiewicz interpolation, see for example [9], to strengthen this result to

$$|\Lambda(\chi_{E_1}, \ldots, \chi_{E_k}, f_{k+1}, \ldots, f_n)| \lesssim \prod_{j \leq k} \|\chi_{E_j}\|_{\frac{n-1}{n-2}} \prod_{j > k} \|f_j\|_{p_j}. \tag{2.3}$$

By permuting the indices we arrive at the estimate of the theorem.

The remaining parts of Theorem 1.2 can be seen from examples which we now present.

Example 2.2. Let us assume that inequality (1.5) holds for the dilated functions $\phi_1(\frac{\cdot}{R}), \dots, \phi_n(\frac{\cdot}{R})$ for all $R > 0$. Then

$$\left| \int \frac{\phi_1(\frac{x_1}{R}), \dots, \phi_n(\frac{x_n}{R})}{\det\left(\begin{smallmatrix} 1 & 1 & \dots & 1 \\ x_1 & x_2 & \dots & x_n \end{smallmatrix}\right)} \, \mathrm{d}x_1 \, \mathrm{d}x_2 \dots \mathrm{d}x_n \right| \lesssim \left\| \phi_1\left(\frac{\cdot}{R}\right) \right\|_{p_1} \cdots \left\| \phi_n\left(\frac{\cdot}{R}\right) \right\|_{p_n}$$

so

$$\left| \int \frac{\phi_1(\frac{x_1}{R}), \dots, \phi_n(\frac{x_n}{R})}{\det\left(\begin{smallmatrix} 1 & 1 & \cdots & 1 \\ \frac{x_1}{R} & \frac{x_2}{R} & \cdots & \frac{x_n}{R} \end{smallmatrix}\right) \cdot R^{n-1}} \frac{\mathrm{d}x_1}{R^{n-1}} \frac{\mathrm{d}x_2}{R^{n-1}} \cdots \frac{\mathrm{d}x_n}{R^{n-1}} \cdot R^{n(n-1)} \right| \lesssim$$

$$\lesssim \left(\prod_{i=1}^{n} R^{\frac{n-1}{p_i}} \right) \|\phi_1\|_{p_1} \cdots \|\phi_n\|_{p_n}$$

so

$$R^{(n-1)^2} \lesssim R^{\sum (n-1) \frac{1}{p_i}}$$

so

$$\sum_{i=1}^{n} \frac{1}{p_i} = n - 1. \tag{2.4}$$

Example 2.3. As stated above we get for $n = 2$ that $\Lambda(f, g) = \pi \langle Hf, g \rangle$ where Hf is the Hilbert transform of f. Thus by well-known properties we see that

$$|\Lambda(f, g)| \lesssim \|f\|_{p_1} \|g\|_{p_2} \quad \text{if} \quad \frac{1}{p_1} + \frac{1}{p_2} = 1 \quad \text{and} \quad 1 < p_1, p_2 < \infty. \tag{2.5}$$

Aside from the endpoints, this is the best estimate we could hope for in the light of the previous example.

Example 2.4. When $n \geq 3$ there is a further restriction on the values of p_j for which (1.5) can hold.

To see this let us first of all note that there exist non-empty open cones $\mathcal{C}_1, \dots, \mathcal{C}_n$ with vertices at the origin in $\mathbb{R}^{n-1}$ such that if $x_1 \in \mathcal{C}_1, \dots, x_n \in \mathcal{C}_n$ then

$$\det \begin{pmatrix} 1 & \cdots & 1 \\ x_1 & \cdots & x_n \end{pmatrix} > 0.$$

To construct these cones we can for example take $\mu_1, \dots, \mu_n \in S^{n-1}$ to be the vertices of a regular simplex with centre at the origin. We shall denote the simplex whose vertices are $\nu_1, \dots, \nu_n$ by $\mathcal{T}_{(\nu_i)}$. Now, the signed volume of $\mathcal{T}_{(\mu_i)}$ is given by

$$\det \begin{pmatrix} 1 & \cdots & 1 \\ \mu_1 & \cdots & \mu_n \end{pmatrix} \tag{2.6}$$

which can therefore not equal zero and we may furthermore assume that we have carried out the numbering of the μ's in such a way that this determinant is positive.

Let us note that if the origin lies in the interior of a simplex $\mathcal{T}_{(\nu_i)}$ then it also lies in the interior of $\mathcal{T}_{(r_i\nu_i)}$ for any positive scalars r_i. We can prove this iteratively if we know that this holds when all of the r_i's except one equal 1. We may then further assume that this exceptional r_i is r_1.

Now, the origin lies in the interior of $\mathcal{T}_{(\nu_i)}$ if and only if the line connecting ν_1 and the origin intersects the interior of the facet opposite ν_1 and this intersection lies beyond the origin. If we replace ν_1 by $r\nu_1$ for $r > 0$ then this line and the opposite facet remain unaltered and the intersection will still lie beyond the origin.

Now let M_i be a small neighbourhood in S^{n-1} around ν_i such that for any tuple $(\tilde{\mu}_i)$ in $M_1 \times \cdots \times M_n$ we have that the determinant in (2.6) is positive and that the origin lies in the interior of the simplex $\mathcal{T}_{(\tilde{\mu}_i)}$.

By what we have said it is now clear that we may take $\mathcal{C}_i$ to be the smallest cone with vertex at the origin which contains M_i.

With this set-up in hand we let $\phi_1, \ldots, \phi_n$ be non-negative C_c^∞ functions such that $\operatorname{supp}\phi_i \subset \mathcal{C}_i$. We also insist that ϕ_i is supported in $|x| < \frac{1}{10}$ for $i = 1, \ldots, k$ while ϕ_i is supported in $\frac{1}{2} < |x| < 1$ for $i = k+1, \ldots, n$. Here k is an integer between 1 and n. These conditions will continue to hold if we replace all the ϕ_i's for $i \le k$ by $\phi_{i,\epsilon} : x \mapsto \phi_i(\frac{x}{\epsilon})$ for $\epsilon < 1$. Now,

$$\det\begin{pmatrix} 1 & \cdots & 1 \\ x_1 & \cdots & x_n \end{pmatrix} = \det(x_2 - x_1, \ldots, x_n - x_1) \le |x_2 - x_1| \cdots |x_n - x_1|$$

by Hadamard's theorem so

$$\begin{aligned}\Lambda(\phi_{1,\epsilon}, \ldots, &\phi_{k,\epsilon}, \phi_{k+1}, \ldots, \phi_n) \gtrsim \\ &\gtrsim \int \cdots \int \frac{\phi_1(\frac{x_1}{\epsilon}) \ldots \phi_k(\frac{x_k}{\epsilon})\phi_{k+1}(x_{k+1})\phi_n(x_n)}{|x_2 - x_1| \cdots |x_k - x_1||x_{k+1}| \cdots |x_n|}\, \mathrm{d}x_1 \ldots \mathrm{d}x_n\end{aligned}$$

because we have $|x_i - x_1| \sim |x_i|$ for all $i > k$. We then have

$$\begin{aligned}&\Lambda(\phi_{1,\epsilon}, \ldots, \phi_{k,\epsilon}, \phi_{k+1}, \ldots, \phi_n) \gtrsim \\ &\gtrsim \epsilon^{(n-1)k-(k-1)} \int \cdots \int \frac{\phi_1(\frac{x_1}{\epsilon}) \ldots \phi_k(\frac{x_k}{\epsilon})\phi_{k+1}(x_{k+1}) \ldots \phi_n(x_n)}{|\frac{x_2}{\epsilon} - \frac{x_1}{\epsilon}| \cdots |\frac{x_k}{\epsilon} - \frac{x_1}{\epsilon}||x_{k+1}| \cdots |x_n|} \\ &\qquad\qquad\qquad\qquad\qquad\qquad \cdot \frac{\mathrm{d}x_1}{\epsilon^{n-1}} \cdots \frac{\mathrm{d}x_k}{\epsilon^{n-1}}\, \mathrm{d}x_{k+1} \ldots \mathrm{d}x_n \\ &\gtrsim \epsilon^{(n-1)k-(k-1)} \int \cdots \int \frac{\phi_1(x_1) \ldots \phi_n(x_n)}{|x_2 - x_1| \cdots |x_k - x_1||x_{k+1}| \cdots |x_n|}\, \mathrm{d}x_1 \ldots \mathrm{d}x_n\end{aligned}$$

and

$$\prod_{i=1}^{k} \|\phi_{i,\epsilon}\|_{p_i} = \epsilon^{(n-1)\sum_{i=1}^{k} \frac{1}{p_i}} \prod_{i=1}^{n-1} \|\phi_i\|_{p_i},$$

so we must have

$$\epsilon^{(n-1)k-(k-1)} \lesssim \epsilon^{(n-1)\sum_{i=1}^{k} \frac{1}{p_i}} \quad \text{for } \epsilon < 1,$$

so

$$\frac{(n-2)k+1}{n-1} \geq \sum_{i=1}^{k} \frac{1}{p_i}.$$

In particular, for $k = n-1$, this tells us that

$$\frac{n^2-3n+3}{n-1} \geq \sum_{i=1}^{n-1} \frac{1}{p_i}$$

and this together with (2.4) and renaming of the variables gives us that, for all $i = 1, \ldots, n$,

$$\frac{1}{p_i} \geq \frac{n-2}{n-1}. \tag{2.7}$$

The polyhedron defined by (2.4) and (2.7) has the permutations of the n-tuple $\left(\frac{n-2}{n-1}, \ldots, \frac{n-2}{n-1}, 1\right)$ as vertices so we see that (1.5) can only hold at points in $\mathcal{S}$.

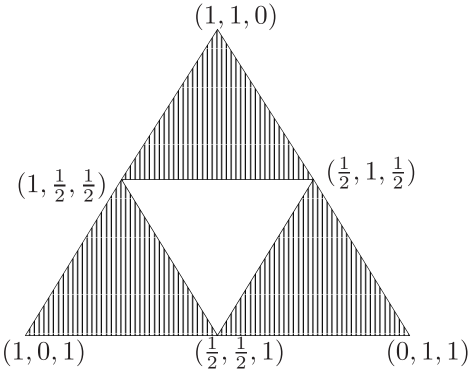


Figure 1: $n = 3$. The estimate (1.5) holds in the open unshaded region and fails in the open shaded region.

Example 2.5. Let us see that we cannot hope to strengthen the estimates on the boundary of $\mathcal{S}$ to strong-type estimates.

We let $\mathcal{C}_i$ be as in the previous example and take ϕ_i to be non-negative functions supported in $\mathcal{C}_i$. Assume that ϕ_i is supported in $|x| < 1$ for $i < n$ and ϕ_n is supported in $|x| > 10$. As before we can estimate by Hadamard's theorem and get

$$\Lambda(\phi_1, \ldots, \phi_n) \gtrsim \int \cdots \int \frac{\phi_1(x_1) \cdots \phi_n(x_n)}{|x_2 - x_1| \cdots |x_{n-1} - x_1||x_n|} \,\mathrm{d}x_1 \cdots \mathrm{d}x_n. \tag{2.8}$$

Let us now assume that ϕ_n has the form $\phi_n(x) = \phi_\omega(\omega)\phi_r(r)$ with $x = r\omega$ in polar coordinates where $\phi_r(r) = (r^{n-2} \log r)^{-1}$ for $10 < r < b$. Then the right hand side of (2.8) contains a factor larger than

$$\int_{10}^{b} \frac{1}{(r^{n-2}\log r)r} r^{n-2} \,\mathrm{d}r = \int_{10}^{b} \frac{1}{\log r} \frac{\mathrm{d}r}{r} = \log b - \log 10.$$

On the other hand we see that

$$\|\phi_n\|_{\frac{n-1}{n-2}} = C\Big(\int_{10}^{b}\Big(\frac{1}{r^{n-2}\log r}\Big)^{\frac{n-1}{n-2}} r^{n-2}\,\mathrm{d}r\Big)^{\frac{n-2}{n-1}} = C\Big(\int_{10}^{b}\Big(\frac{1}{\log r}\Big)^{\frac{n-1}{n-2}}\frac{\mathrm{d}r}{r}\Big)^{\frac{n-2}{n-1}},$$

which is less than a constant independent of b. Since b can be arbitrarily large we get a contradiction unless

$$\frac{1}{p_j} > \frac{n-2}{n-1}.$$

Remark 2.6. It is clear that we can adapt Example 2.2 for $R < 1$ and Example 2.4 to the form Λ_S. This proves the negative part of Theorem 1.4.

Let us see that the positive part of the theorem follows from proving the estimate for points $(\frac{1}{p_1},\ldots,\frac{1}{p_n})$ in the interior of $\mathcal{S}$ (relative to Π). Take $\bar{p} \in \tilde{\mathcal{S}}$. By reordering the indices, we may assume that $\frac{1}{p_1} \geq \cdots \geq \frac{1}{p_n}$. Our aim is to find a point $\bar{q} = (\frac{1}{q_1},\ldots,\frac{1}{q_n})$ in the interior of $\mathcal{S}$ such that $q_i \leq p_i$. The main argument in the proof of Theorem 1.4 applies to the point $\bar{q}$ and the result for $\bar{p}$ follows since the underlying space S^{n-1} is compact.

Take $\bar{q}$ such that there exists a i_0 so that $q_i = p_i$ for $i < i_0$ and $q_i = q_{i_0}$ for $i \geq i_0$. Furthermore we require $p_{i_0} \geq q_{i_0} > p_{i_0-1}$ and $\sum \frac{1}{q_i} = n-1$. Note that either $q_1 = p_1 > 1$, where the strict inequality follows from the defining inequality for $\tilde{\mathcal{S}}$ in the case $k = 1$, $A = \{1\}$, or $q_i = \frac{n-1}{n} > 1$ for all i. Also, the defining inequality gives in the case $k = i_0 - 1$, $A = \{1,\ldots,i_0-1\}$ that

$$\sum_{i=1}^{i_0-1}\frac{1}{q_i} = \sum_{i=1}^{i_0-1}\frac{1}{p_i} < \frac{(n-2)(i_0-1)+1}{n-1}$$

so we get that

$$\frac{1}{q_{i_0}} = \frac{1}{n-i_0+1}\sum_{i=i_0}^{n}\frac{1}{q_i} > \frac{1}{n-i_0+1}\Big(n-1-\frac{(n-2)(i_0-1)+1}{n-1}\Big) = \frac{n-2}{n-1}.$$

This shows that $\bar{q}$ is in the interior of $\mathcal{S}$ as required.

Proof of Theorem 2.1. The proof will be based on Theorem 1.4 and Lemma 2.7 below. First of all we note that

$$\begin{aligned}
|\Lambda(\chi_{E_1},\ldots,\chi_{E_{n-1}},f_n)| &= \Big|\int \frac{\chi_{E_1}(x_1)\cdots\chi_{E_{n-1}}(x_{n-1})f_n(x_n)}{\det\left(\begin{smallmatrix}1 & \cdots & 1\\ x_1 & \cdots & x_n\end{smallmatrix}\right)}\,\mathrm{d}x_1\cdots\mathrm{d}x_n\Big| \\
&\leq \|f_n\|_1 \sup_{x_n}\Big|\int \frac{\chi_{E_1}(x_1)\cdots\chi_{E_{n-1}}(x_{n-1})}{\det\left(\begin{smallmatrix}1 & \cdots & 1\\ x_1 & \cdots & x_n\end{smallmatrix}\right)}\,\mathrm{d}x_1\cdots\mathrm{d}x_{n-1}\Big| \\
&\leq \|f_n\|_1 \sup_{x_n}\Big|\int \frac{\chi_{E_1}(x_1)\cdots\chi_{E_{n-1}}(x_{n-1})}{\det\left(\begin{smallmatrix}1 & \cdots & 1 & 1\\ x_1-x_n & \cdots & x_{n-1}-x_n & 0\end{smallmatrix}\right)}\,\mathrm{d}x_1\cdots\mathrm{d}x_{n-1}\Big| \\
&= \|f_n\|_1 \sup_{x_n}\Big|\int \frac{\chi_{\tilde{E}_1}(x_1)\cdots\chi_{\tilde{E}_{n-1}}(x_{n-1})}{\det(x_1,\ldots,x_{n-1})}\,\mathrm{d}x_1\cdots\mathrm{d}x_{n-1}\Big|
\end{aligned}$$

where $\tilde{E}_i := E_i - x_n$. Since $\|\chi_{\tilde{E}_i}\|_p = \|\chi_{E_i}\|_p$ we will drop these tildes. Let us then define $\tilde{\Lambda}(\chi_{E_1}, \dots \chi_{E_{n-1}})$ to be the quantity inside the modulus signs on the far right hand side of the last chain of inequalities. We change to polar coordinates, $x_i = r_i\omega_i$ with $r_i \in \mathbb{R}_+$ and $\omega_i \in S^{n-2}$. Then $\det(x_1, \dots, x_{n-1}) = r_1 \cdots r_{n-1} \det(\omega_1, \dots, \omega_{n-1})$ and $\mathrm{d}x_i = r_i^{n-2}\,\mathrm{d}r_i\,\mathrm{d}\omega_i$ ($\mathrm{d}\omega_i$ is the unnormalised induced Lebesgue measure on the sphere) so

$$\begin{aligned}\tilde{\Lambda}(\chi_{E_1}, \dots, \chi_{E_{n-1}}) &= \int \frac{\chi_{E_1}(r_1\omega_1)\cdots\chi_{E_{n-1}}(r_{n-1}\omega_{n-1})}{r_1\cdots r_{n-1}\det(\omega_1,\dots,\omega_{n-1})} \\ &\qquad\qquad \cdot (r_1\cdots r_{n-1})^{n-2}\,\mathrm{d}r_1\cdots\mathrm{d}r_{n-1}\,\mathrm{d}\omega_1\cdots\mathrm{d}\omega_{n-1} \\ &= \int \frac{F_{n-1}(\chi_{E_1})(\omega_1)\cdots F_{n-1}(\chi_{E_{n-1}})(\omega_{n-1})}{\det(\omega_1,\dots,\omega_{n-1})}\,\mathrm{d}\omega_1\cdots\mathrm{d}\omega_{n-1} \\ &= \Lambda_S(F_{n-1}(\chi_{E_1}),\dots,F_{n-1}(\chi_{E_{n-1}})) \qquad (2.9)\end{aligned}$$

where $F_{n-1}(f)(\omega) = \int_{\mathbb{R}_+} f(r\omega)r^{n-3}\,\mathrm{d}r$ and in (2.9) we have that Λ_S acts on functions on S^{n-2}. Thus we have separated $\tilde{\Lambda}$ into a radial part, F_{n-1}, and an angular part. By Theorem 1.4 we can estimate (2.9) by a constant multiple of

$$\|F_{n-1}(\chi_{E_1})\|_{L^{\frac{n-1}{n-2}}(S^{n-2})} \cdots \|F_{n-1}(\chi_{E_{n-1}})\|_{L^{\frac{n-1}{n-2}}(S^{n-2})}$$

so Theorem 1.2 will follow from the following lemma. □

Lemma 2.7.

$$\|F_{n-1}(\chi_E)\|_{L^{\frac{n-1}{n-2}}(S^{n-2})} \lesssim \|\chi_E\|_{L^{\frac{n-1}{n-2}}(\mathbb{R}^{n-1})}. \qquad (2.10)$$

Remark 2.8. We note that the estimate in this lemma does not hold for general functions as can be seen by testing on the function $f(r\omega) = (r^{n-2}\log r)^{-1}$ similarly to Example 2.5.

Proof of Lemma 2.7. If $n = 3$ we want to prove that

$$\left\| \int_{\mathbb{R}_+} \chi_E(r\omega)\,\mathrm{d}r \right\|_{L^2(S^1)} \lesssim \|\chi_E\|_{L^2(\mathbb{R}^2)}$$

which is equivalent to

$$\int_{S^1} \left| \int_{\mathbb{R}_+} \chi_E(r\omega)\,\mathrm{d}r \right|^2 \mathrm{d}\omega \lesssim \int_{S^1}\int_{\mathbb{R}_+} |\chi_E(r\omega)|^2 r\,\mathrm{d}r\,\mathrm{d}\omega.$$

Define $E_\omega := \{r \in \mathbb{R}_+ :\ r\omega \in E\}$. We see that it is enough to prove that $|E_\omega|^2 \lesssim \int_{E_\omega} r\,\mathrm{d}r$ holds for each $\omega \in S^1$. The left hand side in this inequality depends only on the measure of E_ω and the infimum of the right hand side, for sets of fixed measure, is clearly attained when $E_\omega = [0, |E_\omega|]$. In this case $\int_{E_\omega} r\,\mathrm{d}r = \frac{1}{2}|E_\omega|^2$ so $|E_\omega|^2 \le 2\int_{E_\omega} r\,\mathrm{d}r$.

More generally, the same reasoning shows that $|E_\omega|^m \lesssim \int_{E_\omega} r^{m-1}\,\mathrm{d}r$. It follows that

$$\begin{aligned}\int_{E_\omega} r^{n-3}\,\mathrm{d}r &\le \Big(\int_{E_\omega} (r^{n-3})^{\frac{n-2}{n-3}}\,\mathrm{d}r\Big)^{\frac{n-3}{n-2}} \Big(\int_{E_\omega} \mathrm{d}r\Big)^{1-\frac{n-3}{n-2}} \qquad \text{(by Hölder)}\\ &= \Big(\int_{E_\omega} r^{n-2}\,\mathrm{d}r\Big)^{\frac{n-3}{n-2}} \Big(\int_{E_\omega} \mathrm{d}r\Big)^{\frac{1}{n-2}}\\ &\lesssim \Big(\int_{E_\omega} r^{n-2}\,\mathrm{d}r\Big)^{\frac{n-3}{n-2}} \Big(\int_{E_\omega} r^{n-2}\,\mathrm{d}r\Big)^{\frac{1}{(n-2)(n-1)}}\\ &= \Big(\int_{E_\omega} r^{n-2}\,\mathrm{d}r\Big)^{\frac{n-2}{n-1}}\end{aligned}$$

which is to say that

$$\Big(\int_{E_\omega} r^{n-3}\,\mathrm{d}r\Big)^{\frac{n-1}{n-2}} \lesssim \int_{E_\omega} r^{n-2}\,\mathrm{d}r.$$

Then we see that

$$\int_{S^{n-2}} \Big|\int_{\mathbb{R}_+} \chi_E(r\omega) r^{n-3}\,\mathrm{d}r\Big|^{\frac{n-1}{n-2}} \mathrm{d}\omega \lesssim \int_{S^{n-2}}\int_{\mathbb{R}_+} \chi_E(r\omega) r^{n-2}\,\mathrm{d}r\,\mathrm{d}\omega$$

so

$$\|F_{n-1}(\chi_E)\|_{L^{\frac{n-1}{n-2}}(S^{n-2})} \lesssim \|\chi_E\|_{L^{\frac{n-1}{n-2}}(\mathbb{R}^{n-1})}.$$

This completes the proof of the lemma. □

Proof of Theorem 1.4. For $n = 2$ we see that

$$\Lambda_S(f_1, f_2) = \int_{S^1}\int_{S^1} \frac{f_1(\omega_1) f_2(\omega_2)}{\sin(\omega_1 - \omega_2)} \lesssim \|f_1\|_{L^{p_1}(S^1)} \|f_2\|_{L^{p_2}(S^1)}$$

provided that $p_1, p_2 > 1$ and $\frac{1}{p_1} + \frac{1}{p_2} = 1$ since $(\sin(\omega_1 - \omega_2))^{-1} = \frac12 \tan\frac12(\omega_1 - \omega_2)$ $+\frac12 \cot\frac12(\omega_1 - \omega_2)$ so the left hand side is the sum of two Hilbert transforms and the result is known.

So that we have a clearer relation with the proof of Theorem 2.1 we shall now change our indexing and in effect increase n by one. We will proceed by using induction and will assume that we have some $n \ge 4$ and that we have proved Corollary 1.6 on S^{n-3}, that is

$$|\Lambda_S(f_1, \dots, f_{n-2})| \lesssim \|f_1\|_{L^{\frac{n-2}{n-3}}(S^{n-3})} \cdots \|f_{n-2}\|_{L^{\frac{n-2}{n-3}}(S^{n-3})} \tag{2.11}$$

and we are interested in proving

$$|\Lambda_S(f_1, \dots, f_{n-1})| \lesssim \|f_1\|_{L^{p_1}(S^{n-2})} \cdots \|f_{n-1}\|_{L^{p_{n-1}}(S^{n-2})} \tag{2.12}$$

with $\big(\frac{1}{p_1}, \dots, \frac{1}{p_{n-1}}\big)$ in the interior of $\mathcal{S}$ (with n replaced by $n-1$). Again by multilinear interpolation, it is enough to prove the estimate for characteristic functions at the vertex ($n-1$-tuple) $\big(\frac{n-3}{n-2}, \dots, \frac{n-3}{n-2}, 1\big)$.

We proceed in the following manner. By definition, $\Lambda_S(f_1,\ldots,f_{n-1})$ equals

$$\frac{1}{2}\int \frac{(f_1(\omega_1)-f_1(\omega_1^*))f_2(\omega_2)\cdots f_{n-1}(\omega_{n-1})}{\det(\omega_1,\ldots,\omega_{n-1})}\,\mathrm{d}\omega_1\cdots\mathrm{d}\omega_{n-1}\,, \tag{2.13}$$

where ω_1^* is the reflection of ω_1 in the great hypercircle containing ω_2 up to ω_{n-1}. We bound this by

$$\|f_{n-1}\|_{L^1(S^{n-2})}\sup_{\omega_{n-1}}\left|\int \frac{(f_1(\omega_1)-f_1(\omega_1^*))f_2(\omega_2)\cdots f_{n-2}(\omega_{n-2})}{2\det(\omega_1,\ldots,\omega_{n-1})}\,\mathrm{d}\omega_1\cdots\mathrm{d}\omega_{n-2}\right|.$$

We thus want to show that

$$\begin{aligned}\sup_{\omega_{n-1}}\left|\frac{1}{2}\int \frac{(f_1(\omega_1)-f_1(\omega_1^*))f_2(\omega_2)\cdots f_{n-2}(\omega_{n-2})}{\det(\omega_1,\ldots,\omega_{n-1})}\,\mathrm{d}\omega_1\cdots\mathrm{d}\omega_{n-2}\right| &\lesssim \\ \lesssim \|f_1\|_{L^{\frac{n-2}{n-3}}(S^{n-2})}\cdots\|f_{n-2}\|_{L^{\frac{n-2}{n-3}}(S^{n-2})}&\end{aligned} \tag{2.14}$$

holds for all f_j being characteristic functions.

By rotational invariance, we can take ω_{n-1} to be the north pole $\binom{1}{0}$. We then split the integral in each of the variables $\omega_1\ldots\omega_{n-2}$ into two integrals, one over each hemisphere.

Because

$$\det(\omega_1,\ldots,-\omega_i,\ldots,\omega_{n-1}) = -\det(\omega_1,\ldots,\omega_i,\ldots,\omega_{n-1})$$

it is enough to consider the integral over the northern hemispheres

$$S_+^{n-2} := \{\omega_0=(\omega_{01},\ldots,\omega_{0,n-1})\in S^{n-2} : \omega_{01}>0\}.$$

Since ω_1^* is the reflection of ω_1 in a great hypercircle containing the north pole we see that ω_1 and ω_1^* will always lie in the same hemisphere. To work with the integral over S_+^{n-2} we change variables from S_+^{n-2} to $\{1\}\times\mathbb{R}^{n-2}$. Specifically, we write $\omega_0\in S_+^{n-2}$ as $(\cos\theta_0,\widetilde{\omega}_0\sin\theta_0)$ where $0\le\theta_0<\frac{\pi}{2}$ and $\widetilde{\omega}_0\in S^{n-3}$. Define

$$\psi(\omega_0) := \frac{1}{\cos\theta_0}(\cos\theta_0,\widetilde{\omega}_0\sin\theta_0) = (1,\widetilde{\omega}_0\tan\theta_0).$$

Since $\widetilde{\omega}_0\in S^{n-3}$, the expression $\widetilde{\omega}_0\sin\theta_0$ for a fixed $\widetilde{\omega}_0$ parametrises an $(n-3)$ dimensional sphere of radius $\sin\theta_0$ and the expression $\widetilde{\omega}_0\tan\theta_0$ parametrises a similar sphere of radius $\tan\theta_0$. This contributes a factor

$$\left(\frac{\sin\theta_0}{\tan\theta_0}\right)^{n-3} = \cos^{n-3}\theta_0$$

to $(J\psi^{-1})(\psi(\omega_0))$. Also,

$$\frac{\partial\psi}{\partial\theta_0}(\omega_0) = \frac{\partial}{\partial\theta_0}\tan\theta_0 = \frac{1}{\cos^2\theta_0}\,,$$

so $(J\psi^{-1})(\psi(\omega_0)) = \cos^{n-1}\theta_0$. The integral (2.14) thus becomes

$$\int_{(\{1\}\times\mathbb{R}^{n-2})^{n-2}} \frac{f_1(\psi^{-1}(\widetilde{x}_1))\dots f_{n-2}(\psi^{-1}(\widetilde{x}_{n-2}))}{\det\left(\frac{\widetilde{x}_1}{|\widetilde{x}_1|},\dots,\frac{\widetilde{x}_{n-2}}{|\widetilde{x}_{n-2}|}\,\begin{smallmatrix}1\\0\end{smallmatrix}\right)}\left(\prod_{i=1}^{n-2}\cos\theta_{i0}\right)^{n-1}\mathrm{d}\widetilde{x}_1\cdots\mathrm{d}\widetilde{x}_{n-2}$$

and we can pull $|\widetilde{x}_i| = \sqrt{1+\tan^2\theta_{i0}} = (\cos\theta_{i0})^{-1}$ out of the determinant.

Let $\widetilde{x}_i = \binom{1}{y_i}$ and $\psi^{-1}(\widetilde{x}_i) = \widetilde{\psi}^{-1}(y_i)$. Then since

$$\cos\theta_{i0} = \frac{1}{|\widetilde{x}_i|} = \frac{1}{(1+|y_i|^2)^{\frac{1}{2}}}$$

we see that the integral becomes

$$\int_{(\mathbb{R}^{n-2})^{n-2}} \frac{f_1(\widetilde{\psi}^{-1}(y_1))\dots f_{n-2}(\widetilde{\psi}^{-1}(y_{n-2}))}{\det(y_1,\dots,y_{n-2})}\prod_{i=1}^{n-2}\frac{1}{(1+|y_i|^2)^{\frac{n-2}{2}}}\,\mathrm{d}y_1\cdots\mathrm{d}y_{n-2}$$

$$= \int_{(S^{n-3})^{n-2}}\int_{(\mathbb{R}_+)^{n-2}} \frac{f_1(\widetilde{\psi}^{-1}(\widetilde{r}_1\widetilde{\omega}_1))\dots f_{n-2}(\widetilde{\psi}^{-1}(\widetilde{r}_{n-2}\widetilde{\omega}_{n-2}))}{\widetilde{r}_1\dots\widetilde{r}_{n-2}\det(\widetilde{\omega}_1,\dots,\widetilde{\omega}_{n-2})}$$

$$\cdot\prod_{i=1}^{n-2}\frac{1}{(1+\widetilde{r}_i^2)^{\frac{n-2}{2}}}(\widetilde{r}_1\dots\widetilde{r}_{n-2})^{n-3}\,\mathrm{d}\widetilde{r}_1\cdots\mathrm{d}\widetilde{r}_{n-2}\,\mathrm{d}\widetilde{\omega}_1\cdots\mathrm{d}\widetilde{\omega}_{n-2}\,,$$

where we have changed to polar coordinates again. By the induction hypothesis we can estimate the angular part of this by

$$\prod_{i=1}^{n-2}\left\|\int\frac{f_i(\widetilde{\psi}^{-1}(r\cdot))}{(1+r)^{\frac{n-2}{2}}}r^{n-4}\,\mathrm{d}r\right\|_{L^{\frac{n-2}{n-3}}(S^{n-3})}.$$

We want to bound this by

$$\prod_{i=1}^{n-2}\|f_i\|_{L^{\frac{n-2}{n-3}}(S^{n-2})}$$

for characteristic functions f_i.

Similarly to the proof of Lemma 2.7 this boils down to proving

$$\left(\int_E\frac{r^{n-4}}{(1+r^2)^{\frac{n-2}{2}}}\,\mathrm{d}r\right)^{\frac{n-2}{n-3}} \lesssim \int_E\frac{r^{n-3}}{(1+r^2)^{\frac{n-1}{2}}}\,\mathrm{d}r$$

for all measurable $E\subseteq\mathbb{R}_+$.

To prove this we note first the following:

$$\left(\int_E\frac{1}{1+r^2}\,\mathrm{d}r\right)^{m+1} \lesssim \int_E\frac{r^m}{(1+r^2)^{\frac{m+2}{2}}}\,\mathrm{d}r.$$

To see this let $r=\tan\alpha$, then $\frac{\mathrm{d}r}{1+r^2} = \mathrm{d}\alpha$ and $(1+r^2)^{-1/2} = (1+\tan^2\alpha)^{-1/2} = \cos\alpha$, so what we want to prove is

$$\left(\int_{\widetilde{E}}\mathrm{d}\alpha\right)^{m+1} \lesssim \int_{\widetilde{E}}\tan^m\alpha\,\cos^m\alpha\,\mathrm{d}\alpha = \int_{\widetilde{E}}\sin^m\alpha\,\mathrm{d}\alpha.$$

In fact, we only have to prove this for $\widetilde{E} \subseteq (0, c)$ where $c > 0$ is small. In that case we can substitute the first term in its Taylor series for $\sin^m \alpha$ and then the result follows from the proof of Lemma 2.7. Now this already proves the result for $n = 4$ (take $m = 1$).

For $n > 4$ we calculate using Hölder's inequality:

$$\int_{\widetilde{E}} \frac{r^{n-4}}{(1+r^2)^{\frac{n-2}{2}}} \,\mathrm{d}r \leq \left(\int_{\widetilde{E}} \frac{r^{n-3}}{(1+r^2)^{\frac{n-1}{2}}} \,\mathrm{d}r \right)^{\frac{n-4}{n-3}} \left(\int_{\widetilde{E}} \frac{1}{1+r^2} \,\mathrm{d}r \right)^{\frac{1}{n-3}}$$
$$\lesssim \left(\int_{\widetilde{E}} \frac{r^{n-3}}{(1+r^2)^{\frac{n-1}{2}}} \,\mathrm{d}r \right)^{\frac{n-4}{n-3}} \left(\int_{\widetilde{E}} \frac{r^{n-4}}{(1+r^2)^{\frac{n-2}{2}}} \,\mathrm{d}r \right)^{\frac{1}{(n-3)^2}}$$

and the result follows.

Now Theorem 1.4 follows for all $n \geq 3$ by induction. □

Proof of Theorem 1.7. The proof follows the induction step in the proof of Theorem 1.4 closely. We will only indicate the differences.

We may immediately reduce ourselves to the case where the functions f_i are defined on $\mathbb{H}^{n-2}_+$, the upper sheet of the hyperboloid.

Next, inequality (2.14) contains a supremum over $q_{n-1} \in \mathbb{H}^{n-2}_+$. Instead of rotational invariance, we use invariance under the action of $O(1, n-2)$ to show that we may take q_{n-1} as $\binom{1}{0}$. Here it is important to note that if q_{n-1} is in the upper sheet and $A \in O(1, n-2)$ takes q_{n-1} to $\binom{1}{0}$ then $\det(A) = 1$ so

$$\det(Aq_1, \dots, Aq_n) = \det(A) \det(q_1, \dots, q_n) = \det(q_1, \dots, q_n).$$

For the definition of the map ψ we note that $q_0 \in \mathbb{H}^{n-2}_+$ can be written as $(\cosh\theta_0, \widetilde{\omega}_0 \sinh\theta_0)$ where $0 \leq \theta_0$ and $\widetilde{\omega}_0 \in S^{n-3}$. Then we define

$$\psi(q_0) := \frac{1}{\cosh\theta_0} (\cosh\theta_0, \widetilde{\omega}_0 \sinh\theta_0) = (1, \widetilde{\omega}_0 \tanh\theta_0).$$

With the trigonometric functions replaced by the corresponding hyperbolic functions the ensuing calculations go through. We note that

$$\cosh\theta_{i0} = \frac{1}{(1-|y_i|^2)^{\frac{1}{2}}}$$

and see that eventually we wish to prove the estimate

$$\prod_{i=1}^{n-2} \left\| \int_0^1 \frac{f_i(\widetilde{\psi}^{-1}(r\cdot))}{(1-r^2)^{\frac{n-2}{2}}} r^{n-4} \,\mathrm{d}r \right\|_{L^{\frac{n-2}{n-3}}(S^{n-3})} \lesssim \prod_{i=1}^{n-2} \|f_i\|_{L^{\frac{n-2}{n-3}}(\mathbb{H}^{n-2})}$$

for characteristic functions f_i. We note that in the integration on the left hand side, the variable r arose as $|y_i| = \tanh\theta_0 < 1$ and this gives the limit of integration. As before this reduces to proving

$$\left(\int_E \frac{r^{n-4}}{(1-r^2)^{\frac{n-2}{2}}} \,\mathrm{d}r \right)^{\frac{n-2}{n-3}} \lesssim \int_E \frac{r^{n-3}}{(1-r^2)^{\frac{n-1}{2}}} \,\mathrm{d}r$$

for all measurable $E \subseteq [0, 1-\epsilon]$ which follows the same lines as before and the bound does not depend on ϵ.

With this we have reduced integration over $\mathbb{H}^{n-2}$ to an integration over S^{n-3}, so now the statement of Theorem 1.4 applies. □

Finally, let us return to the question how the forms are defined and prove Lemma 1.1.

Proof of Lemma 1.1. To begin we take $n = 3$, the case $n = 2$ which is the Hilbert transform is of course well known. We thus want to show that

$$\frac{1}{2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\left|\frac{(f_1(x_1)-f_1(x_1^*))f_2(x_2)f_3(x_3)}{\det\left(\begin{smallmatrix}1 & 1 & 1\\ x_1 & x_2 & x_3\end{smallmatrix}\right)}\right| \mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3$$

is bounded. We can write this as

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\frac{|f_1(x_1)-f_1(x_1^*)|}{|x_1-x_1^*|}\,\mathrm{d}x_1\left|\frac{f_2(x_2)f_3(x_3)}{D(x_2,x_3)}\right|\mathrm{d}x_2\,\mathrm{d}x_3\,,$$

where $D(x_2, x_3)$ is the distance between x_2 and x_3. We see that the x_1 integral is bounded as if x_1^* is close to x_1 we can estimate the integrand by $f_1'(x_1)$ and otherwise we can estimate it by a multiple of $|f_1(x_1)| + |f_1(x_1^*)|$.

For the other integrals we see that it is enough to show that

$$\int_{B_R(0)}\int_{B_R(0)}\frac{1}{|D(x_2,x_3)|}\,\mathrm{d}x_2\,\mathrm{d}x_3$$

is bounded where $B_R(0)$ denotes the ball of radius R around the origin. By letting $x_3 = x_2 + y$ we can estimate this by

$$C\int_{B_{2R}(0)}\frac{\mathrm{d}y}{|y|}$$

and by changing to polar coordinates $y = r\theta$ we can estimate this by

$$C\int_{r\le 2R}\frac{r\,\mathrm{d}r}{r}\,,$$

which is clearly bounded.

For the general case we proceed in the same way and we reduce our problem to showing that

$$\int_{B_R(0)}\cdots\int_{B_R(0)}\frac{1}{|D(x_2,\dots,x_n)|}\,\mathrm{d}x_2\dots\mathrm{d}x_n \tag{2.15}$$

is bounded, where $B_R(0)$ is a ball in $\mathbb{R}^{n-1}$ and $D(x_2, \dots, x_n)$ is the $n-2$ dimensional volume of the simplex whose vertices are $x_2, \dots, x_n$ in the hyperplane of $\mathbb{R}^{n-1}$ in which these points lie. As in our main argument, the boundedness of this can be shown by changing variables to separate out the contribution from x_2, changing to polar coordinates in the other variables, bounding the radial part directly and finally changing variables in the angular part to reduce to (2.15) again but with one less variable. The same argument works for Λ_S and thus we have shown the first part of the lemma.

For the second part we wish to show that

$$\int \frac{(f_1(x_1)f_2(x_2^*) - f_1(x_1^*)f_2(x_2))f_3(x_3)\cdots f_n(x_n)}{\det\left(\begin{smallmatrix} 1 & 1 & \dots & 1 \\ x_1 & x_2 & \dots & x_n \end{smallmatrix}\right)}\,\mathrm{d}x_1\cdots\mathrm{d}x_n = 0. \tag{2.16}$$

We note that almost every tuple $(x_3, \dots, x_n)$ lies in a uniquely determined affine plane in $\mathbb{R}^{n-1}$ of codimension 2 and we can write $x_1 = x_{10} + r(\cos(\theta)e_1 + \sin(\theta)e_2)$ and $x_2 = x_{20} + s(\cos(\phi)e_1 + \sin(\phi)e_2)$ where x_{10}, x_{20} lie in this plane and e_1 and e_2 are orthogonal unit vectors orthogonal to the plane. With these definitions we get that

$$\det\begin{pmatrix} 1 & 1 & \dots & 1 \\ x_1 & x_2 & \dots & x_n \end{pmatrix} = D(x_3, \dots, x_n) rs\sin(\theta - \phi)$$

where now $D(x_3, \dots, x_n)$ denotes the $n-3$ dimensional volume of the simplex whose vertices are $x_3, \dots, x_n$. With this we can write the integral in (2.16) as

$$\int \frac{f(x_3)\cdots f(x_n)}{D(x_3,\dots,x_n)}\left(\int \frac{(f_1(x_1)f_2(x_2^*) - f_1(x_1^*)f_2(x_2))}{rs\sin(\theta-\phi)}\,\mathrm{d}x_1\,\mathrm{d}x_2\right)\mathrm{d}x_3\cdots\mathrm{d}x_n. \tag{2.17}$$

As above we can justify that the quantity outside of the inner integral is integrable. Let us therefore study the inner integral more carefully. We define $A_\epsilon = \{(x_1, x_2) : |\sin(\theta - \phi)| > \epsilon\}$. This definition depends on the variables $x_3, \dots, x_n$ but we shall suppress that. Note that $\lim_{\epsilon\to 0} A_\epsilon = (\mathbb{R}^{n-1})^2$ almost everywhere.

Let us study the inner integral in (2.17) restricted to the set A_ϵ. First of all note that

$$\int_{A_\epsilon}\left|\frac{f_1(x_1)f_2(x_2^*)}{rs\sin(\theta-\phi)}\right|\mathrm{d}x_1\,\mathrm{d}x_2 \le C\int_{\{r<R\}\cap\{s<R\}\cap A_\epsilon}\left|\frac{1}{rs\sin(\theta-\phi)}\right| rs\,\mathrm{d}r\,\mathrm{d}s\,\mathrm{d}\theta\,\mathrm{d}\phi\,,$$

where we have carried out the x_{10} and x_{20} integrations and used the assumption that f_1 and f_2 are compactly supported. We note that the last integral is clearly bounded although the bound depends on ϵ.

For the whole inner integral restricted to A_ϵ we are therefore justified in calculating

$$\begin{aligned}\int_{A_\epsilon} &\frac{(f_1(x_1)f_2(x_2^*) - f_1(x_1^*)f_2(x_2))}{rs\sin(\theta-\phi)}\,\mathrm{d}x_1\,\mathrm{d}x_2 = \\ &= \int_{A_\epsilon}\frac{f_1(x_1)f_2(x_2^*)}{rs\sin(\theta-\phi)}\,\mathrm{d}x_1\,\mathrm{d}x_2 - \int_{A_\epsilon}\frac{f_1(x_1^*)f_2(x_2)}{rs\sin(\theta-\phi)}\,\mathrm{d}x_1\,\mathrm{d}x_2.\end{aligned}$$

A change of variables $x_2 \mapsto x_2^*$ in the first integral and $x_1 \mapsto x_1^*$ in the second yields

$$\int_{A_\epsilon}\frac{f_1(x_1)f_2(x_2)}{-rs\sin(\theta-\phi)}\,\mathrm{d}x_1\,\mathrm{d}x_2 - \int_{A_\epsilon}\frac{f_1(x_1)f_2(x_2)}{-rs\sin(\theta-\phi)}\,\mathrm{d}x_1\,\mathrm{d}x_2 = 0\,.$$

Since the integral in (2.16) is absolutely integrable we get by letting ϵ pass to 0 and an application of the dominated convergence theorem that (2.17) holds. This completes the proof of the second part of the lemma and the third part is proved similarly. □

3. The fractional integral

Proof of Theorem 1.8. Let us introduce the Steiner symmetrisation of a function. For $E \subseteq \mathbb{R}^n$ of finite Lebesgue measure we define the symmetric rearrangement of E as the open ball centred at the origin that has the same measure as E. We denote this by E^*. We then define the Steiner symmetrisation, $\mathcal{R}_j f = f^{*j}$, of a function f with respect to the j-th coordinate direction as

$$f^{*j}(x_1, \ldots, x_n) = \int_0^\infty \chi_{\{|f(x_1,\ldots,x_{j-1},\cdot,x_{j+1},\ldots,x_n)|>t\}^*}(x_j)\,\mathrm{d}t.$$

We can see that f^{*j} is a non-negative measurable function which decreases as the absolute value of the j-th coordinate increases. Also, f and f^* have the same distribution functions and therefore $\|f\|_p = \|f^*\|_p$ for all $1 \leq p \leq \infty$. Finally, we can see that the map $f \mapsto f^*$ is order preserving, in the sense that if f and g are two non-negative functions and $f(x) \leq g(x)$ for all x then also $f^*(x) \leq g^*(x)$ for all x.

We would now like to estimate $\Lambda_\alpha(f_1, \ldots, f_n)$ by $\Lambda_\alpha(f_1^*, \ldots, f_n^*)$. Since

$$\det \begin{pmatrix} 1 & \ldots & 1 \\ x_1 & \ldots & x_n \end{pmatrix}$$

is not a linear combination of the x_i's we cannot apply the rearrangement inequality of Brascamp, Lieb and Luttinger [3] directly. There exists a generalisation of it by Christ [7] which is applicable. However, in order to find all of the optimisers we need to study the cases of equality in the inequality and the argument of Brascamp, Lieb and Luttinger and the extension of Christ do not seem suitable for that study. We shall proceed more directly in order to be able to use the results of Burchard [4], see also [5].

Let us split each of the n integrals over $\mathbb{R}^{n-1}$ into integrals over $\mathbb{R}^{n-2} \times \mathbb{R}$ by separating out the integration in the j-th coordinate. Write $x_i \in \mathbb{R}^{n-1}$ as $(x_{i\hat{\jmath}}, x_{ij})$ where x_{ij} is the j-th coordinate of x_i. Then we can write $\Lambda_\alpha(f_1, \ldots, f_n)$ as

$$\int_{(\mathbb{R}^{n-2})^n} \left(\int_{\mathbb{R}^n} \frac{f_1(x_{1\hat{\jmath}}, x_{1j}) \ldots f_n(x_{n\hat{\jmath}}, x_{nj})}{\left|\det \left(\begin{smallmatrix} 1 & \cdots & 1 \\ x_1 & \cdots & x_n \end{smallmatrix}\right)\right|^\alpha} \,\mathrm{d}x_{1j} \ldots \mathrm{d}x_{nj} \right) \mathrm{d}x_{1\hat{\jmath}} \ldots \mathrm{d}x_{n\hat{\jmath}}. \tag{3.1}$$

We can work with the term in parentheses with the additional assumption that the $x_{i\hat{\jmath}}$'s are fixed for all i's and then

$$\det \begin{pmatrix} 1 & \ldots & 1 \\ x_1 & \ldots & x_n \end{pmatrix}$$

is a linear combination of $x_{1j}, \ldots, x_{nj}$.

We now recall that if we define

$$I(f_1,\dots,f_{m+1})=\int\cdots\int f_1(x_1)\dots f_m(x_m)f_{m+1}\Big(\sum_{j=1}^m b_jx_j\Big)\,\mathrm{d}x_1\dots\mathrm{d}x_m, \tag{3.2}$$

then

$$I(f_1,\dots,f_{m+1})\le I(f_1^*,\dots,f_{m+1}^*). \tag{3.3}$$

This is in fact a special case of the inequality of Brascamp, Lieb and Luttinger. However, for this inequality Burchard, [4], has determined the cases of equality as follows.

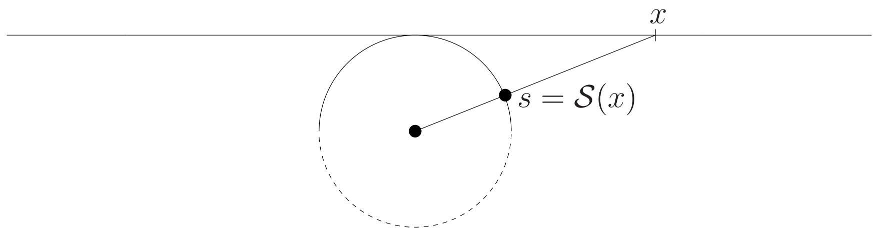
Lemma 3.1. *Assume that $f_1,\dots,f_{m+1}$ are non-negative functions on $\mathbb{R}^n$, f_{m+1} is symmetric decreasing and we have equality in* (3.2)*. Then there are vectors $a_1,\dots,a_m\in\mathbb{R}^n$ such that $\sum b_ia_i=0$ and $f_i(x_i)=f_i^*(x_i-a_i)$ for all $i=1,\dots,m$.*

Burchard states her result with each $b_j=1$ but by making the change of variables $x_j\mapsto b_jx_j$ the theorem reduces to that case.

We take the functions to be $f_1(x_{1\hat{\jmath}},\cdot),\dots,f_n(x_{n\hat{\jmath}},\cdot)$, and $|\cdot|^{-\alpha}$. Now, $|\cdot|^{-\alpha}$ is a symmetric decreasing function so $(|\cdot|^{-\alpha})^*=|\cdot|^{-\alpha}$ and $f_i(x_{i\hat{\jmath}},\cdot)^*=f_i^{*j}(x_{i\hat{\jmath}},\cdot)$ where, as before, f^{*j} denotes the Steiner symmetrisation of f with respect to the j-th coordinate direction. Inequality (3.3) then tells us that

$$\Lambda_\alpha(f_1,\dots,f_n)\le\Lambda_\alpha(f_1^{*j},\dots,f_n^{*j}) \tag{3.4}$$

for any $1\le j\le n-1$.



Let $\mathcal{S}:\mathbb{R}^{n-1}\to S_+^{n-1}$ be the stereographic projection from $\mathbb{R}^{n-1}$ to the northern hemisphere S_+^{n-1}. To a function f on $\mathbb{R}^{n-1}$ we associate a function F on S_+^{n-1} defined by

$$F(s)=|J_{\mathcal{S}^{-1}}(s)|^{\frac{1}{p}}f(\mathcal{S}^{-1}(s)) \tag{3.5}$$

where $J_{\mathcal{S}^{-1}}$ is the Jacobian determinant of the map $\mathcal{S}^{-1}$. Then $\|f\|_p=\|F\|_p$ and it is easily seen that

$$\int_{(\mathbb{R}^{n-1})^n}\frac{f_1(x_1)\dots f_n(x_n)}{\left|\det\begin{pmatrix}1&\cdots&1\\x_1&\cdots&x_n\end{pmatrix}\right|^\alpha}\,\mathrm{d}x_1\dots\mathrm{d}x_n=\int_{(S_+^{n-1})^n}\frac{F_1(s_1)\dots F_n(s_n)}{|\det(s_1\dots s_n)|^\alpha}\,\mathrm{d}s_1\dots\mathrm{d}s_n. \tag{3.6}$$

Here the relationship $1/p=1-\alpha/n$ is key.

We can rotate the hemisphere, by rotating the whole sphere and sending points that are rotated to the southern hemisphere to their antipodal points that lie in the northern hemisphere. The rotated functions give the same value for the integral but correspond to new functions on $\mathbb{R}^{n-1}$. We will use $U^j_\gamma f$ to denote the function we get by rotating F by the rotation that leaves all basis vectors except the j-th and the n-th ones fixed and rotates the plane spanned by those two by γ. We will require that γ is not a rational multiple of π. We note that $f \mapsto U^j_\gamma f$ is order preserving.

For a function f we define a sequence $(f^m)_{m\geq 0}$ in the following way:

$$f^0 = f, \quad f^1 = \mathcal{R}_{n-1}\ldots\mathcal{R}_1 U^1_\gamma f^0, \quad f^2 = \mathcal{R}_1\mathcal{R}_{n-1}\ldots\mathcal{R}_2 U^2_\gamma f^1, \ldots$$
$$\ldots, f^{n-1} = \mathcal{R}_{n-2}\ldots\mathcal{R}_1\mathcal{R}_{n-1} U^{n-1}_\gamma f^{n-2}, \quad f^n = \mathcal{R}_{n-1}\ldots\mathcal{R}_1 U^1_\gamma f^{n-1}\ldots$$

We want to find the L^p limit of this sequence. First, let us assume that f is a bounded function which vanishes outside a bounded set. These functions are clearly dense in L^p. With this assumption we can find a constant C such that

$$f(x) \leq C k_f(x) \tag{3.7}$$

where $k_f(x)$ is a multiple of $k(x)$ from (1.20) scaled such that $\|f\|_p = \|k_f\|_p$. We notice that $k_f(x)$ is a symmetric decreasing function which corresponds to a constant function K on on S^{n-1}_+. It is thus unaffected by $\mathcal{R}_j$ and U^j_γ. Since $f(x) > 0$ and both $\mathcal{R}_j$ and U^j_γ preserve orderings of non-negative functions we have that

$$f^m(x) \leq C k^m_f(x) = C k_f(x) \tag{3.8}$$

for all x and m so the whole sequence (f^m) is dominated by an L^p function. Since

$$\|k_f - U^j_\gamma f\|_p = \|U^j_\gamma k_f - U^j_\gamma f\|_p = \|k_f - f\|_p \tag{3.9}$$

and

$$\|k_f - \mathcal{R}_j f\|_p = \|\mathcal{R}_j k_f - \mathcal{R}_j f\|_p \leq \|k_f - f\|_p \tag{3.10}$$

(since rearrangements are contractive in L^p space) we have that

$$\lim_{m\to\infty} \|k_f - f^m\|_p \tag{3.11}$$

exists and is equal to

$$\inf_m \|k_f - f^m\|_p. \tag{3.12}$$

We call this number A. It is finite since $\|k_f - f\|_p \leq \|k_f\|_p + \|f\|_p < \infty$.

We make the following definition:

Definition 3.2. Let f be a non-negative function. We say that f has the *outward decreasing property* if for all $x, y \in \mathbb{R}^{n-1}$ such that $|x_i| \leq |y_i|$ for all $1 \leq i \leq n$ then $f(x) \geq f(y)$.

Note that a function which has the outward decreasing property is invariant under a reflection along any coordinate hyperplane.

Lemma 3.3. *The functions f^m have the outward decreasing property for $m \geq 1$.*

Proof. It is clear that it is enough to show that $g = \mathcal{R}_1\mathcal{R}_2 f$ has the property that if $0 \leq x_1 \leq y_1$, $0 \leq x_2 \leq y_2$ and $x_i = y_i \geq 0$ for $i \geq 3$ then $g(x) \geq g(y)$.

Furthermore, since $g = \mathcal{R}_1(\mathcal{R}_2 f)$ it is clear that if we also assume that $x_1 = y_1$ since increasing the value of the first variable while keeping the others fixed will not increase the value of g since g is the image of a Steiner rearrangement in the first variable. So it is enough to study the case $x_1 = y_1$, $x_i = y_i$ for $i \geq 3$ and $x_2 \leq y_2$. Obviously, in this case,

$$\mathcal{R}_2 f(x) \geq \mathcal{R}_2 f(y). \tag{3.13}$$

Now set $\lambda := g(y)$. Then

$$\big|\{t : g(t, y_2, \ldots, y_{n-1}) \geq \lambda\}\big| = 2y_1\,,$$

so

$$\big|\{t : \mathcal{R}_2 f(t, y_2, \ldots, y_{n-1}) \geq \lambda\}\big| = 2y_1.$$

Since $x_2 \leq y_2$ we have that

$$\mathcal{R}_2 f(t, x_2, y_3, \ldots, y_{n-1}) \geq \mathcal{R}_2 f(t, y_2, y_3, \ldots, y_{n-1})$$

for all $t, y_3, \ldots, y_{n-1}$ so

$$\big|\{t : \mathcal{R}_2 f(t, x_2, y_3, \ldots, y_{n-1}) \geq \lambda\}\big| \geq 2y_1\,,$$

which is

$$\big|\{t : \mathcal{R}_2 f(t, x_2, x_3, \ldots, x_{n-1}) \geq \lambda\}\big| \geq 2x_1\,,$$

and this tells us that

$$g(x) = \mathcal{R}_1\mathcal{R}_2 f(x_1, \ldots, x_{n-1}) \geq \lambda$$

so $g(x) \geq g(y)$. This completes the proof of the lemma. □

Using this property and Helly's selection principle we can find a subsequence f^{m_j} which converges to some h almost everywhere. We can also impose the condition that $(n-1)$ divides m_j for all j. It is clear that this h will also have the outward decreasing property. Since all the functions f^m are dominated by the L^p function Ch_f we see that h belongs to L^p and

$$A = \lim_{j\to\infty} \|f^{m_j} - k_f\|_p = \|h - k_f\|_p. \tag{3.14}$$

However, we also have

$$A = \lim_{j\to\infty} \|f^{m_j+1} - k_f\|_p = \|\mathcal{R}_{n-1} \ldots \mathcal{R}_1 U^1_\gamma h - k_f\|_p$$

so

$$A \le \|\mathcal{R}_{n-1}\dots\mathcal{R}_1 U^1_\gamma h - k_f\|_p = \|\mathcal{R}_{n-1}\dots\mathcal{R}_1 U^1_\gamma h - \mathcal{R}_{n-1}\dots\mathcal{R}_1 U^1_\gamma k_f\|_p$$
$$\le \|U^1_\gamma h - U^1_\gamma k_f\|_p = \|h - k_f\|_p = A\,,$$

which tells us that we must have equality everywhere in the chain. In particular,

$$\|\mathcal{R}_1 U^1_\gamma h - k_f\|_p = \|U^1_\gamma h - k_f\|_p.$$

Equality can only hold here, see e.g. [6], provided that that for almost every $x_2, \dots, x_{n-1}$ we have that

$$\mathcal{R}_1 U^1_\gamma h(x) = U^1_\gamma h(x).$$

Thus we have shown that both h and $U^1_\gamma h$ are invariant under the reflection

$$h(x_1, x_2, \dots, x_{n-1}) \mapsto \mathcal{T}^1 h := h(-x_1, x_2, \dots, x_{n-1})$$

and since $U^1_{-\gamma} h = \mathcal{T}^1 U_\gamma \mathcal{T}^1 h$ we see that $U^1_{-\gamma} h = U^1_\gamma h$ so $U^1_{2\gamma} h = U^1_\gamma U^1_\gamma h = U^1_\gamma U^1_{-\gamma} h = h$. Since γ is not a rational multiple of π we see that H, the function on the northern hemisphere associated to h, is constant along curves which are intersections of the northern hemisphere and translates of the $x_1 x_n$ coordinate plane. This also tells us that

$$h = U^1_\gamma h = \mathcal{R}_1 U^1_\gamma h \quad \text{a.e.} \tag{3.15}$$

Now we can use the chain of equalities

$$\|\mathcal{R}_{n-1}\dots\mathcal{R}_1 U^1_\gamma h - k_f\|_p = \dots = \|\mathcal{R}_1 U^1_\gamma h - k_f\|_p = \|U^1_\gamma h - k_f\|_p \tag{3.16}$$

to see that

$$\mathcal{R}_{n-1}\dots\mathcal{R}_1 U^1_\gamma h = \dots = \mathcal{R}_1 U^1_\gamma h = U^1_\gamma h = h \quad \text{a.e.} \tag{3.17}$$

We also have

$$\mathcal{R}_2 U^2_\gamma \mathcal{R}_{n-1}\dots\mathcal{R}_1 U^1_\gamma h = U^2_\gamma \mathcal{R}_{n-1}\dots\mathcal{R}_1 U^1_\gamma h \tag{3.18}$$

so the same argument tells us that the function on the northern hemisphere associated to $\mathcal{R}_{n-1}\dots\mathcal{R}_1 U^1_\gamma h$ is constant along curves which are intersections of the northern hemisphere and translates of the $x_2 x_n$ coordinate plane. Since $\mathcal{R}_{n-1}\dots\mathcal{R}_1 U^1_\gamma h = h$ a.e. we see that H is a.e. constant on 3-spaces which are parallel to the $x_1 x_2 x_n$-coordinate 3-space.

From this discussion the induction is evident and the result will be that H is a.e. constant on the northern hemisphere and since h has the outward decreasing property we see that H must be constant everywhere and h must have the form Ck_f for some C. Since $\|h\|_p = \lim_{j\to\infty} \|f^{m_j}\|_p = \|f\|_p = \|k_f\|_p$ we see that $C = 1$ and $h = k_f$.

This tells us that $A = 0$ and since $(\|k_f - f^m\|_p)_{m=0}^{\infty}$ is a decreasing sequence with a subsequence which tends to 0 we see that the whole sequence (f^m) tends to k_f. We have thus shown that for any f in the dense class of L^p functions we started with that $f^m \to k_f$. Since $\|k_f - k'_f\|_p \le \|f - f'\|_p$ for any $f, f' \in L^p$ we see that for any $f \in L^p$ we have that $f^m \to k_f$ in L^p.

Now

$$\mathcal{H}(f_1^m, \dots, f_n^m) \le \mathcal{H}(f_1^{m+1}, \dots, f_n^{m+1}) \tag{3.19}$$

for every $m \ge 0$ so

$$\mathcal{H}(f_1, \dots, f_n) \le \mathcal{H}(k_f, \dots, k_f) = \mathcal{H}(k, \dots, k). \tag{3.20}$$

This tells us that $(k, \dots, k)$ is an optimiser for Λ_α.

Now let us find all the non-negative functions which furnish the best constant. Using Lemma 3.1 we can see that

$$\Lambda_\alpha(f_1, \dots, f_n) = \Lambda_\alpha(f_1^{*j}, \dots, f_n^{*j})$$

can hold only if $f_i(x) = f_i^{*j}(x - a_i e_j)$ where e_j is the j-th coordinate vector and the a_i's satisfy

$$\det \begin{pmatrix} 1 & \dots & 1 \\ x_{11} & \dots & x_{n1} \\ \vdots & & \vdots \\ x_{1,j-1} & \dots & x_{n,j-1} \\ a_1 & \dots & a_n \\ x_{1,j+1} & \dots & x_{n,j+1} \\ \vdots & & \vdots \\ x_{1,n-1} & \dots & x_{n,n-1} \end{pmatrix} = 0. \tag{3.21}$$

This conclusion holds provided that all the adjoint matrices of the a_i's are nonzero and that is true for almost any $x_1, \dots, x_n \in \mathbb{R}^{n-1}$.

Now, let us say that for some $x_{2\hat{\jmath}}, \dots, x_{n\hat{\jmath}}$, where we do not specify the j-th coordinate in each vector, we have found that $f_i(x_{i\hat{\jmath}}, \cdot)$ has centre at a_i for $2 \le i \le n$. Then we can see that for any $x_{i\hat{\jmath}}$ the centre of $f_1(x_{1\hat{\jmath}}, \cdot)$ must be at the point a_1 such that all the $(x_{i\hat{\jmath}}, a_i)$ lie in some $(n-2)$-dimensional hyperplane. Then, by moving the $x_{i\hat{\jmath}}$'s around one by one for $2 \le i \le n$ we can see that there must exist a hyperplane where all the points $(x_{i\hat{\jmath}}, a_i)$ lie.

This tells us that if $(f_1, \dots, f_n)$ is an optimiser for our operator then the functions have the form $f_i(x_i) = h_i(Mx_i + b)$ where the h_i's have the outward decreasing property, M is an $(n-1) \times (n-1)$ matrix with determinant 1 and $b \in \mathbb{R}^{n-1}$.

Now, the transformations $f \mapsto U_\alpha^j f$ and $f \mapsto f(M \cdot + b)$ span a group G. It is now clear that for an optimiser $(f_1, \dots, f_n)$ the rearrangements $\mathcal{R}_j f_i$ will be of the form $T_g f$ for some $g \in G$ and thus the whole sequence $(f^m)_{m \ge 0}$ will be of the form $T_{g_m} f_i$ for some $g_m \in G$, the same g_m for each i.

Since the elements of G are isometries of L^p we have that

$$0 = \lim_{m\to\infty} \|f_i^m - k_{f_i}\|_p = \lim_{m\to\infty} \|f_i - T_{g_m^{-1}} k_{f_i}\|_p.$$

We shall see that for any $g \in G$ we have

$$T_g k_f(x) = \left(\begin{pmatrix} x \\ 1 \end{pmatrix}^T A^T A \begin{pmatrix} x \\ 1 \end{pmatrix} \right)^{-\frac{n}{2p}} \tag{3.22}$$

for some real $n \times n$ matrix with determinant 1.

Let $f_A(x) = \|A\left(\begin{smallmatrix} x \\ 1 \end{smallmatrix}\right)\|^{-\frac{n}{p}}$. Then $f_A(Mx+b) = \left\|A\left(\begin{smallmatrix} Mx+b \\ 1 \end{smallmatrix}\right)\right\|^{-\frac{n}{p}} = \|A'\left(\begin{smallmatrix} x \\ 1 \end{smallmatrix}\right)\|^{-\frac{n}{p}}$ with $A' = A\left(\begin{smallmatrix} M & b \\ 0 & 1 \end{smallmatrix}\right)$ so A' is again a real $n \times n$ matrix with determinant 1.

Now consider U_α^j for some j. Without loss of generality we can take $j = 1$. Then

$$U_\alpha^1 f_A(x) = \left(\frac{1}{1+|x|^2} \frac{1+|w|^2}{\|A\left(\begin{smallmatrix} w \\ 1 \end{smallmatrix}\right)\|^2} \right)^{\frac{n}{2p}}$$

where $\left(\begin{smallmatrix} w \\ 1 \end{smallmatrix}\right)$ is the point in $\mathbb{R}^n$ we get by starting with $\left(\begin{smallmatrix} x \\ 1 \end{smallmatrix}\right)$, projecting it to the hemisphere, that is, to $\frac{1}{\sqrt{1+|x|^2}}\left(\begin{smallmatrix} x \\ 1 \end{smallmatrix}\right)$, then rotating in the x_1x_n-plane by $-\alpha$, this sends

$$\frac{1}{\sqrt{1+|x|^2}} \begin{pmatrix} x \\ 1 \end{pmatrix} = \frac{1}{\sqrt{1+|x|^2}} \begin{pmatrix} x_1 \\ \hat{x}_1 \\ 1 \end{pmatrix}$$

to

$$\frac{1}{\sqrt{1+|x|^2}} \begin{pmatrix} \cos\alpha x_1 + \sin\alpha \\ \hat{x}_1 \\ -\sin\alpha x_1 + \cos\alpha \end{pmatrix},$$

and finally projecting this point to the plane $\left(\begin{smallmatrix} w \\ 1 \end{smallmatrix}\right)$, which sends it to

$$\begin{pmatrix} (\cos\alpha x_1 + \sin\alpha)/(-\sin\alpha x_1 + \cos\alpha) \\ \hat{x}_1/(-\sin\alpha x_1 + \cos\alpha) \\ 1 \end{pmatrix};$$

so $w = (-\sin\alpha x_1 + \cos\alpha)^{-1}\left(\begin{smallmatrix} \cos\alpha x_1 + \sin\alpha \\ \hat{x}_1 \end{smallmatrix}\right) =: w_n^{-1}\left(\begin{smallmatrix} w_1 \\ \hat{x}_1 \end{smallmatrix}\right)$. Since $w_1^2 + w_n^2 = x_1^2 + 1$ we have that

$$\left(\frac{1}{1+|x|^2} \frac{1+|w|^2}{\|A\left(\begin{smallmatrix} w \\ 1 \end{smallmatrix}\right)\|^2} \right)^{\frac{n}{2p}} = \left(\frac{1}{1+|x|^2} \frac{1+x_1^2+|\hat{x}_1|^2}{\left\|A\left(\begin{smallmatrix} w_1 \\ \hat{x}_1 \\ w_n \end{smallmatrix}\right)\right\|^2} \right)^{\frac{n}{2p}} = \left\|A\left(\begin{smallmatrix} w_1 \\ \hat{x}_1 \\ w_n \end{smallmatrix}\right)\right\|^{-\frac{n}{p}},$$

and since

$$\begin{pmatrix} w_1 \\ \hat{x}_1 \\ w_n \end{pmatrix} = \begin{pmatrix} \cos\alpha x_1 + \sin\alpha \\ \hat{x}_1 \\ -\sin\alpha x_1 + \cos\alpha \end{pmatrix} = \begin{pmatrix} \cos\alpha & 0 & \sin\alpha \\ 0 & I & 0 \\ -\sin\alpha & 0 & \cos\alpha \end{pmatrix} \begin{pmatrix} x_1 \\ \hat{x}_1 \\ 1 \end{pmatrix}$$

we get

$$U^1_\alpha f_A(x) = \frac{1}{\|A' \left(\begin{smallmatrix} x \\ 1 \end{smallmatrix}\right)\|^{\frac{n}{p}}}$$

with

$$A' = A \begin{pmatrix} \cos\alpha & 0 & \sin\alpha \\ 0 & I & 0 \\ -\sin\alpha & 0 & \cos\alpha \end{pmatrix}$$

and again A' has determinant 1.

Since the set of functions

$$\left\{ f_A(x) = \|A \left(\begin{smallmatrix} x \\ 1 \end{smallmatrix}\right)\|^{-\frac{n}{p}} | A \text{ an } n \times n \text{ matrix, } \det A = 1 \right\}$$

is closed in L^p and k_f belongs to this set we have shown that all optimisers have the form prescribed in the theorem.

Let us now see that all functions of the prescribed form are optimisers. It is clear that we can take $c_i = 1$. Let us therefore again take $f_A(x) = \|A \left(\begin{smallmatrix} x \\ 1 \end{smallmatrix}\right)\|^{-\frac{n}{p}}$. Then it is enough to show that $\|f_A\|_p = \|f_I\|_p$ and

$$\Lambda_\alpha(f_A, \dots, f_A) = \Lambda_\alpha(f_I, \dots, f_I)\,,$$

where I is the $n \times n$ identity matrix because we know that $(f_I)^n_{i=1}$ is an optimiser. To prove the equality we note first of all that $\Lambda_\alpha(f_1, \dots, f_n)$ is invariant under the transformation $(f_1, \dots, f_n) \mapsto (f_1(M \cdot + b), \dots, f_n(M \cdot + b))$ where as before M is an $(n-1) \times (n-1)$ matrix with determinant 1 and $b \in \mathbb{R}^{n-1}$. We note also that these transformations preserve the L^p-norm of the functions. By using this invariance we may make the additional assumption that A has the form $\left(\begin{smallmatrix} d_1 & 0 \\ 0 & d_2^2 I \end{smallmatrix}\right)$ with positive scalars d_1 and d_2 where I denotes the identity matrix of size $n-1$. Since we have that $\det A = 1$ we get the relation $d_1(d_2)^{2(n-1)} = \det A = 1$.

So we want to consider $\Lambda_\alpha(f_A, \dots, f_A)$ which equals

$$\int \frac{\left(\left(\frac{1}{d_2^{2(n-1)}} + \|d_2 x_1\|^2\right) \cdots \left(\frac{1}{d_2^{2(n-1)}} + \|d_2 x_n\|^2\right)\right)^{-\frac{n}{2p}}}{\left|\left(\begin{smallmatrix} 1 & \dots & 1 \\ x_1 & \dots & x_n \end{smallmatrix}\right)\right|^\alpha} \,\mathrm{d}x_1 \dots \mathrm{d}x_n.$$

We make the change of variables $d_2^n x_i = y_i$. Then $d_2^{n(n-1)}\,\mathrm{d}x_i = \mathrm{d}y_i$ and

$$\det \begin{pmatrix} 1 & \dots & 1 \\ y_1 & \dots & y_n \end{pmatrix} = d_2^{n(n-1)} \det \begin{pmatrix} 1 & \dots & 1 \\ x_1 & \dots & x_n \end{pmatrix}.$$

We thus get

$$\begin{aligned} \Lambda_\alpha(f_A, \dots, f_A) &= \int \frac{\left(\frac{1}{d_2^{2n(n-1)}}\right)^{-\frac{n}{2p}} \left((1 + \|y_1\|^2) \dots (1 + \|y_n\|^2)\right)^{-\frac{n}{2p}}}{d_2^{-n(n-1)\alpha} \left|\det \left(\begin{smallmatrix} 1 & \dots & 1 \\ y_1 & \dots & y_n \end{smallmatrix}\right)\right|^\alpha} \frac{\mathrm{d}y_1 \dots \mathrm{d}y_n}{d_2^{n^2(n-1)}} \\ &= d_2^{(n-1)\left(\frac{n^2}{p} - n^2 + n\alpha\right)} \Lambda_\alpha(f_I, \dots, f_I). \end{aligned}$$

Now note that $\frac{n^2}{p} - n^2 + n\alpha = n^2(1 - \frac{\alpha}{n}) - n^2 + n\alpha = 0$ so that we have the desired equality of the forms.

Finally, we calculate

$$\begin{aligned}\|f_A\|_p &= \int_{\mathbb{R}^{n-1}} \left(\frac{1}{d_2^{2(n-1)}} + \|d_2 x\|^2 \right)^{-\frac{n}{2}} \mathrm{d}x \\ &= \int_{\mathbb{R}^{n-1}} \left(1 + \|d_2^n x\|^2\right)^{-\frac{n}{2}} d_2^{n(n-1)} \, \mathrm{d}x = \|f_I\|_p\end{aligned}$$

where we have used the same change of variables as above. This completes the proof. □

Let us now examine the form $\Lambda_{S,\alpha}$ defined in (1.12). We have for any functions F_i defined on S^{n-1} that

$$\Lambda_{S,\alpha}(F_1, \dots, F_n) = \int_{(S_+^{n-1})^n} \frac{\tilde{F}_1(s_1) \dots \tilde{F}_n(s_n)}{|\det(s_1 \dots s_n)|^\alpha} \, \mathrm{d}s_1 \dots \mathrm{d}s_n =: \tilde{\Lambda}_{S,\alpha}(\tilde{F}_1, \dots, \tilde{F}_n),$$

where $\tilde{F}_i(s_i) = F_i(s_i) + F_i(\bar{s}_i)$ and $\bar{s}_i$ is the antipodal point of s_i.

We note from the preceding proof that $(f_1, \dots, f_n)$ is an optimiser for Λ_α if and only if $(\tilde{F}_1, \dots, \tilde{F}_n)$ is an optimiser for $\tilde{\Lambda}_{S,\alpha}$ where f_i and $\tilde{F}_i$ are related by (3.5). Furthermore, for any $s = (s_1, \dots, s_n) \in S_+^{n-1}$ we have that $\mathcal{S}^{-1}(s) = s/s_n$ and $|J_{\mathcal{S}^{-1}}(s)| = s_n^{-n}$ by the same calculation as in the previous section so if f_i has the form $f_i(x) = c_i \|A \left(\begin{smallmatrix} x \\ 1 \end{smallmatrix}\right)\|^{-\frac{n}{p}}$ as in (1.21) then the corresponding $\tilde{F}_i$ has the form $\tilde{F}_i(s) = c_i \|As\|^{-\frac{n}{p}}$.

Finally, we note that since $p > 1$ then

$$\|\tilde{F}_i(s)\|_{L^p(S_+^{n-1})} = \|F_i(s) + F_i(\bar{s})\|_{L^p(S_+^{n-1})} \le 2^{1-\frac{1}{p}} \|F_i(s)\|_{L^p(S^{n-1})}$$

and there is equality here if and only if $F(s) = F(\bar{s})$ for almost all $s \in S^{n-1}$. Thus we can state the analogue of Theorem 1.8 for $\Lambda_{S,\alpha}$ as follows.

Theorem 3.4. *The tuple $(F_1, \dots, F_n)$ of non-negative functions is an optimiser for $\Lambda_{S,\alpha}$ if and only if there exists an $n \times n$ matrix A with determinant 1 and $c_i \ge 0$ for $1 \le i \le n$ such that*

$$F_i(s) = c_i \|As\|^{-\frac{n}{p}} \quad \text{for each } 1 \le i \le n. \tag{3.23}$$

Note that taking A as the identity makes each F_i constant.

For $\Lambda_{\mathbb{H},\alpha}$ similar reasoning gives:

Theorem 3.5. *The tuple $(F_1, \dots, F_n)$ of non-negative functions is an optimiser for $\Lambda_{\mathbb{H},\alpha}$ if and only if there exists an $n \times n$ matrix A with determinant 1 and $c_i \ge 0$ for $1 \le i \le n$ such that*

$$F_i(s) = c_i \|As\|^{-\frac{n}{p}} \quad \text{for each } 1 \le i \le n. \tag{3.24}$$

Note that $\|As\|$ denotes the Euclidean norm of As viewed as an element of $\mathbb{R}^n$.

3.1. Inequality (1.23)

Provided that the integrability conditions are satisfied, it is easy to see that the only modifications we need to make to the argument above in order for it to apply to (1.23) are in Lemma 3.1 which is not general enough to apply to this case.

We generalise it as follows:

Lemma 3.6. *Assume that $f_1, \ldots, f_N$ and $g_1, \ldots, g_s$ are non-negative functions on $\mathbb{R}^n$ and that $g_1, \ldots, g_s$ are symmetric decreasing. Define*

$$I(f_1, \ldots, f_N; g_1, \ldots, g_s) = \int_{(\mathbb{R}^n)^N} \prod_{k=1}^{N} f_k(x_k) \prod_{i=1}^{s} g_i\Big(\sum_{k=1}^{N} b_{ik} x_k\Big)\, \mathrm{d}x_1 \cdots \mathrm{d}x_N.$$

Then

$$I(f_1, \ldots, f_N; g_1, \ldots, g_s) \le I(f_1^*, \ldots, f_N^*; g_1, \ldots, g_s) \tag{3.25}$$

and if each g_σ is strictly symmetric decreasing and if for each k there exists an i such that $b_{ik} \neq 0$ then there is equality here if and only if there exist vectors $a_1, \ldots, a_N \in \mathbb{R}^n$ such that

$$\sum_{k=1}^{N} b_{ik} a_k = 0$$

for all $i = 1, \ldots, s$ and $f_k(x_k) = f_k^*(x_k - a_k)$ for all $k = 1, \ldots, N$.

Note that inequality (3.25) is just a special case of the Brascamp–Lieb–Luttinger inequality but what is new here is the determination of the cases of equality.

Outline of proof. By writing

$$f_k(x_k) = \int_0^\infty \chi_{\{f_k > t_k\}}(x_k)\, \mathrm{d}t_k$$

we may assume that each $f_k = \chi_{A_k}$ is the characteristic function of an open bounded interval. We are assuming we have equality in (3.25) which means that

$$I(\chi_{A_1}, \ldots, \chi_{A_N}; g_1, \ldots, g_s) = I(\chi_{A_1^*}, \ldots, \chi_{A_N^*}; g_1, \ldots, g_s).$$

If we decompose

$$g_i = g_{i1} + g_{i2} = (g_i - \delta)\chi_{\{g_i > \delta\}} + (g_i \chi_{\{g_i \le \delta\}} + \delta \chi_{\{g_i > \delta\}}),$$

then both constituents of this sum are symmetric decreasing so that inequality (3.25) holds with g_i replaced by either constituent.

Thus, in order for equality to hold in (3.25) there must be equality when we replace $g_2, \ldots, g_s$ with $g_{22}, \ldots, g_{s2}$. By choosing δ small enough, we may assume that if $x_k \in A_k$ then $g_i(\sum_k b_{ik} x_k) > \delta$ for all i.

This means that for $i > 1$, g_{i2} is constant whenever $x_k \in A_k$ so we get

$$I(\chi_{A_1}, \dots, \chi_{A_N}; g_1, g_{22}, \dots, g_{s2}) = \delta^{s-1} \int_{(\mathbb{R}^n)^N} \prod_{k=1}^{m} f_k(x_k)\, g_1\Big(\sum_{k=1}^{m} b_{ik} x_k\Big) \mathrm{d}x_1 \cdots \mathrm{d}x_m.$$

and Lemma 3.1 applies to this case to give us that for every k such that $b_{1k} \neq 0$ then there exists a vector $a_k \in \mathbb{R}^n$ such that $\sum_k b_{1k} a_k = 0$ and $f_k(x_k) = f_k^*(x_k - a_k)$. The result follows. □

With this lemma we may deduce as before that if

$$\Lambda_\gamma(f_1, \dots, f_N) = \Lambda_\gamma(f_1^{*j}, \dots, f_N^{*j})$$

then we must have $f_k(x_k) = f_k^{*j}(x_k - a_k e_j)$, where the a_k's satisfy a system of equations of the same form as equation (3.21) where the column indices in each equation are those of some $P \in \mathcal{P}$. Then, given that I_γ cannot be factorised, we get the same conclusion as before, namely, that all points $(x_{k\hat{j}}, a_k)$ lie in some $(n-2)$-dimensional hyperplane.

The rest of the argument carries through unmodified.

Finally, let us show why conditions (1.26) are necessary and sufficient for the kernel of (1.25) to be locally integrable. The condition $\alpha < 1$ is necessary, since $\det\left(\begin{smallmatrix} 1 & 1 & 1 \\ x & y & z \end{smallmatrix}\right)^{-1}$ is not locally integrable. (It is the kernel of the singular integral operator we studied in Section 2 in the case $n = 3$.)

If $\alpha + \beta + \gamma + \delta \geq 2$ then let us consider the integral of the kernel when x lies in a small ball around the origin and the other variables lie in the first quadrant, in a thin annulus of radius one, centered at the origin. Then we note that the triangle whose vertices are y, z and w is covered by the other triangles whose areas appear in the kernel. So assume to begin with that $\delta = 0$. Let $y = x + r_1\theta_1$, $z = x + r_2\theta_2$ and $w = x + r_3\theta_3$. Then

$$\int_E \frac{\mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z\, \mathrm{d}w}{\left|\det\left(\begin{smallmatrix} 1 & 1 & 1 \\ x & y & z \end{smallmatrix}\right)\right|^\alpha \left|\det\left(\begin{smallmatrix} 1 & 1 & 1 \\ x & y & w \end{smallmatrix}\right)\right|^\beta \left|\det\left(\begin{smallmatrix} 1 & 1 & 1 \\ x & z & w \end{smallmatrix}\right)\right|^\gamma} =$$

$$= C \int_E \frac{r_1 r_2 r_3\, \mathrm{d}r_1\, \mathrm{d}r_2\, \mathrm{d}r_3\, \mathrm{d}\theta_1\, \mathrm{d}\theta_2\, \mathrm{d}\theta_3}{r_1^{\alpha+\beta} r_2^{\alpha+\gamma} r_3^{\beta+\gamma} |\sin(\theta_1 - \theta_2)|^\alpha |\sin(\theta_1 - \theta_3)|^\beta |\sin(\theta_2 - \theta_3)|^\gamma}.$$

If we further restrict attention to the set where the triangles formed by x and two of the other three variables are all comparable in size, which follows from assuming that $\theta_1 = \phi_1 + \theta_3$ and $\theta_2 = \phi_2 + \theta_3$ where $(\phi_1, \phi_2) = (\phi\cos\eta, \phi\sin\eta)$ and $-\frac{\pi}{6} > \eta > -\frac{\pi}{3}$ then we can estimate this integral from below by a multiple of

$$\int_{E'} \frac{\mathrm{d}\phi_1\, \mathrm{d}\phi_2}{|\phi_1 - \phi_2|^\alpha |\phi_1|^\beta |\phi_2|^\gamma} > C \int \frac{\phi}{\phi^{\alpha+\beta+\gamma}}\, \mathrm{d}\phi\, \mathrm{d}\eta$$

and this is finite only if $\alpha + \beta + \gamma < 2$.

Note that the singularity occurred in a region where the triangle formed by y, z and w is smaller than (a fixed multiple of) of any of the other triangles formed.

Therefore, if we had had the full kernel, with δ non-zero, then we would have had the result that the integral could be finite only if $\alpha+\beta+\gamma+\delta > 2$.

To prove the sufficiency of the conditions, we see that by convexity, Hölder's inequality and symmetry it is enough to establish that the kernel is bounded in the case $\gamma = \delta = 0$, $\alpha, \beta < 1$. In this case, the substitution $y = x + r_1\theta_1$, $z = x + r_2\theta_2$ and $w = x + r_3\theta_3$ gives the integral

$$\int_E \frac{r_1 r_2 r_3 \, \mathrm{d}r_1 \, \mathrm{d}r_2 \, \mathrm{d}r_3 \, \mathrm{d}\theta_1 \, \mathrm{d}\theta_2 \, \mathrm{d}\theta_3}{r_1^{\alpha+\beta} r_2^{\alpha} r_3^{\beta} |\sin(\theta_1-\theta_2)|^{\alpha} |\sin(\theta_1-\theta_3)|^{\beta}}$$

and this is clearly bounded.

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