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Some examples of C^∞ extension by linear operators

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Abstract. For two kinds of sets X in $\mathbb{R}^n$, we prove the existence of linear continuous operators extending C^∞ functions on X to C^∞ functions on $\mathbb{R}^n$. The sets we consider are: (a) sequences of points in the real line converging to 0 at a polynomial rate, (b) flag-shaped sets in the plane, which are unions of half-lines with slopes as in (a).

Introduction

Given a subset X of $\mathbb{R}^n$, set $\mathcal{I}_X = \big\{f \in C^\infty(\mathbb{R}^n) : f_{|X} = 0\big\}$, and let $C^\infty(X) = C^\infty(\mathbb{R}^n)/\mathcal{I}_X$ be the space of C^∞-functions restricted to X, endowed with the quotient topology. For notational convenience, we will not assume that X is closed; nevertheless, $C^\infty(X) = C^\infty(\overline{X})$. If X is bounded, $C^\infty(X)$ has the structure of a Fréchet space, defined by the quotient norms of $\mathcal{D}^m(X) = \mathcal{D}^m(B)/\{f : f_{|X} = 0\}$, $m \in \mathbb{N}$, where B is any closed ball containing X in its interior.

Definition 0.1. *We call* C^∞ extension operator *a continuous linear operator* $\mathcal{E}\colon C^\infty(X) \longrightarrow C^\infty(\mathbb{R}^n)$ *such that* $(\mathcal{E}\varphi)_{|X} = \varphi$ *for every* $\varphi \in C^\infty(X)$.

This is equivalent to saying that $\mathcal{I}_X$ admits a closed complementary subspace in $C^\infty(\mathbb{R}^n)$, namely $\mathcal{E}\big(C^\infty(X)\big)$.

For finite orders of differentiability, C. Fefferman proved the existence of C^k extension operators for every k and every X, with a norm which only depends on k and on the dimension of the ambient space [4].

On the contrary, there are examples of sets which do not admit C^∞ extension operators. The simplest example is the set of pairs (x, y) with $x \ge 0$ and $|y| \le e^{-1/x}$ in $\mathbb{R}^2$ [9].

The earliest result on the existence of a C^∞ extension operator is due to R. T. Seeley [7] for half-spaces, followed by E. M. Stein's result for Lipschitz domains [8].

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In [5], J. Mather improved a theorem of G. Schwarz in [6] −stating that every C^∞-function $f(x)$ on $\mathbb{R}^n$, invariant under a compact group K of linear transformations, can be expressed by a C^∞-function $g\big(p_1(x),\dots,p_k(x)\big)$ in the fundamental K-invariant polynomials p_j− by showing that g can be chosen to depend linearly and continuously on f. In other words, a C^∞ extension operator exists for the set $X\subset\mathbb{R}^k$ which is the image in $\mathbb{R}^k$ of the Hilbert map $P=(p_1,\dots,p_k)$.

More recent results are due to E. Bierstone [2] for Hölder domains and subanalytic sets with dense interior, and to E. Bierstone and G. Schwarz [3] for Nash subanalytic sets.

Despite this long history, various basic questions appear, to the best of our knowledge, to have been disregarded in the literature.

In this paper we first construct a C^∞ extension operator for a class of sets X consisting of a sequence of points in the line converging to the origin at a polynomial rate, e.g., $X=\{1/n^a, n>0\}$, $a>0$. This result is then used to construct a C^∞ extension operator on fan-shaped subsets of $\mathbb{R}^2$, i.e. unions of infinitely many half-lines emanating from the origin, with slopes in one of the sets $X\subset\mathbb{R}$ described above.

The motivation for this problem came from questions in harmonic analysis, related to the characterization of the spherical Fourier transforms of U_d-invariant Schwartz functions on the $(2d+1)$-dimensional Heisenberg group [1]. The domain on which these spherical Fourier transforms are defined is the so-called *Heisenberg fan* in $\mathbb{R}^2$ (corresponding to the choice of $(2n+d)^{-1}$ for the slopes of the half-lines). From this point of view, the existence of a C^∞ extension operator on the Heisenberg fan provides an analogue of Mather's theorem, giving a partial answer to a question left open in [1].

1. Sequences of points in the line

In this section we prove the following theorem.

Theorem 1.1. *Let $(\lambda_n)_{n\in\mathbb{N}}$ be a positive sequence, monotonically converging to 0, with $\lambda_n-\lambda_{n+1}\sim n^{-a-1}$ for some $a>0$. Then the set $X=\{\lambda_n : n\in\mathbb{N}\}$ admits a C^∞ extension operator. The same holds for the union $X=X_1\cup(-X_2)$ of two sets X_1 and X_2 as above.*

We give the proof for the set $X=\{1/n\colon n>0\}$, leaving the details for the general case to the reader.

We begin by decomposing X dyadically. For $\nu\in\mathbb{N}$, we set $I_\nu=[2^{-\nu-1},2^{-\nu+1}]$ and $X_\nu=X\cap I_\nu$. We first define extension operators on each X_ν.

Proposition 1.2. *Let m be a positive integer with $m<\#X_\nu$. There exists a linear operator $\mathcal{E}_{\nu,m}$ mapping functions on X_ν to C^∞-functions on I_ν and such that, for every function φ on X_ν,*

(i) $(\mathcal{E}_{\nu,m}\varphi)_{|X_\nu}=\varphi$;

(ii) *if $\varphi=P_{|X_\nu}$, with P a polynomial of degree at most m, then $\mathcal{E}_{\nu,m}(\varphi)=P$;*

(iii) *for every k,*

$$\max_{x\in I_\nu}\left|\frac{d^k}{dx^k}(\mathcal{E}_{\nu,m}\varphi)(x)\right| \le C_{m,k}2^{2\nu k}\max_{X_\nu}|\varphi|\,, \tag{1.1}$$

with $C_{m,k}$ independent of ν.

The proof makes use of the following simple lemma.

Lemma 1.3. *Let I be an interval and $E\subset I$ a set of $\bar m+1$ points, almost equally spaced (i.e., the distance between two consecutive points in E is comparable to $|I|/\bar m$). Given $\varphi : E\longrightarrow \mathbb{C}$, let P be its Lagrange interpolation polynomial,*

$$P(x)=\sum_{t\in E}\varphi(t)\prod_{t'\in E\setminus\{t\}}\frac{x-t'}{t-t'}\,.$$

Then P depends linearly on φ and, for every $m\le \bar m$,

$$\max_{x\in I}|P^{(m)}(x)|\le A_{\bar m}|I|^{-m}\max|\varphi|\,. \tag{1.2}$$

Proof. Linearity is obvious. To prove (1.2), we can assume, by scaling and translation invariance, that $I=[0,1]$. Since $|t-t'|\ge \delta/\bar m$ for any $t\ne t'$ in E, it follows easily that, for $x\in I$,

$$|P^{(m)}(x)|\le A_{\bar m}\max_E|\varphi|\,.$$

□

Proof of Proposition 1.2. There exists $\delta>0$, independent of ν, such that the distance between two consecutive points of X_ν is greater than $\delta 2^{-2\nu}$.

Denoting by η a fixed C^∞-function supported on $[-1,1]$ with $\eta(0)=1$, we set, for $t\in X_\nu$,

$$\eta_t(x)=\eta\big(\delta^{-1}2^{2\nu}(x-t)\big)\,.$$

Then η_t vanishes at all points of X_ν other than t, $\eta_t(t)=1$, and there are constants B_m such that

$$\max_{m'\le m}|\eta_t^{(m')}(x)|\le B_m 2^{2\nu m'}\,, \tag{1.3}$$

for every x and m.

Let Y_ν be a subset of X_ν consisting of $m+1$ almost equally spaced points in I_ν.

Given a function φ defined on X_ν, let P_φ be the Lagrange interpolation polynomial of $\varphi_{|Y_\nu}$ and set

$$(\mathcal{E}_{\nu,m}\varphi)(x)=P_\varphi(x)+\sum_{t\in X_\nu}\big(\varphi(t)-P_\varphi(t)\big)\eta_t(x)\,. \tag{1.4}$$

It is clear that $\mathcal{E}_{\nu,m}$ is linear, that $\mathcal{E}_{\nu,m}\varphi$ is C^∞, and that (i), (ii) hold.

Since P_φ has degree at most m, (iii) follows easily from (1.2) and (1.3). □

We glue together these dyadic extensions operators by fixing a partition of unity

$$1=\sum_{\nu=0}^{\infty}\theta_\nu$$

on $(0,1]$, with $\theta_\nu(x)=\theta(2^\nu x)$ smooth and supported in I_ν.

Given a sequence $(m_\nu)_{\nu\ge 0}$ of integers satisfying the assumption of Proposition 1.2, we define, for $\varphi: X\longrightarrow\mathbb{C}$,

$$\mathcal{E}\varphi(x)=\sum_{\nu=0}^{\infty}\theta_\nu(x)\mathcal{E}_{\nu,m_\nu}(\varphi_{|X_\nu})(x)\,. \tag{1.5}$$

By Proposition 1.2 and finite overlapping, $\mathcal{E}\varphi$ is a C^∞-function on $(0,1]$.

Proposition 1.4. *The integers m_ν can be chosen increasing to infinity slowly enough that, for every k and every C^{2k+1}-function f on $[0,1]$, $\mathcal{E}(f_{|X})$ is C^k on $[0,1]$ and*

$$\|\mathcal{E}(f_{|X})\|_{C^k}\le C_m\|f\|_{C^{2k+1}}.$$

Proof. By $C_{m,k}$ we denote the constants in Proposition 1.2, assuming, as we may, that they are increasing in both indices.

Denote by T the Taylor polynomial of f at 0 of order $2k$. Given a sequence $(m_\nu)_{\nu\ge0}$ increasing to infinity, fix $\nu_0=\nu_0(k)$ such that $m_{\nu_0}\ge 2k$. By Proposition 1.2, $\mathcal{E}_{\nu,m_\nu}(T_{|X_\nu})=T$ for $\nu\ge\nu_0$, and therefore

$$\begin{aligned}\mathcal{E}(f_{|X})(x)&=\sum_{\nu<\nu_0}\theta_\nu(x)\mathcal{E}_{\nu,m_\nu}(f_{|X_\nu})(x)+\sum_{\nu\ge\nu_0}\theta_\nu(x)\mathcal{E}_{\nu,m_\nu}\big((f-T)_{|X_\nu}\big)(x)\\&\quad+T(x)\Theta_{\nu_0}(x)\\&=s_1(x)+s_2(x)+s_3(x)\,,\end{aligned}$$

where

$$\Theta_{\nu_0}=\sum_{\nu\ge\nu_0}\theta_\nu.$$

The s_1 term contains a finite sum and is in fact C^∞ on the whole line. By (1.1) and (1.3), for $j\le k$,

$$\begin{aligned}\Big|\frac{d^j}{dx^j}s_1(x)\Big|&\le\sum_{\nu<\nu_0}\sum_{j_1+j_2=j}\binom{j}{j_1}\Big|\frac{d^{j_1}}{dx^{j_1}}\theta_\nu(x)\Big|\,\Big|\frac{d^{j_2}}{dx^{j_2}}(\mathcal{E}_{\nu,m_\nu}f_{|X_\nu})(x)\Big|\\&\le\alpha_j C_{m_{\nu_0},k}2^{2\nu_0 j}\|f\|_\infty\,.\end{aligned}$$

The s_3 term equals T in a right neighborhood of 0 and its j-th derivative is smaller than $\alpha_j 2^{j\nu_0}\|f\|_{C^{2k}}$.

Consider now the crucial term s_2 and single out the ν-th summand. Since $k\le m_\nu$ and

$$|f(x)-T(x)|\le\alpha_k\,\|f\|_{C^{2k+1}}\,|x|^{2k+1}\,,$$

for $j \le k$ we have

$$\begin{aligned}
&\Big|\frac{d^j}{dx^j}\big(\theta_\nu(x)\,(\mathcal{E}_{\nu,m_\nu}(f-T)_{|X_\nu})(x)\big)\Big| \le \\
&\qquad \le \sum_{j_1+j_2=j}\binom{j}{j_1}\Big|\frac{d^{j_1}}{dx^{j_1}}\theta_\nu(x)\Big|\Big|\frac{d^{j_2}}{dx^{j_2}}(\mathcal{E}_{\nu,m_\nu}(f-T)_{|X_\nu})(x)\Big| \\
&\qquad \le \alpha_k\, C_{m_\nu,k}\,\max_{X_\nu}|f-T|\sum_{j_1+j_2=j}2^{\nu j_1}2^{2\nu j_2} \\
&\qquad \le \alpha_k\, C_{m_\nu,k}\,2^{2\nu j-\nu(2k+1)}\|f\|_{C^{2k+1}} \\
&\qquad \le \alpha_k\, C_{m_\nu,k}\,2^{-\nu}\|f\|_{C^{2k+1}}\,.
\end{aligned}$$

Choosing the m_ν growing to infinity so slowly that

$$\max_{2m\le m_\nu} C_{m,m_\nu} < 2^{\nu/2}\,,$$

we can sum on ν and conclude the proof. □

Proposition 1.4 shows that $\mathcal{E}$ is an extension operator from $C^\infty(X)$ to $C^\infty\big([0,1]\big)$. We can conclude the proof of Theorem 1.1 using Seeley's theorem [7] to extend functions from $[0,1]$ to $\mathbb{R}$.

2. Fans in $\mathbb{R}^2$

If X is a set as in Theorem 1.1, we set

$$F_X = \big\{(x,y) : y=\lambda_n x\,,\ x>0\ ,\ \lambda_n\in X\big\}\subset\mathbb{R}^2\,.$$

Theorem 2.1. *F_X admits a C^∞ extension operator.*

As a model set, we take

$$F = \big\{(x,y) : y = x/n\ ,\ x>0\ ,\ n>0\big\}\,.$$

We denote by $\mathcal{E}^X$ a fixed extension operator as in (1.5), satisfying the estimates of Proposition 1.4. We split F into dyadic pieces F_ν, by restricting the x-variable to the intervals $I_\nu = [2^{-\nu-1},2^{-\nu+1}]$ with $\nu\in\mathbb{Z}$.

Lemma 2.2. *Let $Y = I_0\times\{1/n : n>0\}$ and $R = I_0\times[0,1]$. Given an integer m, there exists a linear operator $\mathcal{E}_m$ mapping $\varphi\in C^\infty(Y)$ into $C^\infty(R)$ with the following properties:*

(i) $(\mathcal{E}_m\varphi)_{|Y} = \varphi$;

(ii) *if P is a polynomial of degree at most m in each variable, then $\mathcal{E}_m(P_{|Y}) = P$;*

(iii) *for every $f\in C^\infty(R)$ and every $(\alpha_1,\alpha_2)\in\mathbb{N}^2$,*

$$\max_{(x,y)\in R}\big|\partial_x^{\alpha_1}\partial_y^{\alpha_2}\mathcal{E}_m(f_{|Y})(x,y)\big| \le C_{m,|\alpha|}\|f\|_{C^{2|\alpha|+1}}\,. \tag{2.1}$$

Proof. Fix the grid $(t_j, u_k) = (j/2m, 1/k)$ with $m \le j, k \le 2m$ and let

$$P_\varphi(x,y) = \sum_{j,k} \varphi(t_j,u_k) \prod_{j'\neq j} \frac{x - t_{j'}}{t_j - t_{j'}} \prod_{k'\neq k} \frac{y - u_{k'}}{u_k - u_{k'}}$$

be the Lagrange interpolation polynomial of a given function φ restricted to the grid. With $\varphi_x(y) = \varphi(x,y)$, we set

$$(\mathcal{E}_m\varphi)(x,y) = P_\varphi(x,y) + \big(\mathcal{E}^X(\varphi - P_\varphi)_x\big)(y)\,.$$

Then (i) and (ii) are obviously satisfied. As to (iii), observe that, for $\varphi = f_{|_Y}$,

$$\partial_x^{\alpha_1}\big(\mathcal{E}^X(\varphi - P_\varphi)_x\big) = \mathcal{E}^X\big(\partial_x^{\alpha_1}(\varphi - P_\varphi)\big)_x\,,$$

by the continuity of $\mathcal{E}^X$ and the smoothness of f. Therefore, for $(x,y)\in R$,

$$\begin{aligned}\big|\partial_x^{\alpha_1}\partial_y^{\alpha_2}\mathcal{E}_m\varphi(x,y)\big| &\le \|P_\varphi\|_{C^{|\alpha|}} + \big\|\mathcal{E}^X\big(\partial_x^{\alpha_1}(\varphi - P_\varphi)\big)_x\big\|_{C^{\alpha_2}}\\ &\le A_m\|f\|_{C^0} + C_{\alpha_2}\big\|\big(\partial_x^{\alpha_1}(\varphi - P_\varphi)\big)_x\big\|_{C^{2\alpha_2+1}}\\ &\le C_{m,|\alpha|}\,\|f\|_{C^{2|\alpha|+1}}\,,\end{aligned}$$

by Proposition 1.4. □

For $\nu\in\mathbb{Z}$, the map

$$\Psi_\nu(x,y) = (2^{-\nu}x, 2^{-\nu}xy)$$

maps Y onto the dyadic piece F_ν of F and R onto the trapezoid

$$R_\nu = \{(x,y) : x\in I_\nu\ ,\ y\le x\}.$$

We define $\mathcal{E}_{\nu,m} : C^\infty(F_\nu) \longrightarrow C^\infty(R_\nu)$ by conjugating $\mathcal{E}_m$ by Ψ_ν, i.e.,

$$\mathcal{E}_{\nu,m}\varphi(x,y) = \big(\mathcal{E}_m(\varphi\circ\Psi_\nu)\big)\Big(2^\nu x\,,\,\frac{y}{x}\Big)\,.$$

Lemma 2.3. *The following properties hold for* $\varphi\in C^\infty(F_\nu)$*:*

(i) $(\mathcal{E}_{\nu,m}\varphi)_{|_{F_\nu}} = \varphi$*;*

(ii) *if* P *is a polynomial of degree at most* m*, then* $\mathcal{E}_{\nu,m}(P_{|_{F_\nu}}) = P$*;*

(iii) *for every* $f\in C^\infty(R_\nu)$ *and every* $\alpha = (\alpha_1,\alpha_2)\in\mathbb{N}^2$*,*

$$\max_{(x,y)\in R_\nu} \big|\partial_x^{\alpha_1}\partial_y^{\alpha_2}\mathcal{E}_{\nu,m}(f_{|_{F_\nu}})(x,y)\big| \le C_{m,|\alpha|}2^{\nu|\alpha|}\|f\|_{C^{2|\alpha|+1}}\,. \tag{2.2}$$

Proof. It follows directly from Lemma 2.2. To prove (ii) it must be observed that if a polynomial $P(x,y)$ has degree at most m, then $P(x,xy)$ has degree at most m in each separate variable. For $\nu = 1$, (2.2) is an obvious consequence of the fact that Ψ_1 is a diffeomorphism of R onto R_1. By scaling, the conclusion follows for general ν. □

We can now proceed as in the previous section. Given a sequence $(m_\nu)_{\nu\ge 0}$ of positive integers, we define, for $\varphi \in C^\infty(F)$,

$$\begin{aligned}\mathcal{E}\varphi(x,y) &= \sum_{\nu\ge 0}\theta_\nu(x)\mathcal{E}_{\nu,m_\nu}(\varphi_{|F_\nu})(x,y) + \sum_{\nu<0}\theta_\nu(x)\mathcal{E}_{\nu,0}(\varphi_{|F_\nu})(x,y)\\ &= \mathcal{E}^+\varphi(x,y) + \mathcal{E}^-\varphi(x,y)\,,\end{aligned}$$

where $\{\theta_\nu\}_{\nu\in\mathbb{Z}}$ is a partition of unity on the positive half-line with $\theta_\nu(x) = \theta(2^\nu x)$ smooth and supported in I_ν. It is quite clear that $(\mathcal{E}\varphi)_{|F} = \varphi$ and that $\mathcal{E}^-$ is continuous. For $\mathcal{E}^+$ we have the following analogue of Proposition 1.4.

Proposition 2.4. *The integers m_ν can be chosen increasing to infinity slowly enough that, for every $f \in C^\infty(\mathbb{R}^2)$, $\mathcal{E}^+(f_{|F})$ is C^∞ and, for every k, there is C_k such that $\|\mathcal{E}^+(f_{|F})\|_{C^k} \le C_k\|f\|_{C^{3k+2}}$.*

Proof. We proceed as in the proof of Proposition 1.4, borrowing the same notation from there. Let $C_{m,k}$ be the constants in Proposition 2.3, assumed to be increasing in both indices.

By T we denote the Taylor polynomial of f at 0 of order $3k+1$. We now choose $\nu_0 = \nu_0(k)$ such that $m_{\nu_0} \ge 3k+1$. Then,

$$\begin{aligned}\mathcal{E}^+ f_{|F}(x) = &\sum_{0\le\nu<\nu_0}\theta_\nu(x)\mathcal{E}_{\nu,m_\nu}(f_{|F_\nu})(x,y) + \sum_{\nu\ge\nu_0}\theta_\nu(x)\mathcal{E}_{\nu,m_\nu}\big((f-T)_{|F_\nu}\big)(x,y)\\ &+ T(x,y)\Theta_{\nu_0}(x)\\ = &\; s_1(x) + s_2(x) + s_3(x)\,.\end{aligned}$$

Once again, the crucial term is s_2. For $|\alpha| \le k$, the ν-th summand in s_2 satisfies the estimate

$$\begin{aligned}&\Big|\partial_x^{\alpha_1}\partial_y^{\alpha_2}\big(\theta_\nu(x)\,(\mathcal{E}_{\nu,m_\nu}(f-T)_{|F_\nu})(x,y)\big)\Big| \le\\ &\qquad\le \sum_{j_1+j_2=\alpha_1}\binom{\alpha_1}{j_1}|\theta_\nu^{(j_1)}(x)|\big|\partial_x^{j_2}\partial_y^{\alpha_2}(\mathcal{E}_{\nu,m_\nu}(f-T)_{|F_\nu})(x,y)\big|\\ &\qquad\le A_k\,C_{m_\nu,k}\,2^{|\alpha|\nu}\,\|f-T\|_{C^{2k+1}}\,,\end{aligned}$$

by Lemma 2.3. But, for $|\beta| \le 2k+1$ and $(x,y)\in R_\nu$,

$$\big|\partial_x^{\beta_1}\partial_y^{\beta_2}\big(f(x,y)-T(x,y)\big)\big| \le A'_k\,\|f\|_{C^{3k+2}}\,2^{-\nu(3k+2-|\beta|)}\,,$$

so that

$$\Big|\partial_x^{\alpha_1}\partial_y^{\alpha_2}\big(\theta_\nu(x)(\mathcal{E}_{\nu,m_\nu}(f-T)_{|F_\nu})(x,y)\big)\Big| \le A''_k\,C_{m_\nu,k}\,2^{-\nu}\,\|f\|_{C^{3k+2}}\,.$$

As before, it is sufficient to choose the m_ν growing to infinity so slowly that $C_{m_\nu,k} < 2^{\nu/2}$. □

This gives an extension operator from $C^\infty(F)$ to $C^\infty(S)$, $S = \big(\bigcup_\nu R_\nu\big)^-$, and an application of Stein's theorem [8] allows to conclude the proof of Theorem 2.1.

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