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On the uniform perfectness of the groups of diffeomorphisms of even-dimensional manifolds

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Abstract. We show that the identity component $\mathrm{Diff}^r(M^{2m})_0$ of the group of C^r diffeomorphisms of a compact $(2m)$-dimensional manifold M^{2m} $(1 \le r \le \infty,\ r \ne 2m+1)$ is uniformly perfect for $2m \ge 6$, i.e., any element of $\mathrm{Diff}^r(M^{2m})_0$ can be written as a product of a bounded number of commutators. It is also shown that for a compact connected manifold M^{2m} $(2m \ge 6)$, the identity component $\mathrm{Diff}^r(M^{2m})_0$ of the group of C^r diffeomorphisms of M^{2m} $(1 \le r \le \infty,\ r \ne 2m+1)$ is uniformly simple, i.e., for elements f and g of $\mathrm{Diff}^r(M^{2m})_0 \setminus \{\mathrm{id}\}$, f can be written as a product of a bounded number of conjugates of g or g^{-1}.

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1. Introduction

For an n-dimensional manifold M^n, let $\mathrm{Diff}^r_c(M^n)$ denote the group of C^r diffeomorphisms of M^n with compact support $(1 \le r \le \infty)$. Here, the *support* of a diffeomorphism f of M^n is defined to be the *closure* of $\{x \in M \mid f(x) \ne x\}$. For a compact manifold M^n, $\mathrm{Diff}^r_c(M^n)$ coincides with the group $\mathrm{Diff}^r(M^n)$ of C^r diffeomorphisms of M^n. Let $\mathrm{Diff}^r_c(M^n)_0$ denote the identity component of $\mathrm{Diff}^r_c(M^n)$. Here $\mathrm{Diff}^r_c(M^n)$ is equipped with the C^r topology ([16], [23]). By the results of Herman, Mather and Thurston ([11], [14], [16], [23], [2]), for an n-dimensional manifold M^n, $\mathrm{Diff}^r_c(M^n)_0$ is a perfect group if $r = 0$ or $1 \le r \le \infty$ and $r \ne n+1$. Here, a group is said to be *perfect* if it coincides with its commutator subgroup. In other words, a group is perfect if any element can be written as a product of commutators. The perfectness of a group is equivalent to the vanishing of first homology group of the group. The homological properties of the group $\mathrm{Diff}^r_c(M^n)_0$ has been studied in connection with the theory of foliations ([23]).

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In general, for an element g of the commutator subgroup $[G, G]$ of a group G, its commutator length is defined to be the minimum number of commutators whose product is equal to g. It is natural to ask whether the commutator length function $\mathrm{cl}\colon [G, G] \to \mathbb{Z}$ is bounded. When the commutator length is not bounded, then it is very interesting to know about the stable commutator length defined by $\mathrm{scl}(g) = \lim_{n\to\infty} \mathrm{cl}(g^n)/n$ in Bavard [3]. The stable commutator length function is related with the bounded cohomology groups $H_b^*(G)$ of the group G defined by Gromov ([7]). Namely, the homomorphism $H_b^2(G) \to H^2(G)$ is injective if and only if the stable commutator length function vanishes on $[G, G]$. This is formulated as the Bavard duality theorem which describes the stable commutator length in terms of homogeneous quasimorphisms ([3]). In recent years, the stable commutator length and the quasimorphisms appear as an important key to study infinite groups (see for example [5] and its references).

We say that a group is *uniformly perfect* if any element can be written as a product of a *bounded* number of commutators. It is easy to see that the uniform perfectness implies the vanishing of stable commutator length function, and hence the injectivity of the map from the second bounded cohomology group to the usual one.

For the question of uniform perfectness of the group of diffeomorphisms, the following results are shown in [4], [30] and [31].

Theorem 1.1 (Burago–Ivanov–Polterovich [4], Tsuboi [30], [31]).

(1) *For the interior M^n of a compact n-dimensional manifold which admits a handle decomposition only with handles of indices not greater than $(n-1)/2$, any element of* $\mathrm{Diff}_c^r(M^n)_0$ $(1 \le r \le \infty, r \ne n+1)$ *can be written as a product of two commutators.*

(2) *For a compact even-dimensional manifold M^{2m} which has a handle decomposition without handles of the middle index m, any element of* $\mathrm{Diff}^r(M^{2m})_0$ $(1 \le r \le \infty, r \ne 2m+1)$ *can be written as a product of four commutators.*

(3) *For an arbitrary compact odd-dimensional manifold M^{2m+1}, any element of* $\mathrm{Diff}^r(M^{2m+1})_0$ $(1 \le r \le \infty, r \ne 2m+2)$ *can be written as a product of five commutators.*

Now the result of this paper concerns the remaining cases.

Theorem 1.2. *The identity component* $\mathrm{Diff}^r(M^{2m})_0$ *of the group of C^r diffeomorphisms* $\mathrm{Diff}^r(M^{2m})$ *of the compact $(2m)$-dimensional manifold M^{2m}* $(1 \le r \le \infty, r \ne 2m+1)$ *is uniformly perfect for $2m \ge 6$, i.e., any element of* $\mathrm{Diff}^r(M^{2m})_0$ *can be written as a product of a bounded number of commutators.*

Here the bound for the number of commutators may depend on manifolds. For the manifolds of dimensions 2 and 4, the problem of uniform perfectness of the identity component of the group of diffeomorphisms is still open. The vanishing of the stable commutator length of these groups is not known either. It is interesting to find some other approach to study the stable commutator length of diffeomorphism groups which might solve the remaining cases (see [4], [12]).

The argument deducing the simplicity of $\mathrm{Diff}^r(M^n)_0$ from the proof of its perfectness ([8], [23], [2]) applies to showing the uniform simplicity from the proof of its uniformly perfectness ([31]). We say that a group G is *uniformly simple* if, for elements f and g of $G \setminus \{\mathbf{1}\}$, f can be written as a product of a bounded number of conjugates of g or g^{-1}.

Corollary 1.3. *For a compact connected* $(2m)$*-dimensional manifold* M^{2m} $(2m \geq 6)$*, the identity component* $\mathrm{Diff}^r(M^{2m})_0$ *of the group* $\mathrm{Diff}^r(M^{2m})$ *of* C^r *diffeomorphisms of* M^{2m} $(1 \leq r \leq \infty, r \neq 2m+1)$ *is uniformly simple.*

The main part of the proof of Theorem 1.2 is a decomposition of an isotopy into a bounded number of isotopies with controlled support. Then the theorem follows from Theorem 1.1 (1) in a way similar to the proof of Theorem 1.1 (2) and (3) in [30] and in [31]. For the decomposition, we give a technique to find the Whitney disks which guide to separate two stratified subsets of the middle dimension m. The condition $2m \geq 6$ on the dimension implies that the Whitney disks can be disjointly embedded in the manifold and enables us to show Theorem 1.2.

We review the proof of Theorem 1.1 in Section 2 and there we give lemmas about the general position of two stratified subsets which were not correctly stated in [31]. Then we give the proof of Theorem 1.2 in Section 3. The proof of lemmas used in Section 3 is given in Sections 4 and 6. We show Corollary 1.3 in Section 5.

The author is grateful to the referee for patient and careful reading and for pointing out several errors in the earlier versions, one of which is a misleading statement on relationship between the decomposition by the stable manifolds of a gradient flow of a Morse function and a cellular decomposition of the manifold (see Section 6).

2. Decomposition of isotopies

The proof of our Theorem 1.2 relies on the general position argument for differentiable maps between manifolds with stratified subsets. In [30] and [31], we looked at the general position of the differentiable mappings from a cellular complex to a manifold with differentiable cellular decomposition.

The argument in [30] and [31] works for differentiable manifolds with stratified subsets which are defined as follows: Let M^n be an n-dimensional manifold. A subset X of M^n is an m-dimensional stratified subset if there is a filtration

$$X = X^{(m)} \supset X^{(m-1)} \supset \cdots \supset X^{(1)} \supset X^{(0)},$$

such that, for $k = 0, \dots, m$,

(1) $X^{(k)}$ is a closed subset,

(2) $X^{(k)} \setminus X^{(k-1)}$ is a k-dimensional submanifold of M^n,

(3) for the closure $\overline{X^{(k)} \setminus X^{(k-1)}}$ of $X^{(k)} \setminus X^{(k-1)}$,

$$\overline{X^{(k)} \setminus X^{(k-1)}} \setminus (X^{(k)} \setminus X^{(k-1)}) \subset X^{(k-1)}.$$

The subset $X^{(k)}$ is called the k-dimensional skeleton of X. This definition of the stratified subsets is a weak one ([36], [24]).

First we show the following lemma which is the necessary generalization of Lemma 4.3 in [30] or Lemma 2.3 in [31].

Lemma 2.1. *Let M^n be an n-dimensional manifold with a compact k-dimensional stratified subset K^k, and N^m be an m-dimensional manifold with a compact ℓ-dimensional stratified subset L^ℓ. Let $f: N^m \to M^n$ be a differentiable map. If $k + \ell + 1 \leq n$, then there is an isotopy $\{\Phi_t : M^n \to M^n\}_{t\in[0,1]}$ ($\Phi_0 = \mathrm{id}$) such that $\Phi_1(K^k) \cap f(L^\ell) = \emptyset$.*

Proof. We construct the isotopy Φ_t, skeleton by skeleton. Let $K^{(u)}$ denote the u-dimensional skeleton of K^k;

$$K^k = K^{(k)} \supset \cdots \supset K^{(1)} \supset K^{(0)}.$$

Assume that for $u - 1 \leq k - 1$, there is an isotopy $\{\Phi_t^{u-1}\}_{t\in[0,1]}$ ($\Phi_0^{u-1} = \mathrm{id}$) such that

$$\Phi_1^{u-1}(K^{(u-1)}) \cap f(L^\ell) = \emptyset.$$

Then there is a neighborhood U_{u-1} of $K^{(u-1)}$ such that $\Phi_1^{u-1}(U_{u-1}) \cap f(L^\ell) = \emptyset$.

Now for $u \leq k$, we construct an isotopy $\{\Phi_t^u\}_{t\in[0,1]}$ ($\Phi_0^u = \mathrm{id}$) such that $\Phi_1^u(K^{(u)}) \cap f(L^\ell) = \emptyset$. Since $K^{(u)}$ is closed in K^k, $K^{(u)} \setminus U_{u-1}$ is compact and is covered by finitely many coordinate neighborhoods $\{(D^u \times D^{n-u})_i\}_{i=1}^{k_u}$ of M^n of the form $D^u \times D^{n-u}$, where D^u and D^{n-u} are the closed balls of radius 1 in $\mathbb{R}^u$ and $\mathbb{R}^{n-u}$, respectively, and

$$(K^{(u)} \setminus U_{u-1}) \cap (D^u \times D^{n-u})_i \subset (D^u \times \{0\})_i.$$

Moreover we can take such neighborhoods that the family

$$\{(\mathrm{int}(D^u_{1/2}) \times \mathrm{int}(D^{n-u}_{1/2}))_i\}_{i=1}^{k_u}$$

still covers $K^{(u)} \setminus U_{u-1}$, and

$$K^{(u)} \setminus U_{u-1} \subset \bigcup_{i=1}^{k_u} (\mathrm{int}(D^u_{1/2}) \times \{0\})_i,$$

where $D^u_{1/2}$ and $D^{n-u}_{1/2}$ are the images of the closed balls of radius $1/2$ in $\mathbb{R}^u$ and $\mathbb{R}^{n-u}$, respectively, and "int" denotes the interior.

Now assume that for $i-1 \le k_u - 1$, we have an isotopy $\{\Phi_t^{u,i-1}\}_{t\in[0,1]}$ ($\Phi_0^{u,i-1} =$ id) with support in $\bigcup_{j=1}^{i-1}(D^u \times D^{n-u})_j$ such that

$$K^{(u)} \cap (\Phi_1^{u-1} \circ \Phi_1^{u,i-1})^{-1}(f(L^\ell)) \subset \bigcup_{j=i}^{k_u}(\mathrm{int}(D^u_{1/2}) \times \mathrm{int}(D^{n-u}_{1/2}))_j .$$

On the neighborhood $(D^u \times D^{n-u})_i$, we have the projection

$$p_i = \mathrm{proj}_2 \colon (D^u \times D^{n-u})_i \longrightarrow D^{n-u}.$$

Put $L^\ell_{i-1} = (\Phi_1^{u-1} \circ \Phi_1^{u,i-1})^{-1}(f(L^\ell))$. Since $p_i(L^\ell_{i-1} \cap (D^u \times D^{n-u})_i)$ is a finite union of images of manifolds of dimension $\le \ell \le n-k-1 \le n-u-1$ under differentiable maps of class C^r ($r \ge 1$), it is a measure zero subset of D^{n-u} by the Sard theorem. Moreover, since L^ℓ is compact, $p_i(L^\ell_{i-1} \cap (D^u \times D^{n-u})_i)$ is a nowhere dense closed subset of D^{n-u}. Take a point q_i close to 0 in the complement of $p_i(L^\ell_{i-1} \cap (D^u \times D^{n-u})_i)$. Let $\{\Phi_t'^{u,i} \colon M^n \to M^n\}_{t\in[0,1]}$ ($\Phi_0'^{u,i} =$ id) be an isotopy with support in $(\mathrm{int}(D^u) \times \mathrm{int}(D^{n-u}))_i$ such that $\Phi_t'^{u,i}(x,0) = (x, t\mu(x)q_i)$ on $(D^u \times D^{n-u})_i$, where $\mu \colon \mathrm{int}(D^u) \to [0,1]$ is a C^∞ function with compact support such that $\mu(x) = 1$ for $x \in D^u_{1/2}$. Since we took q_i in the complement of $p_i(L^\ell_{i-1} \cap (D^u \times D^{n-u})_i)$,

$$L^\ell_{i-1} \cap \Phi_1'^{u,i}(K^{(u)}) \cap (\mathrm{int}(D^u_{1/2}) \times \mathrm{int}(D^{n-u}_{1/2}))_i = \emptyset,$$

hence

$$(\Phi_1'^{u,i})^{-1}(L^\ell_{i-1}) \cap K^{(u)} \cap (\mathrm{int}(D^u_{1/2}) \times \mathrm{int}(D^{n-u}_{1/2}))_i = \emptyset.$$

Since we took q_i sufficiently close to $0 \in D^{n-u}$,

$$(\Phi_t'^{u,i})^{-1}(L^\ell_{i-1}) \cap \Big(K^{(u)} \cup \bigcup_{j=1}^{i-1}(\mathrm{int}(D^u_{1/2}) \times \mathrm{int}(D^{n-u}_{1/2}))_j\Big) = \emptyset \quad (t \in [0,1]).$$

Thus we found the isotopy $\{\Phi_t^{u,i} = \Phi_t^{u,i-1} \circ \Phi_t'^{u,i}\}_{t\in[0,1]}$ ($\Phi_0{}^{u,i} =$ id) with support in $\bigcup_{j=1}^{i}(D^u \times D^{n-u})_j$ such that

$$K^{(u)} \cap (\Phi_1^{u-1} \circ \Phi_1^{u,i})^{-1}(f(L^\ell)) \subset \bigcup_{j=i+1}^{k_u}(\mathrm{int}(D^u_{1/2}) \times \mathrm{int}(D^{n-u}_{1/2}))_j .$$

Let Φ_t^u be the composition $\Phi_t^{u-1} \circ \Phi_t^{u,k_u}$, then $\{\Phi_t^u\}_{t\in[0,1]}$ ($\Phi_0^u =$ id) satisfies that $\Phi_1^u(K^{(u)}) \cap f(L^\ell) = \emptyset$. Then $\Phi_t = \Phi_t^k$ satisfies $\Phi_1(K^k) \cap f(L^\ell) = \emptyset$. $\square$

We use Lemma 2.1 to show the following theorem ([30], [31]).

Theorem 2.2. *Let M^n be a compact n-dimensional manifold. Let P^p and Q^q be p-dimensional and q-dimensional stratified subsets in M^n, respectively. Assume that $p+q+2 \leq n$ and that $P^p \cap Q^q = \emptyset$. Then any element $f \in \mathrm{Diff}^r(M^n)_0$ $(1 \leq r \leq \infty)$ can be written as a product $f = g \circ h$ such that $g \in \mathrm{Diff}^r_c(M^n \setminus k(Q^q))_0$ and $h \in \mathrm{Diff}^r_c(M^n \setminus P^p)_0$, where $k \in \mathrm{Diff}^r_c(M^n \setminus P^p)_0$ is a diffeomorphism of M^n with support in a small neighborhood of Q^q, and $\mathrm{Diff}^r_c(M^n \setminus k(Q^q))_0$ and $\mathrm{Diff}^r_c(M^n \setminus P^p)_0$ are considered as subgroups of $\mathrm{Diff}^r(M^n)_0$, respectively.*

The statement of Theorem 2.2 means that, by moving Q by a small isotopy k, the diffeomorphism g of M^n obtained in Theorem 2.2 is isotopic to the identity by an isotopy which is the identity on a neighborhood of $k(Q^q)$, and h is isotopic to the identity by an isotopy which is the identity on a neighborhood of P^p.

For the completeness, we include the proof of Theorem 2.2.

Proof of Theorem 2.2. Let $\{f_t\}_{t\in[0,1]}$ be the isotopy such that $f_0 = \mathrm{id}$ and $f_1 = f$. Let $F\colon [0,1] \times M^n \to M^n$ be the trace of the isotopy: $F(t,x) = f_t(x)$. Here, $[0,1] \times M^n$ contains the $(p+1)$-dimensional stratified subset $[0,1] \times P^p$.

We look at the image $F([0,1] \times P^p) \subset M^n$. As $p+1+q \leq n-1$, by Lemma 2.1, there is an isotopy $\{k_s\}_{s\in[0,1]}$ $(k_0 = \mathrm{id}, k_1 = k)$ such that $F([0,1] \times P^p) \cap k(Q^q) = \emptyset$.

Then the following lemma implies Theorem 2.2 by putting $P_0 = \emptyset$ and replacing Q^q by $k(Q^q)$. ☐

Lemma 2.3. *Let M^n be a compact n-dimensional manifold. Let P^p and Q^q be p-dimensional and q-dimensional stratified subsets of M^n, respectively. Let P_0 be a subset of P^p. Let $\{f_t\} \subset \mathrm{Diff}^r(M^n)_0$ $(f_0 = \mathrm{id})$ be an isotopy which is the identity on a neighborhood of P_0. Assume that $f_t(P^p \setminus P_0) \cap Q^q = \emptyset$ $(t \in [0,1])$. Then $f_1 \in \mathrm{Diff}^r(M^n)_0$ can be written as a product $f_1 = g_1 \circ h_1$, where $\{g_t\}_{t\in[0,1]} \subset \mathrm{Diff}^r_c(M^n \setminus Q^q)_0$ $(g_0 = \mathrm{id})$ and $\{h_t\}_{t\in[0,1]} \subset \mathrm{Diff}^r_c(M^n \setminus P^p)_0$ $(h_0 = \mathrm{id})$.*

Proof. Let $F\colon [0,1] \times M^n \to M^n$ be the trace of the isotopy: $F(t,x) = f_t(x)$.

Let W be a neighborhood of P_0 in M^n where f_t is the identity. Let U be a neighborhood of $F([0,1] \times (P^p \setminus W \cap P^p))$ and V be a neighborhood of Q^q such that $U \cap V = \emptyset$.

Let ξ be the vector field on $[0,1] \times M^n$ given by $\frac{\partial}{\partial t} + \big(\frac{\mathrm{d} f_{t+s}(x)}{\mathrm{d}s}\big)_{s=0}$ at $(t, f_t(x))$. This ξ generates the isotopy f_t. Let η be a vector field on $[0,1] \times M^n$ with support in $[0,1] \times U$ such that $\eta = \xi$ on a neighborhood of

$$\{(t, f_t(x_0)) \,\big|\, x_0 \in P^p \setminus W \cap P^p,\ t \in [0,1]\}.$$

Then $\eta = \partial/\partial t$ on $[0,1]\times(V\cup W)$ which is a neighborhood of $[0,1]\times(Q^q\cup P_0)$. Then η generates an isotopy $\{g_t\}_{t\in[0,1]}$ such that g_t is the identity on the neighborhood $V\cup W$ of $Q^q\cup P_0$ and $g_t(x) = f_t(x)$ for x in a neighborhood of $P^p = (P^p \setminus W \cap P^p)\cup(W\cap P^p)$. Here, for $x\in W$, $g_t(x) = x = f_t(x)$. Put $h = {g_1}^{-1} f_1$, then h is the identity on a neighborhood of P^p, and it is isotopic to the identity as an element of $\mathrm{Diff}^r(M^n)$. For, put $h_t = {g_t}^{-1}\circ f_t$. Then h_t is the identity on a neighborhood of P^p.

Thus we can write $f = g\circ h$, where $g\in \mathrm{Diff}^r_c(M^n\setminus Q^q)_0$, $h\in \mathrm{Diff}^r_c(M^n\setminus P^p)_0$. □

To use Theorem 2.2, we looked at the stratifications of a compact manifold M^n given by the stable manifolds or by the unstable manifolds of the gradient flow of a Morse function associated with a handle decomposition.

A function $f\colon M^n\to\mathbb{R}$ on a compact n-dimensional manifold M^n without boundary is called a *Morse function* if the critical points are nondegenerate, that is, the Hessian matrices of f at the critical points are nondegenerate. For such a function f, the set of critical points is a finite set. The index of the Hessian matrix of f at a critical point is called the *index* of the critical point.

Any compact n-dimensional manifold M^n without boundary admits a Morse function $f\colon M^n\to\mathbb{R}$ such that $f(M^n) = [0,n]$, the set of critical points of index k is contained in $f^{-1}(k)$ $(k = 0,\dots,n)$. Such a Morse function is called *self-indexing*. If M^n is a compact connected n-dimensional manifold M^n without boundary, there is a self-indexing Morse function $f\colon M^n\to\mathbb{R}$ such that $f^{-1}(0)$ and $f^{-1}(n)$ are one point sets ([19]).

For $a\in[0,n]$, put $M_a = f^{-1}(a)$. Then M_a is a codimension 1 submanifold of M^n if a is not an integer. Put $W_k = f^{-1}([0,k+1/2])$, and then this W_k is a compact manifold with boundary $\partial W_k = M_{k+1/2} = f^{-1}(k+1/2)$. Let c_k be the number of critical points of index k. Then the manifold W_k is diffeomorphic to the manifold obtained from W_{k-1} by attaching c_k handles of index k $(k = 0,\dots,n)$. This means the following.

Let $D^k\times D^{n-k}$ be the product of the k-dimensional disk D^k and the $(n-k)$-dimensional disk D^{n-k}. Let $\varphi_i\colon(\partial D^k)\times D^{n-k}\to\partial W_{k-1}$ $(i = 1,\dots,c_k)$ be diffeomorphisms with disjoint images. Let

$$W'_k = W_{k-1}\cup_{\bigsqcup_{i=1}^{c_k}\varphi_i}\bigsqcup_{i=1}^{c_k}(D^k\times D^{n-k})_i$$

be the space obtained from the disjoint union $W_{k-1}\sqcup\bigsqcup_{i=1}^{c_k}(D^k\times D^{n-k})_i$ by identifying $x\in(\partial D^k)\times D^{n-k}\subset(D^k\times D^{n-k})_i$ with $\varphi_i(x)\in\partial W_{k-1}\subset W_{k-1}$. The image of $D^k\times D^{n-k}$ in W'_k is called a *handle* of index k. We will simply write the handle of index k as $(D^k\times D^{n-k})_i$. Then W'_k is a manifold with boundary and the corner which is the image $\bigsqcup_{i=1}^{c_k}\varphi_i((\partial D^k)\times(\partial D^{n-k}))$. By smoothing along the corner, we obtain W_k'' from W_k' and W_k'' has a differentiable structure which is diffeomorphic to W_k, and we say W_k is obtained from the manifold W_{k-1} by attaching c_k handles of index k ($k = 0, \dots, n$).

In fact, we can consider W_k' as a submanifold with corner of W_k, W_k'' is obtained by taking the union of W_k' and a neighborhood of corner of W_k', and $W_k \setminus W_k''$ is diffeomorphic to $(-\infty, k + 1/2] \times \partial W_k$. We have the sequence of submanifolds

$$\begin{aligned} W_0 \subset W_1' \subset W_1'' \subset W_1 \subset \cdots \subset W_{k-1} \subset W_k' \subset W_k'' \subset W_k \\ \subset \cdots \subset W_{n-1} \subset W_n' = W_n'' = W_n = M^n. \end{aligned}$$

Then, when we identify W_k' with W_k, M^n is decomposed into the union of the handles $(D^k \times D^{n-k})_i$ ($i = 1, \dots, c_k$; $k = 0, \dots, n$) and this decomposition into handles is called a *handle decomposition* of M. However, hereafter we do not identify W_k' or W_k'' with W_k. We call the image of $D^k \times \{0\}$ the *core disk* of the handle $(D^k \times D^{n-k})_i$ of index k. The boundary of the core disk of the handle of index k is an embedded $(k-1)$-dimensional sphere in $\partial W_{k-1} = M_{k-1/2}$ and it is called the *attaching sphere*.

For the above self-indexing Morse function $f : M^n \to \mathbb{R}$ and the constant function n, the function $n - f$ is a Morse function, and the critical points of index k of the Morse function f are nothing but the critical points of index $n - k$ of the Morse function $n - f$. Hence this gives rise to a handle decomposition of M^n called the dual handle decomposition. That is for

$$W_{n-k}^* = (n - f)^{-1}([0, n - k + 1/2]) = f^{-1}([k - 1/2, n]),$$

$$\begin{aligned} M^n = W_n^* = W_n^{*\prime\prime} = W_n^{*\prime} \supset W_{n-1}^* \\ \supset \cdots \supset W_{n-k}^* \supset W_{n-k}^{*\prime\prime} \supset W_{n-k}^{*\prime} \supset W_{n-k-1}^* \\ \supset \cdots \supset W_1^* \supset W_1^{*\prime\prime} \supset W_1^{*\prime} \supset W_0^*. \end{aligned}$$

Then $W_{n-k}^{*\prime}$ is obtained from W_{n-k-1}^* by attaching c_k handles of index $n - k$. The core disk of the handle of index $n - k$ for this handle decomposition is called the *cocore disk* of the handle decomposition for f. The boundary of the cocore disk of the handle of index k is an embedded $(n-k-1)$-dimensional sphere in $\partial W_{n-k-1}^* = \partial W_k = M_{k+1/2}$ and it is called the *belt sphere*.

By choosing a Riemannian metric on the manifold M^n, the Morse function f defines the gradient vector field and the gradient flow Ψ_t. The singular points of the gradient vector field are precisely the critical points of f. The local stable manifold and the local unstable manifold of the singular point p of the gradient flow Ψ_t correspond to the core disk and the cocore disk of the handle containing p of a handle decomposition of M^n, respectively ([18], [19]). Let e_i^k and e_i^{*n-k} denote the global stable manifold and the global unstable manifold, respectively, for the singular point p_i^k which is a critical point of index k of f ($i = 1, \dots, c_k$). Then e_i^k and e_i^{*n-k} are diffeomorphic to $\mathbb{R}^k$ and $\mathbb{R}^{n-k}$, respectively. Let

$$X^{(k)} = \bigcup_{j \le k} \bigcup_{i=1}^{c_j} e_i^j \quad (k = 0, \dots, n).$$

Then

$$M^n = X^{(n)} \supset X^{(n-1)} \supset \cdots \supset X^{(1)} \supset X^{(0)}$$

is a stratification of M^n ([18]). That is, $X^{(k)}$ is a closed subset, $\bigcup_{i=1}^{c_k} e_i^k$ is a k-dimensional submanifold, and $\overline{\bigcup_{i=1}^{c_k} e_i^k} \setminus \bigcup_{i=1}^{c_k} e_i^k \subset X^{(k-1)}$. We call this the stratification by the stable manifolds (for the gradient flow of the Morse function). We also have the stratification by the unstable manifolds (for the gradient flow of the Morse function):

$$M^n = X^{*(n)} \supset X^{*(n-1)} \supset \cdots \supset X^{*(1)} \supset X^{*(0)},$$

where $X^{*(n-k)} = \bigcup_{j \ge k} \bigcup_{i=1}^{c_j} e_i^{*n-j}$ $(k = 0, \dots, n)$. This is the stratification by the stable manifolds for the gradient flow of the Morse function $n - f$.

We look at the k-dimensional skeleton $X^{(k)}$ of the stratification by the stable manifolds and the $(n-k-1)$-dimensional skeleton $X^{*(n-k-1)}$ of the stratification by the unstable manifolds. The boundary $\partial W_k = M_{k+1/2}$ of W_k is transverse to the gradient flow Ψ_t, and hence $M \setminus (X^{(k)} \cup X^{*(n-k-1)})$ is diffeomorphic to $\partial W_k \times \mathbb{R}$ by the map

$$\partial W_k \times \mathbb{R} \ni (x, t) \longmapsto \Psi_t(x) \in M \setminus (X^{(k)} \cup X^{*(n-k-1)}).$$

Moreover $\Psi_t(\partial W_k)$ converges to $X^{(k)}$ as $t \to -\infty$ and to $X^{*(n-k-1)}$ as $t \to \infty$. Hence, $M \setminus X^{*(n-k-1)}$ is diffeomorphic to the interior $\mathrm{int}(W_k)$ of W_k, and any small neighborhood of $X^{(k)}$ contains a deformation retract of both W_k and $M \setminus X^{*(n-k-1)}$:

$$X^{(k)} \subset \mathrm{int}(W_k) \subset W_k \subset M \setminus X^{*(n-k-1)}.$$

Using the gradient flow Ψ_t, for any neighborhood V of $X^{(k)}$ in $\mathrm{int}(W_k)$ and for any compact subset A in $\mathrm{int}(W_k)$, we can construct an isotopy $\{G_t : \mathrm{int}(W_k) \to \mathrm{int}(W_k)\}_{t \in [0,1]}$ with compact support such that $G_0 = \mathrm{id}_{\mathrm{int}(W_k)}$, $G_t(X^{(k)}) \subset X^{(k)}$ $(t \in [0, 1])$ and $G_1(A) \subset V$. A similar statement is true for $X^{(k)} \subset M \setminus X^{*(n-k-1)}$.

Remark 2.4. For our Morse function there is a Riemannian metric on M^n such that the stable manifolds e_i^k and the unstable manifolds $e_{i'}^{*k'}$ intersect transversely ([21]). As we shall see in Section 6 (Proposition 6.2), for a carefully chosen Riemannian metric, there is a cellular complex structure compatible with the stratification by stable manifolds.

Now for the interior M^n of a compact manifold with boundary $\bar{M}^n$ which admits a Morse function such that $W_m = \bar{M}^n$ for $2m < n$, we have the following lemma (see [30], Lemma 4.5).

Lemma 2.5. *Let M^n be the interior of a compact n-dimensional manifold which admits a handle decomposition only with handles of indices not greater than $(n-1)/2$. Let $X^{(m)}$ be the m-dimensional skeleton of the stratification by the stable manifolds for the gradient flow of the Morse function on M^n adapted to the handle decomposition ($2m < n$). Then there are an isotopy $\{F_t \colon M^n \to M^n\}_{t\in[0,1]}$ with compact support ($F_0 = \mathrm{id}$) and an open neighborhood U of $X^{(m)}$ such that $(F_1)^\ell(U)$ ($\ell \in \mathbb{Z}$) are disjoint.*

Proof. Let V_0 be a small neighborhood of $X^{(m)} \subset M^n$. We apply Lemma 2.1 to the identity map $M^n \to M^n$ of M^n with stratified subset $X^{(m)}$. Then there is an isotopy $\{h_t\}_{t\in[0,1]}$ such that $h_0 = \mathrm{id}$ and $h_1(X^{(m)}) \cap X^{(m)} = \emptyset$. We may assume that the support of the isotopy $\{h_t\}_{t\in[0,1]}$ is contained in V_0. Take a neighborhood V_1 of $X^{(m)}$ and V_2 of $h_1(X^{(m)})$ such that $V_1 \cap V_2 = \emptyset$. Then $V_3 = V_1 \cap (h_1)^{-1}(V_2)$ is a neighborhood of $X^{(m)}$ such that $V_3 \cap h_1(V_3) = \emptyset$. Here we can take V_1 and V_2 such that their closures $\overline{V_1}$ and $\overline{V_2}$ are compact, and then $\overline{V_3}$ is compact.

For V_3 and $h_1(\overline{V_3})$, by using the flow lines of the gradient flow Ψ_t, we have an isotopy $\{G_t \colon M^n \to M^n\}_{t\in[0,1]}$ with support in V such that $G_0 = \mathrm{id}$, $G_t|X^{(m)} = \mathrm{id}_{X^{(m)}}$ and $G_1(h_1(\overline{V_3})) \subset V_3$.

Let F_t be the composition of G_t and h_t: $F_t = G_t \circ h_t$. Then $F_1(\overline{V_3}) \subset V_3$. For $U = V_3 \setminus F_1(\overline{V_3})$, $(F_1)^\ell(U)$ ($\ell \in \mathbb{Z}$) are disjoint. □

We give the proof of Theorem 1.1 (1).

Proof of Theorem 1.1 (1). For the manifold M^n, we take the m-dimensional stratified set $X^{(m)}$ ($2m < n$) given in Lemma 2.5. Let $f \in \mathrm{Diff}^r_c(M^n)_0$ ($r \neq n+1$). By the result of Herman, Mather and Thurston ([11], [14], [16], [23], [2]), f can be written as a product of commutators.

$$f = [a_1, b_1] \cdots [a_k, b_k], \quad a_1, b_1, \dots, a_k, b_k \in \mathrm{Diff}^r_c(M^n)_0,$$

where $[a_i, b_i] = a_i b_i {a_i}^{-1} {b_i}^{-1}$. Let C be a compact subset of M^n such that the supports of a_i, b_i as well as the supports of the isotopies $\{a_{it}\}_{t\in[0,1]}$ ($a_{i0} = \mathrm{id}$ and $a_{i1} = a_i$), $\{b_{it}\}_{t\in[0,1]}$ ($b_{i0} = \mathrm{id}$ and $b_{i1} = b_i$) are contained in C.

By using the flow lines of the gradient flow Ψ_t, we have an isotopy $\{G'_t\}_{t\in[0,1]}$ with compact support such that $G'_1(C) \subset U$, where U is the open neighborhood taken in Lemma 2.5. Then by Lemma 2.5, for F_t in Lemma 2.5 and $g = (G'_1)^{-1} \circ F_1 \circ G'_1$, $g^\ell((G'_1)^{-1}(U))$ ($\ell \in \mathbb{Z}$) are disjoint.

Put

$$H = \prod_{i=1}^{k} g^{k-i}([a_1, b_1] \cdots [a_i, b_i]) g^{i-k}.$$

Then H is an element of $\mathrm{Diff}^r_c(M^n)_0$. Now the conjugate of H by g is as follows:

$$\begin{aligned} gHg^{-1} &= \prod_{i=1}^{k} g^{k-i+1}([a_1,b_1]\cdots[a_i,b_i])g^{i-k-1} \\ &= \prod_{i=0}^{k-1} g^{k-i}([a_1,b_1]\cdots[a_{i+1},b_{i+1}])g^{i-k}. \end{aligned}$$

Hence

$$\begin{aligned} H^{-1}gHg^{-1} &= ([a_1,b_1]\cdots[a_k,b_k])^{-1}\prod_{i=0}^{k-1} g^{k-i}[a_{i+1},b_{i+1}]g^{i-k} \\ &= f^{-1}\prod_{i=0}^{k-1} g^{k-i}[a_{i+1},b_{i+1}]g^{i-k} \\ &= f^{-1}\Big[\prod_{i=0}^{k-1} g^{k-i}a_{i+1}g^{i-k}, \prod_{i=0}^{k-1} g^{k-i}b_{i+1}g^{i-k}\Big]. \end{aligned}$$

Put

$$A = \prod_{i=0}^{k-1} g^{k-i}a_{i+1}g^{i-k} \quad \text{and} \quad B = \prod_{i=0}^{k-1} g^{k-i}b_{i+1}g^{i-k},$$

then A and B are elements of $\mathrm{Diff}^r_c(\mathbb{R}^n)_0$. Thus f can be written as a product of two commutators: $f = [A,B][g,H^{-1}]$. □

Proof of Theorem 1.1 (2). For an even-dimensional compact manifold M^{2m} which has a handle decomposition without handles of the middle index m, Theorem 2.2 together with Theorem 1.1 (1) implies Theorem 1.1 (2) (see [30]). □

For the decomposition of an isotopy on an odd dimensional manifold, we used the following lemma (see [30], Remark 4.4).

Lemma 2.6. *In Lemma* 2.1*, let* $K^k = K^{(k)} \supset K^{(k-1)} \supset \cdots \supset K^{(1)} \supset K^{(0)}$ *and* $L^\ell = L^{(\ell)} \supset L^{(\ell-1)} \supset \cdots \supset L^{(1)} \supset L^{(0)}$ *be the stratifications. Then there is an isotopy* $\{\Phi_t : M^n \to M^n\}_{t\in[0,1]}$ $(\Phi_0 = \mathrm{id})$ *with support in a neighborhood of* K^k *such that* $\Phi_1(K^{(a)}) \cap f(L^{(b)}) = \emptyset$ *for* $a+b+1 = n$*, and the intersection* $\Phi_1(K^{(a)}) \cap f(L^{(b)})$ *consists of finitely many transverse intersection points for* $a+b=n$.

Proof. We proceed as in the proof of Lemma 2.1. Assume that for $u-1 \le k-1$, there is an isotopy $\{\widehat{\Phi}^{u-1}_t\}_{t\in[0,1]}$ $(\widehat{\Phi}^{u-1}_0 = \mathrm{id})$ such that $\widehat{\Phi}^{u-1}_1(K^{(a)}) \cap f(L^{(b)}) = \emptyset$ for $a+b+1 = n$ and $a \le u-1$, and the intersection $\widehat{\Phi}^{u-1}_1(K^{(a)}) \cap f(L^{(b)})$ consists of finitely many transverse points for $a + b = n$ and $a \leq u - 1$. Then there is a neighborhood U_{u-1} of $K^{(u-1)}$ such that $\widehat{\Phi}_1^{u-1}(U_{u-1}) \cap f(L^{(n-u)}) = \emptyset$. We cover $K^{(u)} \setminus U_{u-1}$ by finitely many coordinate neighborhoods $\{(D^u \times D^{n-u})_i\}_{i=1}^{k_u}$ such that

$$(K^{(u)} \setminus U_{u-1}) \cap (D^u \times D^{n-u})_i \subset (D^u \times \{0\})_i$$

and $\{(\mathrm{int}(D^u_{1/2}) \times \mathrm{int}(D^{n-u}_{1/2}))_i\}_{i=1}^{k_u}$ still covers $K^{(u)} \setminus U_{u-1}$.

By the proof of Lemma 2.1, we have isotopies $\{\Phi_t^{u,i}\}_{t\in[0,1]}$ ($\Phi_0^{u,i} = \mathrm{id}$, $i = 1, \dots, k_u$) with support in $\bigcup_{j=1}^{i}(D^u \times D^{n-u})_j$ such that

$$K^{(u)} \cap (\widehat{\Phi}_1^{u-1} \circ \Phi_1^{u,i})^{-1}(f(L^{(n-u-1)})) \subset \bigcup_{j=i+1}^{k_u} (\mathrm{int}(D^u_{1/2}) \times \mathrm{int}(D^{n-u}_{1/2}))_j,$$

and for $\Phi_t^u = \widehat{\Phi}_t^{u-1} \circ \Phi_t^{u,k_u}$, $\Phi_1^u(K^{(u)}) \cap f(L^{(n-u-1)}) = \emptyset$.

We modify Φ_t^u to obtain $\widehat{\Phi}_t^u$ such that $\widehat{\Phi}_1^u(K^{(u)}) \cap f(L^{(n-u)})$ consists of finitely many transverse intersection points.

Since $\Phi_1^u(K^{(u)}) \cap f(L^{(n-u-1)}) = \emptyset$, $(f|L^{(n-u)})^{-1}(\Phi_1^u(K^{(u)}))$ is a closed subset hence is a compact subset in $L^{(n-u)}$. Thus it is compact subset in $L^{(n-u)} \setminus L^{(n-u-1)}$.

Now assume that, for $i \leq k_u$, we have an isotopy $\{\widehat{\Phi}_t^{u,i-1}\}_{t\in[0,1]}$ ($\widehat{\Phi}_0^{u,i-1} = \mathrm{id}$) with support in $\bigcup_{j=1}^{i-1}(D^u \times D^{n-u})_j$ such that

$$K^{(u)} \cap (\Phi_1^u \circ \widehat{\Phi}_1^{u,i-1})^{-1}(f(L^{(n-u)})) \cap \bigcup_{j=1}^{i-1}(\mathrm{int}(D^u_{1/2}) \times \mathrm{int}(D^{n-u}_{1/2}))_j.$$

consists of transverse intersection points. Then for

$$L'^{n-u}_{i-1} = (\Phi_1^u \circ \widehat{\Phi}_1^{u,i-1})^{-1}(f(L^{(n-u)})),$$

we look at $p_i(L'^{n-u}_{i-1} \cap (D^u \times D^{n-u})_i)$ in D^{n-u}. More precisely, we look at the map

$$\begin{aligned} &p_i \circ (\Phi_1^u \circ \widehat{\Phi}_1^{u,i-1})^{-1} \circ f : \\ &\quad (L^{(n-u)} \setminus L^{(n-u-1)}) \cap f^{-1}\big((\Phi_1^u \circ \widehat{\Phi}_1^{u,i-1})\big((\mathrm{int}(D^u) \times \mathrm{int}(D^{n-u})\big)_i)\big) \longrightarrow D^{n-u}. \end{aligned}$$

Then by the Sard theorem for C^r mappings between the manifolds of the same dimension ($r \geq 1$), the critical value of $p_i \circ (\Phi_1^u \circ \widehat{\Phi}_1^{u,i-1})^{-1} \circ f$ is measure zero in D^{n-u}. We choose a regular value q'_i close to 0.

Let $\{\widehat{\Phi}'^{u,i}_t\}_{t\in[0,1]}$ be the isotopy with support in $(\mathrm{int}(D^u) \times \mathrm{int}(D^{n-u}))_i$ such that $\widehat{\Phi}'^{u,i}_t(x, 0) = (x, t\mu(x)q'_i)$ ($\widehat{\Phi}'^{u,i}_0 = \mathrm{id}$). Then since q'_i is a regular value,

$$L'^{n-u}_{i-1} \cap \widehat{\Phi}'^{u,i}_1(K^{(u)}) \cap (\mathrm{int}(D^u_{1/2}) \times \mathrm{int}(D^{n-u}_{1/2}))_i$$

or

$$(\widehat{\Phi}'^{u,i}_1)^{-1}L'^{n-u}_{i-1} \cap K^{(u)} \cap (\mathrm{int}(D^u_{1/2}) \times \mathrm{int}(D^{n-u}_{1/2}))_i$$

consists of transverse intersection points. Since q'_i is close to zero, the transversality in $\bigcup_{j=1}^{i-1}(\mathrm{int}(D^u_{1/2}) \times \mathrm{int}(D^{n-u}_{1/2}))$ is preserved. Hence for $\{\widehat{\Phi}^{u,i}_t = \widehat{\Phi}^{u,i-1}_t \circ \widehat{\Phi}'^{u,i}_t\}_{t\in[0,1]}$,

$$K^{(u)} \cap (\Phi^{u-1}_1 \circ \widehat{\Phi}^{u,i}_t)^{-1}(f(L^{(n-u-1)})) = \emptyset$$

and

$$K^{(u)} \cap (\Phi^{u-1}_1 \circ \widehat{\Phi}^{u,i}_t)^{-1}(f(L^{(n-u)})) \cap \bigcup_{j=1}^{i}(\mathrm{int}(D^u_{1/2}) \times \mathrm{int}(D^{n-u}_{1/2}))_j$$

consists of transverse intersection points.

Then for $\widehat{\Phi}^u_t = \Phi^{u-1}_1 \circ \widehat{\Phi}^{u,k_u}_t$,

$$K^{(u)} \cap (\widehat{\Phi}^u_t)^{-1}(f(L^{(n-u-1)})) = \emptyset$$

and

$$K^{(u)} \cap (\widehat{\Phi}^u_t)^{-1}(f(L^{(n-u)}))$$

consists of transverse intersection points. Since $K^{(u)} \cap (\widehat{\Phi}^u_t)^{-1}(f(L^{(n-u)}))$ is compact, this is a finite set.

Put $\Phi_t = \widehat{\Phi}^k_t$. Then Φ_t is the desired isotopy. □

In the rest of this section, we sketch the proof of Theorem 1.1 (3). We need three more lemmas whose proofs are omitted because they are either straightforward or given by rewriting those in [30].

By using Lemma 2.6 and the argument of the proof of Theorem 2.2, we obtain the following lemma.

Lemma 2.7 ([30], Lemma 6.3). *Let M^n be a compact n-dimensional manifold. Let P^p and Q^q be p-dimensional and q-dimensional stratified subsets of M^n, respectively. Assume that $p+q+1=n$ and that $P^p \cap Q^q = \emptyset$. Let $P^p = P^{(p)} \supset P^{(p-1)} \supset \cdots \supset P^{(0)}$ and $Q^q = Q^{(q)} \supset Q^{(q-1)} \supset \cdots \supset Q^{(0)}$ be the stratifications. Then any element $f \in \mathrm{Diff}^r(M^n)_0$ can be written as a product $f = g \circ h$ such that $g \in \mathrm{Diff}^r_c(M^n \backslash k(Q^q))_0$ and $h \in \mathrm{Diff}^r_c(M^n \backslash P^{(p-1)})_0$, where $k \in \mathrm{Diff}^r_c(M^n \backslash P^p)_0$ is a diffeomorphism of M^n with support in a small neighborhood of Q^q. Moreover there is an isotopy $\{h_t\}_{t\in[0,1]}$ such that $h_0 = \mathrm{id}$, $h_1 = h$, h_t is the identity on a neighborhood of $P^{(p-1)}$, and for $H(t,x) = h_t(x)$, $H([0,1] \times P^p) \cap k(Q^{(q-1)}) = \emptyset$ and $H([0,1] \times (P^p \setminus P^{(p-1)})) \cap k(Q^q \setminus Q^{(q-1)})$ consists of finitely many transverse intersection points.*

For an odd dimensional compact manifold M^{2m+1}, we considered a handle decomposition of M^{2m+1} in [30]. Let $M^{2m+1} = P^{(2m+1)} \supset \cdots \supset P^{(0)}$ be the stratification by the stable manifolds for the gradient flow for the corresponding Morse function, and $M^{2m+1} = Q^{(2m+1)} \supset \cdots \supset Q^{(0)}$ be the stratification by the unstable manifolds for the gradient flow. We look at the stratified subsets $P^m = P^{(m)}$ and $Q^m = Q^{(m)}$, and we have the following lemma.

Lemma 2.8 ([30], Lemma 6.4). *Let $\{h_t\}_{t\in[0,1]}$ ($h_0 = \mathrm{id}$) be a C^r isotopy which is the identity on a neighborhood of $P^{(m-1)}$ and $H([0,1]\times P^m)\cap k(Q^{(m-1)}) = \emptyset$ for $H(t,x) = h_t(x)$. Let $V^m \subset P^m$ be the complement of a neighborhood of $P^{(m-1)}$ where $h_t = \mathrm{id}$. Then there is a C^∞ isotopy $\{\bar{h}_t\}_{t\in[0,1]}$ ($\bar{h}_0 = \mathrm{id}$) fixing a neighborhood of $P^{(m-1)}$ such that its trace $\bar{H}\colon [0,1]\times M^{2m+1} \to M^{2m+1}$ is C^r close to $H\colon [0,1]\times M^{2m+1} \to M^{2m+1}$ and $\bar{H}|[0,1]\times V^m$ is an immersion outside of a finite subset. Moreover the image*

$$\bar{H}([0,1]\times V^m) \subset M^{2m+1} \setminus (P^{(m-1)} \cup k(Q^{(m-1)}))$$

has finitely many double point curves which is in general position with respect to the curves $\bar{H}([0,1]\times\{v\})$ ($v \in V^m$). If $m \geq 2$ these double point curves are disjoint, and if $m = 1$, there are at most finitely many triple points and cusps.

Then, using the idea of Burago, Ivanov and Polterovich ([4]), we constructed an isotopy $\{a_t\}_{t\in[0,1]}$ ($a_0 = \mathrm{id}$) with support in a union of disjointly embedded $(2m+1)$-dimensional open balls embedded in M^{2m+1} such that $(a_t \circ \bar{h}_t)(P^m) \cap k(Q^m) = \emptyset$ ($t \in [0,1]$), and we showed the following lemma.

Lemma 2.9 ([30], Lemma 6.5). *For the generic diffeomorphism*

$$\bar{h} = \bar{h}_1 \in \mathrm{Diff}^\infty_c(M^{2m+1} \setminus P^{(m-1)})_0$$

given by Lemma 2.8, $\bar{h}$ can be decomposed as $\bar{h} = a\circ\bar{g}\circ\bar{h}'$, where $a \in \mathrm{Diff}^\infty_c(\bigsqcup_i U_i)_0$, $\bigsqcup_i U_i$ is a union of $(2m+1)$-dimensional open balls U_i disjointly embedded in M^{2m+1}, $\bar{g} \in \mathrm{Diff}^\infty_c(M^{2m+1}\setminus k(Q^m))_0$ and $\bar{h}' \in \mathrm{Diff}^\infty_c(M^{2m+1}\setminus P^m)_0$.

Proof of Theorem 1.1 (3). Note that the element $\bar{h}^{-1}\circ h \in \mathrm{Diff}^r(M^{2m+1})_0$ is close to the identity and it can be decomposed as $\bar{h}^{-1}\circ h = \hat{h}\circ\hat{g}$ with $\hat{h} \in \mathrm{Diff}^r_c(M^{2m+1}\setminus P^m)_0$ and $\hat{g} \in \mathrm{Diff}^r_c(M^{2m+1}\setminus k(Q^m))_0$ (Remark 5.4 in [30], see Remark 2.10). Then by Lemmas 2.7 and 2.9,

$$\begin{aligned} f &= g\circ h = g\circ\bar{h}\circ(\bar{h}^{-1}\circ h) \\ &= g\circ a\circ\bar{g}\circ\bar{h}'\circ\hat{h}\circ\hat{g} \\ &= (g\circ a\circ g^{-1})\circ(g\circ\bar{g}\circ\hat{g})\circ(\hat{g}^{-1}\circ\bar{h}'\circ\hat{h}\circ\hat{g}) \end{aligned}$$

and $g \circ a \circ g^{-1} \in \mathrm{Diff}_c^r(g(\bigsqcup_i U_i))_0$, $g \circ \bar{g} \circ \hat{g} \in \mathrm{Diff}_c^r(M^{2m+1} \setminus k(Q^m))_0$ and $\hat{g}^{-1} \circ \bar{h}' \circ \hat{h} \circ \hat{g} \in \mathrm{Diff}_c^r(M^{2m+1} \setminus \hat{g}^{-1}(P^m))_0$. Noticing that a can be taken as a commutator with support in $\bigsqcup_i U_i$, Theorem 1.1 (1) implies Theorem 1.1 (3) (see [30]). □

It is worth noticing again that, for any compact manifold M^n, there is a neighborhood of the identity of $\mathrm{Diff}^r(M^n)_0$ ($1 \leq r \leq \infty, r \neq n+1$) whose element can be written as a product of four or six commutators([30], Remark 5.4).

Remark 2.10. For a compact manifold M, we have a self-indexing Morse function $F\colon M^n \to [0,n]$. By choosing a Riemannian metric on M^n, we have the stratification $\{X^{(k)}\}_{k=0}^n$ by the stable manifolds for the gradient flow of the Morse function F, and the stratification $\{X^{*(n-k)}\}_{k=0}^n$ by the unstable manifolds. For a compact odd-dimensional manifold M^{2m+1}, M^{2m+1} is covered by two open sets $U_1 = F^{-1}([0, m+2/3))$ and $U_2 = F^{-1}((m+1/3, 2m+1])$, where any neighborhood of $X^{(m)} \subset U_1$ contains a deformation retract of U_1 and any neighborhood of $X^{*(m)} \subset U_2$ contains a deformation retract of U_2. Then by the fragmentation lemma ([2]), there is a neighborhood $\mathcal{N}$ of the identity in $\mathrm{Diff}^r(M^{2m+1})_0$ such that any element f of $\mathcal{N}$ can be written as a product $f = g \circ h$, where $g \in \mathrm{Diff}_c^r(U_1)_0$ and $h \in \mathrm{Diff}_c^r(U_2)_0$. Hence by Theorem 1.1 (1), any element f of $\mathcal{N}$ can be written as a product of four commutators of elements of $\mathrm{Diff}^r(M^{2m+1})_0$ ($1 \leq r \leq \infty$, $r \neq 2m+2$). For a compact even-dimensional manifold M^{2m}, M^{2m} is covered by three open sets U_1, U_2 and U_3. Here, U_3 is a union of disjointly embedded open balls which is a neighborhood of the set of critical points of index m. Let V_3 be a smaller neighborhood of the critical points of index m such that $\bar{V}_3 \subset U_3$. Then we can put $U_1 = (M^{2m} \setminus \bar{V}_3) \cap F^{-1}([0, m+\varepsilon))$ and $U_2 = (M^{2m} \setminus \bar{V}_3) \cap F^{-1}((m-\varepsilon, 2m])$ for a small positive real number ε. Here, we can choose V_3 so that any neighborhood of $X^{(m-1)} \subset U_1$ contains a deformation retract of U_1 and any neighborhood of $X'^{(m-1)} \subset U_2$ contains a deformation retract of U_2. Then by the fragmentation lemma, there is a neighborhood $\mathcal{N}$ of the identity in $\mathrm{Diff}^r(M^{2m})_0$ such that any element f of $\mathcal{N}$ can be written as a product $f = a \circ g \circ h$, where $g \in \mathrm{Diff}_c^r(U_1)_0$, $h \in \mathrm{Diff}_c^r(U_2)_0$ and $a \in \mathrm{Diff}_c^r(U_3)_0$. Hence by Theorem 1.1 (1), any element f of $\mathcal{N}$ can be written as a product of six commutators of elements of $\mathrm{Diff}^r(M^{2m})_0$ ($1 \leq r \leq \infty, r \neq 2m+1$).

3. Proof of the main theorem

For an even dimensional compact manifold M^{2m}, we proceed as follows to prove Theorem 1.2. (The proofs of lemmas are given in the next section.)

For the manifold M^{2m}, we consider any smooth triangulation P of it (for the existence of smooth triangulations, see [33], [37], [20], [6]). Let $P^{(k)}$ denote the

k-dimensional skeleton of P. Then the $(m-1)$-dimensional skeleton $P^{(m-1)}$ of the triangulation P has the following property:

> For each m-dimensional simplex σ^m of $P^{(m)}$, let $(P^{(m-1)} \cup \sigma^m)/\sigma^m$ denote the $(m-1)$-dimensional cell complex obtained from $P^{(m-1)} \cup \sigma^m$ by identifying σ^m to a point. Then there is an embedding ι of $(P^{(m-1)} \cup \sigma^m)/\sigma^m$ in M^{2m} such that, for any neighborhood U of $\iota((P^{(m-1)} \cup \sigma^m)/\sigma^m)$, there is a diffeomorphism of M^{2m} isotopic to the identity which maps $P^{(m-1)} \cup \sigma^m$ into U.

For any smooth triangulation P of M^{2m}, there are a Morse function on M^{2m} and a Riemannian metric on M^{2m} such that the stratification by the stable manifolds of the gradient flow is homeomorphic to P. Here, in a neighborhood of the barycenter b_{σ^k} of the simplex σ^k, we can take a coordinate neighborhood $(U, (x_1, \dots, x_n))$ such that σ^k is locally given as $x_{k+1} = \dots = x_n = 0$, and the Morse function in a neighborhood of b_{σ^k} is given by $k - {x_1}^2 - \dots - {x_k}^2 + {x_{k+1}}^2 + \dots + {x_n}^2$. The homeomorphism can be constructed so that it sends the stable manifold of the barycenter b_{σ^k} differentiably to the interior of the simplex σ^k. Moreover the homeomorphism can be constructed so that it sends the stratification Q by the unstable manifolds of the gradient flow to the cell decomposition P^* dual to P. We show this fact in Section 6 (Proposition 6.1). In this section, we identify the stratification by the stable manifolds with the triangulation P by the homeomorphism and it is denoted by P, and then, we call the stratification Q by the unstable manifolds the cell stratification dual to P. We call the stable manifolds of P *simplices* and the unstable manifold of Q *dual cells*.

Remark 3.1. We may use a cellular complex associated with a handle decomposition of M^{2m} if it has the above property for each m-dimensional cell σ^m. The number N of the m-dimensional cells of such a cellular decomposition of M^{2m} appears in the estimate of the bound for the number of commutators at the end of the proof of Theorem 1.2. We discuss the relationship between the handle decomposition and the cellular decomposition in Section 6 (Proposition 6.2).

Now we look at the m-dimensional skeletons P^m and Q^m of the triangulation P and its dual cell stratification Q. These P^m and Q^m intersect transversely at the barycenters of m-dimensional simplices of P. Then by an isotopy f_t ($t \in [0,1]$), the intersection $f_t(P^m) \cap Q^m$ becomes very complicated. However, we can treat it as follows.

For the manifold M^{2m}, the statement of Lemma 2.7 is written as follows.

Lemma 3.2. *Let P^m denote the m-dimensional skeleton of a triangulation of a $(2m)$-dimensional manifold M^{2m}, and Q^m, the m-dimensional skeleton of the dual cell stratification. Let $P^{(i)}$ and $Q^{(i)}$ denote the i-dimensional skeletons ($i = m-2$, $m-1$) of P^m and Q^m, respectively. Then any element $f \in \mathrm{Diff}^r(M^{2m})_0$ can be written as a product $f = g \circ h$ such that $g \in \mathrm{Diff}^r_c(M^{2m} \setminus k(Q^m))_0$ and $h \in \mathrm{Diff}^r_c(M^{2m} \setminus P^{(m-2)})_0$, where $k \in \mathrm{Diff}^r_c(M^{2m} \setminus P^m)_0$ is a diffeomorphism* *of* M^n *with support in a small neighborhood of* Q^m*. Moreover there is an isotopy* $\{h_t\}_{t\in[0,1]}$ *which has the following properties:*

(1) $h_0 = \mathrm{id}$*,* $h_1 = h$*, and* h_t *is the identity on a neighborhood of* $P^{(m-2)}$*.*

(2) *For* $H(t,x) = h_t(x)$*,*
$H([0,1]\times P^{(m-1)}) \cap k(Q^{(m-1)}) = \emptyset$ *and* $H([0,1]\times P^m)\cap k(Q^{(m-2)}) = \emptyset$*.*

(3) *For each* $(m-1)$*-dimensional simplex* σ^{m-1} *of* $P^{(m-1)}$ *and each* m*-dimensional cell* τ^m *of* Q^m*, the intersection* $H([0,1]\times\sigma^{m-1})\cap k(\tau^m)$ *is transverse. Thus* $H([0,1]\times P^{(m-1)})\cap k(Q^m)$ *is a finite set.*

Then, if $2m \geq 4$, we can separate the image $H([0,1]\times P^{(m-1)})$ from $k(Q^m)$ by an argument similar to the proof of Lemmas 2.8 and 2.9.

First, we approximate the isotopy H by a generic one, say $\bar{H}$. Let

$$\{\bar{h}_t\}_{t\in[0,1]} \subset \mathrm{Diff}_c^\infty(M^{2m}\setminus P^{(m-2)}) \quad (\bar{h}_0 = \mathrm{id})$$

be a C^∞ approximation of $\{h_t\}_{t\in[0,1]} \subset \mathrm{Diff}_c^r(M^{2m}\setminus P^{(m-2)})$ generic with respect to P^m and $k(Q^m)$ such that $\bar{h}_t$ is the identity on a neighborhood of $P^{(m-2)}$. Then $\bar{H}(t,x) = \bar{h}_t(x)$ has the following properties:

(0) $\bar{H}\colon [0,1]\times M^{2m}\to M^{2m}$ is close to $H\colon [0,1]\times M^{2m}\to M^{2m}$ and $\bar{h}_t$ is the identity on a neighborhood of $P^{(m-2)}$.

(1) The restriction

$$\bar{H}|([0,1]\times V^{m-1})\colon [0,1]\times V^{m-1} \longrightarrow M^{2m}$$

is an immersion, where $V^{m-1}(\subset P^{(m-1)})$ is the complement of a neighborhood of $P^{(m-2)}\subset P^{(m-1)}$ where $\bar{h}_t$ is the identity.

(2) $\bar{H}([0,1]\times P^{(m-1)})\cap k(Q^{(m-1)}) = \emptyset$ and $\bar{H}([0,1]\times P^m)\cap k(Q^{(m-2)}) = \emptyset$.

(3) $\bar{H}([0,1]\times P^{(m-1)})\cap k(Q^m)$ is a finite set:

$$\bar{H}([0,1]\times P^{(m-1)})\cap k(Q^m) = \{\bar{H}(s_i,v_i) \mid i = 1, \ldots, r\}.$$

(4) $\bar{H}([0,1]\times\{v_i\})\cap k(Q^m) = \bar{H}(s_i,v_i)$ $(i = 1,\ldots,r)$.

(5) $\bar{H}([0,1]\times\{v_i\})$ does not contain double points of $\bar{H}([0,1]\times P^{(m-1)})$ $(i = 1,\ldots,r)$.

(6) $\bar{H}|[0,1]\times P^{(m-1)}$ restricted to a neighborhood of $[0,1]\times\{v_i\}$ in $[0,1]\times P^{(m-1)}$ is an embedding $(i = 1,\ldots,r)$, and

(7) $\bar{H}([s_i,1]\times\{v_i\})$ $(i = 1,\ldots,r)$ are disjoint.

Here, the statements (1)–(7) hold for generic $\bar{H}$ (or the properties (1)–(7) are generic in the space of isotopies). In particular, the statement (5) holds because the inverse image of the double point set of $\bar{H}([0,1] \times P^{(m-1)})$ is a finite set which is in general position with respect to $[0,1] \times \{v_i\}$ $(i = 1, \dots, r)$ and $2m \geq 4$.

Note that for the proof of uniform perfectness, we can approximate the diffeomorphism for a bounded number of times. In fact in this case, $f_1 = g_1 \circ h_1 = g_1 \circ \bar{h}_1 \circ (\bar{h}_1{}^{-1} \circ h_1)$ and $\bar{h}_1{}^{-1} \circ h_1 \in \mathrm{Diff}^r(M^{2m})$ is close to the identity. By Remark 2.10, $\bar{h}_1{}^{-1} \circ h_1$ can be written as a product of six commutators.

For the above disjoint curves $\bar{H}([s_i, 1] \times \{v_i\})$, we can construct isotopies as in Lemma 2.9 which was used to prove Theorem 1.1 (3).

Lemma 3.3. *For the above generic isotopy $\{\bar{h}_t\}_{t \in [0,1]}$, there is a neighborhood U_i $(i = 1, \dots, r)$ of the curve $\bar{H}([s_i, 1] \times \{v_i\}) \subset M^{2m}$ diffeomorphic to a $(2m)$-dimensional ball such that U_i are disjoint and there is an isotopy $\{a_t\}_{t \in [0,1]}$ $(a_0 = \mathrm{id})$ with support in $\bigsqcup_{i=1}^r U_i$ such that, for $h'_t = a_t \circ \bar{h}_t$,*

$$h'_t(P^{(m-1)}) \cap k(Q^m) = \emptyset \quad (t \in [0,1]).$$

Note that $a_t \in \mathrm{Diff}^r_c(\bigsqcup_{i=1}^r U_i)_0$ can be taken as one commutator with support in $\bigsqcup_{i=1}^r U_i$ (see [31]).

Since $h'_t(P^{(m-1)}) \cap k(Q^m) = \emptyset$ $(t \in [0,1])$, by Lemma 2.3, there are isotopies $\{g'_t\}_{t \in [0,1]} \subset \mathrm{Diff}^r_c(M^{2m} \setminus k(Q^m))$ and $\{h''_t\}_{t \in [0,1]} \subset \mathrm{Diff}^r_c(M^{2m} \setminus P^{(m-1)})$ such that $h'_1 = g'_1 \circ h''_1$. In other words, g'_t and h''_t $(t \in [0,1])$ are the identity on neighborhoods of $k(Q^m)$ and $P^{(m-1)}$, respectively. Note that, by taking h''_t generically on P^m, $h''_t(P^m) \cap k(Q^{(m-2)}) = \emptyset$.

Put $h^{(0)}_t = h''_t$. Then $h^{(0)}_t$ is the identity on a neighborhood of $P^{(m-1)}$ and $h^{(0)}_t(P^m) \cap k(Q^{(m-2)}) = \emptyset$ $(t \in [0,1])$.

We look at the intersection $h^{(0)}_t(P^m) \cap k(Q^m)$. At time 0, the intersection $h^{(0)}_0(P^m) \cap k(Q^m)$ is the set of the points near the barycenters of m-dimensional simplices. The image under the isotopy $h^{(0)}_t$ of an m-dimensional simplex σ^m intersects $k(Q^{(m-1)})$ and $k(Q^m)$. We assume $2m \geq 6$ and we are going to construct an isotopy with support in the union of disjointly embedded balls which removes the intersection of σ^m and $k(Q^m)$ except on the dual m-dimensional cell.

This is the main part of the proof of our Theorem 1.2.

In fact, for an m-dimensional simplex σ^m, we can remove the intersection of the image of the isotopy of σ^m and $k(Q^{(m-1)})$ in a way similar to Lemma 3.3, and then we can remove the intersection of the resultant isotopy of σ^m and $k(Q^m \setminus \sigma^{m*})$, where σ^{m*} is the m-dimensional cell of Q^m dual to σ^m. For the latter process, we will find the Whitney disks which guide the construction of isotopy to reduce the order of the intersection point set. After removing the intersection of an m-dimensional simplex σ^m and $k(Q^m \setminus \sigma^{m*})$, we continue the process for other m-dimensional simplices.

More precisely, we construct the isotopies inductively, in Lemmas 3.4–3.7.

Let σ_i^m $(i = 1, \ldots, N)$ be the m-dimensional simplices of P^m. For $0 \le j \le N$, assume that we have an isotopy

$$\{h_t^{(j)}\}_{t\in[0,1]} \subset \mathrm{Diff}^r(M^{2m})_0 \quad (h_0^{(j)} = \mathrm{id})$$

such that $h_t^{(j)}$ is the identity on a neighborhood of $P^{(m-1)} \cup \bigcup_{i=1}^j \sigma_i^m$. Let $\bar{h}_t^{(j)}$ be a C^∞ approximation of $h_t^{(j)}$ generic with respect to P^m and $k(Q^m)$ such that $\bar{h}_t^{(j)}$ is the identity on a neighborhood of $P^{(m-1)} \cup \bigcup_{i=1}^j \sigma_i^m$. Then $\bar{H}^{(j)}(t, x) = \bar{h}_t^{(j)}(x)$ has the following properties:

(0) $\bar{H}^{(j)}\colon [0, 1] \times M^{2m} \to M^{2m}$ is close to $H^{(j)}\colon [0, 1] \times M^{2m} \to M^{2m}$ defined by $H^{(j)}(t, x) = h_t^{(j)}(x)$ and $\bar{h}_t^{(j)}$ is the identity on a neighborhood of $P^{(m-1)} \cup \bigcup_{i=1}^j \sigma_i^m$.

(1) The restriction

$$\bar{H}^{(j)}|[0, 1] \times V_{(j)}^m\colon [0, 1] \times V_{(j)}^m \longrightarrow M^{2m}$$

is an immersion outside of a 1-dimensional subset (a codimension m subset) of $[0, 1] \times V_{(j)}^m$, where $V_{(j)}^m (\subset P^m)$ is the complement of a neighborhood of $P^{(m-1)}$ in P^m where $\bar{h}_t^{(j)}$ is the identity.

(2) $\bar{H}^{(j)}([0, 1] \times P^{(m-1)}) \cap k(Q^{(m-1)}) = \emptyset$ and $\bar{H}^{(j)}([0, 1] \times P^m) \cap k(Q^{(m-2)}) = \emptyset$.

(3) $\bar{H}^{(j)}([0, 1] \times P^m) \cap k(Q^{(m-1)})$ is a finite set:

$$\bar{H}^{(j)}([0, 1] \times P^m) \cap k(Q^{(m-1)}) = \{\bar{H}^{(j)}(s_i^{(j)}, v_i^{(j)}) \mid i = 1, \ldots, r^{(j)}\}.$$

(4) $\bar{H}^{(j)}([0, 1] \times \{v_i^{(j)}\}) \cap k(Q^{(m-1)}) = \bar{H}^{(j)}(s_i^{(j)}, v_i^{(j)})$ $(i = 1, \ldots, r^{(j)})$.

(5) $\bar{H}^{(j)}([0, 1] \times \{v_i^{(j)}\})$ does not contain double points of $\bar{H}^{(j)}([0, 1] \times P^m)$ $(i = 1, \ldots, r^{(j)})$.

(6) $\bar{H}^{(j)}|[0, 1] \times P^m$ restricted to a neighborhood of $[0, 1] \times \{v_i^{(j)}\}$ in $[0, 1] \times P^m$ is an embedding $(i = 1, \ldots, r^{(j)})$, and

(7) $\bar{H}^{(j)}([s_i^{(j)}, 1] \times \{v_i^{(j)}\})$ are disjoint.

Here, the statements (1)–(7) hold for generic $\bar{H}^{(j)}$. In particular, for the statement (1), we notice that the set of rank m matrices in the space of $(m + 1) \times (2m)$ matrices is codimension m ([22]). The statement (6) holds because the inverse image of the double point set of $\bar{H}^{(j)}([0, 1] \times P^m)$ is 2-dimensional in $[0, 1] \times P^m$ which is in general position with respect to $[0, 1] \times \{v_i^{(j)}\}$ $(i = 1, \ldots, r^{(j)})$ and $2m \ge 6$.

Lemma 3.4. *For the above generic isotopy* $\{\bar{h}_t^{(j)}\}_{t\in[0,1]}$*, there is a neighborhood* $U_i^{(j)}$ $(i = 1, \dots, r^{(j)})$ *of the curve* $\bar{H}^{(j)}([s_i^{(j)}, 1]\times\{v_i^{(j)}\}) \subset M^{2m}$ *diffeomorphic to a* $(2m)$*-dimensional ball such that* $U_i^{(j)}$ *are disjoint and there is an isotopy* $\{a_t^{(j+1)}\}_{t\in[0,1]}$ $(a_0^{(j+1)} = \mathrm{id})$ *with support in* $\bigsqcup_{i=1}^{r^{(j)}} U_i^{(j)}$ *such that, for* $h'^{(j)}_t = a_t^{(j+1)} \circ \bar{h}_t^{(j)}$*,*

$$h'^{(j)}_t(P^m) \cap k(Q^{(m-1)}) = \emptyset \quad (t \in [0,1]).$$

Note again that $a_t^{(j+1)} \in \mathrm{Diff}_c^r(\bigsqcup_{i=1}^{r^{(j)}} U_i^{(j)})_0$ can be taken as one commutator with support in $\bigsqcup_{i=1}^{r^{(j)}} U_i^{(j)}$ (see [31]).

The isotopy $h'^{(j)}_t$ given by Lemma 3.4 has the following properties.

(0) $h'^{(j)}_t$ is the identity on a neighborhood of $P^{(m-1)} \cup \bigcup_{i=1}^{j} \sigma_i^m$.

(1) $H'^{(j)}([0,1] \times P^m) \cap k(Q^{(m-1)}) = \emptyset$.

(2) $h'^{(j)}_t$ is generic with respect to P^m and $k(Q^m)$.

Now we look at the intersection $h'^{(j)}_t(P^m) \cap k(Q^m)$. Since $h'^{(j)}_t$ is the identity on a neighborhood of $P^{(m-1)} \cup \bigcup_{i=1}^{j} \sigma_i^m$, the intersection $h'^{(j)}_t(\sigma_i^m) \cap k(Q^m)$ for $i \leq j$ is always the one point set $\sigma_i^m \cap k(\sigma_i^{m*})$, where σ_i^{m*} is the m-dimensional cell of Q^m dual to σ_i^m $(i \leq j)$. For the simplex σ_{j+1}^m, the intersection $h'^{(j)}_t(\sigma_{j+1}^m) \cap k(Q^m)$ is a finite set which vary with respect to the parameter t. If $2m \geq 6$, we can find the Whitney disks which guide to reduce the order of intersection point set $h'^{(j)}_t(\sigma_{j+1}^m) \cap k(Q^m \setminus \sigma_{j+1}^{m*})$, where σ_{j+1}^{m*} is the m-dimensional cell of Q^m dual to σ_{j+1}^m as we explain now.

For the m-dimensional simplex σ_{j+1}^m of P^m, the intersection of σ_{j+1}^m and $k(Q^m)$ is just one point which is the intersection of σ_{j+1}^m and $k(\sigma_{j+1}^{m*})$, Then the behavior of the intersection $h'^{(j)}_t(\sigma_{j+1}^m) \cap k(\sigma_{j+1}^{m*})$ it rather complicated. Hence we look at $H'^{(j)}([0,1] \times \sigma_{j+1}^m) \cap k(Q^m \setminus \sigma_{j+1}^{m*})$ or $h'^{(j)}_t(\sigma_{j+1}^m) \cap k(Q^m \setminus \sigma_{j+1}^{m*})$. First, note that $h'^{(j)}_t(\sigma_{j+1}^m) \cap k(Q^m \setminus \sigma_{j+1}^{m*})$ is the empty set for small t, and since $h'^{(j)}_t(\sigma_{j+1}^m) \cap k(Q^{(m-1)}) = \emptyset$ (and $h'^{(j)}_t(P^{(m-1)}) \cap k(Q^m) = \emptyset$), the algebraic intersection number of the two m-dimensional cells $h'^{(j)}_t(\sigma_{j+1}^m)$ and $k(\tau^m)$ $(t \in [0,1])$ is always 0 for each m-dimensional cell τ^m of the dual cell complex Q^m other than σ_{j+1}^{m*}.

If we look at the movement of the intersection $h'^{(j)}_t(\sigma_{j+1}^m) \cap k(\tau^m)$ with respect to the parameter t, there happen a finite number of generations of pairs of intersection points and cancellations of pairs of intersection points. For generic $H'^{(j)}$ or $h'^{(j)}_t$, the values of the parameters t of generations and cancellations are different. This genericity argument follows from the following well known lemma.

Lemma 3.5. *Consider the space of* C^∞ *maps* $F : \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}^m$*. Then, for generic* F*, the inverse image of a generic point* $y \in \mathbb{R}^m$ *consists of regular points and fold* *points for* $F_t = F(t, \bullet)$*. At a fold point* x *for* F_t*, by changing the coordinates of* $\mathbb{R}^m$ *(both of the second factor of* $\mathbb{R} \times \mathbb{R}^m$ *and the target* $\mathbb{R}^m$*),* F_t *is locally written as*

$$F_t(x_1, \dots, x_m) = (x_1, \dots, x_{m-1}, y_m(t, x_1, \dots, x_m)),$$

where $\frac{\partial y_m}{\partial x_m} = 0$*,* $\frac{\partial y_m}{\partial t} \neq 0$ *and* $\frac{\partial^2 y_m}{\partial x_m{}^2} \neq 0$ *at* x*. The fold points are discrete in* $F^{-1}(y)$ *and correspond to the generations or cancellations of pairs of intersection points.*

We use this Lemma 3.5 in the following way. We take a tubular neighborhood of $k(\tau^m)$ and the projection $p_{k(\tau^m)}$ to the fiber which is an m-dimensional disk, and look at the map $p_{k(\tau^m)} \circ (H'^{(j)}|[0,1] \times \sigma^m_{j+1})$. Then for generic $H'^{(j)}$, by using Lemma 3.5, there are only finitely many generations and cancellations of pairs of intersections in the family $\{h'^{(j)}_t(\sigma^m_{j+1}) \cap k(\tau^m)\}_{t \in [0,1]}$.

We are going to construct the disks associated with the intersection $H'^{(j)}([0,1] \times \sigma^m_{j+1}) \cap k(\tau^m)$ for an m-dimensional cell τ^m of Q^m other than σ^{m*}_{j+1}.

For a generation of a pair of intersection points, the intersection points near the generation point are written as $h'^{(j)}_t(x_t)$ and $h'^{(j)}_t(y_t)$ ($t \in [t_0, t_0 + \varepsilon_0)$), where $h'^{(j)}_{t_0}(x_{t_0}) = h'_{t_0}{}^{(j)}(y_{t_0})$ is the generation point. Here, x_t and y_t are continuous functions written as $x_t = (c_1, \dots, c_{m-1}, \sqrt{t - t_0})$ and $y_t = (c_1, \dots, c_{m-1}, -\sqrt{t - t_0})$, respectively, for a suitable choice of coordinate around $(t_0, x_{t_0}) = (t_0, y_{t_0}) \in [0,1] \times \sigma^m_{j+1}$, where $c_1, \dots, c_{m-1}$ are constants.

We take a flat metric on the m-dimensional simplex σ^m_{j+1} and we draw the geodesic segment $\overline{x_t y_t}$ in σ^m_{j+1} joining the intersection points x_t and y_t ($t \in [t_0, t_0 + \varepsilon_0)$).

Once we choose the pair of intersection points to be joined by the geodesic segment, we continue joining them as the parameter t increases unless one of these intersection points meets a cancellation point.

For a cancellation of a pair of intersections, the intersection points near the cancellation point are written as $h'^{(j)}_t(x_t)$ and $h'^{(j)}_t(y_t)$ ($t \in (t_0 - \varepsilon_0, t_0]$), where $h'^{(j)}_{t_0}(x_{t_0}) = h'^{(j)}_{t_0}(y_{t_0})$ is the cancellation point. Here, x_t and y_t are continuous functions written as $x_t = (c_1, \dots, c_{m-1}, \sqrt{-t + t_0})$ and $y_t = (c_1, \dots, c_{m-1}, -\sqrt{-t + t_0})$, respectively, for a suitable choice of coordinate around $(t_0, x_{t_0}) = (t_0, y_{t_0}) \in [0,1] \times \sigma^m_{j+1}$, where $c_1, \dots, c_{m-1}$ are constants.

Assume that we have chosen geodesic segments for the intersection points such that $t < t_0$. Let x'_t ($t \in (t_0 - \varepsilon_0, t_0)$) be the other endpoint of the geodesic segment containing x_t, and y'_t ($t \in (t_0 - \varepsilon_0, t_0)$) be the other endpoint of the geodesic segment containing y_t. There are two cases. In the case where $x'_{t_0} \neq y'_{t_0}$, that is, if it is a cancellation of intersection points belonging to different geodesic segments $\overline{x_t x'_t}$ and $\overline{y_t y'_t}$ in $\{t\} \times \sigma^m_{j+1}$ ($t \in (t_0 - \varepsilon_0, t_0)$), we draw the geodesic triangle joining the 3 points $x_{t_0} = y_{t_0}$, x'_{t_0} and y'_{t_0} in $\{t_0\} \times \sigma^m_{j+1}$, and continue to draw the geodesic segment $\overline{x'_t y'_t}$ joining x'_t and y'_t in $\{t\} \times \sigma^m_{j+1}$ ($t \in (t_0, t_0 + \varepsilon_0)$). In the case where $x'_{t_0} = y'_{t_0}$, that is, if it is a cancellation of intersection points of the same geodesic segment $\overline{x_t y_t}$ in $\{t\} \times \sigma^m_{j+1}$ ($t \in (t_0 - \varepsilon_0, t_0)$, $x'_t = y_t$ and $y'_t = x_t$), we add the auxiliary band

$$\bigcup_{t\in[t_0-\varepsilon,t_0]} [t,1] \times \{x_t\} \cup \bigcup_{t\in[t_0-\varepsilon,t_0]} [t,1] \times \{y_t\},$$

which contains the curve $[t_0, 1] \times \{x_{t_0}\} = [t_0, 1] \times \{y_{t_0}\}$, where ε ($< \varepsilon_0$) is a small positive real number. Note that the image of the auxiliary band does not contain double points of $H'^{(j)}([0,1] \times \sigma^m_{j+1})$ for generic $H'^{(j)}$, and hence $H'^{(j)}$ restricted to the auxiliary band is an embedding into $M^{2m} \setminus k(Q^{(m-1)})$.

Now we have a family of geodesic segments in σ^m_{j+1} moving with respect to the parameter t and there are only finitely many times t_i ($i = 1, \ldots, \bar{r}^{(j)}$) when there appear geodesic triangles.

We are assuming that $2m \geq 6$, and for generic $h'^{(j)}_t$, the family of geodesic segments satisfies the following properties because the preimage of the double points of $h'^{(j)}_t(P^m)$ is 1-dimensional in $[0,1] \times \sigma^m_{j+1}$.

(1) The geodesic segments in σ^m_{j+1} joining the pairs of intersection points in $(h'^{(j)}_t)^{-1}(k(\tau^m))$ never contain the preimage of double points of $(h'^{(j)}_t)(P^m)$.

(2) The geodesic triangles never contain the preimage of double points of $(h'^{(j)}_t)(P^m)$.

For t_i ($i = 1, \ldots, \bar{r}^{(j)}$), let Y be the union of the geodesic triangle with the three vertices $x_{t_i} = y_{t_i}$, x'_{t_i} and y'_{t_i} in $\{t_i\} \times \sigma^m_{j+1}$, the geodesic segments $\overline{x_t x'_t}$ and $\overline{y_t y'_t}$ in $\{t\} \times \sigma^m_{j+1}$, ($t \in (t_i - \varepsilon_i, t_i)$) and the geodesic segments $\overline{x'_t y'_t}$ in $\{t\} \times \sigma^m_{j+1}$ ($t \in (t_i, t_i + \varepsilon_i)$):

$$\begin{aligned} Y = &\Big(\bigcup_{t\in(t_i-\varepsilon_i,t_i)} \{t\} \times \overline{x_t x'_t} \Big) \cup \Big(\bigcup_{t\in(t_i-\varepsilon_i,t_i)} \{t\} \times \overline{y_t y'_t} \Big) \\ &\cup \big(\{t_i\} \times \triangle x_{t_i} x'_{t_i} y'_{t_i} \big) \cup \Big(\bigcup_{t\in(t_i,t_i+\varepsilon_i)} \{t\} \times \overline{x'_t y'_t} \Big) \\ &\subset (t_i - \varepsilon_i, t_i + \varepsilon_i) \times \sigma^m_{j+1}. \end{aligned}$$

We deform it to obtain a 2-dimensional manifold Y' embedded in $(t_i - \varepsilon_i, t_i + \varepsilon_i) \times \sigma^m_{j+1}$ such that

$$\begin{aligned} \partial Y' = \partial Y &= \{(t, x'_t)\}_{t\in(t_i-\varepsilon_i,t_i+\varepsilon_i)} \cup \{(t, y'_t)\}_{t\in(t_i-\varepsilon_i,t_i+\varepsilon_i)} \\ &\quad \cup \{(t, x_t)\}_{t\in(t_i-\varepsilon_i,t_i]} \cup \{(t, y_t)\}_{t\in(t_i-\varepsilon_i,t_i]} \\ &\subset (t_i - \varepsilon_i, t_i + \varepsilon_i) \times \sigma^m_{j+1}, \end{aligned}$$

and Y' coincides with Y for $|t - t_i| \geq \varepsilon_i/2$ and the intersection of Y' and $\{t\} \times \sigma^m_{j+1}$ is a union of two disjoint differentiable curves near the original geodesic segments for $t \in [t_i - \varepsilon_i/2, t_i)$ and is one differentiable curve near the geodesic triangle for $t \in [t_i, t_i + \varepsilon_i/2]$.

Now we look at the union Z of geodesic segments which are not modified by the above operation and the manifolds Y' for all t_i $(i = 1, \dots, \bar{r}^{(j)})$. If there are auxiliary bands we add them to Z and modify it to make Z an embedded 2-dimensional manifold with boundary in $[0,1] \times \sigma^m_{j+1}$.

For a generic choice of the isotopy $H'^{(j)}$ and manifolds Y', if $2m \geq 8$, Z is a union of disjointly embedded 2-dimensional disks in $[0,1] \times \sigma^m_{j+1}$. If $2m = 6$, the 2-dimensional disks may intersect in $[0,1] \times \sigma^3_{j+1}$ creating finitely many double points.

For $2m \geq 8$, the fact that a connected component of the union Z is diffeomorphic to a 2-dimensional disk can be seen as follows: Consider the space obtained from Z by identifying the points in each connected component of $Z \cap (\{t\} \times \sigma^m_{j+1})$. Then it is a graph with vertices corresponding to the generation points and cancellation points. The generation points correspond to the vertices of valency 1 and the cancellation points correspond to the vertices of valency 3 except the cancellation points with auxiliary bands. For the cancellation points with auxiliary bands, the auxiliary bands become edges ending at $\{1\} \times \sigma^m_{j+1}$. Thus each connected component of the graph is a tree rooted at time $t = 1$ which grows in the negative direction in t. Hence each connected component of Z is a 2-dimensional disk.

In the case where $2m = 6$, we see in a similar way that $Z \subset [0,1] \times \sigma^3_{j+1}$ is an immersed image of 2-dimensional disks which has generically a finite number of double points. That is, the curves joining the pairs of intersection points in $(h'^{(j)}_t)^{-1}(k(\tau^3))$ may intersect at finitely many points $(\hat{t}_\ell, \hat{x}_\ell)$ $(\ell = 1, \dots, \hat{r}^{(j)})$. Then for generic $H'^{(j)}$, $\hat{t}_\ell$ are not the time of generations or cancellations. When two geodesic curves $\gamma^{(t)}_1$ and $\gamma^{(t)}_2$ intersect at the time $\hat{t}_\ell$, we modify one of the family $\{\gamma^{(t)}_2\}$ of geodesic curves near $\hat{t}_\ell$ by a family $\{\gamma'^{(t)}_2\}$ of curves which does not intersect $\{\gamma^{(t)}_1\}$ near $\hat{t}_\ell$.

More concretely, for a small positive real number $\hat{\varepsilon}_\ell$, we can find a neighborhood of $\gamma^{(\hat{t}_\ell)}_1 \cup \gamma^{(\hat{t}_\ell)}_2 \subset [0,1] \times \sigma^m$ which is diffeomorphic to $(\hat{t}_\ell - \hat{\varepsilon}_\ell, \hat{t}_\ell + \hat{\varepsilon}_\ell) \times X$, where X is a neighborhood of $[-1,1] \times \{0\} \times \{0\} \cup \{0\} \times [-1,1] \times \{0\}$ in $\mathbb{R}^3$,

$$\gamma^{(\hat{t}_\ell)}_1 = \{\hat{t}_\ell\} \times [-1,1] \times \{0\} \times \{0\}$$

and

$$\gamma^{(\hat{t}_\ell)}_2 = \{\hat{t}_\ell\} \times \{0\} \times [-1,1] \times \{0\}.$$

We can choose the parametrization in this neighborhood so that

$$\gamma^{(\hat{t}_\ell + s)}_1(u) = (\hat{t}_\ell + s, u, 0, s)$$

and

$$\gamma^{(\hat{t}_\ell + s)}_2(u) = (\hat{t}_\ell + s, v_1 s, u + v_2 s, v_3 s)$$

for a vector $(v_1, v_2, v_3) \in \mathbb{R}^3$ $(v_3 \neq 1)$. By using a smooth bump function $\mu\colon [-1,1] \to [0,1]$ such that $\mu(x) = \mu(-x)$, $\mu|[0,1/3] = 1$ and $\mu|[2/3,1] = 0$, we modify $\gamma_2^{(t)}$. Put

$$\gamma_2'^{(\hat{t}_\ell+s)}(u) = (\hat{t}_\ell + s, (1+c_\ell)\mu(s/\hat{\varepsilon}_\ell)\mu(u/\delta_\ell) + v_1 s, u + v_2 s, v_3 s),$$

where c_ℓ and δ_ℓ are small positive real numbers such that the image of $\gamma_2'^{(\hat{t}_\ell+s)}$ is contained in our neighborhood X. Then the curves $\gamma_1^{(t)}$ and $\gamma_2'^{(t)}$ $(t \in (\hat{t}_\ell - \hat{\varepsilon}_\ell, \hat{t}_\ell + \hat{\varepsilon}_\ell))$ do not intersect in σ_{j+1}^m.

Thus for $2m \geq 6$, using the above family of curves if necessary, we have the union Z' of a finite number of disjointly embedded 2-dimensional disks in $[0,1] \times \sigma_{j+1}^m$ such that

$$(H'^{(j)}|[0,1] \times \sigma_{j+1}^m)^{-1}(k(\tau^m)) \subset Z'.$$

Since $2m \geq 6$, the images under generic $H'^{(j)}$ of these 2-dimensional disks are disjointly embedded in $M^{2m} \setminus k(Q^{(m-1)})$. The images of these disks are called the Whitney disks.

We have been looking at the intersection point set $h_t'^{(j)}(\sigma_{j+1}^m) \cap k(\tau^m)$ for one m-dimensional cell τ^m of Q^m other than σ^{m*}. These considerations can be applied to the intersection point sets $h_t'^{(j)}(\sigma_{j+1}^m) \cap k(\tau^m)$ for all (finitely many) m-dimensional cells τ^m of Q^m other than σ^{m*} simultaneously. This is because, if $2m \geq 8$, the embedded 2-dimensional disks Z' are disjoint for different τ^m for generic $H'^{(j)}$, and if $2m = 6$, we can remove the intersection of the embedded 2-dimensional disks Z' for different τ^m in a way similar to what we did for the intersection of Z for the same τ^m. Thus we obtained the union Z' of a finite number of disjointly embedded 2-dimensional disks in $[0,1] \times \sigma_{j+1}^m$ such that

$$(H'^{(j)}|[0,1] \times \sigma_{j+1}^m)^{-1}(k(Q^m \setminus \sigma_{j+1}^{m*})) \subset Z',$$

and $H'^{(j)}|Z'$ is an embedding.

If $2m \geq 8$, then the Whitney disks $H'^{(j)}(Z')$ do not contain double points of $H'^{(j)}([0,1] \times P^m)$ for generic $H'^{(j)}$. This is because the inverse image of the double point set of $H'^{(j)}([0,1] \times P^m)$ is 2-dimensional in $[0,1] \times P^m$ and $m+1 \geq 5$.

If $2m = 6$, then the Whitney disks $H'^{(j)}(Z')$ may intersect the double point set of $H'^{(j)}([0,1] \times P^3)$. Then, for generic $H'^{(j)}$, the intersection is a finite set and we pick up the points of Whitney disks which are in the image of $h_t'^{(j)}(P^3)$ with larger t;

$$H'^{(j)}(t_i^{(j)}, w_i^{(j)}) = H'^{(j)}(t_i'^{(j)}, w_i'^{(j)}) \quad (i = 1, \dots, r'^{(j)}),$$

where $(t_i^{(j)}, w_i^{(j)})$ is a point $Z' \subset [0,1] \times \sigma_{j+1}^m$, $(t_i'^{(j)}, w_i'^{(j)}) \in [0,1] \times P^3$ and $t_i^{(j)} < t_i'^{(j)}$. Then, for generic $H'^{(j)}$, the curve $H'^{(j)}([t_i'^{(j)}, 1] \times \{w_i'^{(j)}\})$ is embedded in $M^{2m} \setminus k(Q^m)$ and does not contain double points of $H'^{(j)}([0,1] \times P^3)$ other than

$H'^{(j)}(t'^{(j)}_i, w'^{(j)}_i)$. Hence if $2m = 6$, we have the Whitney disks $H'^{(j)}(Z')$ together with the curves $H'^{(j)}([t'^{(j)}_i, 1] \times \{w'^{(j)}_i\})$ $(i = 1, \ldots, r'^{(j)})$.

Using the Whitney disks $H'^{(j)}(Z')$ and curves $H'^{(j)}([t'^{(j)}_i, 1] \times \{w'^{(j)}_i\})$ $(i = 1, \ldots, r'^{(j)})$, we prove the following lemmas in the next section.

Lemma 3.6. *For $h'^{(j)}_t$, there is an isotopy $\{b^{(j+1)}_t\}_{t\in[0,1]}$ $(b^{(j+1)}_0 = \mathrm{id})$ with support in a union of disjointly embedded open balls such that for $h''^{(j)}_t = b^{(j+1)}_t \circ h'^{(j)}_t$, $h''^{(j)}_t$ is the identity on a neighborhood of $P^{(m-1)} \cup \bigcup_{i=1}^{j} \sigma^m_i$ and $h''^{(j)}_t(\sigma^m_{j+1}) \cap k(Q^m \setminus \sigma^{m*}_{j+1}) = \emptyset$.*

Lemma 3.7. *For $h''^{(j)}_t$ given by Lemma 3.6, there are isotopies*

$$\{g^{(j+1)}_t\}_{t\in[0,1]} \subset \mathrm{Diff}^r_c(M^{2m} \setminus k(Q^m \setminus \sigma^{m*}_{j+1})) \quad (g^{(j+1)}_0 = \mathrm{id})$$

and

$$\{h^{(j+1)}_t\}_{t\in[0,1]} \subset \mathrm{Diff}^r_c\Big(M^{2m} \setminus \Big(P^{(m-1)} \cup \bigcup_{i=1}^{j+1} \sigma^m_i\Big)\Big) \quad (h^{(j+1)}_0 = \mathrm{id})$$

such that $h''^{(j)}_t = g^{(j+1)}_t \circ h^{(j+1)}_t$.

Now we complete the proof of our main Theorem 1.2.

Proof of Theorem 1.2. Let f be an element of $\mathrm{Diff}^r(M^{2m})_0$. By Lemma 3.2, there are $g \in \mathrm{Diff}^r_c(M^{2m} \setminus k(Q^m))_0$ and $h \in \mathrm{Diff}^r_c(M^{2m} \setminus P^{(m-2)})_0$ such that $f = g \circ h$. Then by using the approximation $\bar{h}$ of h,

$$f = g \circ \bar{h} \circ (\bar{h}^{-1} \circ h).$$

By Lemmas 3.3 and 2.3, there are a diffeomorphism a with support in a union of disjointly embedded open balls, $g' \in \mathrm{Diff}^r_c(M^{2m} \setminus k(Q^m))_0$ and $h'' \in \mathrm{Diff}^r_c(M^{2m} \setminus P^{(m-1)})_0$ such that

$$\bar{h} = a^{-1} \circ (a \circ \bar{h}) = a^{-1} \circ g' \circ h''.$$

Put $h^{(0)} = h'' \in \mathrm{Diff}^r_c(M^{2m} \setminus P^{(m-1)})_0$, and for $h^{(j)} \in \mathrm{Diff}^r_c(M^{2m} \setminus (P^{(m-1)} \cup \bigcup_{i=1}^{j} \sigma^m_i))_0$ $(j = 0, \ldots, N-1)$, we use its approximation $\bar{h}^{(j)}$ and by Lemmas 3.4, 3.6 and 3.7, there are diffeomorphisms $a^{(j+1)}$ and $b^{(j+1)}$ with support in unions of disjointly embedded open balls, $g^{(j+1)} \in \mathrm{Diff}^r_c(M^{2m} \setminus k(Q^m \setminus \sigma^{m*}_{j+1}))_0$ and $h^{(j+1)} \in \mathrm{Diff}^r_c(M^{2m} \setminus (P^{(m-1)} \cup \bigcup_{i=1}^{j+1} \sigma^m_i))_0$ such that

$$\begin{aligned} h^{(j)} &= \bar{h}^{(j)} \circ ((\bar{h}^{(j)})^{-1} \circ h^{(j)}) \\ &= (a^{(j+1)})^{-1} \circ (a^{(j+1)} \circ \bar{h}^{(j)}) \circ ((\bar{h}^{(j)})^{-1} \circ h^{(j)}) \\ &= (a^{(j+1)})^{-1} \circ (b^{(j+1)})^{-1} \circ g^{(j+1)} \circ h^{(j+1)} \circ ((\bar{h}^{(j)})^{-1} \circ h^{(j)}). \end{aligned}$$

Hence,

$$\begin{aligned} f &= g \circ \bar{h} \circ (\bar{h}^{-1} \circ h) \\ &= g \circ a^{-1} \circ g' \circ h^{(0)} \circ (\bar{h}^{-1} \circ h) \\ &= g \circ a^{-1} \circ g' \circ (a^{(1)})^{-1} \circ (b^{(1)})^{-1} \circ g^{(1)} \circ h^{(1)} \circ ((\bar{h}^{(0)})^{-1} \circ h^{(0)}) \circ (\bar{h}^{-1} \circ h) \\ &= g \circ a^{-1} \circ g' \circ (a^{(1)})^{-1} \circ (b^{(1)})^{-1} \circ g^{(1)} \circ \cdots \circ (a^{(N)})^{-1} \circ (b^{(N)})^{-1} \circ g^{(N)} \\ &\qquad \circ h^{(N)} \circ ((\bar{h}^{(N-1)})^{-1} \circ h^{(N-1)}) \circ \cdots \circ ((\bar{h}^{(0)})^{-1} \circ h^{(0)}) \circ (\bar{h}^{-1} \circ h). \end{aligned}$$

Here, note that

$$h^{(N)} \in \mathrm{Diff}^r_c(M^{2m} \setminus (P^{(m-1)} \cup \bigcup_{i=1}^{N} \sigma_i^m))_0 = \mathrm{Diff}^r_c(M^{2m} \setminus P^m)_0.$$

Since

$$((\bar{h}^{(N-1)})^{-1} \circ h^{(N-1)}) \circ \cdots \circ ((\bar{h}^{(0)})^{-1} \circ h^{(0)}) \circ (\bar{h}^{-1} \circ h) \in \mathrm{Diff}^r(M^{2m})$$

is close to the identity, by Remark 2.10, it can be written as $\hat{h} \circ \hat{a} \circ \hat{g}$, where $\hat{h} \in \mathrm{Diff}^r_c(M^{2m} \setminus P^m)_0$, $\hat{g} \in \mathrm{Diff}^r_c(M^{2m} \setminus k(Q^m))_0$ and $\hat{a}$ is with support in a union of disjointly embedded open balls which is a neighborhood of the union of m handles. Thus

$$\begin{aligned} f = g \circ a^{-1} \circ g' \circ (a^{(1)})^{-1} \circ (b^{(1)})^{-1} \circ g^{(1)} \circ \cdots \\ \cdots \circ (a^{(N)})^{-1} \circ (b^{(N)})^{-1} \circ g^{(N)} \circ h^{(N)} \circ \hat{h} \circ \hat{a} \circ \hat{g}. \end{aligned}$$

Now by the construction, each of a^{-1}, $(a^{(1)})^{-1}$, …, $(a^{(N)})^{-1}$, $(b^{(1)})^{-1}$, …, $(b^{(N)})^{-1}$ can be written as one commutator with support in a union of disjointly embedded open balls. The diffeomorphism $\hat{a}$ can be written as a product of two commutators by Theorem 1.1 (1). The diffeomorphism $h^{(N)} \circ \hat{h} \in \mathrm{Diff}^r_c(M^{2m} \setminus P^m)_0$ can be written as a product of two commutators in $\mathrm{Diff}^r_c(M^{2m} \setminus P^m)_0$ by Theorem 1.1 (1). Each of the diffeomorphisms g, g' and $\hat{g} \in \mathrm{Diff}^r_c(M^{2m} \setminus k(Q^m))_0$ can also be written as a product of two commutators in $\mathrm{Diff}^r_c(M^{2m} \setminus k(Q^m))_0$ by Theorem 1.1 (1). By the property of the triangulation, the diffeomorphism $g^{(j)} \in \mathrm{Diff}^r_c(M^{2m} \setminus k(Q^m \setminus \sigma^{m*}_{j+1}))_0$ is supported on an open set which can be deformed in a neighborhood of the embedded $(m-1)$-dimensional complex $\iota((P^{(m-1)} \cup \sigma_j^m)/\sigma_j^m)$, and hence $g^{(j)}$ can be written as a product of two commutators in $\mathrm{Diff}^r_c(M^{2m} \setminus k(Q^m \setminus \sigma^{m*}_{j+1}))_0$ by Theorem 1.1 (1). Thus f can be written as a product of $4N + 11$ commutators. □

4. Proofs of the lemmas

We now give the proofs of the lemmas we used in the previous section to show Theorem 1.2.

Proof of Lemma 3.2. This follows from Lemma 2.7. □

Proof of Lemma 3.3. The construction of a_t is essentially due to Burago, Ivanov and Polterovich ([4]) and we wrote it in the proof of Lemma 2.9 which is Lemma 6.5 in [30]. However, we write it again here, for, we use this argument later again.

For $\bar{H}(s_i, v_i)$, we take a small neighborhood U_i of $\bar{H}([s_i, 1] \times \{v_i\})$ diffeomorphic to the $(2m)$-dimensional ball. We can take these U_i to be disjoint.

The intersection of U_i and $\bar{H}([0,1] \times P^{(m-1)})$ or $k(Q^m)$ is described as follows. We put a coordinate

$$(x_1, x_2, \ldots, x_m, x_{m+1}, \ldots, x_{2m}) \in (-2, 2)^{2m}$$

on U_i such that, for $\varepsilon_i > 0$,

$$\begin{aligned} k(Q^m) \cap U_i &= \{0\} \times \{0\}^{m-1} \times (-2, 2)^m, \\ \bar{H}((s_i - 2\varepsilon_i(1 - s_i), 1] \times \{v_i\}) \cap U_i &= (-2, 1] \times \{0\}^{2m-1}, \quad \text{and} \\ \bar{h}_{s_i + t(1-s_i)}(P^{(m-1)}) \cap U_i &= \{t\} \times (-2, 2)^{m-1} \times \{0\}^m \quad (t \in [-\varepsilon_i, 1]). \end{aligned}$$

Take an isotopy $\{a_t\}_{t \in [0,1]}$ with support in $\bigsqcup_{i=1}^r U_i$ such that, on each U_i, $a_0 = \mathrm{id}$ and, for $(x_1, x_2, \ldots, x_{2m}) \in [-\varepsilon_i, 1] \times [-1, 1]^{2m-1} \subset (-2, 2)^{2m}$,

$$a_t(x_1, x_2, \ldots, x_{2m}) = (x_1 - (1 + \varepsilon_i)t, x_2, \ldots, x_{2m}).$$

Now $(a_1 \circ \bar{h}_1)(P^{(m-1)}) \cap k(Q^m) = \emptyset$. Moreover, by changing the time parameter of the above a_t, we obtain an isotopy a_t ($a_0 = \mathrm{id}$) with support in $\bigsqcup_{i=1}^r U_i$ such that for $h'_t = a_t \circ \bar{h}_t$,

$$h'_t(P^{(m-1)}) \cap k(Q^m) = \emptyset \quad (t \in [0, 1]).$$

In fact, if we put

$$t = s_i + u_i(1 - s_i) \in [s_i - \varepsilon_i(1 - s_i), 1], \quad \text{i.e., } u_i \in [-\varepsilon_i, 1],$$

and look at $a_{(u_i+\varepsilon_i)/(1+\varepsilon_i)} \circ \bar{h}_{s_i+u_i(1-s_i)}$, then on U_i,

$$\begin{aligned} &(a_{(u_i+\varepsilon_i)/(1+\varepsilon_i)} \circ \bar{h}_{s_i+u_i(1-s_i)})(\{-\varepsilon_i\} \times [-1, 1]^{m-1} \times \{0\}^m) \\ &= a_{(u_i+\varepsilon_i)/(1+\varepsilon_i)}(\{u_i\} \times [-1, 1]^{m-1} \times \{0\}^m) \\ &= \{u_i - (u_i + \varepsilon_i)\} \times [-1, 1]^{m-1} \times \{0\}^m \\ &= \{-\varepsilon_i\} \times [-1, 1]^{m-1} \times \{0\}^m. \end{aligned}$$

Hence by using the above a_t with appropriate time change, we obtain the desired isotopy a_t.

Note that $a_1 \in \mathrm{Diff}^r_c(\bigsqcup_{i=1}^r U_i)_0$ can be taken as one commutator with support in $\bigsqcup_{i=1}^r U_i$ ([31]). □

Proof of Lemma 3.4. The proof is similar to that of Lemma 3.3.

For $\bar{H}^{(j)}(s_i^{(j)}, v_i^{(j)})$, we take a small neighborhood $U_i^{(j)}$ of $\bar{H}([s_i^{(j)}, 1] \times \{v_i^{(j)}\})$ diffeomorphic to the $(2m)$-dimensional ball. We can take these $U_i^{(j)}$ to be disjoint.

The intersection of $U_i^{(j)}$ and $\bar{H}^{(j)}([0,1] \times P^m)$ or $k(Q^{(m-1)})$ is described as follows. We put a coordinate

$$(x_1, x_2, \dots, x_{m+1}, x_{m+2}, \dots, x_{2m}) \in (-2, 2)^{2m}$$

on $U_i^{(j)}$ such that, for $\varepsilon_i^{(j)} > 0$,

$$\begin{aligned} k(Q^{(m-1)}) \cap U_i^{(j)} &= \{0\} \times \{0\}^m \times (-2,2)^{m-1}, \\ \bar{H}((s_i^{(j)} - 2\varepsilon_i^{(j)}(1 - s_i^{(j)}), 1] \times \{v_i^{(j)}\}) \cap U_i^{(j)} &= (-2, 1] \times \{0\}^{2m-1}, \quad \text{and} \\ \bar{h}^{(j)}_{s_i^{(j)} + t(1-s_i^{(j)})}(P^m) \cap U_i^{(j)} &= \{t\} \times (-2,2)^m \times \{0\}^{m-1} \quad (t \in [-\varepsilon_i^{(j)}, 1]). \end{aligned}$$

Take an isotopy $\{a_t^{(j+1)}\}_{t \in [0,1]}$ with support in $\bigsqcup_{i=1}^{r(j)} U_i^{(j)}$ such that, on each $U_i^{(j)}$, $a_0^{(j+1)} = \mathrm{id}$ and, for $(x_1, x_2, \dots, x_{2m}) \in [-\varepsilon_i^{(j)}, 1] \times [-1,1]^{2m-1} \subset (-2,2)^{2m}$,

$$a_t^{(j+1)}(x_1, x_2, \dots, x_{2m}) = (x_1 - (1 + \varepsilon_i^{(j)})t, x_2, \dots, x_{2m}).$$

Now $(a_1^{(j+1)} \circ \bar{h}_1^{(j)})(P^m) \cap k(Q^{(m-1)}) = \emptyset$. Moreover, by changing the time parameter, we obtain an isotopy $a_t^{(j+1)}$ ($a_0^{(j+1)} = \mathrm{id}$) with support in $\bigsqcup_{i=1}^r U_i^{(j)}$ such that, for $h'^{(j)}_t = a_t^{(j+1)} \circ \bar{h}_t^{(j)}$,

$$h'^{(j)}_t(P^m) \cap k(Q^{(m-1)}) = \emptyset \quad (t \in [0,1]).$$

In fact, if we put

$$t = s_i^{(j)} + u_i^{(j)}(1 - s_i^{(j)}) \in [s_i^{(j)} - \varepsilon_i^{(j)}(1 - s_i^{(j)}), 1], \quad \text{i.e., } u_i^{(j)} \in [-\varepsilon_i^{(j)}, 1],$$

and look at $a^{(j+1)}_{(u_i^{(j)} + \varepsilon_i^{(j)})/(1+\varepsilon_i^{(j)})} \circ \bar{h}^{(j)}_{s_i^{(j)} + u_i^{(j)}(1-s_i^{(j)})}$, then on $U_i^{(j)}$,

$$\begin{aligned} &(a^{(j+1)}_{(u_i + \varepsilon_i)/(1+\varepsilon_i)} \circ \bar{h}^{(j)}_{s_i^{(j)} + u_i^{(j)}(1-s_i^{(j)})})(\{-\varepsilon_i^{(j)}\} \times [-1,1]^m \times \{0\}^{m-1}) \\ &\quad = a^{(j+1)}_{(u_i^{(j)} + \varepsilon_i^{(j)})/(1+\varepsilon_i^{(j)})}(\{u_i^{(j)}\} \times [-1,1]^m \times \{0\}^{m-1}) \\ &\quad = \{u_i^{(j)} - (u_i^{(j)} + \varepsilon_i^{(j)})\} \times [-1,1]^m \times \{0\}^{m-1} \\ &\quad = \{-\varepsilon_i^{(j)}\} \times [-1,1]^m \times \{0\}^{m-1}. \end{aligned}$$

Hence by using the above $a_t^{(j+1)}$ with appropriate time change, we obtain the desired isotopy $a_t^{(j+1)}$.

Note again that $a_1^{(j+1)} \in \mathrm{Diff}_c^r(\bigsqcup_{i=1}^{r(j)} U_i^{(j)})_0$ can be taken as one commutator with support in $\bigsqcup_{i=1}^{r(j)} U_i^{(j)}$ ([31]). □

Proof of Lemma 3.5. For

$$F(t, x_1, \dots, x_m) = (f_1(t, x_1, \dots, x_m), \dots, f_m(t, x_1, \dots, x_m)),$$

put

$$\frac{\partial F}{\partial t} = \begin{pmatrix} \frac{\partial f_1}{\partial t} \\ \vdots \\ \frac{\partial f_m}{\partial t} \end{pmatrix} \quad \text{and} \quad \frac{\partial F}{\partial x} = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_m} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_m} \end{pmatrix}.$$

On the 2-jet bundle $J^2(\mathbb{R} \times \mathbb{R}^m, \mathbb{R}^m)$, we consider the subbundle E_1 defined by $\operatorname{rank}\begin{pmatrix} \frac{\partial F}{\partial t} & \frac{\partial F}{\partial x} \end{pmatrix} = m - 1$ and the subbundle E_2 defined by the two equations, $\operatorname{rank}\begin{pmatrix} \frac{\partial F}{\partial x} \end{pmatrix} = m - 1$ and $\operatorname{rank}\begin{pmatrix} \frac{\partial F}{\partial x} \\ \frac{\partial}{\partial x} \det \frac{\partial F}{\partial x} \end{pmatrix} = m - 1$, where

$$\frac{\partial}{\partial x} \det \frac{\partial F}{\partial x} = \begin{pmatrix} \frac{\partial}{\partial x_1} \det \frac{\partial F}{\partial x} & \cdots & \frac{\partial}{\partial x_n} \det \frac{\partial F}{\partial x} \end{pmatrix}.$$

Then E_1 and E_2 are codimension 2 subbundles. The closures of these subbundles are the sets determined by the inequalities expressing the ranks are not greater than $m - 1$.

By the jet transversality theorem, the jet of a generic map F intersects these subbundles transversely. Hence the set

$$\{(t, x) \mid J^2_{(t,x)} F \in E_1 \cup E_2\}$$

is an $(m - 1)$-dimensional subset and its image in $\mathbb{R}^m$ is nowhere dense. We take a point y in $\mathbb{R}^m$ in the complement of this image and consider its inverse image $F^{-1}(y)$. Then for a point $x \in F^{-1}(y)$, either $\operatorname{rank}\begin{pmatrix} \frac{\partial F}{\partial x} \end{pmatrix} = m$ holds or the three equations $\operatorname{rank}\begin{pmatrix} \frac{\partial F}{\partial x} \end{pmatrix} = m - 1$, $\operatorname{rank}\begin{pmatrix} \frac{\partial F}{\partial t} & \frac{\partial F}{\partial x} \end{pmatrix} = m$ and $\operatorname{rank}\begin{pmatrix} \frac{\partial F}{\partial x} \\ \frac{\partial}{\partial x} \det \frac{\partial F}{\partial x} \end{pmatrix} = m$ hold.

If $\operatorname{rank}\begin{pmatrix} \frac{\partial F}{\partial x} \end{pmatrix} = m$ at x, then x is a regular point of $F_t = F(t, \bullet)$ and the inverse image is locally a 1-dimensional manifold transverse to $\{t\} \times \mathbb{R}^m$.

Assume that the three equations hold. Since $\operatorname{rank}\begin{pmatrix} \frac{\partial F}{\partial x} \end{pmatrix} = m - 1$, by the implicit function theorem, we can change the local coordinate $(x_1, \dots, x_m)$ of the second factor of the source to $(x'_1, \dots, x'_m)$ and that $(y_1, \dots, y_m)$ of the target to $(y'_1, \dots, y'_m)$ so that

$$F(t, x'_1, \dots, x'_m) = (x'_1, \dots, x'_{m-1}, y'_m(t, x'_1, \dots, x'_m)).$$

Then $\det\left(\frac{\partial F}{\partial x}\right) = \frac{\partial y'_m}{\partial x'_m}$ and the matrix $\begin{pmatrix} \frac{\partial F}{\partial x} \\ \frac{\partial}{\partial x}\det\frac{\partial F}{\partial x}\end{pmatrix}$ with respect to these coordinates is written as

$$\begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & 0 \\ 0 & \cdots & 0 & 1 & 0 \\ \frac{\partial y'_m}{\partial x'_1} & \cdots & \cdots & \frac{\partial y'_m}{\partial x'_{m-1}} & \frac{\partial y'_m}{\partial x'_m} \\ \frac{\partial^2 y'_m}{\partial x'_m \partial x'_1} & \cdots & \cdots & \frac{\partial^2 y'_m}{\partial x'_m \partial x'_{m-1}} & \frac{\partial^2 y'_m}{\partial {x'_m}^2} \end{pmatrix}$$

and the matrix $\begin{pmatrix}\frac{\partial F}{\partial t} & \frac{\partial F}{\partial x}\end{pmatrix}$ with respect to these coordinates is written as

$$\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 1 & 0 \\ \frac{\partial y'_m}{\partial t} & \frac{\partial y'_m}{\partial x'_1} & \cdots & \cdots & \frac{\partial y'_m}{\partial x'_{m-1}} & \frac{\partial y'_m}{\partial x'_m} \end{pmatrix}.$$

Hence, $\frac{\partial y'_m}{\partial x'_m} = 0$, $\frac{\partial y'_m}{\partial t} \neq 0$ and $\frac{\partial^2 y'_m}{\partial {x'_m}^2} \neq 0$ at x.

Thus at $x \in F^{-1}(y)$, either $\det\left(\frac{\partial F}{\partial x}\right) \neq 0$ or F is locally written as

$$F(t, x'_1, \dots, x'_m) = (x'_1, \dots, x'_{m-1}, y'_m(t, x'_1, \dots, x'_m)),$$

where $\frac{\partial y'_m}{\partial x'_m} = 0$, $\frac{\partial y'_m}{\partial t} \neq 0$ and $\frac{\partial^2 y'_m}{\partial {x'_m}^2} \neq 0$. □

The proof of Lemma 3.6 is divided into two cases.

Proof of Lemma 3.6 *in the case where* $2m \geq 8$. If $2m \geq 8$, the Whitney disks guide the way to construct the isotopy $b_t^{(j+1)}$ with support in a union of disjoint open balls. In fact, the support of $b_t^{(j+1)}$ is in a neighborhood of the union of the Whitney disks. The construction of the isotopy $b_t^{(j+1)}$ is possible because the neighborhood of one of the Whitney disks can be considered as a neighborhood of a tree growing in the negative direction in t in $[0, 1] \times \sigma_{j+1}^m$.

The construction of $b_t^{(j+1)}$ is as follows. Take a vector field of the form $\frac{\partial}{\partial t} + \zeta(t, v)$ on the union of disks $Z' \subset [0, 1] \times \sigma_{j+1}^m$ which is tangent to Z' and transverse to the boundary $\partial Z' \subset Z'$, where $\zeta(t, v)$ is a vector field in the direction of σ_{j+1}^m. Such a vector field $\frac{\partial}{\partial t} + \zeta(t, v)$ exists because Z' deforms to a tree which grows in the negative direction in t by shrinking the connected components of $Z' \cap (\{t\} \times \sigma_{j+1}^m)$ to a point. We extend $\zeta(t, \bullet)$ on σ_{j+1}^m so that the support is contained in a small neighborhood of Z'. Let ${b'}_t^{(j+1)}$ denote the isotopy generated by $\frac{\partial}{\partial t} + \zeta(t, v)$. Then the support of ${b'}_t^{(j+1)}$ is contained in a neighborhood $U'^{(j)}$ of the union of the Whitney disks $H'^{(j)}(Z')$. Since $H'^{(j)}(Z')$ does not contain double points of $H'^{(j)}([0,1] \times P^m)$, the support of ${b'}_t^{(j+1)}$ intersects $H'^{(j)}([0,1] \times P^m)$ only in $U'^{(j)}$. Here, $U'^{(j)}$ is a union of disjointly embedded open balls in M^{2m}. Moreover, $({h'}_t^{(j)})_*\zeta(t, \bullet)$ is tangent to the union of the Whitney disks $H'^{(j+1)}(Z')$ in M^{2m} and

$$({b'}_t^{(j+1)})^{-1}({h'}_t^{(j)}(\sigma_{j+1}^m)) \cap k(Q^m \setminus \sigma_{j+1}^{m*}) = \emptyset \quad (t \in [0,1]).$$

Put $b_t^{(j+1)} = ({b'}_t^{(j+1)})^{-1}$, then

$$(b_t^{(j+1)} \circ {h'}_t^{(j)})(\sigma_{j+1}^m) \cap k(Q^m \setminus \sigma_{j+1}^{m*}) = \emptyset \quad (t \in [0,1]).$$

Note that $b_1^{(j+1)} \in \mathrm{Diff}_c^r(U'^{(j)})_0$ can be taken as one commutator with support in $U'^{(j)}$ ([31]). □

Proof of Lemma 3.6 in the case where $2m = 6$. If $2m = 6$, then we also consider the curves $H'^{(j)}([{t'}_i^{(j)}, 1] \times \{{w'}_i^{(j)}\})$ $(i = 1, \ldots, r'^{(j)})$.

First take a small neighborhood $U'^{(j)}$ of the union of the Whitney disks which is a union of disjointly embedded open balls in M^6, and construct $b_t^{(j+1)}$ as in the case where $2m \geq 8$. Then we modify it by using an isotopy.

We take a small neighborhood ${U'}_i^{(j)}$ of the curve $H'^{(j)}([{t'}_i^{(j)}, 1] \times \{{w'}_i^{(j)}\})$ $(i = 1, \ldots, r'^{(j)})$. We put a coordinate

$$(x_1, x_2, x_3, x_4, x_5, x_6) \in (-2, 3) \times (-2, 2)^5$$

on ${U'}_i^{(j)}$ such that, for ${\varepsilon'}_i^{(j)} > 0$,

$$H'^{(j)}(({t'}_i^{(j)} - 2{\varepsilon'}_i^{(j)}(1 - {t'}_i^{(j)}), 1] \times \{{w'}_i^{(j)}\}) \cap {U'}_i^{(j)} = (-2, 1] \times \{0\}^5,$$

and

$$h'_{{t'}_i^{(j)} - 2{\varepsilon'}_i^{(j)}(1 - {t'}_i^{(j)})}(P^3) \cap {U'}_i^{(j)} = \{t\} \times (-2,2)^3 \times \{0\}^2 \quad (t \in [-{\varepsilon'}_i^{(j)}, 1]).$$

We take an isotopy $\{{a'}_t^{(j+1),i}\}_{t \in [0,1]}$ with support in ${U'}_i^{(j)}$ such that ${a'}_0^{(j+1),i} = \mathrm{id}$ and, for $(x_1, x_2, x_3, x_4, x_5, x_6) \in [-{\varepsilon'}_i^{(j)}, 1] \times [-1, 1]^5 \subset (-2, 3) \times (-2, 2)^5$,

$${a'}_t^{(j+1),i}(x_1, x_2, x_3, x_4, x_5, x_6) = (x_1 + t(1 + {\varepsilon'}_i^{(j)}), x_2, x_3, x_4, x_5, x_6).$$

Put $\bar{a} = \prod_{i=1}^{r'^{(j)}} {a'}_1^{(j+1),i}$. Then $\bar{a} \circ b_1^{(j+1)} \circ \bar{a}^{-1}$ is isotopic to the identity by the isotopy with support in the union of disjoint 6-dimensional open balls $\bar{a}(U'^{(j)})$. By the construction,

$$((\bar{a} \circ b_1^{(j+1)} \circ \bar{a}^{-1}) \circ \bar{h}_1)(\sigma_{j+1}^3) \cap k(Q^3 \setminus \sigma_{j+1}^{3*}) = \emptyset.$$

Moreover, by an appropriate change of time parameter on each ${U'}_i^{(j)}$, we obtain an isotopy $\bar{a}_t$ ($t \in [0, 1]$) such that

$$((\bar{a}_t \circ b_t^{(j+1)} \circ \bar{a}_t^{-1}) \circ \bar{h}_t)(\sigma_{j+1}^3) \cap k(Q^3 \setminus \sigma_{j+1}^{3*}) = \emptyset$$

and the support of the isotopy $\bar{a}_t \circ b_t^{(j+1)} \circ \bar{a}_t^{-1}$ is contained in $U'^{(j)} \cup \bigsqcup_{i=1}^{r'(j)} {U'}_i^{(j)}$ which is a union of disjointly embedded open balls in M^{2m}. Thus we obtained the desired isotopy.

Note that $\bar{a} \circ b_1^{(j+1)} \circ \bar{a}^{-1}$ can be taken as one commutator with support in a union of disjointly embedded open balls. □

Proof of Lemma 3.7. This follows from Lemmas 3.6 and 2.3. □

5. Uniform simplicity

We prove Corollary 1.3. In Theorem 2.2 of [31], we showed the following theorem.

Theorem 5.1 ([31]). *Let M^n be the interior of a compact n-dimensional manifold with handle decomposition with handles of indices not greater than $(n-1)/2$. Let c be the order of the set of indices appearing in the handle decomposition. Then any element of* $\mathrm{Diff}_c^r(M^n)_0$ ($1 \leq r \leq \infty$, $r \neq n+1$) *can be written as a product of two commutators. Moreover, if M^n is connected, any element of* $\mathrm{Diff}_c^r(M^n)_0$ *can be written as a product of* $4c+1$ *commutators with support in embedded open balls.*

In Section 3, we showed that any element $f \in \mathrm{Diff}^r(M^{2m})_0$ can be written as

$$f = g \circ a^{-1} \circ g' \circ (a^{(1)})^{-1} \circ (b^{(1)})^{-1} \circ g^{(1)} \circ \cdots$$
$$\cdots \circ (a^{(N)})^{-1} \circ (b^{(N)})^{-1} \circ g^{(N)} \circ h^{(N)} \circ \hat{h} \circ \hat{a} \circ \hat{g}.$$

Since a compact subset of a union of disjointly embedded open balls is contained in a larger embedded open ball, each of diffeomorphisms a^{-1}, $(a^{(1)})^{-1}, \dots, (a^{(N)})^{-1}$, $(b^{(1)})^{-1}, \dots, (b^{(N)})^{-1}$ can be written as one commutator with support in an embedded open ball and the diffeomorphism $\hat{a}$ can be written as a product of two commutators with support in an embedded open ball. Now by Theorem 5.1, each of the diffeomorphisms $h^{(N)} \circ \hat{h} \in \mathrm{Diff}_c^r(M^{2m} \setminus P^m)_0$, g, g' and $\hat{g} \in \mathrm{Diff}_c^r(M^{2m} \setminus k(Q^m))_0$, $g^{(j)} \in \mathrm{Diff}_c^r(M^{2m} \setminus k(Q^m \setminus \sigma_{j+1}^{m*}))_0$ is written as a product of $4m+1$ commutators with support in embedded open balls. Hence f is written as a product of $4(N+4)m + 3N + 7$ commutators with support in embedded open balls.

Now Corollary 1.3 follows from the following lemma ([31], Lemma 3.1).

Lemma 5.2 ([31]). *Let M^n be a connected n-dimensional manifold. Let g be a nontrivial element of* $\mathrm{Diff}_c^r(M^n)_0$. *Assume that* $f \in \mathrm{Diff}_c^r(M^n)_0$ *is written as a* *product of commutators* $[a_i, b_i]$ $(i = 1, \dots, k)$; $f = [a_1, b_1] \cdots [a_k, b_k]$, *where* a_i *and* b_i *are with support in an embedded open ball* $U_i \subset \overline{U_i} \subset M^n$. *Then* f *can be written as a product of* $4k$ *conjugates of* g *or* g^{-1}.

Proof of Corollary 1.3. Let g be a nontrivial element of $\mathrm{Diff}^r(M^{2m})_0$ $(1 \le r \le \infty, r \ne 2m+1)$. Since any element f of $\mathrm{Diff}^r(M^{2m})_0$ can be written as a product of $4(N+4)m + 3N + 7$ commutators with support in embedded open balls, by Lemma 5.2, f can be written as a product of $16(N+4)m + 12N + 28$ conjugates of g or g^{-1}. □

Remark 5.3. We showed in [31] that, for a compact connected n-dimensional manifold M^n with handle decomposition without handles of the middle index $n/2$, for any elements f and g of $\mathrm{Diff}^r(M^n)_0 \setminus \{\mathrm{id}\}$, f can be written as a product of at most $16n + 28$ conjugates of g or g^{-1}. For such manifolds, the bound for the number of conjugates depends only on the dimension n. In Corollary 1.3, however, the bound for the number of conjugates may depend on the topology of M^{2m}.

6. Appendix

In this section, we show two propositions. The first proposition constructs the Morse function adapted to a smooth triangulation of a compact manifold. The second proposition constructs a cellular decomposition adapted to a Morse function.

Proposition 6.1. *Let* P *be a smooth triangulation of a compact* n*-dimensional manifold* M^n. *Let* $\mathrm{bsd}(P)$ *denote the barycentric subdivision of* P *and* P^* *be the cell decomposition dual to* P *of* M^n. *Then there is a Morse function* f *on* M^n *and a Riemannian metric on* M^n *such that, for the gradient flow* φ_t *of* f, *there is a homeomorphism of* M^n *which sends the stratification by the stable manifolds of the critical points of* f *and that by the unstable manifolds of the critical points of* f *to* P *and* P^*, *respectively.*

First we prepare a Morse type function on each simplex of $\mathrm{bsd}(P)$.

Let e_i $(i = 1, \dots, n)$ be the basis of $\mathbb{R}^n$. Let

$$\Delta^n = \left\{(t_1, \dots, t_n) = \textstyle\sum_{i=1}^n t_i e_i \in \mathbb{R}^n \;\middle|\; 1 \ge t_1 \ge \cdots \ge t_n \ge 0\right\}$$

be the standard simplex. Let $f : \mathbb{R}^n \to \mathbb{R}$ be the function defined by

$$f(t_1, \dots, t_n) = n - \sum_{i=1}^n \cos(t_n/\pi).$$

The function f is a Morse function such that the vertex $(1, \dots, 1, 0, \dots, 0) = \sum_{i=1}^j e_i$ of Δ^n is the critical point of index j $(j = 0, 1, \dots, n)$ and $f(\sum_{i=1}^j e_i) = j$.

Let $X_{\mathbb{R}^n} = \mathrm{grad}(f)$ denote the gradient vector field of f with respect to the Euclidean metric. The standard simplex is invariant under the flow generated by $X_{\mathbb{R}^n}$.

Let $\mathrm{bsd}(P)$ be the barycentric subdivision of P. An n-dimensional simplex of $\mathrm{bsd}(P)$ is the simplex with vertices $b_{\sigma^0}, \dots, b_{\sigma^n}$, where $\sigma^0 \prec \sigma^1 \prec \cdots \prec \sigma^{n-1} \prec \sigma^n$, b_{σ^j} is the barycenter of the j-dimensional simplex σ^j and "$\sigma^i \prec \sigma^j$"means that "σ^i is a face of σ^j".

Let $g \colon M^n \to \Delta^n$ be the map which sends each n-dimensional simplex with vertices $b_{\sigma^0}, \dots, b_{\sigma^n}$ of $\mathrm{bsd}(P)$ linearly to Δ^n so that $g(b_{\sigma^j}) = \sum_{i=0}^{j} e_i$. Then $f \circ g$ is a piecewise smooth function on M^n which looks like a Morse function on M^n and $X = g_*^{-1} X_{\mathbb{R}^n}$ is a Lipschitz continuous piecewise smooth vector field on M^n.

We show that there are a Morse function $\hat{f} \colon M^n \to \mathbb{R}$ and a Riemannian metric on M^n such that there is a homeomorphism of M^n sending the stratification by the stable manifolds for the gradient flow of the critical points of $\hat{f}$ to the triangulation P and the stratification by the unstable manifolds of the critical points of $\hat{f}$ to the dual cell decomposition P^*.

Since the function $f \circ g$ is transverse to the triangulation outside a neighborhood of the set of vertices of $\mathrm{bsd}(P)$, $(f \circ g)^{-1}(k + 1/2)$ is a piecewise smooth codimension 1 submanifold of M "transverse" to each simplex of $\mathrm{bsd}(P)$ and is transverse to the vector field X.

We are going to modify $(f \circ g)^{-1}(k + 1/2)$ to a smooth manifold $M_{k+1/2}$ transverse to each simplex of $\mathrm{bsd}(P)$ and to the vector field X.

Let $\mathrm{bsd}(P)^{(i)}$ denote the i-dimensional skeleton of $\mathrm{bsd}(P)$. First, we modify $(f \circ g)^{-1}(k + 1/2)$ in a neighborhood of the intersection $(f \circ g)^{-1}(k + 1/2) \cap \mathrm{bsd}(P)^{(1)}$ and obtain $M^{(1)}_{k+1/2} \subset M^n$ approximating $(f \circ g)^{-1}(k+1/2)$ which is smooth near the 1-dimensional skeleton $\mathrm{bsd}(P)^{(1)}$ and transverse to $\mathrm{bsd}(P)^{(1)}$ and to the vector field X. After obtaining $M^{(i)}_{k+1/2} \subset M^n$ which is smooth near the i-dimensional skeleton $\mathrm{bsd}(P)^{(i)}$ and transverse to $\mathrm{bsd}(P)^{(i)}$ and to the vector field X, we obtain $M^{(i+1)}_{k+1/2}$ approximating $M^{(i)}_{k+1/2}$ in a neighborhood of the intersection $M^{(i)}_{k+1/2} \cap \mathrm{bsd}(P)^{(i+1)}$ which is smooth near the $(i + 1)$-dimensional skeleton $\mathrm{bsd}(P)^{(i+1)}$ and transverse $\mathrm{bsd}(P)^{(i+1)}$ and to the vector field X. Finally, put $M_{k+1/2} = M^{(n-1)}_{k+1/2}$. Then $M_{k+1/2}$ is a smooth codimension 1 submanifold transverse to $\mathrm{bsd}(P)$ and X.

The codimension 1 submanifold $M_{k+1/2}$ divides M^n into two compact manifolds W_k and W^*_{n-k} which are obtained from $(f \circ g)^{-1}([0, k + 1/2])$ and $(f \circ g)^{-1}([k + 1/2, n])$ by smoothing, respectively.

We are going to show that W_k is diffeomorphic to the manifold obtained from W_{k-1} by attaching handles of index k for k-dimensional simplices of P and by smoothing along the corner. That is, for each k-dimensional simplex σ^k, we can define a handle $D^k_{\sigma^k} \times D^{n-k}_{\sigma^k}$ of index k such that W_k is diffeomorphic to the manifold obtained from W_{k-1} by attaching $D^k_{\sigma^k} \times D^{n-k}_{\sigma^k}$ for all k-dimensional simplices σ^k of P and by smoothing along the corner. The reason is as follows. First, for each k-dimensional simplex σ^k of P, since the intersection $S^{k-1}_{\sigma^k} = \sigma^k \cap M_{k-1/2}$ approximates $\sigma^k \cap (f \circ g)^{-1}(k-1/2)$, it is diffeomorphic to the $(k-1)$-dimensional sphere S^{k-1} which bounds a k-dimensional disk $D^k_{\sigma^k}$ in σ^k. Then by choosing a Riemannian metric in a neighborhood of σ^k and using the exponential map, we obtain a diffeomorphism from a neighborhood of the zero section of the normal bundle of the k-dimensional disk $D^k_{\sigma^k}$ to a neighborhood of $D^k_{\sigma^k}$. By an appropriate choice of the metric, this defines an embedding $D^k_{\sigma^k} \times D^{n-k}_{\sigma^k} \subset M^n$ such that $\partial D^k_{\sigma^k} \times D^{n-k}_{\sigma^k} \subset M_{k-1/2}$. Then we obtain

$$W'_k = W_{k-1} \cup \bigcup_{\sigma^k} (D^k_{\sigma^k} \times D^{n-k}_{\sigma^k}) \quad (\subset W_k)$$

We can add a neighborhood of the corner of W'_k and obtain W''_k such that the orbits of ψ_t on $W_k - \mathrm{int}(W''_k)$ are transverse to $M_{k+1/2} = \partial W_k$ and $\partial W''_k$. Here each orbit of ψ_t intersects both $\partial W''_k$ and ∂W_k. Since this transversality is preserved when we approximate X by a smooth vector field $\widehat{X}$, $W_k - \mathrm{int}(W''_k)$ is diffeomorphic to $M_{k+1/2} \times [0,1]$. Thus this gives the $(n-k-1)$-dimensional sphere $S^{n-k-1}_{\sigma^k}$ on $M_{k+1/2} = \partial W_k$ corresponding to $\{0\} \times \partial D^{n-k}_{\sigma^k}$ which will be used as the belt sphere.

Now we define a smooth vector field $\widehat{X}$ on M^n which generates the flow $\hat{\psi}_t$ satisfying the following conditions.

(1) $\widehat{X}$ restricted to a neighborhood of b_{σ^k} is of the form

$$-\sum_{i=1}^{k} x_i \frac{\partial}{\partial x_i} + \sum_{i=k+1}^{n} x_i \frac{\partial}{\partial x_i}$$

and the stable manifold $L^s_{b(\sigma^k)}$ of b_{σ^k} of the flow $\hat{\psi}_t$ contains $D^k_{\sigma^k} \times \{0\} \subset \sigma^k$.

(2) The orbits of $\hat{\psi}_t$ are transverse to $M_{k+1/2}$ $(k = 0, \ldots, n-1)$.

(3) An orbit of $\hat{\psi}_t$ in $W_k \setminus \mathrm{int}(W_{k-1})$ is one of the following types.

- An orbit crossing through both $M_{k-1/2}$ and $M_{k+1/2}$.
- An orbit contained in the stable manifold of b_{σ^k} and crossing through $M_{k-1/2}$ at a point of $S^{k-1}_{\sigma^k}$.
- An orbit contained in the unstable manifold of b_{σ^k} and crossing through $M_{k+1/2}$ at a point of $S^{n-k-1}_{\sigma^k}$.

(4) For two simplices σ^k and σ^{k+1} of P, if $\sigma^k \prec \sigma^{k+1}$, then $S^{n-k-1}_{\sigma^k}$ and $S^k_{\sigma^{k+1}}$ $(\subset M_{k+1/2})$ intersect transversely at a point. Conversely, if $S^{n-k-1}_{\sigma^k}$ and $S^k_{\sigma^{k+1}}$ $(\subset M_{k+1/2})$ intersect, they intersect transversely, and $\sigma^k \prec \sigma^{k+1}$.

The statement (4) implies that there is a unique orbit of $\hat{\psi}_t$ connecting b_{σ^k} and $b_{\sigma^{k+1}}$ if and only if $\sigma^k \prec \sigma^{k+1}$.

By [21], for this $\hat{\psi}_t$, we can define a Morse function $\hat{f}$ and a Riemannian metric such that $\mathrm{grad}(\hat{f}) = \hat{X}$. These are the desired Morse function and Riemannian metric for our Proposition 6.1.

Proof of Proposition 6.1. We show that the configuration of stable manifolds and unstable manifolds of $\hat{\psi}_t$ is homeomorphic to the configuration of the triangulation P and its dual cell decomposition P^*.

We explain how we take W_k' and W_k'' such that $W_{k-1} \subset W_k' \subset W_k'' \subset W_k$ related to the flow $\hat{\psi}_t$.

First, each connected component of W_0 is in the unstable manifold of a 0-dimensional simplex σ^0. On $M_{1/2} = \partial W_0$ we have a number of $S^0_{\sigma^1}$ for 1-dimensional simplices σ^1. Hence the stable manifold $L^s_{b(\sigma^1)}$ of b_{σ^1} consists of b_{σ^1} and the two orbits of $\hat{\psi}_t$ connecting b_{σ^1} and σ_i^0 $(i = 1, 2)$ such that $\sigma_i^0 \prec \sigma^1$.

For a 1-dimensional simplex σ^1, in a neighborhood of b_{σ^1}, the unstable manifold $L^u_{b(\sigma^1)}$ of b_{σ^1} divides the neighborhood into two parts which are the subsets of the unstable manifolds of $\sigma_1^0 \prec \sigma^1$ and $\sigma_2^0 \prec \sigma^1$. We can take the union of W_0 and appropriate closed neighborhoods of $L^s_{b(\sigma^1)}$ for 1-dimensional simplices σ^1 of P as $W_1' = W_0 \cup \bigcup_{\sigma^1} D^1_{\sigma^1} \times D^{n-1}_{\sigma^1}$, where the flow $\hat{\psi}_t$ on $D^1_{\sigma^1} \times D^{n-1}_{\sigma^1}$ is of the form $\hat{\psi}_t(x_1, \dots, x_n) = (e^{-t}x_1, e^t x_2, \dots, e^t x_n)$. Then we can take W_1'' which is obtained from W_1' by smoothing along the corner and there is an isotopy sending W_1'' to W_1 along the orbits of $\hat{\psi}_t$. There is a homeomorphism h_1 sending W_1 to $(f \circ g)^{-1}([0, 1 + 1/2])$ such that h_1 sends the stable manifold $L^s_{b(\sigma^1)}$ of b_{σ^1} to σ^1 and the unstable manifold $L^u_{b(\sigma^1)}$ of b_{σ^1} to $\sigma^{1*} \cap (f \circ g)^{-1}([0, 1+1/2])$, respectively.

Now we look at a 2-dimensional simplex σ^2. On $M_{1+1/2}$, we have $S^1_{\sigma^2}$ for each 2-dimensional simplex σ^2 of P and $S^{n-2}_{\sigma^1}$ for each 1-dimensional simplex σ^1 of P. A 2-dimensional simplex σ^2 of P has three faces σ_i^1 $(i = 1, 2, 3)$, hence we have three orbits of $\hat{\psi}_t$ which pass through $S^1_{\sigma^2} \cap S^{n-2}_{\sigma_i^1}$ and connect $b_{\sigma_i^1}$ and b_{σ^2} $(i = 1, 2, 3)$. Each component of $S^1_{\sigma^2} \setminus \bigcup_{i=1}^3 S^1_{\sigma^2} \cap S^{n-2}_{\sigma_i^1}$ is sent by the flow $\hat{\psi}_t$ in the negative time direction to one of the components of W_0. The component of W_0 is necessarily the one which contains one of the three vertices of σ^2 and the stable manifold $L^s_{b(\sigma^2)}$ of b_{σ^2} is bounded by the union of stable manifolds of $b_{\sigma_i^1}$ $(i = 1, 2, 3)$ and the vertices of σ^2. Thus the stable manifold $L^s_{b(\sigma^2)}$ is homeomorphic to a 2-dimensional simplex and the union $\bigcup_{i \leq 2} L^s_{b(\sigma^i)}$ is homeomorphic to the 2-dimensional skeleton $P^{(2)}$. Then the stable manifold $L^s_{b(\sigma^2)}$ as well as a neighborhood of $L^s_{b(\sigma^2)}$ is divided by the union of the unstable manifolds of $b_{\sigma_i^1}$ $(i = 1, 2, 3)$ and b_{σ^2} into three parts, each of which is contained in the stable manifold of one of the vertices of σ^2. We can take the union of W_1 and closed neighborhoods of $L^s_{b(\sigma^2)}$ for 2-dimensional simplices σ^2 of P as $W_2' = W_1 \cup \bigcup_{\sigma^2}(D^2_{\sigma^2} \times D^{n-2}_{\sigma^2})$, where the flow $\hat{\psi}_t$ on $D^2_{\sigma^2} \times D^{n-2}_{\sigma^2}$ is of the form $\hat{\psi}_t(x_1, \dots, x_n) = (e^{-t}x_1, e^{-t}x_2, e^t x_3, \dots, e^t x_n)$. We can take W_2'' which is obtained from W_2' by smoothing along the corner and there is an isotopy sending W_2'' to W_2 along the orbits of $\hat{\psi}_t$. Then there is a homeomorphism h_2 sending W_2 to $(f \circ g)^{-1}([0, 2+1/2])$ extending h_1 such that h_2 sends the stable manifold $L^s_{b(\sigma^2)}$ of b_{σ^2} to σ^2 and the unstable manifold $L^u_{b(\sigma^2)}$ of b_{σ^2} to $\sigma^{2*} \cap (f \circ g)^{-1}([0, 2+1/2])$, respectively.

Inductively, assume that we showed that

(1) for a $(j-1)$-dimensional simplex σ^{j-1} of P, the stable manifold of $b_{\sigma^{j-1}}$ is bounded by the union of the stable manifolds $L^s_{b(\sigma^i)}$ of b_{σ^i} such that $\sigma^i \prec \sigma^{j-1}$,

(2) $L^s_{b(\sigma^{j-1})}$ is homeomorphic to a $(j-1)$-dimensional simplex,

(3) the union $\bigcup_{i \le j-1} L^s_{b(\sigma^i)}$ is homeomorphic to the $(j-1)$-dimensional skeleton $P^{(j-1)}$,

(4) $L^s_{b(\sigma^{j-1})}$ as well as a neighborhood of $L^s_{b(\sigma^{j-1})}$ is divided by the union of the unstable manifolds $L^u_{b(\sigma^i)}$ of b_{σ^i} such that $\sigma^i \prec \sigma^{j-1}$ into j parts each of which is contained in the unstable manifold of one of the vertices of σ^{j-1}, and

(5) there is a homeomorphism h_{j-1} sending W_{j-1} to $(f \circ g)^{-1}([0, j-1/2])$ such that h_{j-1} sends the stable manifold $L^s_{b(\sigma^i)}$ of b_{σ^i} to σ^i and the unstable manifold $L^u_{b(\sigma^i)}$ of b_{σ^i} to $\sigma^{i*} \cap (f \circ g)^{-1}([0, j-1/2])$, respectively.

Consider a j-dimensional simplex σ^j. On $M_{j-1/2}$, we have $S^{j-1}_{\sigma^j}$ for each j-dimensional simplex σ^j of P and $S^{n-j}_{\sigma^{j-1}}$ for each $(j-1)$-simplex σ^{j-1} of P. A j-dimensional simplex σ^j of P has $j+1$ $(j-1)$-dimensional faces σ^{j-1}_i ($i = 1$, ..., $j+1$), hence we have $j+1$ orbits of $\hat{\psi}_t$ which pass through $S^{j-1}_{\sigma^j} \cap S^{n-j}_{\sigma^{j-1}_i}$ and connect $b_{\sigma^{j-1}_i}$ and b_{σ^j} ($i = 1$, ..., $j+1$). Any point on $S^{j-1}_{\sigma^j}$ is in an unstable manifold $L^u_{b(\sigma^k)}$ of b_{σ^k} for a k-dimensional simplex, where $k \le j-1$. If $k = j-1$, it is one of the points $S^{j-1}_{\sigma^j} \cap S^{n-j}_{\sigma^{j-1}_i}$. The flow $\hat{\psi}_t$ transverse to $M_{j-1/2}$ sends a neighborhood of $W_{j-1} \cup P^{(j)}$ to a neighborhood of W_j. Hence a neighborhood of $S^{j-1}_{\sigma^j} \cap S^{n-j}_{\sigma^{j-1}_i} \in M_{j-1/2}$ is divided by the union of the unstable manifolds $L^u_{b(\sigma^i)}$ of b_{σ^i} such that $\sigma^i \prec \sigma^{j-1}$ into j parts, each of which is contained in the un stable manifold of one of the vertices of σ^{j-1}. This means that the closure of the stable manifold of b_{σ^j} contains the union of the stable manifolds $L^s_{b(\sigma^i)}$ of b_{σ^i} such that $\sigma^i \prec \sigma^j$. Since $\bigcup_{\sigma^i \prec \sigma^j} L^s_{b(\sigma^i)}$ is homeomorphic to $\partial \Delta^j$, by looking at the flow $\hat{\psi}_t$, we see that the stable manifold of b_{σ^j} is bounded by the union of the stable manifolds $L^s_{b(\sigma^i)}$ of b_{σ^i} such that $\sigma^i \prec \sigma^j$. We see then that $L^s_{b(\sigma^j)}$ is homeomorphic to a j-dimensional simplex and $L^s_{b(\sigma^j)}$ as well as a neighborhood of $L^s_{b(\sigma^j)}$ is divided by the union of the unstable manifolds of b_{σ^i} such that $\sigma^i \prec \sigma^j$ into $j+1$ parts each of which is contained in the unstable manifold of one of the vertices of σ^j. We can take the union of W_{j-1} and closed neighborhoods of $L^s_{b(\sigma^j)}$ for j-dimensional simplices σ^j of P as $W'_j = W_{j-1} \cup \bigcup_{\sigma^j}(D^j_{\sigma^j} \times D^{n-j}_{\sigma^j})$, where the flow $\hat{\psi}_t$ on $D^j_{\sigma^j} \times D^{n-j}_{\sigma^j}$ is of the form

$$\hat{\psi}_t(x_1, \dots, x_n) = (e^{-t}x_1, \dots, e^{-t}x_j, e^t x_{j+1}, \dots, e^t x_n).$$

We can take W''_j which is obtained from W'_j by smoothing along the corner and there is an isotopy sending W''_j to W_j along the orbits of $\hat{\psi}_t$. Then there is a homeomorphism h_j sending W_j to $(f \circ g)^{-1}([0, j+1/2])$ extending h_{j-1} such that h_j sends the stable manifold $L^s_{b(\sigma^i)}$ of b_{σ^i} to σ^i and the unstable manifold $L^u_{b(\sigma^i)}$ of b_{σ^i} to $\sigma^{i*} \cap (f \circ g)^{-1}([0, j+1/2])$, respectively.

Thus we see that the configuration of stable manifolds and unstable manifolds of $\hat{\psi}_t$ is homeomorphic to the configuration of the triangulation P and its dual cell decomposition P^*. □

Now we construct a cellular decomposition adapted to a Morse function.

Let M^n be a compact n-dimensional manifold. Let $F\colon M^n \to [0, n]$ be a self-indexing Morse function. Then there is a Riemannian metric such that the gradient flow φ_t at a critical point of F of index k is of the form

$$\varphi_t(x_1, \dots, x_n) = (e^{-t}x_1, \dots, e^{-t}x_k, e^t x_{k+1}, \dots, e^t x_n)$$

in a coordinate neighborhood and the stable manifolds and unstable manifolds of critical points of F are transverse.

For such a gradient flow we have the following proposition.

Proposition 6.2. *For a k-dimensional stable manifold L of a critical point (of index k) of F, there is a continuous map $h\colon D^k \to M^n$ such that $h|\mathrm{Int}(D^k)$ is a diffeomorphism to L and $h(\partial D^k) \subset P^{(k-1)}$, where $P^{(k-1)}$ is the $(k-1)$-dimensional skeleton of the stratification by the stable manifolds of φ_t.*

This proposition is shown by Laudenbach in [13]. The author is grateful to the referee for indicating him this reference. We include the proof of Proposition 6.2 for completeness.

To show Proposition 6.2, we need to use the fact that the stratification by the stable manifolds of such φ_t satisfy a much stronger condition, namely, the closure of a stable manifold is a submanifold with conical singularities (smcs) which is defined in [13].

An m-dimensional stratified subset $X = X^{(m)} \supset \cdots \supset X^{(0)}$ of M^n defined in Section 2 is called a submanifold with conical singularities (smcs) if, for $1 \le k \le m$ and any $x \in X^{(k)} \setminus X^{(k-1)}$, there are a neighborhood V of x diffeomorphic to $D^k \times D^{n-k}$ and an $(m-k)$-dimensional smcs $T = T^{(m-k)} \supset \cdots \supset T^{(0)}$ in D^{n-k} such that $V \cap X$ is diffeomorphic to $D^k \times T$, and for $x \in X^{(0)}$, there is a C^1 embedded n-dimensional ball B centered at x such that $X' = X \cap \partial B$ is an $(m-1)$-dimensional smcs in the $(n-1)$-dimensional sphere and $(B, B \cap X^{(m)}, \cdots, B \cap X^{(1)})$ is diffeomorphic to $(B, CX'^{(m-1)}, \cdots, CX'^{(0)})$, where C denotes the cone with respect to the linear structure of the C^1 parametrization for B.

Roughly speaking Proposition 6.2 is shown in the following way. Let $p^j_1, \ldots, p^j_{c_j}$ be the critical points of F of index j. Let $S^{j-1}_{p^j_i}$ denote the attaching sphere which is the intersection of the stable manifold $L^s_{p^j_i}$ and $M_{j-1/2}$, and is the boundary of the core disk $D^j_{p^j_i} = L^s_{p^j_i} \cap F^{-1}([j-1/2, j+1/2])$. Let $S^{n-j-1}_{p^j_i}$ denote the belt sphere which is the intersection of the unstable manifold $L^u_{p^j_i}$ and $M_{j+1/2}$, and is the boundary of the cocore disk $D^{n-j}_{p^j_i} = L^u_{p^j_i} \cap F^{-1}([j-1/2, j+1/2])$.

We look at $\overline{L} \cap M_{j+1/2}$ for $j = k-1, \ldots, 0$. and we show that $\overline{L} \cap M_{j+1/2}$ is a $(k-1)$-dimensional smcs of $M_{j+1/2}$. In fact, on $M_{j+1/2}$, there are belt spheres $S^{n-j-1}_{p^j_i}$ $(i = 1, \ldots c_j)$ which intersect transversely to $\overline{L} \cap M_{j+1/2}$. On the cocore disk $D^{n-j}_{p^j_i}$ which is bounded by $S^{n-j-1}_{p^j_i}$, $\overline{L} \cap D^{n-j}_{p^j_i}$ is homeomorphic to the cone over $\overline{L} \cap S^{n-j-1}_{p^j_i}$. $\overline{L}$ restricted to a neighborhood of the cocore disk $D^{n-j}_{p^j_i}$ is homeomorphic to a product of $\overline{L} \cap D^{n-j}_{p^j_i}$ and an open ball of D^j. Using the flow φ_t on $F^{-1}([j - 1/2, j+1/2]) \setminus \bigcup_{i=1}^{c_j} D^{n-j}_{p^j_i}$, we see that $\overline{L} \cap M_{j-1/2}$ is a $(k-1)$-dimensional smcs of $M_{j-1/2}$.

By using this structure we define the homeomorphism h in the proposition.

Now the first step of the proof of Proposition 6.2 is the following lemmas, which show that the closure of a stable manifold of such φ_t is a submanifold with conical singularities (smcs) ([13], Proposition 2).

Lemma 6.3. *Let φ_t be the flow on $D^j \times D^{n-j}$ such that $\varphi_t(\boldsymbol{x}, \boldsymbol{y}) = (e^{-t}\boldsymbol{x}, e^t\boldsymbol{y})$, where $\boldsymbol{x} = (x_1, \ldots, x_j)$ and $\boldsymbol{y} = (x_{j+1}, \ldots, x_n)$. Let $N = N^{(k)} \supset \cdots \supset N^{(0)}$ be a k-dimensional stratified subset of $D^j \times D^{n-j}$ invariant under the flow φ_t such that $N \cap (D^j \times \partial D^{n-j})$ is a $(k-1)$-dimensional smcs of $D^j \times \partial D^{n-j}$ near $\{0\} \times \partial D^{n-j}$ and N is transverse to $\{0\} \times D^{n-j}$. Then there is a neighborhood U of $0 \in D^j$ such that $N \cap (U \times D^{n-j})$ is homeomorphic to $U \times C(N \cap (\{0\} \times \partial D^{n-j}))$, where C denotes the cone.*

Proof. Since $N \cap (D^j \times \partial D^{n-j})$ is a $(k-1)$-dimensional smcs and transverse to $\{0\} \times \partial D^{n-j}$, $N' = N \cap (\{0\} \times \partial D^{n-j})$ is a $(k-1-j)$-dimensional smcs in $\{0\} \times \partial D^{n-j}$ and there is a positive real number ε such that the ε-neighborhood $U = \operatorname{int}(D^j_\varepsilon)$ of $0 \in D^j$ has the following property. There is a mapping $\boldsymbol{v}\colon U \times N' \to \partial D^{n-j}$ such that $\boldsymbol{v}(0, \boldsymbol{y}) = \boldsymbol{y}$,

$$N \cap (U \times \partial D^{n-j}) = \{(\boldsymbol{x}, \boldsymbol{v}(\boldsymbol{x}, \boldsymbol{y})) \mid (\boldsymbol{x}, \boldsymbol{y}) \in U \times N'\},$$

and $\boldsymbol{v}$ is smooth on each product $U \times S$, where S is a stratum of N'. By the invariance under the flow φ_t, the set $\{(\boldsymbol{x}, \boldsymbol{v}(\boldsymbol{x}, \boldsymbol{y})) \mid (\boldsymbol{x}, \boldsymbol{y}) \in U \times S\}$ is contained in the stratum in $N \cap (U \times D^{n-j})$ which is written as

$$\{(\boldsymbol{x}, s\boldsymbol{v}(s\boldsymbol{x}, \boldsymbol{y})) \mid (\boldsymbol{x}, \boldsymbol{y}) \in U \times S,\ s \in [0, 1]\}.$$

In particular, $N \cap (\{0\} \times D^{n-j}) = C(N')$. Hence the map $(\boldsymbol{x}, s\boldsymbol{v}(s\boldsymbol{x}, \boldsymbol{y})) \longmapsto (\boldsymbol{x}, s\boldsymbol{v}(0, \boldsymbol{y}))$ is a homeomorphism sending $N \cap (U \times D^{n-j})$ to $U \times C(N')$. □

Lemma 6.4. *$\overline{L} \cap M_{j+1/2}$ is a $(k-1)$-dimensional smcs of $M_{j+1/2}$ for $j = k-1$, …, 0.*

Proof. The above lemma implies that if $\overline{L} \cap M_{j+1/2}$ is a $(k-1)$-dimensional smcs of $M_{j+1/2}$, then on

$$M'_{j+1/2} = M_{j+1/2} \setminus \bigcup_{i=1}^{c_j} U \times \partial D^{n-j}_{p^j_i} \cup \bigcup_{i=1}^{c_j} \partial U \times D^{n-j}_{p^j_i}$$

smoothened appropriately, $\overline{L} \cap M'_{j+1/2}$ is a $(k-1)$-dimensional smcs of $M'_{j+1/2}$. Since $F^{-1}([j-1/2, j+1/2]) \setminus (U \times D^{n-j}_{p^j_i})$ after smoothing along the corner is diffeomorphic to $[0,1] \times M_{j-1/2}$, where the flow φ_t corresponds to the flow in the direction of $[0,1]$, $\overline{L} \cap M'_{j+1/2}$ is diffeomorphic to $\overline{L} \cap M_{j-1/2}$. Hence $\overline{L} \cap M_{j-1/2}$ is a $(k-1)$-dimensional smcs of $M_{j-1/2}$.

Since $\overline{L} \cap M_{k-1/2}$ is a union of attaching spheres $S^{k-1}_{p^k_i}$ $(i = 1, \ldots, c_k)$, $\overline{L} \cap M_{j+1/2}$ is a $(k-1)$-dimensional smcs of $M_{j+1/2}$ for $i = k-1$, …, 0. □

Let $L = L^s_p$ be the stable manifold of the critical point p of index k. The stable manifold L is diffeomorphic to $\mathbb{R}^k$ and the restriction $\varphi_t | L$ of the flow φ_t is conjugate to the radial contraction ψ_t on $\mathbb{R}^k$ defined by $\psi_t(x_1, \ldots, x_k) = e^{-t}(x_1, \ldots, x_k)$. First we embed $\mathbb{R}^k$ in D^k such that the ray from the origin corresponds to the radial ray in $\operatorname{int}(D^k)$. Let $i : L \to D^k$ denote the embedding. Then we see that the identity map $i(L) \to L$ does not extend to a continuous map $D^k \to \overline{L}$ in general.

In order to define the map $h\colon D^k \to \overline{L}$, we use the construction in the above lemmas. For a subset A of D^k, we write $R(A)$ the radial saturation of A, that is the union of the radial segments of length 1 from the origin 0 passing through the points of A.

Proof of Proposition 6.2. We are going to construct the k-dimensional compact submanifold B_j of D^k with boundary such that

$$B_k \subset B_{k-1} \subset \cdots \subset B_1 \subset B_0 = D^k$$

and the homeomorphisms

$$h_j : B_j \longrightarrow \bar{L} \cap F^{-1}([j-1/2, k+1/2]) \quad (j = k, \dots, 0),$$

such that $h_j|(B_j \cap \operatorname{int}(D^k))$ is a diffeomorphism onto $L \cap F^{-1}([j-1/2, k+1/2])$.

First, for $L = L^s_p$, $L \cap M_{k-1/2}$ is a $(k-1)$-dimensional sphere which is the attaching sphere S^{k-1}_p bounding the core disk D^k_p. Put $B_k = i(D^k_p) \subset D^k$, and we define $h_k : B_k \to \bar{L}$ to be i^{-1}.

Secondly, we look at the finite set $S^{k-1}_p \cap S^{n-k}_{p^{k-1}_i}$. The cone $C_{p^{k-1}_i}(S^{k-1}_p \cap S^{n-k}_{p^{k-1}_i})$ is contained in $\bar{L}$ and we take the closed disk neighborhood $\bar{U}_i$ of $S^{k-1}_p \cap S^{n-k}_{p^{k-1}_i}$ in S^{k-1}_p given by Lemma 6.3 such that $U_i \times C_{p^{k-1}_j}(S^{k-1}_p \cap S^{n-k}_{p^{k-1}_i})$ is a neighborhood of $C_{p^{k-1}_j}(S^{k-1}_p \cap S^{n-k}_{p^{k-1}_i})$ in $\bar{L}$. Then we take the radial saturation $R(i(\bar{U}_i))$ in D^k. The part $R(i(\bar{U}_i)) \setminus \operatorname{int}(i(D^k_p))$ is diffeomorphic to $i(\bar{U}_i) \times [0,1]$, where $i(\bar{U}_i) \times \{0\} \subset \partial D^k$ and $i(\bar{U}_i) \times \{1\} = i(\bar{U}_i)$. Then we define

$$h'_k : i(\bar{U}_i) \times [0,1] \longrightarrow \bar{U}_i \times C_{p^{k-1}_j}(S^{k-1}_p \cap S^{n-k}_{p^{k-1}_i})$$

by $h'_k(x,t) = (i^{-1}(x), t)$, where t is the parameter of the cone such that $t = 0$ corresponds the vertex. Then we take the union $i(D^k_p) \cup \bigcup_{i=1}^{c_{k-1}} R(i(\bar{U}_i))$ and add a neighborhood of $\bigcup_{i=1}^{c_{k-1}} i(\partial \bar{U}_i)$ to obtain a smooth manifold B'_k in D^k. On the other hand, we take the union

$$D^k_p \cup \bigcup_{i=1}^{c_{k-1}} \bar{U}_i \times C_{p^{k-1}_j}(S^{k-1}_p \cap S^{n-k}_{p^{k-1}_i})$$

and add a neighborhood of $\bigcup_{i=1}^{c_{k-1}} \partial \bar{U}_i$ to obtain the subset $A_k \subset \bar{L}$. There is a continuous map $h''_k : B'_k \to A_k \subset \bar{L}$ extending h_k such that $h''_k|(B'_k \cap \operatorname{Int}(D^k))$ is a diffeomorphism onto $L \cap A_k$. Since $\bar{L} \cap F^{-1}([k-3/2, k+1/2]) \setminus A_k$ is invariant under the flow φ_t and the flow φ_t on

$$F^{-1}([k-3/2, k+1/2]) \setminus \bigcup_{i=1}^{c_{k-1}} U_i \times D^{n-k+1}_{p^{k-1}_i}$$

is conjugate to the flow on $[0,1] \times M_{k-3/2}$ in the direction of $[0,1]$, we can perform the following construction. We take a collar neighborhood $\partial B'_k \times [0,1]$ of $\partial B'_k$ in

$D^k \setminus \mathrm{int}(B'_k)$ and let B_{k-1} be the union of B'_k and its collar neighborhood. Using the flow φ_t, we can construct a continuous map

$$h_{k-1}\colon B_{k-1} \longrightarrow \overline{L} \cap F^{-1}([k-3/2, k+1/2])$$

such that $h_{k-1}|(B_{k-1} \cap \mathrm{int}(D^k))$ is a diffeomorphism onto $L \cap F^{-1}([k-3/2, k+1/2])$. We may arrange that B_{k-1} is star-shaped with respect to $0 \in D^k$ in such a way that ∂B_{k-1} and radial segments from 0 to points of ∂D^k are transverse.

Thirdly, assume that we have constructed the k-dimensional compact submanifold B_{j+1} of D^k with boundary and the homeomorphism

$$h_{j+1}\colon B_{j+1} \longrightarrow \overline{L} \cap F^{-1}([j+1/2, k+1/2])$$

such that $h_{j+1}|(B_{j+1} \cap \mathrm{int}(D^k))$ is a diffeomorphism onto $L \cap F^{-1}([j+1/2, k+1/2])$ and B_{j+1} is star-shaped with respect to 0. Then $\overline{L} \cap M_{j+1/2}$ is a $(k-1)$-dimensional smcs of $M_{j+1/2}$ and the belt spheres $S^{n-j-1}_{p_i^j}$ $(\subset M_{j+1/2})$ are transverse to $\overline{L} \cap M_{j+1/2}$ $(i = 1, \dots c_j)$. Hence $\overline{L} \cap S^{n-j-1}_{p_i^j}$ is a $(k-j-1)$-dimensional smcs of $S^{n-j-1}_{p_i^j}$. The cone $C_{p_i^j}(\overline{L} \cap S^{n-j-1}_{p_i^j})$ is contained in $\overline{L}$ and we take the closed disk neighborhood $\overline{U}_i \subset D^j_{p_i^j}$ of p_i^{k-1} given by Lemma 6.3 such that $U_i \times C_{p_i^j}(\overline{L} \cap S^{n-j-1}_{p_i^j})$ is a neighborhood of $C_{p_i^j}(\overline{L} \cap S^{n-j-1}_{p_i^j})$ in $\overline{L}$. We look at $(h_{j+1})^{-1}(\overline{L} \cap S^{n-j-1}_{p_i^j})$ and its closed neighborhood

$$\overline{V}_i^{j+1} = (h_{j+1})^{-1}(\overline{U}_i \times (\overline{L} \cap S^{n-j-1}_{p_i^j}))$$

in ∂B_{j+1}. Then we take the radial saturation $R(\overline{V}_i^{j+1})$ in D^k. This time, the part $R(\overline{V}_i^{j+1}) \setminus \mathrm{int}(B_{j+1})$ and $\overline{V}_i^{j+1} \times [0,1]$ are not diffeomorphic, but homeomorphic. The reason is that $R(\overline{V}_i^{j+1}) \setminus \mathrm{int}(B_{j+1})$ near $\overline{V}_i^{j+1} \cap \partial D^k$ is a manifold with corner along $\overline{V}_i^{j+1} \cap \partial D^k$, and there is a homeomorphism $V_i^{j+1} \times [0,1] \to R(\overline{V}_i^{j+1}) \setminus \mathrm{int}(B_{j+1})$ such that $V_i^{j+1} \times \{0\} \subset \partial D^k$ and $V_i^{j+1} \times \{0\} = V_i^{j+1}$, which straighten the corner along $(\overline{V}_i^{j+1} \cap \partial D^k) \times \{0\}$ and is no longer send the radial segments to the direction of [0, 1] near $(\overline{V}_i^{j+1} \cap \partial D^k) \times \{0\}$. This homeomorphism can be taken to be a diffeomorphism on $V_i^{j+1} \times [0,1)$. Then we take the union $B_{j+1} \cup \bigcup_{i=1}^{c_j} R(\overline{V}_i^{j+1})$ and add a neighborhood of $\bigcup_{i=1}^{c_j} \partial \overline{V}_i^{j+1}$ to obtain a smooth manifold B'_{j+1} in D^k. On the other hand, we take the union

$$\big(\overline{L} \cap F^{-1}([j+1/2, k+1/2])\big) \cup \bigcup_{i=1}^{c_j} \overline{U}_i \times C_{p_i^j}(\overline{L} \cap S^{n-j-1}_{p_i^j})$$

and add a neighborhood of $\bigcup_{i=1}^{c_{k-1}} \bar{U}_i \times (\bar{L} \cap S^{n-j-1}_{p_i^j})$ to obtain the subset $A_{j+1} \subset \bar{L}$. There is a continuous map $h''_{j+1}\colon B'_{j+1} \to A_{j+1} \subset \bar{L}$ extending h_{j+1} such that $h''_{j+1}|(B'_{j+1} \cap \mathrm{Int}(D^k))$ is a diffeomorphism onto $L \cap A_{j+1}$. Since $\bar{L} \cap F^{-1}([j - 1/2, j + 1/2]) \setminus A_{j+1}$ is invariant under the flow φ_t and the flow φ_t on

$$F^{-1}([j-1/2, j+1/2]) \setminus \bigcup_{i=1}^{c_j} U_i \times D^{n-j}_{p_i^j}$$

is conjugate to the flow on $[0,1] \times M_{j-1/2}$ in the direction of $[0,1]$, we can perform the following construction. We take a collar neighborhood $\partial B'_{j+1} \times [0,1]$ of $\partial B'_{j+1}$ in $D^k \setminus \mathrm{int}(B'_{j+1})$ and let B_j be the union of B'_{j+1} and its collar neighborhood. Using the flow φ_t, we can construct a continuous map $h_j\colon B_j \to \bar{L} \cap F^{-1}([j-1/2, k+1/2])$ such that $h_j|(B_j \cap \mathrm{int}(D^k))$ is a diffeomorphism onto $L \cap F^{-1}([j-1/2, k+1/2])$. We may arrange that B_j is star-shaped with respect to $0 \in D^k$ in such a way that ∂B_j and radial segments from 0 to points of ∂D^k are transverse.

Finally, for $j = 0$ in the above construction, we notice that $B'_1 = B_1 \cup \bigcup_{i=1}^{c_0} R(\bar{V}_i^1)$ is D^k itself and the map $h''_1\colon B'_1 \to A_1$ extending h_1 is the desired map. □

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