

The Set of Divergent Infinite Products in a Banach Space is σ -Porous

Simeon Reich and Alexander J. Zaslavski

Abstract. Let K be a bounded closed convex subset of a Banach space. We study several convergence properties of infinite products of non-expansive self-mappings of K . In our recent work we have considered several spaces of sequences of such self-mappings. Endowing them with appropriate topologies, we have shown that the infinite products corresponding to generic sequences converge. In the present paper we prove that the subsets consisting of all sequences of mappings with divergent infinite products are not only of the first Baire category, but also σ -porous.

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1. Introduction

The convergence of infinite products of operators is of interest in many areas of Mathematics and its applications. See, for example, [1, 4 - 6, 8, 15 - 17, 22 - 27, 29] and the references mentioned therein. Given a bounded closed convex subset K of a Banach space and a sequence $\mathbf{A} = \{A_t\}_{t=1}^{\infty}$ of self-mappings of K , we are interested in convergence properties of the sequence of products $\{A_n \cdots A_1 x\}_{n=1}^{\infty}$, where $x \in K$. In the special case of a constant sequence \mathbf{A} we are led to study the convergence of powers of a single operator. In their classical 1976 paper [10] De Blasi and Myjak show that the powers of a generic non-expansive self-mapping of K do converge. Such an approach, when a certain property is investigated for a whole space of operators and not just for a single operator, has already been successfully applied in many areas of Analysis. We mention, for instance, the theory of dynamical systems [11, 30], optimization [14, 28], variational analysis [2], approximation theory [12, 13], calculus of variations [3, 9, 34] and optimal control [35, 36]. In a recent paper [25] we extended the De Blasi and Myjak result in several directions to certain spaces of operator sequences.

In 1989 De Blasi and Myjak [11] refined their 1976 result by using the concept of porosity [7, 11 - 14, 31 - 33] which we now recall.

Both authors: The Technion – Israel Inst. Techn., Dept. Math., 32000 Haifa, Israel
sreich@tx.technion.ac.il and ajzasl@tx.technion.ac.il

Let (Y, d) be a complete metric space. We denote by $B(y, r)$ the closed ball of center $y \in Y$ and radius $r > 0$. A subset $E \subset Y$ is called *porous* in (Y, d) if there exist $\alpha \in (0, 1)$ and $r_0 > 0$ such that for each $r \in (0, r_0]$ and each $y \in Y$ there is a $z \in Y$ for which

$$B(z, \alpha r) \subset B(y, r) \setminus E.$$

A subset of the space Y is called σ -porous in (Y, d) if it is a countable union of porous subsets in (Y, d) .

Remark. It is known that in the above definition of porosity, the point y can be assumed to belong to E . Also other notions of porosity have been used in the literature [7, 31 - 33]. We use the rather strong notion which appears in [11 - 14].

Since porous sets are nowhere dense, all σ -porous sets are of the first category. If Y is a finite-dimensional Euclidean space \mathbb{R}^n , then σ -porous sets are of Lebesgue measure 0. The existence of a non- σ -porous set $P \subset \mathbb{R}^n$, which is of the first Baire category and of Lebesgue measure 0, was established in [31]. It is easy to see that for any σ -porous set $A \subset \mathbb{R}^n$ the set $A \cup P \subset \mathbb{R}^n$ also belongs to the family \mathcal{E} of all the non- σ -porous subsets of \mathbb{R}^n which are of the Baire first category and have Lebesgue measure 0. Moreover, if $Q \in \mathcal{E}$ is a countable union of sets $Q_i \subset \mathbb{R}^n$ ($i \geq 1$), then there is a number $j \in \mathbb{N}$ for which the set Q_j is non- σ -porous. Evidently, this set Q_j also belongs to \mathcal{E} . Therefore, one sees that the family \mathcal{E} is quite large. Also, every complete metric space without isolated points contains a closed nowhere dense set which is not σ -porous [33].

To point out the difference between porous and nowhere dense sets, note that if $E \subset Y$ is nowhere dense, $y \in Y$ and $r > 0$, then there are a point $z \in Y$ and a number $s > 0$ such that $B(z, s) \subset B(y, r) \setminus E$. If, however, E is also porous, then for small enough r we can choose $s = \alpha r$, where $\alpha \in (0, 1)$ is a constant which depends only on E .

In [11] De Blasi and Myjak show that the complement of the set of power convergent non-expansive self-mappings of K is not only of the first Baire category, but also σ -porous. Thus a natural question is whether the results of [25] can also be refined in the spirit of [11] by using the notion of porosity. In the present paper we answer this question in the affirmative.

It turns out that the natural setting for our results is the class of complete hyperbolic metric spaces which includes not only all Banach spaces, but also other spaces of interest such as the Hilbert ball and its powers. We emphasize, however, that all our results are new even in Banach spaces.

To define this class, let (X, ρ) be a metric space and let \mathbb{R} denote the real line. We say that a mapping $c : \mathbb{R} \rightarrow X$ is a metric embedding of \mathbb{R} into X if $\rho(c(s), c(t)) = |s - t|$ for all $s, t \in \mathbb{R}$. The image of \mathbb{R} under a metric embedding is called a metric line. The image of a real interval $[a, b] = \{t \in \mathbb{R} : a \leq t \leq b\}$ under such a mapping is called a metric segment. Assume that (X, ρ) contains a family M of metric lines such that for each pair of distinct points $x, y \in X$ there is a unique metric line in M which passes through x and y . This metric line determines a unique metric segment joining x and y . We denote this segment by $[x, y]$. For each $0 \leq t \leq 1$ there is a unique point $z \in [x, y]$ such that $\rho(x, z) = t\rho(x, y)$ and $\rho(z, y) = (1 - t)\rho(x, y)$. This

point is denoted by $(1 - t)x \oplus ty$. We say that X , or more precisely (X, ρ, M) , is a hyperbolic space if

$$\rho\left(\frac{1}{2}x \oplus \frac{1}{2}y, \frac{1}{2}x \oplus \frac{1}{2}z\right) \leq \frac{1}{2}\rho(y, z)$$

for all $x, y, z \in X$. A set $K \subset X$ is called ρ -convex if $[x, y] \subset K$ for all $x, y \in K$. It is clear that all normed linear spaces are hyperbolic. A discussion of more examples of hyperbolic spaces and in particular of the Hilbert ball can be found, for example, in [24]. In the sequel we will repeatedly use the following fact (cf. [19: pp. 77, 104] and [24]): If (X, ρ, M) is a hyperbolic space, then

$$\rho((1 - t)x \oplus tz, (1 - t)y \oplus tw) \leq (1 - t)\rho(x, y) + t\rho(z, w)$$

for all $x, y, z, w \in X$ and $0 \leq t \leq 1$.

The paper is organized as follows. Section 1 is devoted to weak ergodicity in the sense of population biology (see [25] and the references therein). Our Theorem 1.1 is a refinement of [25: Theorem 2.2] and also includes the 1989 result of De Blasi and Myjak [11]. Theorem 2.1 is concerned with the convergence of infinite products to a (unique) common fixed point. According to [25: Theorem 2.3], the complement of the set of convergent infinite products is of the first Baire category. Here we show that it is, in fact, σ -porous. In Section 3 we let F be a closed ρ -convex subset of K and $Q : K \rightarrow F$ a non-expansive retraction onto it. We consider several spaces of sequences of mappings which fix every point of F . Improving upon [25: Theorems 3.2 and 3.3], we show that the complements of the sets of sequences with convergent infinite products are not only of the first category, but are also σ -porous. To the best of our knowledge, our results provide the first application of the concept of porosity to the study of infinite products.

1. Weak ergodicity

Let (X, ρ, M) be a complete hyperbolic space and let $K \subset X$ be a non-empty bounded closed ρ -convex subset of X . Denote by \mathfrak{A} the set of all continuous mappings $A : K \rightarrow K$. For the space \mathfrak{A} we consider the metric $\rho_{\mathfrak{A}}$ defined by

$$\rho_{\mathfrak{A}}(A, B) = \sup_{x \in K} \rho(Ax, Bx) \quad (A, B \in \mathfrak{A}). \tag{1.1}$$

It is easy to see that the metric space $(\mathfrak{A}, \rho_{\mathfrak{A}})$ is complete. Denote by \mathcal{A} the set of all sequences $\{A_t\}_{t=1}^{\infty}$, where each $A_t \in \mathfrak{A}$. Such a sequence will occasionally be denoted by a boldface \mathbf{A} . For the space \mathcal{A} we consider the metric $\rho_{\mathcal{A}}$ defined by

$$\rho_{\mathcal{A}}(\{A_t\}_{t=1}^{\infty}, \{B_t\}_{t=1}^{\infty}) = \sup_{t \geq 1} \rho_{\mathfrak{A}}(A_t, B_t) \quad (\{A_t\}_{t=1}^{\infty}, \{B_t\}_{t=1}^{\infty} \in \mathcal{A}). \tag{1.2}$$

Clearly, the metric space $(\mathcal{A}, \rho_{\mathcal{A}})$ is complete.

An operator $A : K \rightarrow K$ is called non-expansive if

$$\rho(Ax, Ay) \leq \rho(x, y) \quad (x, y \in K). \tag{1.3}$$

Define

$$\mathfrak{A}_{ne} = \{A \in \mathfrak{A} : A \text{ is non-expansive}\}.$$

It is clear that \mathfrak{A}_{ne} is a closed subset of \mathfrak{A} . Further, define

$$\mathcal{A}_{ne} = \{\{A_t\}_{t=1}^\infty \in \mathcal{A} : A_t \in \mathfrak{A}_{ne}\}.$$

Clearly, \mathcal{A}_{ne} is a closed subset of \mathcal{A} . For each $A \in \mathfrak{A}$, let $\widehat{\mathbf{A}} = \{\widehat{A}_t\}_{t=1}^\infty \in \mathcal{A}$ be the constant sequence, where $\widehat{A}_t = A$ ($t \geq 1$). Set

$$d(K) = \sup_{x,y \in K} \rho(x,y). \tag{1.4}$$

Further, for each $x \in K$ and each $E \subset K$, set

$$\rho(x,E) = \inf_{y \in E} \rho(x,y). \tag{1.5}$$

A sequence $\{A_t\}_{t=1}^\infty \in \mathcal{A}_{ne}$ is called regular if for any $\varepsilon > 0$ there exists a number $N \in \mathbb{N}$ such that, for each $x, y \in K$, each integer $T \geq N$ and each mapping $h : \{1, \dots, T\} \rightarrow \{1, 2, \dots\}$,

$$\rho(A_{h(T)} \cdots A_{h(1)}x, A_{h(T)} \cdots A_{h(1)}y) \leq \varepsilon.$$

A mapping $A \in \mathfrak{A}_{ne}$ is called regular if the sequence $\widehat{\mathbf{A}} = \{A_t\}_{t=1}^\infty$, where $A_t = A$ ($t \geq 1$), is regular. It is easy to verify that if $A \in \mathfrak{A}_{ne}$ is regular, then there exists a unique $x_A \in K$ such that $Ax_A = x_A$ and $A^n x \rightarrow x_A$ as $n \rightarrow \infty$, uniformly on K .

Denote by \mathcal{F} the set of all regular elements of \mathcal{A}_{ne} . We already know by [25: Theorem 2.2] that the complement of \mathcal{F} is of the first Baire category. In our first theorem (Theorem 1.1 below) we show that, in fact, it is σ -porous. This theorem also includes the 1989 result of De Blasi and Myjak [11].

We denote, for each $n \in \mathbb{N}$, by \mathcal{F}_n the set of all sequences $\{A_t\}_{t=1}^\infty \in \mathcal{A}_{ne}$ which have the following property:

There exists an integer $N \in \mathbb{N}$ such that, for each $x, y \in K$, each integer $T \geq N$ and each mapping $h : \{1, \dots, T\} \rightarrow \{1, 2, \dots\}$,

$$\rho(A_{h(T)} \cdots A_{h(1)}x, A_{h(T)} \cdots A_{h(1)}y) \leq \frac{1}{n}.$$

It is not difficult to see that $\mathcal{F} = \bigcap_{n=1}^\infty \mathcal{F}_n$.

Denote by $\mathcal{F}^{(0)}$ the set of all $A \in \mathfrak{A}_{ne}$ such that $\widehat{\mathbf{A}} \in \mathcal{F}$, and for each $n \in \mathbb{N}$ denote by $\mathcal{F}_n^{(0)}$ the set of all $A \in \mathfrak{A}_{ne}$ such that $\widehat{\mathbf{A}} \in \mathcal{F}_n$. Clearly, $\mathcal{F}^{(0)} = \bigcap_{n=1}^\infty \mathcal{F}_n^{(0)}$.

Theorem 1.1.

- (i) *The set $\mathcal{A}_{ne} \setminus \mathcal{F}$ is σ -porous in \mathcal{A}_{ne} .*
- (ii) *The set $\mathfrak{A}_{ne} \setminus \mathcal{F}^{(0)}$ is σ -porous in \mathfrak{A}_{ne} .*

Proof. To establish this theorem, it is sufficient to show that $\mathcal{A} \setminus \mathcal{F}_n$ is porous in \mathcal{A}_{ne} and that $\mathfrak{A}_{ne} \setminus \mathcal{F}_n^{(0)}$ is porous in \mathfrak{A}_{ne} for each $n \in \mathbb{N}$. To this end, let $n \in \mathbb{N}$, fix $\theta \in K$ and choose $\alpha \in (0, 1)$ such that

$$\alpha < (1 - \alpha)(d(K) + 1)^{-1}(8n)^{-1}. \tag{1.6}$$

Assume that $\{A_t\}_{t=1}^\infty \in \mathcal{A}_{ne}$ and $r \in (0, 1]$. Set

$$\gamma = (1 - \alpha)r(2d(K) + 2)^{-1} \tag{1.7}$$

and choose $2 < N \in \mathbb{N}$ such that

$$N\alpha r > 2d(K) + 1. \tag{1.8}$$

For each $t \in \mathbb{N}$, define

$$A_{\gamma t}x = (1 - \gamma)Ax \oplus \gamma\theta \quad (x \in K). \tag{1.9}$$

It is clear that $\{A_{\gamma t}\}_{t=1}^\infty \in \mathcal{A}_{ne}$. Note that if $A_t = A$ ($t \geq 1$) with $A \in \mathfrak{A}_{ne}$, then $A_{\gamma t} = A_\gamma$ ($t \geq 1$) where

$$A_\gamma x = (1 - \gamma)Ax \oplus \gamma\theta \quad (x \in K). \tag{1.10}$$

Clearly,

$$\rho_{\mathcal{A}}(\{A_{\gamma t}\}_{t=1}^\infty, \{A_t\}_{t=1}^\infty) \leq \gamma d(K). \tag{1.11}$$

Now assume that $\{B_t\}_{t=1}^\infty \in \mathcal{A}_{ne}$ and that

$$\rho_{\mathcal{A}}(\{A_{\gamma t}\}_{t=1}^\infty, \{B_t\}_{t=1}^\infty) \leq \alpha r. \tag{1.12}$$

Then (1.11), (1.12) and (1.7) imply

$$\rho(\{A_t\}_{t=1}^\infty, \{B_t\}_{t=1}^\infty) \leq \alpha r + \gamma d(K) \leq \alpha r + \frac{(1 - \alpha)r}{2} = \frac{(1 + \alpha)r}{2} < r. \tag{1.13}$$

We will show that, for each $x, y \in K$, each integer $T \geq N$, and each mapping $h : \{1, \dots, T\} \rightarrow \{1, 2, \dots\}$,

$$\rho(B_{h(T)} \cdots B_{h(1)}x, B_{h(T)} \cdots B_{h(1)}y) \leq \frac{1}{n}. \tag{1.14}$$

To meet this goal, it is sufficient to show that for each $x, y \in K$ and each mapping $r : \{1, \dots, N\} \rightarrow \{1, 2, \dots\}$ there is an integer $m \in \{1, \dots, N\}$ such that

$$\rho(B_{r(m)} \cdots B_{r(1)}x, B_{r(m)} \cdots B_{r(1)}y) \leq \frac{1}{n}. \tag{1.15}$$

Assume that $x, y \in K$ and $r : \{1, \dots, N\} \rightarrow \{1, 2, \dots\}$. We will show that there is an $m \in \{1, \dots, N\}$ such that (1.15) holds. Assume the contrary. Then, for each integer $i \in [1, N]$,

$$\rho(B_{r(i)} \cdots B_{r(1)}x, B_{r(i)} \cdots B_{r(1)}y) > \frac{1}{n} \quad \text{and} \quad \rho(x, y) > \frac{1}{n}. \quad (1.16)$$

Set

$$\begin{aligned} x_0 &= x, & x_{i+1} &= B_{r(i+1)}x_i \\ y_0 &= y, & y_{i+1} &= B_{r(i+1)}y_i \end{aligned} \quad (i = 0, \dots, N - 1). \quad (1.17)$$

Let $i \in \{0, \dots, N - 1\}$. Then, by (1.16) - (1.17),

$$\rho(x_i, y_i) > \frac{1}{n}. \quad (1.18)$$

By (1.9) and (1.3) we have

$$\begin{aligned} &\rho(A_{\gamma r(i+1)}x_i, A_{\gamma r(i+1)}y_i) \\ &= \rho((1 - \gamma)A_{r(i+1)}x_i \oplus \gamma\theta, (1 - \gamma)A_{r(i+1)}y_i \oplus \gamma\theta) \\ &\leq (1 - \gamma)\rho(A_{r(i+1)}x_i, A_{r(i+1)}y_i) \\ &\leq (1 - \gamma)\rho(x_i, y_i). \end{aligned} \quad (1.19)$$

It follows from (1.17), (1.19), (1.12), (1.18), (1.7) and (1.6) that

$$\begin{aligned} \rho(x_{i+1}, y_{i+1}) &= \rho(B_{r(i+1)}x_i, B_{r(i+1)}y_i) \\ &\leq \rho(B_{r(i+1)}x_i, A_{\gamma r(i+1)}x_i) \\ &\quad + \rho(A_{\gamma r(i+1)}x_i, A_{\gamma r(i+1)}y_i) \\ &\quad + \rho(A_{\gamma r(i+1)}y_i, B_{r(i+1)}y_i) \\ &\leq (1 - \gamma)\rho(x_i, y_i) + 2\alpha r \\ &= \rho(x_i, y_i) - \gamma\rho(x_i, y_i) + 2\alpha r \\ &\leq \rho(x_i, y_i) - \frac{\gamma}{n} + 2\alpha r \\ &= \rho(x_i, y_i) + 2\alpha r - (2n)^{-1}(1 - \alpha)r(d(K) + 1)^{-1} \\ &\leq \rho(x_i, y_i) - 2\alpha r. \end{aligned} \quad (1.20)$$

Therefore, by (1.4) and (1.8),

$$\rho(x_N, y_N) \leq \rho(x_0, y_0) - 2N\alpha r \leq 2d(K) - N\alpha r < 0$$

which is a contradiction. The contradiction we have reached yields the existence of an integer m for which (1.15) is true, and the relation $\{B_t\}_{t=1}^\infty \in \mathcal{F}_n$. Thus we have shown that (see (1.13))

$$\begin{aligned} &\left\{ \{B_t\}_{t=1}^\infty \in \mathcal{A}_{ne} : \rho_{\mathcal{A}}(\{A_{\gamma t}\}_{t=1}^\infty, \{B_t\}_{t=1}^\infty) \right. \\ &\quad \left. \leq \alpha r \right\} \subset \{C \in \mathcal{A}_{ne} : \rho(C, \mathbf{A}) < r\} \cap \mathcal{F}_n. \end{aligned}$$

If $A_t = A$ ($t \geq 1$) where $A \in \mathfrak{A}_{ne}$, then $A_{\gamma t} = A_\gamma$ ($t \geq 1$) (see (1.10)) and

$$\{B \in \mathfrak{A}_{ne} : \rho(B, A_\gamma) \leq \alpha r\} \subset \{C \in \mathfrak{A}_{ne} : \rho(C, A) < r\} \cap \mathcal{F}_n^{(0)}.$$

Consequently, the set $\mathcal{A}_{ne} \setminus \mathcal{F}_n$ is porous in \mathcal{A}_{ne} and the set $\mathfrak{A}_{ne} \setminus \mathcal{F}_n^{(0)}$ is porous in \mathfrak{A}_{ne} . This completes the proof of Theorem 1.1 ■

2. Convergence to a common fixed point

We use the notations and definitions introduced in Section 1. Also, we denote by \mathcal{A}_{ne}^* the set of all sequences $\mathbf{A} = \{A_t\}_{t=1}^\infty \in \mathcal{A}_{ne}$ for which there exists $x(\mathbf{A}) \in K$ such that

$$A_t x(\mathbf{A}) = x(\mathbf{A}) \quad (t \geq 1). \tag{2.1}$$

The closure of \mathcal{A}_{ne}^* in the metric space $(\mathcal{A}_{ne}, \rho_{\mathcal{A}})$ will be denoted by $\bar{\mathcal{A}}_{ne}^*$.

Theorem 2.1. *Let \mathcal{F} be the set of all $\mathbf{A} = \{A_t\}_{t=1}^\infty \in \bar{\mathcal{A}}_{ne}^*$ for which the following assertions hold:*

- (i) *There exists $x_* \in K$ such that $A_t x_* = x_*$ ($t \geq 1$).*
- (ii) *For each $\varepsilon > 0$ there exists an $N \in \mathbb{N}$ such that, for each integer $n \geq N$, each mapping $h : \{1, \dots, n\} \rightarrow \{1, 2, \dots\}$ and each $x \in K$,*

$$\rho(A_{h(n)} \cdots A_{h(1)} x, x_*) \leq \varepsilon. \tag{2.2}$$

Then the set $\bar{\mathcal{A}}_{ne}^ \setminus \mathcal{F}$ is σ -porous in $\bar{\mathcal{A}}_{ne}^*$.*

Proof. For each $n \in \mathbb{N}$, denote by \mathcal{F}_n the set of all sequences $\{A_t\}_{t=1}^\infty \in \bar{\mathcal{A}}_{ne}^*$ for which there exist $x^{(n)} \in K$ and an $N \in \mathbb{N}$ such that, for each integer $T \geq N$, each mapping $h : \{1, \dots, T\} \rightarrow \{1, 2, \dots\}$ and each $x \in K$,

$$\rho(A_{h(T)} \cdots A_{h(1)} x, x^{(n)}) \leq \frac{1}{n}.$$

It is not difficult to see that $\mathcal{F} = \bigcap_{n=1}^\infty \mathcal{F}_n$. Let $n \in \mathbb{N}$. We will show that the set $\bar{\mathcal{A}}_{ne}^* \setminus \mathcal{F}_n$ is porous in $\bar{\mathcal{A}}_{ne}^*$. For this choose an $\alpha \in (0, 1)$ such that

$$\alpha < (4n)^{-1} (16(d(K) + 1))^{-1}. \tag{2.3}$$

Assume that $\{\tilde{A}_t\}_{t=1}^\infty \in \bar{\mathcal{A}}_{ne}^*$ and $r \in (0, 1]$. There exists $\{A_t\}_{t=1}^\infty \in \mathcal{A}_{ne}^*$ such that

$$\rho_{\mathcal{A}}(\{\tilde{A}_t\}_{t=1}^\infty, \{A_t\}_{t=1}^\infty) \leq \frac{r}{4}. \tag{2.4}$$

Let $x_{\mathbf{A}} \in K$ be such that

$$A_t x_{\mathbf{A}} = x_{\mathbf{A}} \quad (t \geq 1). \tag{2.5}$$

Set

$$\gamma = 16^{-1} r (d(K) + 1)^{-1} \tag{2.6}$$

and choose an $2 < N \in \mathbb{N}$ such that

$$(1 - \gamma)^N (2d(K) + 2) < (4n)^{-1}. \tag{2.7}$$

For each $t \in \mathbb{N}$ define

$$A_{\gamma t} x = (1 - \gamma) A_t x \oplus \gamma x_{\mathbf{A}} \quad (x \in K). \tag{2.8}$$

Clearly, $\{A_{\gamma t}\}_{t=1}^{\infty} \in \mathcal{A}_{ne}^*$ and

$$\rho_{\mathcal{A}}(\{A_{\gamma t}\}_{t=1}^{\infty}, \{A_t\}_{t=1}^{\infty}) \leq 2\gamma d(K). \tag{2.9}$$

Assume that $\{B_t\}_{t=1}^{\infty} \in \mathcal{A}_{ne}$ and

$$\rho_{\mathcal{A}}(\{A_{\gamma t}\}_{t=1}^{\infty}, \{B_t\}_{t=1}^{\infty}) \leq \alpha r. \tag{2.10}$$

Then (2.4), (2.9), (2.10), (2.6) and (2.3) imply

$$\begin{aligned} \rho_{\mathcal{A}}(\{\tilde{A}_t\}_{t=1}^{\infty}, \{B_t\}_{t=1}^{\infty}) &\leq \rho_{\mathcal{A}}(\{\tilde{A}_t\}_{t=1}^{\infty}, \{A_t\}_{t=1}^{\infty}) \\ &\quad + \rho(\{A_t\}_{t=1}^{\infty}, \{A_{\gamma t}\}_{t=1}^{\infty}) \\ &\quad + \rho(\{A_{\gamma t}\}_{t=1}^{\infty}, \{B_t\}_{t=1}^{\infty}) \\ &\leq \frac{r}{4} + 2\gamma d(K) + \alpha r \\ &= r(\alpha + 4^{-1}) + \frac{r}{4} \\ &< r \end{aligned}$$

and

$$\rho_{\mathcal{A}}(\{\tilde{A}_t\}_{t=1}^{\infty}, \{B_t\}_{t=1}^{\infty}) < r. \tag{2.11}$$

We will show that the following property holds:

(P1) For each $x \in K$, each integer $T \geq N$ and each mapping $h : \{1, \dots, T\} \rightarrow \{1, 2, \dots\}$, $\rho(B_{h(T)} \cdots B_{h(1)}x, x_{\mathbf{A}}) \leq \frac{1}{n}$.

Let $y \in K$ and $t \in \mathbb{N}$. By (2.8) and (1.3) we have

$$\begin{aligned} \rho(A_{\gamma t}y, x_{\mathbf{A}}) &= \rho((1 - \gamma)A_t y \oplus \gamma x_{\mathbf{A}}, x_{\mathbf{A}}) \\ &\leq (1 - \gamma)\rho(A_t y, x_{\mathbf{A}}) \\ &\leq (1 - \gamma)\rho(y, x_{\mathbf{A}}). \end{aligned} \tag{2.12}$$

When combined with (2.10), this inequality implies

$$\rho(B_t y, x_{\mathbf{A}}) \leq \rho(A_{\gamma t}y, x_{\mathbf{A}}) + \rho(B_t y, A_{\gamma t}y) \leq \alpha r + (1 - \gamma)\rho(y, x_{\mathbf{A}}) \tag{2.13}$$

for each $t \in \mathbb{N}$ and each $y \in K$.

Assume that $x \in K$, $T \geq N$, and $h : \{1, \dots, T\} \rightarrow \{1, 2, \dots\}$. Set

$$\begin{aligned} x_0 &= x \\ x_{i+1} &= B_{h(i+1)}x_i \quad (i \geq 0). \end{aligned} \tag{2.14}$$

It follows from (2.13) that, for any integer $i \in \mathbb{N}_0$,

$$\rho(x_{i+1}, x_{\mathbf{A}}) \leq \alpha r + (1 - \gamma)\rho(x_i, x_{\mathbf{A}}).$$

Using induction we see that, for $i = 1, \dots, T$,

$$\rho(x_i, x_{\mathbf{A}}) \leq (1 - \gamma)^i \rho(x_0, x_{\mathbf{A}}) + \alpha r \left(\sum_{t=0}^{i-1} (1 - \gamma)^t \right). \tag{2.15}$$

By (2.14), (2.15), (1.4), (2.6), (2.3) and (2.7),

$$\begin{aligned} \rho(B_{h(T)} \cdots B_{h(1)}x, x_{\mathbf{A}}) &= \rho(x_T, x_{\mathbf{A}}) \\ &\leq (1 - \gamma)^T \rho(x_0, x_{\mathbf{A}}) + \gamma^{-1} \alpha r \\ &\leq (1 - \gamma)^N d(K) + \gamma^{-1} \alpha r \\ &= (1 - \gamma)^N d(K) + 16(d(K) + 1)\alpha \\ &\leq (1 - \gamma)^N d(K) + (4n)^{-1} \\ &< \frac{1}{n} \end{aligned}$$

and

$$\rho(B_{h(T)} \cdots B_{h(1)}x, x_{\mathbf{A}}) < \frac{1}{n}. \tag{2.16}$$

Therefore property (P1) holds.

We have shown that (2.10) implies (2.11) and property (P1). If, in addition, $\{B_t\}_{t=1}^\infty \in \bar{\mathcal{A}}_{ne}^*$, then $\{B_t\}_{t=1}^\infty \in \mathcal{F}_n$. Hence $\bar{\mathcal{A}}_{ne}^* \setminus \mathcal{F}_n$ is porous in $\bar{\mathcal{A}}_{ne}^*$. This completes the proof of Theorem 2.1 ■

3. Convergence to a retraction

We continue to use the notations and definitions introduced in the previous sections. Assume that F is a non-empty closed ρ -convex subset of K . Denote by $\mathfrak{A}^{(F)}$ the set of all $A \in \mathfrak{A}$ such that

$$Ax = x \quad (x \in F) \quad \text{and} \quad \rho(Ay, x) \leq \rho(y, x) \quad (x \in F, y \in K). \tag{3.1}$$

It is clear that $\mathfrak{A}^{(F)}$ is a closed subset of \mathfrak{A} . Denote by $\mathfrak{A}_u^{(F)}$ the set of all uniformly continuous $A \in \mathfrak{A}^{(F)}$ and by $\mathfrak{A}_{ne}^{(F)}$ the set of all $A \in \mathfrak{A}^{(F)}$ such that

$$\rho(Ax, Ay) \leq \rho(x, y) \quad (x, y \in K). \tag{3.2}$$

Clearly, $\mathfrak{A}_u^{(F)}$ and $\mathfrak{A}_{ne}^{(F)}$ are closed subsets of $\mathfrak{A}^{(F)}$. Denote by

- $\mathcal{A}^{(F)}$ the set of all $\{A_t\}_{t=1}^\infty \in \mathcal{A}$ such that $A_t \in \mathfrak{A}^{(F)}$
- $\mathcal{A}_u^{(F)}$ the set of all $\{A_t\}_{t=1}^\infty \in \mathcal{A}$ such that $A_t \in \mathfrak{A}_u^{(F)}$
- $\mathcal{A}_{ne}^{(F)}$ the set of all $\{A_t\}_{t=1}^\infty \in \mathcal{A}$ such that $A_t \in \mathfrak{A}_{ne}^{(F)}$.

Clearly, $\mathcal{A}^{(F)}$, $\mathcal{A}_u^{(F)}$ and $\mathcal{A}_{ne}^{(F)}$ are closed subsets of \mathcal{A} . We consider the metric spaces $(\mathcal{A}^{(F)}, \rho_{\mathcal{A}})$, $(\mathcal{A}_u^{(F)}, \rho_{\mathcal{A}})$ and $(\mathcal{A}_{ne}^{(F)}, \rho_{\mathcal{A}})$. We assume that there exists $Q \in \mathfrak{A}^{(F)}$ such that $Q(K) = F$.

A sequence $\{A_t\}_{t=1}^\infty \in \mathcal{A}^{(F)}$ is called normal if the following two properties hold:

- (i) For each mapping $h : \{1, 2, \dots\} \rightarrow \{1, 2, \dots\}$ there exists an operator $P_h : K \rightarrow F$ such that $\lim_{t \rightarrow \infty} A_{h(T)} \cdots A_{h(1)}x = P_hx$ for all $x \in K$.
- (ii) For each $\varepsilon > 0$ there exists an $N \in \mathbb{N}$ such that, for each integer $T \geq N$, each mapping $h : \{1, 2, \dots\} \rightarrow \{1, 2, \dots\}$ and each $x \in K$, $\rho(A_{h(T)} \cdots A_{h(1)}x, P_hx) \leq \varepsilon$.

Denote by \mathcal{F} the set of all normal sequences in $\mathcal{A}^{(F)}$. Let $n \in \mathbb{N}$. Denote by \mathcal{F}_n the set of all $\{A_t\}_{t=1}^\infty \in \mathcal{A}^{(F)}$ for which there exists an $N \in \mathbb{N}$ such that, for each $x \in K$, each integer $T \geq N$ and each mapping $h : \{1, 2, \dots, T\} \rightarrow \{1, 2, \dots\}$,

$$\rho(A_{h(T)} \cdots A_{h(1)}x, F) < \frac{1}{n}.$$

It is not difficult to see that $\mathcal{F} = \bigcap_{n=1}^\infty \mathcal{F}_n$.

We will prove the following result.

Theorem 3.1.

- (i) The set $\mathcal{A}^{(F)} \setminus \mathcal{F}$ is σ -porous in $(\mathcal{A}^{(F)}, \rho_{\mathcal{A}})$.
- (ii) The set $\mathcal{A}_u^{(F)} \setminus \mathcal{F}$ is σ -porous in $(\mathcal{A}_u^{(F)}, \rho_{\mathcal{A}})$ if $Q \in \mathfrak{A}_u^{(F)}$.
- (iii) The set $\mathcal{A}_{ne}^{(F)} \setminus \mathcal{F}$ is σ -porous in $(\mathcal{A}_{ne}^{(F)}, \rho_{\mathcal{A}})$ if $Q \in \mathfrak{A}_{ne}^{(F)}$.

A mapping $A \in \mathfrak{A}^{(F)}$ is called normal if the constant sequence $\{A_t\}_{t=1}^\infty$ with $A_t = A$ ($t \geq 1$) is normal. Denote by $\mathcal{F}^{(0)}$ the set of all normal mappings $A \in \mathfrak{A}^{(F)}$.

Theorem 3.2.

- (i) The set $\mathfrak{A}^{(F)} \setminus \mathcal{F}^{(0)}$ is σ -porous in $(\mathfrak{A}^{(F)}, \rho_{\mathfrak{A}})$.
- (ii) The set $\mathfrak{A}_u^{(F)} \setminus \mathcal{F}^{(0)}$ is σ -porous in $(\mathfrak{A}_u^{(F)}, \rho_{\mathfrak{A}})$ if $Q \in \mathfrak{A}_u^{(F)}$.
- (iii) The set $\mathfrak{A}_{ne}^{(F)} \setminus \mathcal{F}^{(0)}$ is σ -porous in $(\mathfrak{A}_{ne}^{(F)}, \rho_{\mathfrak{A}})$ if $Q \in \mathfrak{A}_{ne}^{(F)}$.

Proof of Theorems 3.1 and 3.2. Let $n \in \mathbb{N}$ and choose a number $\alpha \in (0, 1)$ such that

$$\alpha < 32^{-1}(4n)^{-1}(d(K) + 1)^{-1}. \tag{3.3}$$

Assume that $\{A_t\}_{t=1}^\infty \in \mathcal{A}^{(F)}$ and $r \in (0, 1]$. Set

$$\gamma = 32^{-1}r(d(K) + 1)^{-1} \tag{3.4}$$

and choose a natural number $N > 2$ such that

$$(1 - \gamma)^N 2(d(K) + 2) < (4n)^{-1}. \tag{3.5}$$

At last, for each $t \in \mathbb{N}$ define

$$A_{\gamma t}x = (1 - \gamma)A_t x \oplus \gamma Qx \quad (x \in K). \tag{3.6}$$

Clearly,

$$\{A_{\gamma t}\}_{t=1}^\infty \in \mathcal{A}^{(F)}$$

if $\{A_t\}_{t=1}^\infty \in \mathcal{A}_u^{(F)}$ and $Q \in \mathfrak{A}_u^{(F)}$, then $\{A_{\gamma t}\}_{t=1}^\infty \in \mathcal{A}_u^{(F)}$

if $\{A_t\}_{t=1}^\infty \in \mathcal{A}_{ne}^{(F)}$ and $Q \in \mathfrak{A}_{ne}^{(F)}$, then $\{A_{\gamma t}\}_{t=1}^\infty \in \mathcal{A}_{ne}^{(F)}$.

Note that if $A_t = A$ ($t \geq 1$) with $A \in \mathfrak{A}$, then $A_{\gamma t}x = (1 - \gamma)Ax \oplus \gamma Qx$ for all $x \in K$ and $t \geq 1$. Evidently,

$$\rho_{\mathcal{A}}(\{A_{\gamma t}\}_{t=1}^\infty, \{A_t\}_{t=1}^\infty) \leq \gamma d(K). \tag{3.7}$$

Assume that $\{B_t\}_{t=1}^\infty \in \mathcal{A}^{(F)}$ and

$$\rho_{\mathcal{A}}(\{A_{\gamma t}\}_{t=1}^\infty, \{B_t\}_{t=1}^\infty) \leq \alpha r. \tag{3.8}$$

Relations (3.7), (3.8), (3.4) and (3.3) imply

$$\begin{aligned} \rho_{\mathcal{A}}(\{B_t\}_{t=1}^\infty, \{A_t\}_{t=1}^\infty) &\leq \rho_{\mathcal{A}}(\{A_t\}_{t=1}^\infty, \{A_{\gamma t}\}_{t=1}^\infty) + \rho_{\mathcal{A}}(\{A_{\gamma t}\}_{t=1}^\infty, \{B_t\}_{t=1}^\infty) \\ &\leq \alpha r + \gamma d(K) \\ &\leq \alpha r + \frac{r}{8} \\ &< r. \end{aligned} \tag{3.9}$$

Let $T \geq N$ be an integer, $x \in K$, and $h : \{1, \dots, T\} \rightarrow \{1, 2, \dots\}$. We will show that

$$\rho(B_{h(T)} \cdots B_{h(1)}x, F) < \frac{1}{n}.$$

It is sufficient to show that

$$\rho(B_{h(N)} \cdots B_{h(1)}x, F) < \frac{1}{n}.$$

Let $y \in K$ and $t \in \mathbb{N}$. By (3.6) and (3.1), for each $z \in F$,

$$\begin{aligned} \rho(A_{\gamma h(t)}y, F) &\leq \rho(A_{\gamma h(t)}y, (1 - \gamma)z \oplus \gamma Qy) \\ &= \rho((1 - \gamma)A_{h(t)}y \oplus \gamma Qy, (1 - \gamma)z \oplus \gamma Qy) \\ &\leq (1 - \gamma)\rho(A_{h(t)}y, z) \\ &\leq (1 - \gamma)\rho(y, z) \end{aligned}$$

and

$$\rho(A_{\gamma h(t)}y, F) \leq (1 - \gamma)\rho(y, z)$$

for all $z \in F$. Therefore,

$$\rho(A_{\gamma h(t)}y, F) \leq (1 - \gamma)\rho(y, F).$$

When combined with (3.8), this inequality implies

$$\begin{aligned} \rho(B_{h(t)}y, F) &\leq \rho(A_{\gamma h(t)}y, F) + \rho(A_{\gamma h(t)}y, B_{h(t)}y) \\ &\leq \alpha r + (1 - \gamma)\rho(y, F). \end{aligned} \tag{3.10}$$

By induction, using (3.10), we obtain for $i = 1, \dots, N$

$$\rho(B_{h(i)} \cdots B_{h(1)}x, F) \leq \rho(x, F)(1 - \gamma)^i + \alpha r \sum_{t=0}^{i-1} (1 - \gamma)^t.$$

It follows from this inequality, (1.4), (3.4), (3.5) and (3.3) that

$$\begin{aligned} \rho(B_{h(N)} \cdots B_{h(1)}x, F) &\leq (1 - \gamma)^N d(K) + \gamma^{-1} \alpha r \\ &\leq (1 - \gamma)^N d(K) + 32(d(K) + 1)\alpha \\ &< \frac{1}{n}. \end{aligned}$$

Thus we have shown that

$$\rho(B_{h(T)} \cdots B_{h(1)}x, F) < \frac{1}{n}$$

for each $x \in K$, each integer $T \geq N$ and each mapping $h : \{1, \dots, T\} \rightarrow \{1, 2, \dots\}$, and hence

$$\{B_t\}_{t=1}^\infty \in \mathcal{F}_n. \quad (3.11)$$

Therefore, for each $\{B_t\}_{t=1}^\infty$ satisfying (3.8), relations (3.9) and (3.11) hold. This implies that $\mathcal{A}^{(F)} \setminus \mathcal{F}_n$ is porous in $(\mathcal{A}^{(F)}, \rho_{\mathcal{A}})$. Since $\mathcal{F} = \bigcap_{n=1}^\infty \mathcal{F}_n$, we conclude that $\mathcal{A}^{(F)} \setminus \mathcal{F}$ is σ -porous in $(\mathcal{A}^{(F)}, \rho_{\mathcal{A}})$. It is not difficult to see that we have also proved the remaining statements of Theorems 3.1 and 3.2 ■

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