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Some Remarks on Complementarity Problems in a Hilbert Space

A. Carbone and P. P. Zabreiko

Abstract. We present a new approach to the analysis of solvability properties for complementarity problems in a Hilbert space. This approach is based on the Skrypnik degree which, in the case of mappings in a Hilbert space, is essentially more general in comparison with the classical Leray-Schauder degree. Namely, the Skrypnik degree allows us to obtain some new results about solvability of complementarity problems in the infinite-dimensional case. The case of generalized solutions is also considered.

Keywords: *Complementarity problems, topological degree, exceptional elements, homotopy, operator of class S_+, quasi-monotone operators*

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This article deals with some topological properties of the classical complementarity problem (more exactly, *explicit complementarity problem*) in a Hilbert space with a completely continuous operator f. It is proved that the natural linear homotopy between the original complementary problem for the operator f and the trivial one for the zero operator possesses a standard alternative property:

For each bounded domain Ω containing 0, under some additional conditions, either every mapping of this homotopy has a zero on the boundary $\partial\Omega$ of the domain Ω or there exists a zero of the limit operator which defines a solution in Ω to the complementarity problem under consideration.

This alternative property implies the corresponding alternative property for the complementarity problem under consideration:

For each $r, 0 < r < \infty$, under some natural conditions, either every complementarity problem with the operator $(1-\lambda)I + \lambda f$ $(0 < \lambda < 1)$ has a solution in the set $\{u : \|u\| = r\}$ or the complementarity problem with the operator f has a solution in the set $\{u : \|u\| \leq r\}$.

Results presented in this article are close to those from [5] (see also [1, 4]). However, we do note use *exceptional families of elements* which are essential for

A. Carbone: Univ. della Calabria, Dip. di Matem., IT-87036 Arcavacata di Rende (Cosenza), Italia; carbonea@unical.it
P. P. Zabreiko: Nat. Acad. Sci., Inst. Math., ul. Surganova 11, Minsk, 220072 Belarus & Belarus State Univ., Fac. Mech. and Math., Prospect Skoriny 4, Minsk, 220050 Belarus; zabreiko@mmf.bsu.unibel.by

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considerations and constructions in [5]. We prefer more standard terminology of the homotopy theory. Moreover, the basic difference between our approach and the approach in [5] is based on the utilization of the Skrypnik degree whereas in [5] the authors use the Leray-Schauder degree. Of course, in the finite-dimensional case our results are equivalent to the results of [5]. We remark also that results of this article give a partial solution to complementarity problems in Hilbert space which were stated in [5].

1. Let X be a Hilbert space, K a closed cone in X, and K^* its dual wedge which is defined by the formula

$$K^* = \{x \in X : (x, \xi) \geq 0 \ (\xi \in K)\};$$

this is a cone if and only if $\overline{K - K} = X$. Further, let P_K be the operator of the best approximation onto K; this operator is defined by the equation

$$\|x - P_K x\| = \inf_{u \in K} \|x - u\|.$$

The operator P_K has been studied in detail by many authors (see, e.g., [6, 8]). The following lemma collects all properties of P_K which will be used in this article.

Lemma 1. *The operator P_K has the following properties:*

(a) *P_K is a non-expansive operator, i.e. $\|P_K x' - P_K x''\| \leq \|x' - x''\| \quad (x', x'' \in X)$.*

(b) *P_K satisfies $\|P_K x' - P_K x''\|^2 \leq (x' - x'', P_K x' - P_K x'') \quad (x', x'' \in X)$.*

(c) *The equality $u = P_K x$ $(x \in X,\ u \in K)$ holds if and only if $(x - u, v) \leq 0 \quad (v \in K)$ and $(x - u, u) = 0$. In particular, $(x - P_K x, v) \leq 0 \quad (v \in K)$ and $(x - P_K x, P_K x) = 0$ for any $x \in X$.*

(d) *The equation $P_K + P_{(-K^*)} = I$ holds; moreover, the equality $x = u + v$ $(u \in K, v \in K^*)$ with $(u, v) = 0)$ holds if and only if $u = P_K x$ and $v = P_{(-K^*)} x$ where $P_{(-K^*)}$ is the operator of the best approximation onto $-K^*$.*

Observe that the operator P_K in general is not weakly sequentially continuous. In the most important case when $X = L_2(\Omega, \mathcal{A}, \mu)$ (here Ω is a set, $\mathcal{A}$ a σ-algebra of subsets and μ is a σ-finite measure) and K is the cone of non-negative functions from X, the operator P_K coincides with the operator $x \to x_+ = \max\{x, 0\}$ and all statements of Lemma 1 are trivial. The corresponding abstract situation is considered, e.g., in [8, 9].

Let F be a operator defined on K and taking its values in X. Recall [3] that the classical complementarity problem for F is the problem of finding elements u such that

$$u \in K, \quad -F(u) \in K^*, \quad -(u, F(u)) = 0. \tag{1}$$

This problem can easily be reduced to the fixed point problem for the operator

$$Ax = P_K x - F(P_K x). \tag{2}$$

More precisely, the following lemma holds [2].

Lemma 2. *Let F be an operator from K into X and $A = P_K - FP_K$. Then the complementarity problem with F is solvable if and only if the operator A has a fixed point in X. Furthermore, if x_* is a fixed point of A, then $u_* = P_K x_*$ is a solution to the complementarity problem under consideration.*

Note that, if $u_* \in K$ is a solution of the complementarity problem with the operator F, then $x_* = u_* - F(u_*)$ is a fixed point of the corresponding operator A.

2. Let f be a completely continuous operator from K into X. Consider the family of vector fields in X

$$\Phi(\lambda)x = x - \lambda(P_K x - f(P_K x)) \qquad (0 \le \lambda \le 1,\ x \in X). \tag{3}$$

This is a linear deformation connecting the vector field $\Phi = I - A$ whose zeros define solutions to the complementarity problem under consideration and the trivial field $\Phi_0 = I$. In what follows we are interested in zeros of the vector fields $\Phi(\lambda)$ $(0 \le \lambda \le 1)$ in the interior and on the boundary of special domains.

Consider the family of sets

$$\Omega_{r,\rho} = \big\{x : \|x\| \le \rho,\ \|P_K x\| \le r\big\} \qquad (0 < r < \rho < \infty). \tag{4}$$

Figure 1

It is evident that $\Omega_{r,\rho}$ is a bounded domain in X, and 0 is its interior point. The boundary $\partial\Omega_{r,\rho}$ of this domain is

$$\partial\Omega_{r,\rho} = \big\{x : \|x\| < \rho,\ \|P_K x\| = r\big\} \cup \big\{x : \|x\| = \rho,\ \|P_K x\| \le r\big\}. \tag{5}$$

Let

$$\partial\Omega^\circ_{r,\rho} = \big\{x : \|x\| < \rho,\ \|P_K x\| = r\big\}. \tag{6}$$

We need a special simple *a priori* estimate for values of the vector fields $\Phi(\lambda)$ $(0 \le \lambda \le 1)$ which shows that the part $\partial\Omega^\circ_{r,\rho}$ of the boundary $\partial\Omega_{r,\rho}$ is fundamental in our constructions. Let

$$\mu(r) = \sup_{\|u\| \le r,\, u \in K} \|u - f(u)\|.$$

The following evident lemma is a modification of the corresponding statement from [5].

Lemma 3. *Let $\rho > \mu(r)$. Then $\|\Phi(\lambda)x\| \geq \rho - \mu(r) \quad (x \in \partial\Omega_{r,\rho} \setminus \partial\Omega^{\circ}_{r,\rho})$. In particular, the zeros of the fields $\Phi(\lambda) \quad (0 \leq \lambda \leq 1)$ which are situated on the boundary $\partial\Omega_{r,\rho}$ lie on $\partial\Omega^{\circ}_{r,\rho}$.*

Proof. It is evident that

$$\|\Phi(\lambda)x\| \geq \|x\| - \|P_K x - f(P_K x)\| \geq \rho - \mu(r) > 0 \qquad (x \in \partial\Omega_{r,\rho} \setminus \partial\Omega^{\circ}_{r,\rho})$$

and the statement is proved ∎

We point out that the statement of this lemma means: *if the inequality $\rho > \mu(r)$ holds, then the zero of $\Phi(\lambda) \quad (0 \leq \lambda \leq 1)$ situated on the boundary $\partial\Omega_{r,\rho}$ really lie on its part $\partial\Omega^{\circ}_{r,\rho}$.* Figure 2 illustrates the situation of the zero $x_* \in \partial\Omega^{\circ}_{r,\rho}$.

Figure 2

Let us consider now the family of complementarity problems

$$u \in K, \quad (1-\lambda)u + \lambda f(u) \in K^*, \quad \big(u, (1-\lambda)u + \lambda f(u)\big) = 0 \quad (0 \leq \lambda \leq 1) \tag{7}$$

which corresponds to the family of the operators $(1-\lambda)I + \lambda f \quad (0 \leq \lambda \leq 1)$. Lemma 2 can be reformulated in the following form:

Lemma 4. *Let f be an operator from K into X and $A(\lambda) = \lambda(P_K - fP_K) \quad (0 \leq \lambda \leq 1)$. Then the complementarity problem with $(1-\lambda)I + \lambda f$ is solvable if and only if the operator $A(\lambda)$ has a fixed point in X. Furthermore, if x_* is a fixed point of $A(\lambda)$, then $u_* = P_K x_*$ is a solution to the complementarity problem with the operator $(1-\lambda)I + \lambda f$.*

Let u_* be a solution to the complementarity problem (7) for $\lambda_* \in [0,1]$ and x_* a fixed point of the operator $A(\lambda_*)$, i.e.

$$x_* = \lambda_*(P_K x_* - f(P_K x_*))$$

or

$$x_* - P_K x_* = -((1-\lambda_*)x_* + \lambda_* f(P_K x_*)).$$

Then

$$f(P_K x_*) = -\tfrac{1-\lambda_*}{\lambda_*}\, P_K x_* - \tfrac{1}{\lambda_*}\,(x_* - P_K x_*).$$

This equation can be written in the form

$$f(u_*) = -\mu u_* + v_* \tag{8}$$

with

$$u_* = P_K x_* \in K, \quad v_* = -\tfrac{1}{\lambda_*}(x_* - P_K x_*) \in K^*, \quad \mu = \tfrac{1-\lambda_*}{\lambda_*}.$$

This means that u_* is an exceptional element in the sense of [5]. (In [5] the corresponding definition is given for the finite-dimensional case; in the general case an element $u_* \in K$ is called *exceptional* if (8) holds with $v_* \in K^*$ satisfying the equation $(u, v_*) = 0 \ \ (u \in \Pi(u_*))$ where $\Pi(u_*) = \big\{u \in K : \ell(u) = 0 \ (\ell \in K^*,\ \ell(u_*) = 0)\big\}$.)

As we remarked above, in what follows we do not use the notion of exceptional elements and families but prefer to formulate our basic results in terms of complementarity problems. Passing to formulations with families of exceptional elements is trivial.

3. Now we return to the family of vector fields (3). Recall that a vector field Φ in a Hilbert space X is called of *class* S_+ if each sequence (x_n) from X, which weakly converges to x_* and satisfies the condition

$$\limsup_{n\to\infty} (\Phi x_n, x_n - x_*) \le 0,$$

converges to x_* in norm. A vector field Φ in a Hilbert space X is called *quasi-monotone* if each sequence (x_n) from X, which weakly converges to x_*, satisfies the condition

$$\liminf_{n\to\infty} (\Phi x_n, x_n - x_*) \ge 0.$$

Each vector field of class S_+ is quasi-monotone; the converse is not true.

The mappings of class S_+ and quasi-monotone mappings were introduced and studied in detail by F. Browder, H. Brézis, I. V. Skrypnik and others; here we follow I. V. Skrypnik (see [10]; see also [7]).

Lemma 5. *Let f be a completely continuous operator from K into X. Then the vector field $\Phi(\lambda)$ $(0 \le \lambda < 1)$ is of class S_+; the vector field $\Phi(1)$ is only quasi-monotone.*

Proof. Let $0 \le \lambda < 1$, (x_n) weakly convergent to x_*, and

$$\limsup_{n\to\infty} \big(\Phi(\lambda)x_n, x_n - x_*\big) \le 0.$$

Then (see Lemma 1)

$$\begin{aligned}
&(1-\lambda)\big(x_n - x_*, x_n - x_*\big)\\
&\quad \le \big(x_n - x_* - P_K x_n + P_K x_*, x_n - x_*\big)\\
&\quad = \big(\Phi(\lambda)x_n, x_n - x_*\big) - \big(x_* - \lambda P_K x_*, x_n - x_*\big) - \lambda\big(f(P_K x_n), x_n - x_*\big).
\end{aligned}$$

Without loss of generality, we can assume that the first summand in the right-hand side of this chain, as $n \to \infty$, has a non-positive limit by the properties of the sequence

(x_n). The second summand tends to 0 as $n \to \infty$ by the weak convergence of (x_n) to 0. The third summand also tends to 0 as $n \to \infty$ since f is a compact operator and, therefore, the sequence $(f(P_K x_n))$ is compact. Thus,

$$\limsup_{n\to\infty}(1-\lambda)\big(x_n - x_*, x_n - x_*\big) \le 0$$

and, since $0 \le \lambda < 1$, the sequence (x_n) tends to x_* in norm.

Now let $\lambda = 1$ and (x_n) weakly converge to x_*. Then (see Lemma 1 again)

$$\begin{aligned}\big(\Phi(1)x_n, x_n - x_*\big) &= \big(x_n - x_* - P_K x_n + P_K x_*, x_n - x_*\big) \\ &\quad + \big(f(P_K x_n), x_n - x_*\big) + \big(x_* - P_K x_*, x_n - x_*\big) \\ &\ge \big(f(P_K x_n), x_n - x_*\big) + \big(x_* - P_K x_*, x_n - x_*\big).\end{aligned}$$

Both summands in the right-hand side of this chain tend to 0 as $n \to \infty$ by the properties of the sequence (x_n) and the operator f. Thus,

$$\liminf_{n\to\infty}\big(\Phi(1)x_n, x_n - x_*\big) \ge 0$$

as claimed ∎

In general the vector field $\Phi(1)$ is not of class S_+. In what follows we can use a special property of $\Phi(1)$, which can be called *zero-closedness* of $\Phi(1)$. More precisely, we say that Φ is *zero-closed* if the convergence in norm of (Φx_n) to 0 implies that there exists a point $x_* \in \overline{\mathrm{co}}\,\{x_n\}$ such that the equality $\Phi x_* = 0$ holds. Vector fields of class S_+ are zero-closed.

We call an operator $f : K \to X$ *regular*, if for each sequence (u_n), $u_n \in K$ $(n \ge 1)$, weakly convergent to u_* and such that the sequence $(f(u_n))$ converges to $v_* \in K^*$ in norm, the equation $f(u_*) = v_*$ holds.

The following statement gives only a sufficient condition for a vector field to be zero-closed.

Lemma 6. *Let f be a regular completely continuous operator from K into X. Then the vector field $\Phi(1)$ is zero-closed.*

Proof. Let the sequence $(\Phi(1)x_n)$ converge in norm to 0 as $n \to \infty$; without loss of generality one can assume that the sequence (x_n) weakly converges to an element x_* and the sequence $(f(P_K x_n))$ converges in norm to v_*. In this case the sequence $(x_n - P_K x_n)$ converges in norm to $-v_*$, since $x_n - P_K x_n = f(P_K x_n) + \Phi(1)x_n$ $(n \ge 1)$. By Lemma 1 we have

$$\big(x_n - P_K x_n - x + P_K x, x_n - x\big) \ge 0 \qquad (x \in X,\ n \ge 1).$$

Passing to the limit in these inequalities as $n \to \infty$ we get

$$\big(-v_* - x + P_K x, x_* - x\big) \ge 0 \qquad (x \in X).$$

Putting $x = x_* + th$ $(h \in X,\ 0 < t < \infty)$ and dividing by t we have

$$\big(- v_* - x_* - th + P_K(x_* + th), h\big) \le 0 \qquad (h \in X,\ 0 < t < \infty).$$

Passing to the limit as $t \to 0$ and using Lemma 1 we obtain

$$\big(- v_* - x_* + P_K x_*, h\big) \le 0 \qquad (h \in X$$

Since $h \in X$ is an arbitrary element in X this means that $v_* = -(x_* - P_K x_*) \in K^*$. Furthermore, the sequence (u_n), $u_n = P_K x_n$ $(n \ge 1)$ weakly converges to $u_* = P_K x_*$ since

$$P_K x_n = x_n - (x_n - P_K x_n) \;\longrightarrow\; x_* + v_* = x_* - x_* + P_K x_* = P_K x_*.$$

Thus, the sequence (u_n) weakly converges to u_* and the sequence $(f(u_n))$ converges in norm to $v_* \in K^*$. By the regularity of f we have $f(u_*) = v_*$. Therefore,

$$\Phi(1)x_* = x_* - P_K x_* + f(P_K x_*) = x_* - P_K x_* + v_* = 0$$

and the lemma is proved ∎

In particular, the vector field $\Phi(1)$ is zero-closed if the operator f is weakly-strongly continuous (i.e., maps weakly convergent sequences into strongly convergent ones).

4. All operators considered below are *demicontinuous*, i.e. map strongly (in norm) convergent sequences into weakly convergent sequences.

The Skrypnik theory [10] (see also [7]) states that for each field Φ of class S_+ (and even zero-closed and quasi-monotone field Φ) defined on a bounded domain Ω and being without zero on the boundary $\partial\Omega$ of the domain Ω there is defined an integer $\gamma(\Phi, \Omega)$ (*the degree of the field* Φ *on the boundary* $\partial\Omega$ *of the domain* Ω), and the function

$$(\Phi, \Omega) \to \gamma(\Phi, \Omega)$$

has the usual properties of Brouwer-Hopf and Leray-Schauder degree. More precisely, this function has the following properties:

I. $\gamma(I, \Omega) = 1$ if $0 \in \Omega$.

II. If $\Omega = \Omega_1 \cup \Omega_2$ and Φ has no zero on the set $\partial\Omega_1 \cup \partial\Omega_2 \cup (\Omega_1 \cap \Omega_2)$, then $\gamma(\Phi, \Omega) = \gamma(\Phi, \Omega_1) + \gamma(\Phi, \Omega_2)$.

III. If Φ_0 and Φ_1 are homotopic on Ω, then $\gamma(\Phi_0, \Omega) = \gamma(\Phi_1, \Omega)$ (vector fields Φ_0 and Φ_1 are *homotopic* on Ω if there exists a family $\Phi(\lambda, \cdot)$ $(0 \le \lambda \le 1)$ of class S_+ (or zero-closed and quasi-monotone), defined on $\overline{\Omega}$ and demicontinuous with respect to both variables such that $\Phi(0, \cdot) = \Phi_0$, $\Phi(1, \cdot) = \Phi_1$, $\Phi(\lambda, x) \ne 0$ $(0 \le \lambda \le 1,\ x \in \partial\Omega)$).

Furthermore, as in the usual degree theory, in the theory of Skrypnik degree the following analogue of the basic existence principle holds:

If Φ has no zero on the boundary $\partial\Omega$ of the domain Ω and the degree $\gamma(\Phi,\Omega)$ of this vector field on the boundary $\partial\Omega$ of Ω is non-zero, then there exists at least one zero x_* of Φ in Ω.

As was proved above, the vector fields $\Phi(\lambda)$ $(0 \le \lambda < 1)$ defined by (3) are of class S_+ and the field $\Phi(1)$ is quasi-monotone; moreover, the field $\Phi(1)$ is zero-closed if f is regular. This means that we can apply the Skrypnik theory for studying fixed points of these fields.

Consider the family of vector fields $\Phi(\lambda)$ $(0 \le \lambda < 1)$ on the domain $\Omega_{r,\rho}$; we will assume that ρ and r are fixed positive reals and $\rho > \mu(r)$. It is evident that the family under our assumptions is demicontinuous with respect to both variables and $\Phi(0) = \Phi_0$, $\Phi(1) = \Phi_1$.

We have two possibilities:

First, there exist $\lambda_* \in (0,1)$ and $x_* \in \partial\Omega_{r,\rho}$ (really $x_* \in \partial\Omega^\circ_{r,\rho}$) such that $\Phi(\lambda_*)x_* = 0$. In this case $u_* = P_K x_*$ is a solution of the complementarity problem with the operator $(1-\lambda_*)I + \lambda_* f$, and this solution is situated on the set $S_r \cap K$ with $S_r = \{u : \|u\| = r\}$.

Second, for all $\lambda \in (0,1)$ the inequalities

$$\Phi(\lambda)x \ne 0 \qquad (x \in \partial\Omega_{r,\rho})$$

hold. In this case all vector fields $\Phi(\lambda)$ $(0 \le \lambda < 1)$ are homotopic on $\Omega_{r,\rho}$ and, therefore, they have the same degree $\gamma(\Phi(\lambda),\Omega_{r,\rho})$ on the boundary $\partial\Omega_{r,\rho}$ of the domain $\Omega_{r,\rho}$. But $\gamma(\Phi(0),\Omega_{r,\rho}) = 1$ since $\Phi(0) = I$ and $0 \in \Omega_{r,\rho}$. Thus, in the second case we have

$$\gamma(\Phi(\lambda),\Omega_{r,\rho}) = 1 \qquad (0 \le \lambda < 1). \tag{9}$$

Moreover, if the vector field $\Phi(1)$ is zero-closed and has no zero on $\partial\Omega_{r,\rho}$, then we have

$$\gamma(\Phi(\lambda),\Omega_{r,\rho}) = 1 \qquad (0 \le \lambda \le 1). \tag{10}$$

If the vector field $\Phi(1)$ is zero-closed, then equation (10) implies the existence of a zero of $\Phi(1)$ in the domain $\Omega_{r,\rho}$ and, therefore, the solvability of the complementarity problem with the operator f in the set $B_r \cap K$ with $B_r = \{u : \|u\| \le r\}$. In the general case equation (9) implies that each vector field $\Phi(\lambda)$ $(0 \le \lambda < 1)$ has at least one zero $x_\lambda \in \Omega_{r,\rho}$:

$$x_\lambda = \lambda\big(P_K x_\lambda - f(P_K x_\lambda)\big);$$

this implies the solvability of all complementarity problems with operators $(1-\lambda)I + \lambda f$ $(0 \le \lambda < 1)$ in the set $B_r \cap K$. Since $B_r \cap K$ is bounded there exists a sequence (λ_n) convergent to 1 such that the sequence (x_n), $x_n = x_{\lambda_n}$, is weakly convergent to an element x_*. It is evident that

$$x_n = \lambda_n\big(P_K x_n - f(P_K x_n)\big)$$

or

$$\Phi(1)x_n = -(1-\lambda_n)\big(P_K x_n - f(P_K x_n)\big).$$

This equation implies that $\|\Phi(1)x_n\| \to 0$.

If the vector field $\Phi(1)$ is zero-closed we can pass to the limit and again get the solvability of the complementarity problem under consideration. In the other case we only have the weakly convergent sequence $x_n \to x_*$ for which the sequence $\Phi(1)x_n$ converges to 0 in norm and, moreover, the equalities

$$f(P_K x_n) = -(x_n - P_K x_n) - \tfrac{1-\lambda_n}{\lambda_n}\, x_n \qquad (n \geq 1)$$

hold which imply the inequalities

$$\big(f(P_K x_n), P_K x_n\big) = -\tfrac{1-\lambda_n}{\lambda_n}\,\big(P_K x_n, P_K x_n\big) \leq 0 \qquad (n \geq 1).$$

Since $\Phi(1)x = x - P_K x + f(P_K x)$ and f is compact, we can repeat all corresponding considerations from the proof of Lemma 6. As a result, we obtain that the sequence (u_n), $u_n = P_K x_n \in K$, converges weakly to $u_* = P_K x_*$ and the sequence $f(P_K x_n)$ converges in norm to $\phi_* = -(x_* - P_K x_*) \in K^*$.

In order to formulate clearer the situation described above we introduce a *special closure* of the operator f. Namely, for each $u \in K$ denote by $\mathcal{Q}(u)$ the set of sequences (u_n), $u_n \in K$, which weakly converge to u, for which the sequence $(f(u_n))$ converges in norm to an element from K^*, and which satisfy the inequalities $(f(u_n), u_n) \leq 0$. Let

$$\widetilde{f}(u) = \Big\{ \lim_{n\to\infty} f(u_n) :\ (u_n) \in \mathcal{Q}(u) \Big\}; \tag{11}$$

the multi-valued operator $\widetilde{f}$ will be called the *special closure* of the (single-valued) operator f. In terms of $\widetilde{f}$ the general situation considered above can be formulated in the following manner:

There exist an element $u_* \in K$ and a value $\phi_* \in \widetilde{f}(u_*) \cap K^*$ such that $(u_*, \phi_*) = 0$.

In other words, in the general case our arguments prove the solvability of the generalized (and multi-valued) complementarity problem

$$u \in K, \quad -\widetilde{f}(u) \cap K^* \neq \emptyset, \quad -(u, \phi) = 0 \;\; (\phi \in \widetilde{f}(u) \cap K^*). \tag{12}$$

We summarize all statements obtained as a result of these considerations in the form of the following two theorems.

Theorem 1. *Let f be a regular completely continuous operator from K into X, and $0 < r < \infty$. Then*

- *either for some $\lambda \in (0,1)$ the complementarity problem with the operator $(1-\lambda)I + \lambda f$ has a solution in the set $S_r \cap K$*
- *or the complementarity problem with the operator f has a solution in the set $B_r \cap K$.*

Theorem 2. *Let f be a completely continuous operator from K into X, and $0 < r < \infty$. Then*

- *either for some $\lambda \in (0,1)$ the complementarity problem with the operator $(1-\lambda)I + \lambda f$ has a solution in the set $S_r \cap K$*

• *or the generalized complementarity problem with the special closure $\widetilde{f}$ of the operator f has a solution in the set $B_r \cap K$.*

5. This article is devoted to the case when the operator f is completely continuous. But this assumption is not natural for the general theory of complementarity problems. Here we restrict ourselves only to a simple remark that both theorems hold if f is assumed to be demicontinuous and either of class S_+ or zero-closed and quasi-monotone; moreover, these assumptions seem to more natural. We omit formulations of the corresponding analogues of Theorems 1 and 2. We also point out that some of our constructions can be generalized to the case when X is a Banach space and f is an operator from X into X^*.

The complementarity problem considered in this article is close to some similar problems in which the disjointness property instead of orthogonality is considered; both classes of problems intersect in the case when $X = L_2(\Omega, \mathcal{A}, \mu)$.

Finally, here we do not study so-called implicit complementarity problems and some other kinds of them. The simplest results on the base of the Skrypnik degree can be formulated without any difficulties; we are going to consider this problem in a separate article.

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