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Existence and Asymptotic Behavior of Positive Solutions of a Non-Autonomous Food-Limited Model with Unbounded Delay

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Abstract. Consider the non-autonomous logistic model

$$\Delta x_n = p_n x_n \left(\frac{1 - x_{n-k_n}}{1 + \lambda x_{n-k_n}}\right)^r \qquad (n \geq 0)$$

where $\Delta x_n = x_{n+1} - x_n$, $\{p_n\}$ is a sequence of positive real numbers, $\{k_n\}$ is a sequence of non-negative integers such that $\{n - k_n\}$ is non-decreasing, $\lambda \in [0,1]$, and r is the ratio of two odd integers. We obtain new sufficient conditions for the attractivity of the equilibrium $x = 1$ of the model and conditions that guarantee the solution to be positive, which improve and generalize some recent results established by Phios and by Zhou and Zhang.

Keywords: *Global attractivity, difference equations, oscillation*

AMS subject classification: Primary 34K20, 39A10, 39A11, 34K45, secondary 92D25

1. Introduction

The asymptotic behavior of solutions of difference equations with unbounded delay was studied in [1 - 3]. In the present paper we consider the non-autonomous logistic model

$$\Delta x_n = p_n x_n \left(\frac{1 - x_{n-k_n}}{1 + \lambda x_{n-k_n}}\right)^r \qquad (n \geq 0) \tag{1}$$

where $\Delta x_n = x_{n+1} - x_n$, $\{p_n\}$ is a sequence of positive real numbers, $\{k_n\}$ is a sequence of non-negative integers such that $\{n - k_n\}$ is non-decreasing, $\lambda \in [0,1]$ and r is the ratio of two odd integers. Let

$$\gamma = -\min\{n - k_n : n \geq 0\} \geq 0$$
$$\sigma_0 = \max\{n : n - k_n < 0\} + 1$$
$$\sigma = \max\{n : n - k_n < \sigma_0\} + 1.$$

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By a solution of equation (1) we mean a sequence $\{x_n\}$ which is defined for $n \geq -\gamma$, satisfies (1) for $n \geq 0$ and which satisfies for given numbers a_i $(-\gamma \leq i \leq 0)$ the initial condition $x_i = a_i > 0$ $\quad (-\gamma \leq i \leq 0)$.

Equation (1) contains as special case the equation

$$\Delta x_n = p_n x_n (1 - x_{n-k_n}) \qquad (n \geq 0)$$

The global attractivity of the equilibrium $x = 1$ of this equation has been well studied in [2, 3]. In most results of these papers it is supposed that the solution $\{x_n\}$ satisfies $x_n > 0$, but we find that this does not always succeed. We give an example as follows: Let

$$\Delta x_n = 1.3 x_n (1 - x_{n-1}) \qquad (n \geq 0)$$

where $p_n \equiv 1.3$, $k_n = 1$, $\lambda = 0$ and the initial condition is $x_{-1} = 0.23$ and $x_0 = 0.2$. Then

$$x_1 = 0.4002, \quad x_2 = 0.8164, \quad x_3 = 1.453, \quad x_4 = 1.7998, \quad x_5 = 0.7399$$

are positive but $x_6 = -0.62938$ ... are negative.

Two problems appear naturally considering equation (1):

1. Under what conditions every solution (x_n) satisfies $x_n > 0$?

2. Under what conditions every positive solution converges to 1 ?

In Section 2 we answer the first problem and in Section 3 the second problem is settled. Our results improve the theorems in [2, 3].

By the way, equation (1) is the discrete type of the differential equation

$$N'(t) = r(t)N(t)\Big(\frac{1 - N(t-\tau)}{1 + \lambda N(t-\tau)}\Big)^r \qquad (t \geq 0)$$

which was called *generalized Food-Limited model*, posed in [4] and studied by many authors (see [2 - 4] and the references cited therein). If $r = 1$, the above equation becomes the well-known Food-Limited ecology mathematical model

$$N'(t) = r(t)N(t)\frac{1 - N(t-\tau)}{1 + \lambda N(t-\tau)} \qquad (t \geq 0)$$

which together with its discrete type

$$x_{n+1} = x_n \exp\Big(r_n \frac{1 - x_{n-k}}{1 + \lambda x_{n-k}}\Big) \qquad (n \geq 0)$$

was studied in [4, 5]. However, to the best of our knowledge, its discrete analogue

$$\Delta x_n = r_n x_n \frac{1 - x_{n-k}}{1 + \lambda x_{n-k}} \qquad (n \geq 0)$$

has not been studied. We call equation (1) the generalized *difference* Food-Limited model.

2. Positivity of solutions

Remember that in equation (1) $\lambda \in [0,1]$ and that $x_i \quad (-\gamma \le i \le 0)$ is the initial condition. In this section, we prove the following

Theorem 1. *Suppose there are numbers $\beta > 0$ and $\theta > 1$ such that:*

(i) $\sum_{j=n-k_n}^{n} p_j \le \alpha \quad (n \ge \sigma_0)$ *and* $p_n \le \beta \quad (0 \le n \le \sigma)$ *where α is a real root of the transcendental equation*

$$e^{\alpha}\Big(\frac{1+\lambda}{1+\lambda e^{-\alpha}}\Big)^{-\frac{1+\lambda}{\lambda}} = \frac{\alpha^{1/r}+1}{\alpha^{1/r}-\lambda} \qquad (\lambda \in (0,1]) \tag{2}$$

or for $\lambda = 0$ of the equation

$$e^{\alpha-1+e^{-\alpha}} = 1 + \frac{1}{\alpha^{1/r}}. \tag{3}$$

(ii) $\frac{\theta\alpha^{1/r}+1}{\theta\alpha^{1/r}-\lambda}(1+\beta)^{\sigma} < \frac{1+\alpha^{1/r}}{-\lambda+\alpha^{1/r}}$.

(iii) $0 < x_i < \frac{\theta\alpha^{1/r}+1}{\theta\alpha^{1/r}-\lambda} \quad (-\gamma \le i \le 0)$.

Then every solution (x_n) of equation (1) satisfies

$$0 < x_n < \frac{\alpha^{1/r}+1}{\alpha^{1/r}-\lambda}$$

for $n \ge 1$.

Proof. By (2) or (3) we see that $\alpha > 1$ and by (1) we get

$$x_{n+1} = x_n\Big[1 + p_n\Big(\frac{1-x_{n-k_n}}{1+\lambda x_{n-k_n}}\Big)^{r}\Big].$$

Since

$$0 < x_{-r}, x_{-r+1}, \ldots, x_0 < \frac{\theta\alpha^{1/r}+1}{\theta\alpha^{1/r}-\lambda} < \frac{\alpha^{1/r}+1}{\alpha^{1/r}-\lambda},$$

then by assumptions (ii) - (iii) we find

$$x_1 = x_0\Big(1 + p_0\Big(\frac{1-x_{-k_0}}{1+\lambda x_{-k_0}}\Big)^{r}\Big)$$

$$\begin{cases} > x_0\left(1 + p_0\left(\frac{1-\frac{\alpha^{1/r}+1}{\alpha^{1/r}-\lambda}}{1+\lambda\frac{\alpha^{1/r}+1}{\alpha^{1/r}-\lambda}}\right)^{r}\right) \ge x_0\left(1 + \alpha\left(\frac{1-\frac{\alpha^{1/r}+1}{\alpha^{1/r}-\lambda}}{1+\lambda\frac{\alpha^{1/r}+1}{\alpha^{1/r}-\lambda}}\right)^{r}\right) = 0 \\ \le \frac{\theta\alpha^{1/r}+1}{\theta\alpha^{1/r}-\lambda}(1+\beta) < \frac{\alpha^{1/r}+1}{\alpha^{1/r}-\lambda}. \end{cases}$$

Similarly we obtain

$$x_2 = x_1\Big(1 + p_1\Big(\frac{1-x_{1-k_1}}{1+\lambda x_{1-k_1}}\Big)^{r}\Big)$$

$$\begin{cases} > x_1\left(1 + p_1\left(\frac{1-\frac{\alpha^{1/r}+1}{\alpha^{1/r}-\lambda}}{1+\lambda\frac{\alpha^{1/r}+1}{\alpha^{1/r}-\lambda}}\right)^{r}\right) \ge x_1\left(1 + \alpha\left(\frac{1-\frac{\alpha^{1/r}+1}{\alpha^{1/r}-\lambda}}{1+\lambda\frac{\alpha^{1/r}+1}{\alpha^{1/r}-\lambda}}\right)^{r}\right) = 0 \\ \le x_1(1+p_0) \le \frac{\theta\alpha^{1/r}+1}{\theta\alpha^{\frac{1}{r}}-\lambda}(1+\beta)^2 < \frac{\alpha^{1/r}+1}{\alpha^{\frac{1}{r}}-\lambda} \end{cases}$$

and so on. Finally we get

$$x_\sigma = x_{\sigma-1}\Big(1+p_{\sigma-1}\Big(\frac{1-x_{\sigma-1-k_{\sigma-1}}}{1+\lambda x_{\sigma-1-k_{\sigma-1}}}\Big)^r\Big)$$

$$\begin{cases} x_{\sigma-1}\left(1+p_{\sigma-1}\left(\frac{1-\frac{\alpha^{1/r}+1}{\alpha^{1/r}-\lambda}}{1+\lambda\frac{\alpha^{1/r}+1}{\alpha^{1/r}-\lambda}}\right)^r\right) \geq x_{\sigma-1}\left(1+\alpha\left(\frac{1-\frac{\alpha^{1/r}+1}{\alpha^{1/r}-\lambda}}{1+\lambda\frac{\alpha^{1/r}+1}{\alpha^{1/r}-\lambda}}\right)^r\right) = 0 \\ \leq \frac{\theta\alpha^{1/r}+1}{\theta\alpha^{1/r}-\lambda}(1+\beta)^\sigma < \frac{\alpha^{1/r}+1}{\alpha^{1/r}-\lambda}. \end{cases}$$

Now it suffices to prove that if $n_0 \geq \sigma$ and

$$0 < x_n < \frac{\alpha^{1/r}+1}{\alpha^{1/r}-\lambda} \qquad (0 \leq n \leq n_0),$$

then

$$0 < x_{n_0+1} < \frac{\alpha^{1/r}+1}{\alpha^{1/r}-\lambda}. \tag{4}$$

In fact,

$$x_{n_0+1} > x_{n_0}\left(1+\alpha\left(\frac{1-\frac{\alpha^{1/r}+1}{\alpha^{1/r}-\lambda}}{1+\lambda\frac{\alpha^{1/r}+1}{\alpha^{1/r}-\lambda}}\right)^r\right) = 0,$$

which is the left inequality in (4). Next we prove the right inequality in (4). Assume that contrary $x_{n_0+1} \geq \frac{\alpha^{1/r}+1}{\alpha^{1/r}-\lambda}$, set $p(t) = p_n$ for $t \in [n, n+1)$ and $0 \leq n \leq n_0$ and

$$x(t) = \begin{cases} x_n & \text{if } t = n \\ x_n(\frac{x_{n+1}}{x_n})^{t-n} & \text{if } n \leq t < n+1. \end{cases}$$

The function x is positive and continuous on the interval $[0, n_0+1]$, $x(n) = x_n$ $(n \geq 0)$ and x is monotone on $[n, n+1)$. Let $[\cdot]$ denote the maximum integer function and let x' stand for the left derivative of x. Then

$$x'(t) = x(t)\ln\left\{1+p(t)\left(\frac{1-x([t-k_{[t]}])}{1+\lambda x([t-k_{[t]}])}\right)^r\right\} \tag{5}$$

for $0 \leq t \leq n_0+1$. Since $x([t-k_{[t]}]) > 0$ for these t, we get

$$x'(t) \leq x(t)\ln(1+p(t)) \leq p(t)x(t) \qquad \text{a.e. on } [0, n_0+1). \tag{6}$$

By $\Delta x_{n_0} = x_{n_0+1} - x_{n_0} > 0$ and (1) we have $x_{n_0-k_{n_0}} < 1$. Then there exists $\xi \in [n_0 - k_{n_0}, n_0+1)$ such that $x(\xi) = 1$ and $x(t) > 1$ for $t \in (\xi, n_0+1]$. When $0 \leq t \leq \xi$, integrating (6) from t to ξ we get

$$x(t) > \exp\left(-\int_t^\xi p(s)\,ds\right) \qquad (0 \leq t \leq \xi).$$

If $\xi \le t < n_0+1$ and $[t-k_{[t]}] \le \xi$, we obtain

$$x([t-k_{[t]}]) \ge \exp\left(-\int_{[t-k_{[t]}]}^{\xi} p(s)\,ds\right).$$

Substituting this into (5), it follows

$$x'(t) < x(t)p(t)\frac{1-\exp\big(-\int_{[t-k_{[t]}]}^{\xi} p(s)\,ds\big)}{1+\lambda\exp\big(-\int_{[t-k[t]]}^{\xi} p(s)\,ds\big)}. \tag{7}$$

If $[t-k_{[t]}] > \xi$, since $[t] \le n_0$, then $n_0 - k_{n_0} \ge [t]-k_{[t]} = [t-k_{[t]}] > \xi$. But this contridicts $\xi \in [n_0-k_{n_0}, n_0+1]$. Hence this case is impossible.

Integrating (7) from ξ to n_0+1, noting that $\lambda \in [0,1]$ and $n_0 \ge \sigma$ implies $n_0 - k_{n_0} \ge \sigma_0$, thus $[t-k_{[t]}] \ge 0$ for $t \in [\xi, n_0+1]$, we get

$$\begin{aligned}
&\ln x(n_0+1)\\
&< \int_{\xi}^{n_0+1} p(t)\frac{1-\exp\big(-\int_{[t-k_{[t]}]}^{\xi} p(s)\,ds\big)}{1+\lambda\exp\big(-\int_{[t-k_{[t]}]}^{\xi} p(s)\,ds\big)}\,dt\\
&\le \int_{\xi}^{n_0+1} p(t)\frac{1-e^{-\alpha}\exp\big(\int_{\xi}^{t} p(s)\,ds\big)}{1+\lambda e^{-\alpha}\exp\big(\int_{\xi}^{t} p(s)\,ds\big)}\,dt\\
&= \int_{\xi}^{n_0+1} p(t)\,dt - (1+\lambda)\int_{\xi}^{n_0+1} p(t)\frac{e^{-\alpha}\exp\big(\int_{\xi}^{t} p(s)\,ds\big)}{1+\lambda\exp\big(-\alpha+\int_{\xi}^{t} p(s)\,ds\big)}\,dt\\
&= \begin{cases} \int_{\xi}^{n_0+1} p(t)\,dt - \frac{1+\lambda}{\lambda}\ln\frac{1+\lambda\exp\left(-\alpha+\int_{\xi}^{n_0+1} p(s)\,ds\right)}{1+\lambda e^{-\alpha}} & \text{if } \lambda \in (0,1]\\ \int_{\xi}^{n_0+1} p(t)\,dt - e^{-\alpha}\big[\exp\big(\int_{\xi}^{n_0} p(s)\,ds\big)-1\big] & \text{if } \lambda = 0. \end{cases}
\end{aligned}$$

Since

$$\int_{\xi}^{n_0+1} p(t)\,dt \le \sum_{j=n_0-k_{n_0}}^{n_0} p_j \le \alpha,$$

the function

$$x - \frac{1+\lambda}{\lambda}\ln\frac{1+\lambda e^{-\alpha+x}}{1+\lambda e^{-\alpha}} \qquad \text{or} \qquad x - e^{-\alpha}(e^x-1)$$

is increasing in $[0,\alpha]$. Then

$$\begin{aligned}
x_{n_0+1} &= x(n_0+1)\\
&< \begin{cases} \exp\left(\alpha - \frac{1+\lambda}{\lambda}\ln\frac{1+\lambda}{1+\lambda e^{-\alpha}}\right) = \frac{\alpha^{1/r}+1}{\alpha^{1/r}-\lambda} & \text{if } \lambda \in (0,1]\\ \alpha - 1 + e^{-\alpha} = 1 + \frac{1}{\alpha^{1/r}} & \text{if } \lambda = 0 \end{cases}
\end{aligned}$$

which contradicts the assumption $x_{n_0+1} \ge \frac{\alpha^{1/r}+1}{\alpha^{1/r}-\lambda}$. This completes the proof ∎

Remark 1. Since $1 < \frac{\theta\alpha^{1/r}+1}{\theta\alpha^{1/r}-\lambda} < \frac{\alpha^{1/r}+1}{\alpha^{\frac{1}{r}}-\lambda}$, Theorem 1 gives sufficient conditions which guarantee $x_n > 0$ for each solution $\{x_n\}$ of equation (1).

3. Global attractivity

In this section we give a sufficient condition that guarantees every positive solution of equation (1) to converge to 1 as $n \to +\infty$. Remember that in (1) $\lambda \in [0, 1]$.

Theorem 2. *Suppose there is a constant $\delta > 0$ such that:*

(i) $\sum_{s=n-k_n}^{n} p_s \leq \delta(1+\lambda)$ *for sufficiently large n.*

(ii) $\sum_{n=1}^{+\infty} p_n = +\infty$.

(ii) $\delta(1+\lambda)\Big(\frac{e^{\delta^2(1+\lambda)^2/2}-1}{\lambda e^{\delta^2(1+\lambda)^2/2}+1}\Big)^r \leq 1$.

Then every positive solution of equation (1) tends to 1 as $n \to +\infty$.

Proof. Suppose that $\{x_n\}$ is a positive solution of equation (1). The following proof of $x_n \to 1$ as $n \to +\infty$ will be given in three steps.

Step 1: If $\{x_n\}$ is eventually greater than 1, we will prove that $x_n \to 1$ as $n \to +\infty$. Choose N_1 such that $x_{n-k_n} > 1$ for $n \geq N_1$. By (1) we see that $\Delta x_n \leq 0$ for $n \geq N_1$, hence $\lim_{n\to+\infty} x_n = \mu$ exists. We prove that $\mu = 1$. Assuming $\mu \neq 1$, we have $\mu > 1$. Then, for $n \geq N_1$,

$$\Delta x_n \leq p_n x_n \Big(\frac{1-\mu}{1+\lambda\mu}\Big)^r$$

and

$$\ln \frac{x_{n+1}}{x_n} \leq \ln \Big[1 + p_n \Big(\frac{1-\mu}{1+\lambda\mu}\Big)^r\Big] \leq p_n \Big(\frac{1-\mu}{1+\lambda\mu}\Big)^r.$$

Hence

$$\ln \frac{x_{n+1}}{x_{N_1}} \leq \Big(\frac{1-\mu}{1+\lambda\mu}\Big)^r \sum_{s=N_1}^{n} p_s.$$

Letting $n \to +\infty$ we get a contradiction.

Step 2: If $\{x_n\}$ is eventually less than 1, we will prove that $x_n \to 1$ as $n \to +\infty$. Choose N_2 such that $x_{n-k_n} < 1$ for $n \geq N_2$. Then $\Delta x_n \geq 0$, so $\lim_{n\to+\infty} x_n = \mu$ exists. We prove that $\mu = 1$. Assuming $\mu \neq 1$, we have $\mu < 1$. We choose $0 < \varepsilon < 1$ such that $\delta(1+\lambda)(\frac{1-\varepsilon}{1+\lambda\varepsilon})^r \leq 1$ and $x_{n-k_n} < \varepsilon$ for $n \geq N_2$. Again, we have

$$\frac{x_{n+1}}{x_n} \geq 1 + p_n \Big(\frac{1-\varepsilon}{1+\lambda\varepsilon}\Big)^r \qquad (n \geq N_2).$$

Since $\ln(1+x) \geq \frac{1}{2}x$ for $x \in [0, 1]$, then

$$\ln \frac{x_{n+1}}{x_n} \geq \frac{1}{2} p_n \Big(\frac{1-\varepsilon}{1+\lambda\varepsilon}\Big)^r \qquad (n \geq N_2).$$

It is now easy to derive a contradiction to our assumption $\mu \neq 1$ but we omit the details.

Step 3: If $\{x_n\}$ is oscillatory about 1, we will also prove that $x_n \to 1$ as $n \to +\infty$. By a method similar to that in Theorem 1, we can prove that $\{x_n\}$ is bounded. Let

$\ln x_n = y_n$ for $n \geq 0$. Then $\{y_n\}$ is oscillatory and bounded, and equation (1) becomes

$$\Delta y_n = \ln\Big(1 + p_n\Big(\frac{1 - e^{y_{n-k_n}}}{1 + \lambda e^{y_{n-k_n}}}\Big)^r\Big) \qquad (n \geq 0). \tag{8}$$

We will prove now that $\lim_{n\to+\infty} y_n = 0$. Let $u = \limsup_{n\to+\infty} y_n$ and $v = \liminf_{n\to+\infty} y_n$. Then $-\infty < v \leq 0 \leq u < +\infty$. For any $\varepsilon > 0$ there is N_3 such that

$$v_1 = v - \varepsilon < y_{n-k_n} < u + \varepsilon = u_1 \qquad (n \geq N_3).$$

Then we get

$$\Delta y_n \begin{cases} \leq \ln(1 + p_n(\frac{1-e^{v_1}}{1+\lambda e^{v_1}})^r) \leq \ln(1 + p_n(1 - e^{v_1})) \\ \geq \ln(1 + p_n(\frac{1-e^{u_1}}{1+\lambda e^{u_1}})^r) \end{cases} \qquad (n \geq N_3). \tag{9}$$

Choose two subsequence of $\{y_n\}$, denoted by $\{y_{n_i}\}$ and $\{y_{m_i}\}$ with $N_3 \leq n_i \uparrow, m_i \uparrow$ such that $0 < y_{n_i} \uparrow u$ and $0 > y_{m_i} \uparrow v$. By (8) one gets $y_{n_i-1-k_{n_i-1}} \leq 0$ and then there is n_i^* with $n_i - 1 - k_{n_i-1} \leq n_i^* \leq n_i - 1$ such that $y_{n_i^*} \leq 0$ and $y_n > 0$ for $n_i^* + 1 \leq n \leq n_i$. Choose a number $\xi_i \in [0, 1)$ such that

$$y_{n_i^*} + \xi_i(y_{n_i^*+1} - y_{n_i^*}) = 0. \tag{10}$$

By the inequality

$$\Big(\prod_{i=1}^m a_i^{\alpha_i}\Big)^{1/\sum_{i=1}^m \alpha_i} \leq \frac{\sum_{i=1}^m \alpha_i a_i}{\sum_{i=1}^m \alpha_i}$$

we get

$$\begin{aligned} -y_{j-k_j} &= -y_{n_i^*} + \sum_{s=j-k_j}^{n_i^*-1} (y_{s+1} - y_s) \\ &= \xi_i(y_{n_i^*+1} - y_{n_i^*}) + \sum_{s=j-k_j}^{n_i^*-1} \ln\Big(1 + p_s\Big(\frac{1 - e^{y_{s-k_s}}}{1 + \lambda e^{y_{s-k_s}}}\Big)^r\Big) \\ &\leq \xi_i \ln\big(1 + p_{n_i^*}(1 - e^{v_1})\big) + \sum_{s=j-k_j}^{n_i^*-1} \ln\big(1 + p_s(1 - e^{v_1})\big) \\ &\leq (n_i^* - j + k_j + \xi_i) \ln\Big[1 + \frac{1 - e^{v_1}}{n_i^* - j + k_j + \xi_i}\Big(\xi_i p_{n_i^*} + \sum_{s=j-k_j}^{n_i^*-1} p_s\Big)\Big]. \end{aligned}$$

Then

$$e^{y_{j-k_j}} \geq \Big[1 + (1 - e^{v_1})\frac{1}{n_i^* - j + k_j + \xi_i}\Big(\xi_i p_{n_i^*} + \sum_{s=j-k_j}^{n_i^*-1} p_s\Big)\Big]^{-(n_i^*-j+k_j+\xi_i)}.$$

By $(1+\frac{x}{n})^{-n} \geq 1-x$ for $n>0$ and $x \geq 0$ we get

$$e^{y_{j-k_j}} \geq 1-(1-e^{v_1})\Big(\xi_i p_{n_i^*}+\sum_{s=j-k_j}^{n_i^*-1} p_s\Big). \tag{11}$$

Thus by (9) - (11) we get

$$\begin{aligned}
y_{n_i} &= y_{n_i^*+1}+\sum_{s=n_i^*+1}^{n_i-1}(y_{s+1}-y_s)\\
&=(1-\xi_i)(y_{n_i^*+1}-y_{n_i^*})+\sum_{n=n_i^*+1}^{n_i-1}\ln\Big(1+p_n\Big(\frac{1-e^{y_{n-k_n}}}{1+\lambda e^{y_{n-k_n}}}\Big)^r\Big)\\
&\leq (1-\xi_i)\ln\big(1+p_{n_i^*}(1-e^{y_{n_i^*-k_{n_i^*}}})\big)+\sum_{n=n_i^*+1}^{n_i-1}\ln\big(1+p_n(1-e^{y_{n-k_n}})\big)\\
&\leq (1-\xi_i)\ln\Big[1+p_{n_i^*}(1-e^{v_1})\Big(\xi_i p_{n_i^*}+\sum_{s=n_i^*-k_{n_i^*}}^{n_i^*-1}p_s\Big)\Big]\\
&\quad+\sum_{n=n_i^*+1}^{n_i-1}\ln\Big[1+p_n(1-e^{v_1})\Big(\xi_i p_{n_i^*}+\sum_{s=n-k_n}^{n_i^*-1}p_s\Big)\Big].
\end{aligned}$$

By assumption (i) we get

$$\begin{aligned}
y_{n_i} &\leq (1-\xi_i)\ln\Big[1+p_{n_i^*}(1-e^{v_1})\big(\delta(1+\lambda)-(1-\xi_i)p_{n_i^*}\big)\Big]\\
&\quad+\sum_{n=n_i^*+1}^{n_i-1}\ln\Big[1+p_n(1-e^{v_1})\Big(\delta(1+\lambda)-\sum_{s=n_i^*+1}^{n}p_s-(1-\xi_i)p_{n_i^*}\Big)\Big]\\
&\leq (n_i-n_i^*-\xi_i)\ln\Big\{1+\frac{1}{n_i-n_i^*-\xi_i}(1-e^{v_1})\\
&\quad\times\Big[(1-\xi_i)p_{n_i^*}\big(\delta(1+\lambda)-(1-\xi_i)p_{n_i^*}\big)\\
&\quad+\sum_{n=n_i^*+1}^{n_i-1}p_n\Big(\delta(1+\lambda)-\sum_{s=n_i^*+1}^{n}p_s-(1-\xi_i)p_{n_i^*}\Big)\Big]\Big\}.
\end{aligned}$$

Supposing $k_n \leq k$, since $n_i-n_i^*-\xi_i \leq k_{n_i-1}+1 \leq k+1$ it results in

$$\begin{aligned}
y_{n_i} &\leq (k+1)\ln\Big\{1+\frac{1}{k+1}(1-e^{v_1})\Big[(1-\xi_i)p_{n_i^*}\big(\delta(1+\lambda)-(1-\xi_i)p_{n_i^*}\big)\\
&\quad+\sum_{n=n_i^*+1}^{n_i-1}p_n\Big(\delta(1+\lambda)-\sum_{s=n_i^*+1}^{n}p_s-(1-\xi_i)p_{n_i^*}\Big)\Big]\Big\}.
\end{aligned}$$

Let

$$d_i = \sum_{n=n_i^*+1}^{n_i-1} p_n + (1-\xi_i)p_{n_i^*}.$$

Then by the inequality

$$\sum_{i=1}^{m} x_s^2 \geq \frac{1}{m}\Big(\sum_{s=1}^{m} x_s\Big)^2$$

we get

$$\begin{aligned} y_{n_i} &\leq (k+1)\ln\Bigg\{1+\frac{1}{k+1}\delta(1+\lambda)(1-e^{v_1})d_i - \frac{1}{k+1}(1-e^{v_1}) \\ &\quad \times \Bigg[(1-\xi_i)^2 p_{n_i^*}^2 + (1-\xi_i)p_{n_i^*}\sum_{n=n_i^*-1}^{n_i-1} p_n + \sum_{n=n_i^*+1}^{n_i-1} p_n \sum_{s=n_i^*+1}^{n_i-1} p_s\Bigg]\Bigg\} \\ &= (k+1)\ln\Bigg\{1+\frac{1}{k+1}\delta(1+\lambda)(1-e^{v_1})d_i - \frac{1}{2(k+1)}(1-e^{v_1})d_i^2 \\ &\quad - \frac{1}{2(k+1)}(1-e^{v_1})\Bigg[\sum_{n=n_i^*+1}^{n_i-1} p_n^2 + (1-\xi_i)^2 p_{n_i^*}^2\Bigg]\Bigg\} \\ &\leq (k+1)\ln\Bigg\{1+\frac{\delta(1+\lambda)}{k+1}(1-e^{v_1})d_i - \frac{1}{2(k+1)}(1-e^{v_1})d_i^2 \\ &\quad - \frac{1}{2(k+1)}(1-e^{v_1})\frac{1}{n_i-n_i^*}d_i^2\Bigg\} \\ &\leq (k+1)\ln\Bigg\{1+\frac{\delta(1+\lambda)}{k+1}(1-e^{v_1})d_i - \frac{k+2}{2(k+1)^2}(1-e^{v_1})d_i^2\Bigg\}. \end{aligned}$$

Since

$$\delta(1+\lambda)x - \frac{k+2}{2(k+1)}x^2 \Big\uparrow \qquad \text{when} \;\; x \leq \frac{k+1}{k+2}\delta(1+\lambda),$$

the maximum point of the function is $x = \frac{k+1}{k+2}\delta(1+\lambda)$. Then

$$y_{n_i} \leq (k+1)\ln\Big(1+\frac{\delta^2(1+\lambda)^2}{2(k+2)}(1-e^{v_1})\Big).$$

It is easy to see that the function $x\ln(1+\frac{\delta^2(1+\lambda^2}{2(x+1)})$ is increasing on $(0,+\infty)$, hence

$$(k+1)\ln\Big(1+\frac{\delta^2(1+\lambda)^2}{2(k+2)}\Big)\Big\uparrow\frac{\delta^2(1+\lambda)^2}{2} \qquad (k\to\infty).$$

Letting $i\to+\infty$ and $\varepsilon\to 0$ we get

$$u \leq (k+1)\ln\Big(1+\frac{\delta^2(1+\lambda)^2}{2(k+2)}(1-e^{v})\Big). \tag{12}$$

Now, let $y_{n_*} = \max\{0, y_n\}$. Again, since $\Delta y_{m_i-1} \le 0$, by (8) we have $y_{m_i-1-k_{m_i-1}} \ge 0$. Then

$$\begin{aligned} y_{m_i} &= y_{m_i-1-k_{m_i-1}} + \sum_{s=m_i-1-k_{m_i-1}}^{m_i-1} \ln\Big(1 + p_s\Big(\frac{1-e^{y_{s-k_s}}}{1+\lambda e^{y_{s-k_s}}}\Big)^r\Big) \\ &\ge \sum_{s=m_i-1-k_{m_i-1}}^{m_i-1} \ln\Big(1 + p_s\Big(\frac{1-e^{y_{(s-k_s)*}}}{1+\lambda e^{y_{(s-k_s)*}}}\Big)^r\Big) \\ &\ge \ln\Big(1 + \sum_{s=m_i-1-k_{m_i-1}}^{m_i-1} p_s\Big(\frac{1-e^{y_{(s-k_s)*}}}{1+\lambda e^{y_{(s-k_s)*}}}\Big)^r\Big) \\ &\ge \ln\Big(1 + \delta(1+\lambda)\Big(\frac{1-e^{u_1}}{1+\lambda e^{u_1}}\Big)^r\Big) \end{aligned}$$

and hence

$$e^{y_{m_i}} \ge 1 + \delta(1+\lambda)\Big(\frac{1-e^{u_1}}{1+\lambda e^{u_1}}\Big)^r.$$

Letting $i \to +\infty$ and $\varepsilon \to 0$, one gets

$$e^{v} \ge 1 + \delta(1+\lambda)\Big(\frac{1-e^{u}}{1+\lambda e^{u}}\Big)^r. \tag{13}$$

If $u \ne 0$, then $u > 0$. By (12) - (13) we get

$$u \le \ln\Big(1 + \frac{\delta^3(1+\lambda)^3}{2(k+2)}\Big(\frac{e^u-1}{1+\lambda e^u}\Big)^r\Big)^{k+1}.$$

From (12),

$$u < \ln\Big(1 + \frac{\delta^2(1+\lambda)^2}{2(k+2)}\Big)^{k+1} = u_0. \tag{14}$$

Let

$$f(u) = u - \ln\Big(1 + \frac{\delta^3(1+\lambda)^3}{2(k+2)}\Big(\frac{e^u-1}{1+\lambda e^u}\Big)^r\Big)^{k+1}.$$

Clearly, $f(0) = 0, f''(u) \le 0$, $f(u)$ has at most two zero points in $[0, +\infty)$ and

$$f(u_0) = \ln\Big(1 + \frac{\delta^2(1+\lambda)^2}{2(k+2)}\Big)^{k+1} - \ln\Big(1 + \frac{\delta^3(1+\lambda)^3}{2(k+2)}\Big(\frac{e^{u_0}-1}{1+\lambda e^{u_0}}\Big)^r\Big)^{k+1}.$$

By (14), $u_0 \uparrow \frac{\delta^2(1+\lambda)^2}{2}$, hence $e^{u_0} \le e^{\delta^2(1+\lambda)^2/2}$. Using assumption (iii) we get $f(u_0) \ge 0$. We see that $f(u) > 0$ for $u \in (0, u_0)$ which contradicts (13). Then $u = 0$ and $v = 0$, which implies $\lim_{n\to+\infty} y_n = 0$. This completes the proof ∎

Corollary 3. *Suppose that assumption* (ii) *of Theorem* 2 *holds and that*

$$\sum_{s=n-k_n}^{n} p_s \le \frac{1}{2}(1+\lambda)$$

for sufficiently large n. *Then every positive solution of equation* (1) *tends to* 1 *as* $n \to +\infty$.

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