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Estimates for Quasiconformal Mappings onto Canonical Domains (II)

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Abstract. In this paper we establish estimates for normal K-quasiconformal mappings $z = g(w)$ of any finitely-connected domain in the extended w-plane onto the interior or exterior of the unit circle or the extended z-plane with n (≥ 0) slits on the circles $|z| = R_j$ $(j = 1, \ldots, n)$. The bounds in the estimates for $R_j, |g(w)|$, etc. are explicitly given. They are sharp or asymptotically sharp and deduced mainly from estimates for the inverse mappings of g in our previous paper [10] based on Carleman's and Grötzsch's inequalities and partly improved here. A generalization of the Schwarz lemma and improvements of some classical inequalities for conformal mappings are shown.

Keywords: *K-quasiconformal mappings, Riemann moduli of a multiply-connected domain*

AMS subject classification: 30C62, 30C75, 30C80, 30C30, 30C35

1. Introduction and notations

This paper is a continuation of our previous paper [11] where estimates for K-quasiconformal mappings (see the definition in [4: p.16]) onto a circular ring $Q < |z| < 1$ with some circular slits are given. Here we shall establish estimates for normal K-quasiconformal mappings $z = g(w)$ of any finitely-connected domain in the extended w-plane onto the interior or exterior of the unit circle or the extended z-plane with some circular slits.

Throughout this paper, we use the following notations. Let $w = f(z)$ be a K-quasiconformal mapping of a domain A in the extended z-plane onto a domain B in the extended w-plane. Put

$$m(R, f) = \{ \min |w| : w \in E(R, f) \}$$
$$M(R, f) = \{ \max |w| : w \in E(R, f) \}$$

where $E(R, f)$ means the set of the w-plane corresponding to the circle $|z| = R$ by f, that may contain some slits or a circle as boundary components of A. Moreover, denote by $S(R, f)$ the inner area of the domain bounded by the above set $E(R, f)$.

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If A contains $z=0$ and $f(0)=0$, then put

$$m'(0,f)=\lim_{R\to 0}\frac{m(R,f)}{R^{\frac{1}{K}}},\quad m^*(0,f)=\lim_{R\to 0}\frac{m(R,f)}{R^K}$$
$$M'(0,f)=\lim_{R\to 0}\frac{M(R,f)}{R^{\frac{1}{K}}},\quad M^*(0,f)=\lim_{R\to 0}\frac{M(R,f)}{R^K}$$
$$S'(0,f)=\lim_{R\to 0}\frac{S(R,f)}{\pi R^{\frac{2}{K}}}.$$

If A contains $z=\infty$ and $f(\infty)=\infty$, then put

$$m'(\infty,f)=\lim_{R\to \infty}\frac{m(R,f)}{R^{\frac{1}{K}}},\quad m^*(\infty,f)=\lim_{R\to \infty}\frac{m(R,f)}{R^K}$$
$$M'(\infty,f)=\lim_{R\to \infty}\frac{M(R,f)}{R^{\frac{1}{K}}},\quad M^*(\infty,f)=\lim_{R\to \infty}\frac{M(R,f)}{R^K}$$
$$S'(\infty,f)=\lim_{R\to \infty}\frac{S(R,f)}{\pi R^{\frac{2}{K}}}.$$

Throughout this paper, we suppose that the introduced limits exist. Clearly, if $E(R,f)$ separates the points 0 and ∞, then

$$m'(0,f)^2\le S'(0,f)\le M'(0,f)^2 \tag{1.1}$$
$$m'(\infty,f)^2\le S'(\infty,f)\le M'(\infty,f)^2. \tag{1.2}$$

We consider now the three following classes of K-quasiconformal mappings onto canonical domains.

Let B_1 be any domain in the disk $|w|<1, 0\in B_1$, bounded by C_1 as the external boundary component with $\max\{|w| : w\in C_1\}=1$ and pn $(p\in\mathbb{N}, n\in\mathbb{N}_0=\mathbb{N}\cup\{0\})$ others $\sigma_1,\dots,\sigma_{pn}$. Suppose that B_1 is transformed into itself by the rotation

$$t=e^{i\frac{2\pi}{p}}w. \tag{1.3}$$

Denote by

S_1 $(\le\pi)$ – the inner area of the domain bounded by C_1
S $(\le S_1)$ – the inner area of the domain B_1
G_1 – the class of all K-quasiconformal mappings $z=g(w)$

each of which maps B_1 onto the disk $|z|<1$ that has been slit along pn circular arcs $L_1(g),\dots,L_{pn}(g)$ concentric with the unit circle such that $|z|=1$ and L_j correspond to C_1 and σ_j $(j=1,\dots,pn)$, respectively, $g(0)=0$, and satisfies the p-fold rotational symmetry

$$g(e^{i\frac{2\pi}{p}}w)=e^{i\frac{2\pi}{p}}g(w) \tag{1.4}$$

for all $w\in B_1$. Clearly, this condition is trivial for $p=1$.

Let B_2 be any domain in $|w|>1$ containing $w=\infty$ bounded by pn boundary components $\sigma_1,\dots,\sigma_{pn}$ and C_2 whose interior contains $|w|<1$ but cannot contain any disk $|w - w_0| < r$ with $r > 1$. Suppose that B_2 is transformed into itself by rotation (1.3). Denote by

S_2 – the external area of the compact set bounded by C_2
G_2 – the class of all K-quasiconformal mappings $z = g(w)$

each of which maps B_2 onto the domain $|z| > 1$ that has been slit along pn circular arcs $L_1(g), \dots, L_{pn}(g)$ concentric with the unit circle such that $|z| = 1$ and L_j correspond to C_2 and σ_j $(j = 1, \dots, pn)$, respectively, $g(\infty) = \infty$, and satisfies (1.4) for all $w \in B_2$.

Let B_3 be any domain in the extended w-plane containing $w = 0$ and $w = \infty$ bounded by pn boundary components $\sigma_1, \dots, \sigma_{pn}$ and transformed into itself by rotation (1.3). Denote by G_3 the class of all K-quasiconformal mappings $z = g(w)$ each of which maps B_3 onto the extended z-plane that has been slit along pn circular arcs $L_1(g), \dots, L_{pn}(g)$ concentric with the unit circle such that L_j corresponds to σ_j $(j = 1, \dots, pn), g(0) = 0, g(\infty) = \infty$, and satisfies (1.4) for all $w \in B_3$. Moreover, suppose $m^*(\infty, g) = 1$ for $g \in G_3$.

For each σ_j as boundary component of B_ν and for each $L_j(g)$ $(g \in G_\nu;\ j = 1, \dots, pn;\ \nu = 1, 2, 3)$ put

$c_j = \min_{w \in \sigma_j} |w|$
$d_j = \max_{w \in \sigma_j} |w|$
$R_j(g) = |z|$ $(z \in L_j(g))$
$R_0(g) = \max_{1 \le j \le pn} R_j(g)$
$s_0 = \min_{1 \le j \le pn} s_j$
$s = \sum_{j=1}^{pn} s_j$

where s_j means the external area of the compact set bounded by σ_j.

The principal aim of this paper is to estimate $|g(w)|$, the radii $R_j(g)$, etc. for $w \in B_\nu, g \in G_\nu$ $(j = 1, \dots, pn;\ \nu = 1, 2, 3)$. For $K = 1$ (conformal mappings) these radii are nothing but the *Riemann moduli* of the domains B_ν (see [5: p. 334]). The obtained estimates are sharp or asymptotically sharp. Their bounds are explicitly given as functions of $|w|, c_j, d_j, s_j$, etc. with the help of two auxiliary functions $R(p, t, s)$ and $T(p, r, s)$ introduced and studied in [11 : pp. 822 - 823] or, more precisely, [6: pp. 102 - 105] where $R(p, t, s) = r_p(t, s)$ and $T(p, r, s) = \tilde{\varrho}_p(r, s)$. They are deduced from the estimates for the classes F_ν of all mappings $f = g^{-1}, g \in G_\nu$ $(\nu = 1, 2, 3)$, that by (1.4) satisfy

$$e^{i\frac{2\pi}{p}} f(z) = f(e^{i\frac{2\pi}{p}} z) \tag{1.5}$$

for all $z \in A_\nu$ with $A_\nu = g(B_\nu)$ $(\nu = 1, 2, 3)$. Therefore the classes F_1 and F_2 introduced here are larger than F_1 and F_2 studied in [10], respectively, whose estimates will be partly improved. Our main tools are two inequalities due to Carleman [1: p. 212] and Grötzsch [2: p. 372] that were generalized and improved in [6 - 9] and especially in [10].

2. Estimates for the classes F_1 and G_1

To establish estimates for the class G_1 we need the following estimates for F_1.

Theorem 1. *Under the above hypotheses and notations we have for every* $f \in F_1, 0 < R < 1, (0 <)\ R_j\ (< 1), j = 1, \dots, pn$ *and* $(0 \neq)\ z \in A_1$

$$(\pi \geq)\ S_1(f) \geq \pi S'(0,f) + \textstyle\sum_{j=1}^{pn} R_j^{-\frac{2}{K}} s_j(f) \tag{2.1}$$

$$(\pi \geq)\ S(f) \geq \pi S'(0,f) + \textstyle\sum_{j=1}^{pn} (R_j^{-\frac{2}{K}} - 1) s_j(f) \tag{2.2}$$

$$(0 \leq)\ S'(0,f) \leq \tfrac{S(f)}{\pi}\ (\leq 1) \tag{2.3}$$

$$(0 \leq)\ p s_j(f) \leq \big[S_1(f) - \pi S'(0,f)\big] R_j^{\frac{2}{K}} \tag{2.4}$$

$$p s_j(f) \leq \big[S(f) - \pi S'(0,f)\big] (R_j^{-\frac{2}{K}} - 1)^{-1} \tag{2.5}$$

$$(0 \leq)\ s(f) \leq \big[S_1(f) - \pi S'(0,f)\big] R_0^{\frac{2}{K}} \tag{2.6}$$

$$s(f) \leq \big[S(f) - \pi S'(0,f)\big] (R_0^{-\frac{2}{K}} - 1)^{-1} \tag{2.7}$$

$$(0 \leq)\ s_0(f) \leq \big[S_1(f) - \pi S'(0,f)\big] \big(\textstyle\sum_{j=1}^{pn} R_j^{-\frac{2}{K}}\big)^{-1} \tag{2.8}$$

$$s_0(f) \leq \big[S(f) - \pi S'(0,f)\big] \big(\textstyle\sum_{j=1}^{pn} R_j^{-\frac{2}{K}} - pn\big)^{-1} \tag{2.9}$$

$$S'(0,f) \pi R^{\frac{2}{K}} \leq S(R,f) \leq S_1(f) R^{\frac{2}{K}} \tag{2.10}$$

$$m(R,f) \leq \sqrt{\tfrac{S_1(f)}{\pi}} R^{\frac{1}{K}} \tag{2.11}$$

$$M(R,f) \geq \sqrt{S'(0,f)} R^{\frac{1}{K}} \tag{2.12}$$

$$m(R,f) \geq 4^{-\frac{1}{p}} m'(0,f) R^{\frac{1}{K}} = m_0\ (\geq 0) \tag{2.13}$$

$$M(R,f) \leq T(p, R^{\frac{1}{K}}, m_0) \leq T(p, R^{\frac{1}{K}}, 0) < 4^{\frac{1}{p}} R^{\frac{1}{K}} \tag{2.14}$$

$$m = 4^{-\frac{1}{p}} m'(0,f) |z|^{\frac{1}{K}} \leq |f(z)| \leq T(p, |z|^{\frac{1}{K}}, m) < 4^{\frac{1}{p}} |z|^{\frac{1}{K}} \tag{2.15}$$

$$m_j = 4^{-\frac{1}{p}} m'(0,f) R_j^{\frac{1}{K}} \leq c_j(f) \leq d_j(f) \leq T(p, R_j^{\frac{1}{K}}, m_j) < 4^{\frac{1}{p}} R_j^{\frac{1}{K}} \tag{2.16}$$

$$(1 \leq)\ \tfrac{d_j(f)}{c_j(f)} < 2^{\frac{4}{p}} m'(0,f)^{-1} \quad \text{if } m'(0,f) > 0 \tag{2.17}$$

where equality in each of relations (2.1) – (2.12) *holds if and only if* $f(z) = az|z|^{\frac{1}{K}-1}$ *with* $|a| = 1$.

Proof. Applying [10: Lemma 2.1] to the mapping $f \in F_1$ of the domain A_1 onto B_1, we have (2.1) and thus (2.2) since $S = S_1 - s$. Because of the p-fold rotational symmetry (1.5) of $f \in F_1$ from (2.1) we obtain (2.4), (2.6) and (2.8). Similarly, from (2.2) we get (2.3), (2.5), (2.7) and (2.9). Applying again [10: Lemma 2.1] to the mapping $f \in F_1$ of the domain $A_1 \cap \{|z| < R\}$ we obtain the lower estimate for $S(R,f)$ in (2.10), while the upper estimate holds by applying [9: Lemma 2.1] to the mapping $f \in F_1$ of the domain $A_1 \cap \{|z| > R\}$. Combining this with the relation

$$\pi m(R,f)^2 \leq S(R,f) \leq \pi M(R,f)^2 \tag{2.18}$$

for $0 < R < 1$ and $f \in F_1$ yields (2.11) and (2.12). Equality in each of relations (2.1) - (2.12) holds if and only if $f(z) = az|z|^{\frac{1}{K}-1} + b$, with $b = 0$ since $f(0) = 0$ and with $|a| = 1$ since $M(1,f) = 1$. The proof of estimates (2.13) - (2.17) is as in [10: p. 372] ∎

Remark 1. Theorem 1 with $S \le S_1 \le \pi$ generalizes and improves [10: Theorem 2.1], where C_1 is the unit circle, i.e. $S = S_1 = \pi$. It generalizes also [7: Theorem 3], where $K = 1$.

Remark 2. The upper estimate for $|f(z)|$ in (2.15) presents a generalization of the Schwarz lemma to the case of quasiconformal mappings of finitely-connected domains. The sharpness of this estimate is open. In the particular case $n = 0$ and $p = 1$, where A_1 is the open unit disk, Hersch and Pfluger [3] showed the sharp upper estimate for $|f(z)|$ that under our notations has the form $|f(z)| \le T(1, r^{\frac{1}{K}}, 0)$ with $r = R(1, |z|, 0), f \in F_1, z \in A_1$. Note that this cannot remain true for $n \ge 1$ by a similar example as in [9: pp. 62 - 63].

Corollary 1. *For $K = 1$ by $S'(0,f) = |f'(0)|^2$ from (2.3) we obtain*

$$|f'(0)| \le \sqrt{\tfrac{S(f)}{\pi}} \qquad (f \in F_1) \tag{2.19}$$

with equality if and only if $f(z) = az$ with $|a| = 1$.

By $S \le \pi$ this improves the classical inequality $|f'(0)| \le 1$ for $f \in F_1$ with $K = 1$ (see [5: p. 352]).

Lemma 1. *Let $w = f(z)$ be a K-quasiconformal mapping of a domain containing $z = 0$ with $f(0) = 0$ and $m'(0,f) > 0$. Then for $g = f^{-1}$ we have*

$$m'(0,f) = M^*(0,g)^{-\frac{1}{K}} \tag{2.20}$$

$$M'(0,f) = m^*(0,g)^{-\frac{1}{K}}. \tag{2.21}$$

Proof. For small $R > 0$ put $C_R = \{z : |z| = R\}$ and $C'_R = f(C_R)$. Clearly, there exist a point $w_1 \in C'_R$ and a point $z_1 \in C_R$ such that

$$m(R,f) = |w_1| = |f(z_1)| = r.$$

Put $L_r = \{w : |w| = r\}$ and $L'_r = g(L_r)$. Noticing that L'_r is situated in $|z| \le R$, we get

$$M(r,g) = |g(w_1)| = |z_1| = R.$$

Thus, since $m'(0,f) > 0$ we conclude

$$m'(0,f) = \lim_{R \to 0} \frac{m(R,f)}{R^{\frac{1}{K}}} = \lim_{r \to 0} \frac{r}{M(r,g)^{\frac{1}{K}}} = \lim_{r \to 0} \left[\frac{M(r,g)}{r^K}\right]^{-\frac{1}{K}} = M^*(0,g)^{-\frac{1}{K}}.$$

Similarly we can prove (2.21) ∎

Remark 3. For $K = 1$, since $m'(0,f) = |f'(0)|$ and $M^*(0,g) = |g'(0)|$, equality (2.20) becomes the well-known relation $|f'(0)| = |g'(0)|^{-1}$.

Theorem 2. *Under the above hypotheses and notations we have for every* $g \in G_1, w \in B_1$ *and* $j = 1, \dots, pn$

$$M^*(0,g) \geq \left(\tfrac{\pi}{S}\right)^{\frac{K}{2}} \ (\geq 1) \tag{2.22}$$

$$M^*(0,g) > 2^{-\frac{4K}{p}} \left(\tfrac{d_j}{c_j}\right)^K \tag{2.23}$$

$$R_j(g) > \left[\frac{ps_j}{ps_j + S - \pi M^*(0,g)^{-\frac{2}{K}}}\right]^{\frac{K}{2}} \quad \textit{with } s_j > 0 \tag{2.24}$$

$$R_0(g) > \left[\frac{s}{S_1 - \pi M^*(0,g)^{-\frac{2}{K}}}\right]^{\frac{K}{2}} \quad \textit{with } s > 0 \tag{2.25}$$

$$(pn <) \ \textstyle\sum_{j=1}^{pn} R_j^{-\frac{2}{K}}(g) < \frac{S_1 - \pi M^*(0,g)^{-\frac{2}{K}}}{s_0} \quad \textit{with } s_0 > 0 \tag{2.26}$$

$$\textstyle\sum_{j=1}^{pn} R_j^{-\frac{2}{K}}(g) < pn + \frac{S - \pi M^*(0,g)^{-\frac{2}{K}}}{s_0} \quad \textit{with } s_0 > 0 \tag{2.27}$$

$$4^{-\frac{K}{p}} d_j^K < R(p, d_j, 0)^K \leq R_j(g) \leq 4^{\frac{K}{p}} M^*(0,g) c_j^K \tag{2.28}$$

$$R_j(g) > \left[\tfrac{ps_j}{Q_j(g)}\right]^{\frac{K}{2}} \tag{2.29}$$

with $s_j > 0$ *and* $Q_j(g) = S_1 - \left(\pi + 2^{-\frac{4}{p}} \sum_{R_\nu \neq R_j} \frac{s_\nu}{c_\nu^2}\right) M^*(0,g)^{-\frac{2}{K}} \ (> 0)$ *and*

$$4^{-\frac{K}{p}} |w|^K < R(p, |w|, 0)^K \leq |g(w)| \leq 4^{\frac{K}{p}} M^*(0,g) |w|^K. \tag{2.30}$$

Equality in relation (2.22) *holds if and only if* $B_1 = B_1^0$, *where* B_1^0 *means the open unit disk that has been slit along pn circular arcs concentric with the unit circle, and* $g(w) = aw|w|^{K-1}$ *with* $|a| = 1$.

Proof. Combining (1.1), (2.3) and (2.20) yields estimate (2.22) with equality if and only if

$$w = f(z) = g^{-1}(z) = bz|z|^{\frac{1}{K}-1} \qquad \text{with } |b| = 1.$$

This implies the above assertion in the case of equality in (2.22). Similarly, from (2.17) estimate (2.23) follows. With the help of (1.1) and (2.20), by (2.5), (2.6), (2.8) and (2.9) we obtain inequalities (2.24) - (2.27), respectively. From (2.16) we get

$$d_j \leq T(p, R_j^{\frac{1}{K}}, m_j) \leq T(p, R_j^{\frac{1}{K}}, 0) = t_j.$$

Thus, by the definitions of the auxyliary functions $T(p,r,s)$ and $R(p,t,s)$ and their monotonys (see [11: pp. 822 – 823]) we conclude

$$R_j^{\frac{1}{K}} = R(p, t_j, 0) \geq R(p, d_j, 0) > 4^{-\frac{1}{p}} d_j,$$

hence the lower estimate for R_j in (2.28) follows, while its upper estimate is deduced easily from (2.16) and (2.20). Writing (2.1) by (1.5) in the form

$$S_1 \geq \pi S'(0,f) + \frac{ps_j}{R_j^{\frac{2}{K}}} + \sum_{R_\nu \neq R_j} \frac{s_\nu}{R_\nu^{\frac{2}{K}}}$$

and using (1.1), (2.20) and the upper estimate for R_ν in (2.28) we obtain (2.29). Estimate (2.30) is deduced from (2.15) and (2.20) similarly as in the proof of (2.28) ∎

Corollary 2. *For $K = 1$ estimate* (2.22) *becomes*

$$|g'(0)| \geq \sqrt{\tfrac{\pi}{S}} \qquad (g \in G_1) \tag{2.31}$$

with equality if and only if $B_1 = B_1^0$ and $g(w) = aw$ with $|a| = 1$.

Estimate (2.31) with $S \leq \pi$ improves the classical inequality $|g'(0)| \geq 1$ for $g \in G_1$ with $K = 1$.

In order to establish an estimate that can sharpen (2.22) and therefore (2.31) we shall prove

Corollary 3. *Putting $C = 2^{-\frac{4}{p}} \sum_{j=1}^{pn} \frac{s_j}{c_j^2}$ (≥ 0), for every $g \in G_1$ we have*

$$M^*(0,g) \geq \left(\tfrac{\pi + C}{S_1}\right)^{\frac{K}{2}} \tag{2.32}$$

with equality if and only if $B_1 = B_1^0$ and $g(w) = aw|w|^{K-1}$ with $|a| = 1$.

Proof. Combining (1.1), (2.1), (2.20) and (2.28) yields

$$S_1 \geq \pi M^*(0,g)^{-\frac{2}{K}} + CM^*(0,g)^{-\frac{2}{K}},$$

hence (2.32) follows with the above assertion in the case of equality ∎

Corollary 4. *In the case $K = 1$, where $M^*(0,g) = |g'(0)|$, estimate* (2.32) *becomes*

$$|g'(0)| \geq \sqrt{\tfrac{\pi + C}{S_1}} \qquad (g \in G_1)$$

with equality if and only if $B_1 = B_1^0$ and $g(w) = aw$ with $|a| = 1$.

3. Estimates for the classes $\boldsymbol{F}_2$ and $\boldsymbol{G}_2$

To establish estimates for the class G_2 we need the following estimates for F_2.

Theorem 3. *Under the hypotheses and notations given in Section 1, for $f \in F_2, z \in A_2, 1 < R < \infty, (1 <)\ R_j\ (< \infty)\ \ (j = 1, \dots, pn)$ we have the estimates*

$$S'(\infty, f) \geq \tfrac{S_2(f)}{\pi} + \sum_{j=1}^{pn} \tfrac{s_j(f)}{\pi R_j^{\frac{2}{K}}} \ \left(\geq \tfrac{S_2}{\pi} \geq 1\right) \tag{3.1}$$

$$p s_j(f) \leq \left[\pi S'(\infty, f) - S_2(f)\right] R_j^{\frac{2}{K}} \tag{3.2}$$

$$s_0(f) \leq \left[\pi S'(\infty, f) - S_2(f)\right] \left(\sum_{j=1}^{pn} R_j^{-\frac{2}{K}}\right)^{-1} \tag{3.3}$$

$$s(f) \leq \left[\pi S'(\infty, f) - S_2(f)\right] R_0^{\frac{2}{K}} \tag{3.4}$$

$$(\pi R^{\frac{2}{K}} \leq)\ S_2(f) R^{\frac{2}{K}} \leq S(R, f) \leq S'(\infty, f) \pi R^{\frac{2}{K}} \tag{3.5}$$

$$M(R, f) \geq \sqrt{\tfrac{S_2(f)}{\pi}} R^{\frac{1}{K}} \tag{3.6}$$

$$m(R,f) \le \sqrt{S'(\infty,f)}\, R^{\frac{1}{K}} \tag{3.7}$$

$$M(R,f) < 4^{\frac{1}{p}} M'(\infty,f) R^{\frac{1}{K}} = M_0 \tag{3.8}$$

$$m(R,f) \ge T(p, R^{-\frac{1}{K}}, M_0^{-1})^{-1} \ge T(p, R^{-\frac{1}{K}}, 0)^{-1} > 4^{-\frac{1}{p}} R^{\frac{1}{K}} \tag{3.9}$$

$$4^{-\frac{1}{p}} |z|^{\frac{1}{K}} < T(p, |z|^{-\frac{1}{K}}, M^{-1})^{-1} \le |f(z)| < 4^{\frac{1}{p}} M'(\infty,f)|z|^{\frac{1}{K}} = M \tag{3.10}$$

$$4^{-\frac{1}{p}} R_j^{\frac{1}{K}} < T(p, R_j^{-\frac{1}{K}}, M_j^{-1})^{-1} \le c_j \le d_j < 4^{\frac{1}{p}} M'(\infty,f) R_j^{\frac{1}{K}} = M_j \tag{3.11}$$

$$(1 \le)\ \tfrac{d_j}{c_j} < 2^{\frac{4}{p}} M'(\infty,f) \tag{3.12}$$

where equality in each of relations (3.1) – (3.7) *holds if and only if* $f(z) = az|z|^{\frac{1}{K}-1}$ *with* $|a| = 1$.

Proof. Applying [10: Lemma 3.1] to the mapping $f \in F_2$ of the domain A_2 onto B_2, we have (3.1) and therefore (3.2) - (3.4). Applying again this lemma to the mapping $f \in F_2$ of the domain $A_2 \cap \{|z| > R\}$, we get the upper estimate for $S(R,f)$ in (3.5), while the lower estimate holds by applying [9: Lemma 2.1] to the mapping $f \in F_2$ of the domain $A_2 \cap \{|z| < R\}$. Thus, by (2.18) for $R > 1$ and $f \in F_2$, we obtain estimates (3.6) and (3.7). The equality in each of relations (3.1) - (3.7) holds if and only if $f(z) = az|z|^{\frac{1}{K}-1} + b$ with $b = 0$ and $|a| = 1$ because of the conditions of C_2. The proof of estimates (3.8) - (3.12) is as in [10: pp. 374 – 375] ∎

Remark 4. Theorem 3 with $S_2 \ge \pi$ generalizes and improves [10: Theorem 3.1], where C_2 is the circle $|w| = 1$, i.e. $S_2 = \pi$. It generalizes also [7: Theorem 5], where $K = 1$.

Lemma 2. *Let* $w = f(z)$ *be a* K*-quasiconformal mapping of a domain containing* $z = \infty$ *with* $f(\infty) = \infty$ *and* $M'(\infty,f) > 0$. *Then for* $g = f^{-1}$ *we have*

$$\begin{aligned} M'(\infty,f) &= m^*(\infty,g)^{-\frac{1}{K}} \\ m'(\infty,f) &= M^*(\infty,g)^{-\frac{1}{K}}. \end{aligned} \tag{3.13}$$

Proof. Similarly to the proof of Lemma 1, we can prove this lemma ∎

Theorem 4. *Under the hypotheses and notations given in Section* 1*, for* $g \in G_3, w \in B_3$ *and* $j = 1, \dots, pn$ *we have the estimates*

$$(0 \le)\ m^*(\infty,g) \le \left(\tfrac{\pi}{S_2}\right)^{\frac{K}{2}}\ (\le 1) \tag{3.14}$$

$$m^*(\infty,g) < 2^{\frac{4K}{p}} \left(\tfrac{c_j}{d_j}\right)^K \tag{3.15}$$

$$4^{-\frac{K}{p}} m^*(\infty,g) d_j^K \le R_j(g) \le R(p, c_j^{-1}, 0)^{-K} < 4^{\frac{K}{p}} c_j^K \tag{3.16}$$

$$R_j(g) > \left[\tfrac{ps_j}{V_j(g)}\right]^{\frac{K}{2}} \tag{3.17}$$

with $s_j > 0$ *and*

$$\begin{aligned} (0 <)\ V_j(g) &= \pi m^*(\infty,g)^{-\frac{2}{K}} - S_2 - \sum_{R_\nu \ne R_j} s_\nu R(p, c_\nu^{-1}, 0)^2 \\ &\le \pi m^*(\infty,g)^{-\frac{2}{K}} - S_2 - 2^{-\frac{4}{p}} \sum_{R_\nu \ne R_j} \frac{s_\nu}{c_\nu^2} \end{aligned}$$

and

$$R_0(g) > \left[\frac{s}{\pi m^*(\infty,g)^{-\frac{2}{K}} - S_2}\right]^{\frac{K}{2}} \quad \textit{with } s > 0 \tag{3.18}$$

$$\textstyle\sum_{j=1}^{pn} R_j^{-\frac{2}{K}}(g) < \frac{\pi m^*(\infty,g)^{-\frac{2}{K}} - S_2}{s_0} \quad \textit{with } s_0 > 0 \tag{3.19}$$

$$4^{-\frac{K}{p}} m^*(\infty,g)|w|^K < |g(w)| < R(p,|w|^{-1},0) < 4^{\frac{K}{p}}|w|^K \tag{3.20}$$

with equality in (3.14) *if and only if* $B_2 = B_2^0$, *where* B_2^0 *means the domain* $|w| > 1$ *that has been slit along* pn *circular arcs concentric with* $|w| = 1$, *and* $g(w) = aw|w|^{K-1}$ *with* $|a| = 1$.

Proof. Combining (1.1), (3.1) and (3.13) yields (3.14) with equality if and only if

$$w = f(z) = g^{-1}(z) = bz|z|^{\frac{1}{K}-1} \qquad \text{with } |b| = 1.$$

This implies the above assertion in the case of equality in (3.14). Estimate (3.15) follows from (3.12) and (3.13). By the definitions of the auxiliary functions $T(p,r,s)$ and $R(p,t,s)$ and their monotony (see [11: p. 822]) we get from (3.11) the equivalence

$$c_j^{-1} \le T(p, R_j^{-\frac{1}{K}}, 0) = t_j \quad \Longleftrightarrow \quad R_j^{-\frac{1}{K}} = R(p,t_j,0) \ge R(p,c_j^{-1},0),$$

hence the upper estimate for $R_j(g)$ in (3.16) follows, while the lower estimate is deduced easily from (3.11) and (3.13). By (1.1), (1.5) and (3.13) relation (3.1) can be represented in the form

$$m^*(\infty,g)^{-\frac{2}{K}} \ge \frac{S_2}{\pi} + \frac{ps_j}{\pi R_j^{\frac{2}{K}}} + \sum_{R_\nu \ne R_j} \frac{s_\nu}{\pi R_\nu^{\frac{2}{K}}}.$$

Thus, using the upper estimate in (3.16) for R_ν, we get (3.17). With the help of (3.13) relations (3.18) and (3.19) are deduced from (3.4) and (3.3), respectively. Similarly to the proof of (3.16), using (3.10) and (3.13) we can show (3.20) ∎

In order to improve estimate (3.14) we shall prove

Corollary 5. *Putting*

$$D = \sum_{j=1}^{pn} s_j R(p,c_j^{-1},0)^2 \ge 2^{-\frac{4}{p}} \sum_{j=1}^{pn} \frac{s_j}{c_j^2} \ (\ge 0)$$

we have for every $g \in G_2$

$$m^*(\infty,g) \le \left(\tfrac{\pi}{S_2+D}\right)^{\frac{K}{2}} \le \left(\tfrac{\pi}{S_2}\right)^{\frac{K}{2}} \ (\le 1), \tag{3.21}$$

with equality if and only if $B_2 = B_2^0$ *and* $g(w) = aw|w|^{K-1}$ *with* $|a| = 1$.

Proof. Combining (1.1), (3.1), (3.13) and (3.16) yields $m^*(\infty,g)^{-\frac{2}{K}} \ge \frac{S_2+D}{\pi}$, hence (3.21) with the above assertion in the case of equality ∎

Corollary 6. *In the case $K = 1$, where*

$$m^*(\infty, g) = \lim_{R\to\infty} \frac{m(R,g)}{R} = \lim_{z\to\infty} \frac{|g(z)|}{|z|} = |g'(\infty)|$$

inequality (3.21) *becomes*

$$|g'(\infty)| \le \sqrt{\tfrac{\pi}{S_2+D}} \qquad (g \in G_2)$$

with equality if and only if $B_2 = B_2^0$ and $g(w) = aw$ with $|a| = 1$.

This sharpens the classical inequality $|g'(\infty)| \le 1$ for $g \in G_2$ with $K = 1$.

4. Estimates for the class G_3

Since estimates for the class F_3 with $M'(\infty, f) = m^*(\infty, g)^{-\frac{1}{K}} = 1$ for $g^{-1} = f \in F_3$ by (3.13) were shown in [10] we can now establish them for the class G_3.

Theorem 5. *Under the hypotheses and notations given in Section 1 we have for every $g \in G_3, w \in B_3$ and $j = 1, \dots, pn$ the estimates*

$$M^*(0,g) \ge 1 \tag{4.1}$$

$$M^*(0,g) \ge 2^{-\frac{4K}{p}} \left(\tfrac{d_j}{c_j}\right)^K \tag{4.2}$$

$$4^{-\frac{K}{p}} d_j^K \le R_j(g) \le 4^{\frac{K}{p}} M^*(0,g) c_j^K \tag{4.3}$$

$$R_j(g) > \left[\tfrac{ps_j}{T_j(g)}\right]^{\frac{K}{2}} > \left(\tfrac{ps_j}{\pi}\right)^{\frac{K}{2}} \tag{4.4}$$

with $s_j > 0$ and $T_j(g) = \pi - \Big(\pi + 2^{-\frac{4}{p}} \sum_{R_\nu \ne R_j} \frac{s_\nu}{c_\nu^2}\Big) M^*(0,g)^{-\frac{2}{K}}$ (> 0) and

$$R_0(g) > \left\{\tfrac{s}{\pi[1 - M^*(0,g)^{-\frac{2}{K}}]}\right\}^{\frac{K}{2}} \quad \text{with } s > 0 \tag{4.5}$$

$$\textstyle\sum_{j=1}^{pn} R_j^{-\frac{2}{K}}(g) \le \dfrac{\pi}{s_0}\left[1 - M^*(0,g)^{-\frac{2}{K}}\right] \quad \text{with } s_0 > 0 \tag{4.6}$$

$$4^{-\frac{K}{p}} |w|^K \le |g(w)| \le 4^{\frac{K}{p}} M^*(0,g)|w|^K \tag{4.7}$$

with equality in (4.1) if and only if $B_3 = B_3^0$, where B_3^0 means the extended w-plane that has been slit along pn circular arcs concentric with $|w| = 1$, and $g(w) = aw|w|^{K-1}$ with $|a| = 1$.

Proof. Applying [10: Lemma 4.1] to the mapping $f \in F_3$ of A_3 onto B_3, we obtain by (1.1) and (2.20) for $g \in G_3$

$$\sum_{j=1}^{pn} \frac{s_j}{\pi R_j^{\frac{2}{K}}(g)} \le 1 - M^*(0,g)^{-\frac{2}{K}}, \tag{4.8}$$

hence (4.1) with the above assertion in the case of equality. Estimate (4.2) follows from [10: Formula 4.13] and (2.20), while (4.3) is deduced from [10: Formula 4.12] and (2.20). By the p-fold rotational symmetry of $g \in G_3$ inequality (4.8) can be written in the form

$$\frac{ps_j}{\pi R_j^{\frac{2}{K}}} + \sum_{R_\nu \neq R_j} \frac{s_\nu}{\pi R_\nu^{\frac{2}{K}}} \leq 1 - M^*(0,g)^{-\frac{2}{K}},$$

hence using upper estimate (4.3) for R_ν, we get (4.4). Estimates (4.5) and (4.6) follow from (4.8). Combining [10: Corollary 4.1] with (2.20) yields (4.7) ∎

In order to improve estimate (4.1) we shall prove

Corollary 7. *Putting*

$$E = (\pi 2^{\frac{4}{p}})^{-1} \sum_{j=1}^{pn} \frac{s_j}{c_j^2} \ (\geq 0)$$

we have for every $g \in G_3$

$$M^*(0,g) \geq (1+E)^{\frac{K}{2}} \tag{4.9}$$

with equality if and only if $B_3 = B_3^0$ and $g(w) = aw|w|^{K-1}$ with $|a| = 1$.

Proof. Combining (4.8) with (4.3) yields (4.9) with the above assertion in the case of equality ∎

Corollary 8. *In the case $K = 1$, where $M^*(0,g) = |g'(0)|$, estimate (4.9) becomes*

$$|g'(0)| \geq \sqrt{1+E} \qquad (g \in G_3)$$

with equality if and only if $B_3 = B_3^0$ and $g(w) = aw$ with $|a| = 1$.

This sharpens the classical inequality $|g'(0)| \geq 1$ for $g \in G_3$ with $K = 1$ (see [5: p. 350]).

Concluding Remark. All estimates obtained in this paper are sharp or asymptotically sharp. This follows from [10: p. 377].

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