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On the Asymptotics of Nonlinear Difference Equations

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Abstract. Solutions of nonlinear difference equations of second order are investigated with respect to their asymptotic behaviour. In particular, seven conjectures of Kulenović and Ladas concerning rational difference equations are verified.

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1. Introduction

The book by Kulenović and Ladas [4] contains a large number of open problems and conjectures concerning the dynamics of rational difference equations

$$x_{n+1} = \frac{\alpha + \beta x_n + \gamma x_{n-1}}{A + Bx_n + Cx_{n-1}} \qquad (n \in \mathbb{N}_0) \tag{1.1}$$

with non-negative parameters ($A + B + C > 0$) and of more general equations. Problems and conjectures concerning the asymptotic behaviour of the solutions x_n of equations (1.1) can be solved by constructing two suitable bounds y_n and z_n with

$$y_n \leq x_n \leq z_n \tag{1.2}$$

for great n. This construction can be realized in the following way (cf. [2: §11]): Choose an asymptotic scale $\varphi_k(n)$ ($k \in \mathbb{N}_0$), i.e. a sequence of positive functions with $\varphi_{k+1}(n) = o(\varphi_k(n))$ for $n \to \infty$, such that all shifts $\varphi_k(n \pm 1)$ and all products $\varphi_l \varphi_m$ possess asymptotic expansions with respect to this scale. In the case $\alpha \neq 0$ also the constant function 1 must possess such an expansion. Then make the ansatz

$$x_{nK} = \sum_{k=0}^{K} c_k \varphi_k(n) \tag{1.3}$$

with a fixed $K \geq 1$, determine the coefficients out of

$$x_{n+1}(A + Bx_n + Cx_{n-1}) - \alpha - \beta x_n - \gamma x_{n-1} = O(\varphi_L(n)) \tag{1.4}$$

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as $n \to \infty$, with $x_n = x_{nK}$ and L as great as possible, and put

$$\left.\begin{aligned} y_n &= x_{n,K-1} + a\varphi_K(n) \\ z_n &= x_{n,K-1} + b\varphi_K(n) \end{aligned}\right\} \tag{1.5}$$

with $a < c_K < b$. Simple examples for possible scales are $\varphi_k = \frac{1}{n^k}$ and $\varphi_k = t^{kn}$ with $0 < t < 1$. After having found the bounds y_n and z_n it remains to show the existence of a solution x_n of equation (1.1) with property (1.2) which shall be done in Section 2.

If we have no idea how to choose the scale φ_k, we can try the following possibility (cf. [2: §15]). Replace equation (1.1) by a differential equation which approximates (1.1) asymptotically as $n \to \infty$ and which can be solved explicitly. Then take its solution (or an asymptotic approximation of it) as $x_{n,K-1}$ in (1.5). In the simplest case the approximating differential equation can be obtained by substituting into (1.1) the first terms of the Taylor expansions for x_{n+1} and x_{n-1}. However, this requires that the derivatives with respect to n (considered as continuous variable) have a smaller order than the functions as it is the case by the functions $\frac{1}{n^k}$, but not by the functions t^{kn}. For more complicated possibilities cf. [2: §14].

If asymptotically two-periodic solutions are sought, then put $u_n = x_{2n-1}$ and $v_n = x_{2n}$, and replace equation (1.1) by the system

$$\left.\begin{aligned} u_{n+1} &= \frac{\alpha + \beta v_n + \gamma u_n}{A + Bv_n + Cu_n} \\ v_{n+1} &= \frac{\alpha + \beta u_{n+1} + \gamma v_n}{A + Bu_{n+1} + Cv_n} \end{aligned}\right\} \tag{1.6}$$

to which the foregoing procedures can be transferred.

In the following we deal on the one hand with the later generalization (2.1) of equation (1.1), and on the other hand with its special cases

$$x_{n+1} = \frac{x_{n-1}}{1 + x_n} \tag{1.7}$$

$$x_{n+1} = \beta + \frac{x_{n-1}}{x_n} \tag{1.8}$$

$$x_{n+1} = \frac{1 + x_{n-1}}{x_n} \tag{1.9}$$

$$x_{n+1} = \frac{\alpha + x_{n-1}}{1 + x_n} \quad (\alpha > 0). \tag{1.10}$$

In particular, we verify the following conjectures:

Conjecture [4: 4.8.2]. Show that equation (1.7) has a solution which converges to zero.

Conjecture [4: 4.8.3]. Show that equation (1.8) has a solution which remains above the equilibrium $\overline{x} = \beta + 1$ for all $n \geq -1$.

Conjecture [4: 5.4.6]. Show that equation (1.9) has a non-trivial positive solution which decreases monotonically to the equilibrium of the equation.

Conjecture [4: 6.10.3]. Show that equation (1.10) has a positive and monotonically decreasing solution.

We also deal with asymptotically periodic solutions of equation (1.7) and we give a partial answer to Open Problem [4: 4.8.4], which among other things demands to investigate the global character of the solution of equation (1.7) in dependence on its initial values x_{-1} and x_0.

Finally, we verify three conjectures of [4] (see Section 6) concerning bounded solutions of equation (1.1), and we refer to a further conjecture of [4] (see beginning of Section 7) concerning the rational difference equation

$$x_{n+1} = \frac{p + x_{n-2}}{x_n} \qquad (n \in \mathbb{N}_0) \tag{1.11}$$

which is not of type (1.1). We shall verify the conjecture for $p = 0$, whereas for $p > 0$ we shall supplement it by an open problem.

For some calculations we have used the DERIVE system.

2. The inclusion theorem

In order to verify inequalities (1.2) we consider the equation

$$x_{n-1} = f(x_n, x_{n+1}) \tag{2.1}$$

which can be either the inversion of equation (1.1) with respect to x_{n-1} (in the case $\gamma + C > 0$) or an equation with an arbitrary f (such that (2.1) is uniquely invertible with respect to x_{n+1}).

Theorem 1. *Let the function $f = f(y,z)$ be continuous and non-decreasing in both arguments, and let (y_n) and (z_n) be sequences with $y_n < z_n$ for $n \geq n_0$ as well as*

$$y_{n-1} \leq f(y_n, y_{n+1}), \quad f(z_n, z_{n+1}) \leq z_{n-1} \qquad (n > n_0). \tag{2.2}$$

Then there exists a solution of equation (2.1) *with property* (1.2) *for $n \geq n_0$.*

Proof. Choosing an arbitrary integer $N > n_0$, then the solution x_n of equation (2.1) with given initial values x_{N+1} and x_N satisfying inequalities (1.2) for $n = N+1$ and $n = N$ can be continued by means of (2.1) to arguments n with $n < N$. Inequalities (2.2) and the monotony of f yield the validity of inequalities (1.2) for all n with $n_0 \leq n \leq N+1$. Let A_N be the non-empty set of all pairs (x_{n_0}, x_{n_0+1}) such that the solutions x_n of equation (2.1) satisfy (1.2) for $n_0 \leq n \leq N+1$. The continuity of f implies that A_N is a closed set, and the monotony of f implies that $A_N \supset A_{N+1}$. Hence, there exists a non-empty set $A = \cap_{N=n_0+1}^{\infty} A_N$ of pairs (x_{n_0}, x_{n_0+1}) such that all attached solutions x_n of equation (2.1) satisfy (1.2) for all $n \geq n_0$ ∎

As the proof shows, the continuity and the monotony of f are only necessary for such arguments which satisfy (1.2) for $n > n_0$.

Theorem 1 can be modified in different ways (cf. [1 - 3, 5]), but we do not need here such modifications. Instead of that we come back to the special cases (1.7) - (1.10) of equation (1.1) and to the verification of the announced four conjectures.

Example 1. For example (1.7) of equation (1.1) inversion (2.1) yields the function $f(x_n, x_{n+1}) = (1 + x_n)x_{n+1}$, which satisfies the assumptions of Theorem 1 for positive arguments. Writing $x_n = x$ and using the approximations $x_{n\pm1} \approx x \pm x'$, we replace (1.7) by the differential equation $(2 + x)x' + x^2 = 0$ with the solution $x = \frac{2}{n+\ln x+C}$. In the case $x \to 0$ as $n \to \infty$ we find $x \sim \frac{2}{n}$ and therefore, choosing $C = -\ln 2$, we find iteratively the asymptotic approximations

$$x^{[0]} = \tfrac{2}{n}, \qquad x^{[1]} = \tfrac{2}{n-\ln n}, \qquad x^{[2]} = \tfrac{2}{n-\ln n+\frac{1}{n}\ln n}.$$

Taking into account that $x^{[2]} = \frac{2}{n} + \frac{2}{n^2}\ln n + \frac{2}{n^3}\ln^2 n + O(\frac{1}{n^3}\ln n)$ we make the ansatz

$$\begin{aligned} y_n &= \tfrac{2}{n} + \tfrac{2}{n^2}\ln n + \tfrac{a}{n^3}\ln^2 n \\ z_n &= \tfrac{2}{n} + \tfrac{2}{n^2}\ln n + \tfrac{b}{n^3}\ln^2 n \end{aligned}$$

with $a < 2 < b$ (cf. (1.5) with $K = 2$). Then we find the asymptotic relation

$$y_{n+1}(1 + y_n) - y_{n-1} \sim \tfrac{2}{n^4}(2 - a)\ln^2 n$$

and an analogous relation with z and b instead of y and a, respectively. These relations show that inequalities (2.2) are satisfied for sufficiently great n. Hence, Theorem 1 can be applied and it yields, in particular, the existence of a solution of equation (1.7) converging to zero, i.e. it verifies conjecture [4: 4.8.2].

The next three examples (1.8) - (1.10) are special cases of

$$x_{n+1} = \frac{\alpha + \beta x_n + x_{n-1}}{A + x_n}. \tag{2.3}$$

Inversion (2.1) yields the function

$$f(x_n, x_{n+1}) = (A + x_n)(x_{n+1} - \beta) + A\beta - \alpha$$

which is continuous and increasing for $x_n > 0$ and $x_{n+1} > \beta$. An equilibrium $\overline{x}$ of (2.3) is a solution of $\overline{x}^2 + (A - \beta - 1)\overline{x} = \alpha$. Here we need the non-negative equilibrium

$$\overline{x} = \tfrac{1}{2}\big(\beta + 1 - A + \sqrt{(\beta + 1 - A)^2 + 4\alpha}\big). \tag{2.4}$$

Making with an unknown $t \in (0, 1)$ the ansatz

$$x_n = \overline{x} + t^n + ct^{2n} + o(t^{2n}) \qquad (n \to \infty) \tag{2.5}$$

we find, according to (1.4),

$$\overline{x} = \frac{1+\beta t - At^2}{(1+t)t}, \quad c = \frac{(1+t)t^3}{(1-t)(1+t+t^2+(A+\beta)t^3)} \tag{2.6}$$

provided that the first equation has a solution $t \in (0,1)$. In this case the ansatz

$$y_n = \overline{x} + t^n + at^{2n}$$
$$z_n = \overline{x} + t^n + bt^{2n}$$

leads to the asymptotic representation

$$f(y_n, y_{n+1}) - y_{n-1} \sim \big(1 - \tfrac{a}{c}\big) t^{2n+1}$$

and an analogous one with z and b instead of y and a, respectively. These representations show that inequalities (2.2) are satisfied for sufficiently great n, since $c > 0$, and Theorem 1 yields the existence of a solution of equation (2.3) with asymptotic behaviour (2.5) which will verify the corresponding three conjectures from [4]. However, it remains to prove that $t \in (0,1)$.

Example 2. Choosing in (2.3) $\alpha = A = 0$ we get example (1.8) of equation (1.1). Equations (2.4) and (2.6) specialize to

$$\overline{x} = \beta + 1 = \frac{1+\beta t}{(1+t)t}, \quad c = \frac{t^3}{(1-t)(1+t-t^2)}$$

and one solution of the first equation is $t = \frac{1}{2(\beta+1)}\big(\sqrt{4\beta+5} - 1\big)$, which satisfies $t \in (0,1)$ even for $\beta > -1$. Hence, there exists a solution of equation (1.8) with property (2.5), i.e. in particular, a solution of (1.8) with $x_n > \overline{x} = \beta + 1$ when $\beta > -1$ and $n \geq n_0$. But there exists also such a solution when $n \geq -1$, namely x_{n+n_0+1}.

Example 3. Choosing in (2.3) $\alpha = 1$ and $\beta = A = 0$ we get example (1.9) of equation (1.1). Equations (2.4) and (2.6) specialize to

$$\overline{x} = \tfrac{1}{2}\big(1+\sqrt{5}\big) = \tfrac{1}{(1+t)t}, \quad c = \frac{(1+t)t^3}{1-t^3}$$

and $t = \frac{1}{2}\big(\sqrt{2\sqrt{5}-1} - 1\big) \approx 0.4317$ is the solution of the first equation contained in $(0,1)$. Hence, there exists a solution of equation (1.9) with property (2.5). This asymptotic relation shows that x_n is eventually monotonically decreasing to $\overline{x}$, and a suitable shift of x_n is decreasing for all $n \geq -1$.

Example 4. Choosing in (2.3) $\beta = 0$ and $A = 1$ we get example (1.10) of equation (1.1). Equations (2.4) and (2.6) specialize to

$$\overline{x} = \sqrt{\alpha} = \frac{1-t}{t}, \quad c = \frac{(1+t)t^3}{1-t^4}$$

and the first equation implies $t = \frac{1}{\sqrt{\alpha}+1} \in (0,1)$. Hence, there exists a solution of equation (1.10) with property (2.5). The validity of the corresponding conjecture [4: 6.10.3] follows analogously as in the foregoing examples.

3. Asymptotically two-periodic solutions

Equation (1.7) possesses the two-periodic solution $x_{2n-1} = 0, x_{2n} = p$ with an arbitrary constant p. Looking for an asymptotically two-periodic solution, we put $u_n = x_{2n-1}$ and $v_n = x_{2n}$ as before and make the ansatz

$$u_n = \sum_{\nu=1}^{\infty} a_\nu c^\nu t^{\nu n}, \qquad v_n = \sum_{\nu=0}^{\infty} b_\nu c^\nu t^{\nu n} \tag{3.1}$$

with $b_0 = p$ and arbitrary c, since (1.6) is an autonomous equation. We choose $c > 0$. In the case of equation (1.7) equations (1.6) specialize to

$$\left.\begin{aligned} (1+v_n)u_{n+1} &= u_n \\ (1+u_{n+1})v_{n+1} &= v_n \end{aligned}\right\}. \tag{3.2}$$

Substitution of (3.1) into these equations and comparing the coefficients yields $t = \frac{1}{p+1}$, $a_1 = b_1$ undetermined, and

$$\left.\begin{aligned} a_\nu &= \tfrac{1}{(p+1)^{\nu-1}-1} \sum_{\mu=1}^{\nu-1} b_\mu a_{\nu-\mu}(p+1)^{\mu-1} \\ b_\nu &= \frac{1}{(p+1)^\nu - 1} \sum_{\mu=0}^{\nu-1} b_\mu a_{\nu-\mu} \end{aligned}\right\} \quad (\nu \geq 2). \tag{3.3}$$

In view of the presence of the arbitrary constant c we can choose $a_1 = b_1 = 1$. The next coefficients read

$$\begin{aligned} a_2 &= \tfrac{1}{p} \\ b_2 &= \tfrac{2}{p(p+2)} \end{aligned} \qquad \text{and} \qquad \begin{aligned} a_3 &= \tfrac{3p+4}{p^2(p+2)^2} \\ b_3 &= \tfrac{p^2+9p+12}{p^2(p+2)^2(p^2+3p+3)}. \end{aligned}$$

For positive p and $0 < t < 1$, and the coefficients a_ν, b_ν are also positive. It can easily be proved by induction that the further coefficients allow the estimates $a_\nu \leq \frac{1}{p^{\nu-1}}$ and $b_\nu \leq \frac{1}{p^{\nu-1}}$ for all $\nu \geq 1$. This means that series (3.1) are not only asymptotic ones as $n \to \infty$, but that they even converge for $t^n < \frac{p}{c}$, i.e. for suitable great n.

Remark.

1. By positive initial values u_0 and v_0 it follows from system (3.2) that all its solutions are also positive and decreasing, hence converging to a non-negative limit. At least one limit equals zero (cf. [4: p. 60]). In (3.1) we have $u_n \to 0$ and $v_n \to p$ as $n \to \infty$.

2. By elimination it can be shown that both solutions of system (3.2) are also solutions of the rational difference equation $w_{n+1} = \frac{w_n + w_n^2}{w_{n-1} + w_n^2} w_n$ which is not of type (1.1).

4. Dependence on the initial values

Next, we want to study the solution of equation (1.7) in dependence on its initial values x_{-1} and x_0.

Proposition 1. *For $n \in \mathbb{N}_0$ and positive x_{-1} and x_0 the solution of system* (1.6) *satisfies the estimates*

$$\left.\begin{aligned} x_{2n} &\le x_0 t^n \\ x_{2n-1} &\ge p + (x_{-1} - p)t^n \end{aligned}\right\} \tag{4.1}$$

with $t = \frac{1}{\sqrt{x_{-1}+1}}$ *and* $p = \sqrt{x_{-1}+1} - 1$ *when*

$$x_0 \le \tfrac{1}{2}\big(\sqrt{x_{-1}+1} - 1\big) \tag{4.2}$$

and the estimates

$$\left.\begin{aligned} x_{2n+1} &\le x_1 t^n \\ x_{2n} &\ge p + (x_0 - p)t^n \end{aligned}\right\} \tag{4.3}$$

with $t = \frac{1}{\sqrt{x_0+1}}$ *and* $p = \sqrt{x_0+1} - 1$ *when*

$$x_1 \le \tfrac{1}{2}\big(\sqrt{x_0+1} - 1\big). \tag{4.4}$$

Proof. We use the foregoing notations $u_n = x_{2n-1}$ and $v_n = x_{2n}$ for which estimates (4.1) read

$$\left.\begin{aligned} v_n &\le v_0 t^n \\ u_n &\ge p + (u_0 - p)t^n \end{aligned}\right\}. \tag{4.5}$$

Since these estimates are valid for $n = 0$ we shall prove them by induction. Hence, according to (3.2), it suffices to show that

$$\frac{v_0 t^n}{1 + p + (u_0 - p)t^{n+1}} \le v_0 t^{n+1}, \qquad \frac{p + (u_0 - p)t^n}{1 + v_0 t^n} \ge p + (u_0 - p)t^{n+1}$$

for $n \in \mathbb{N}_0$, i.e. (for $t > 0, v_0 > 0$ and $0 < p < u_0$)

$$1 \le (1+p)t + (u_0 - p)t^{n+2}, \qquad (u_0 - p)(1 - t) \ge v_0\big(p + (u_0 - p)t^{n+1}\big).$$

The optimal solution of the first inequality for $n \in N_0$ is $t = \frac{1}{p+1}$, so that $0 < t < 1$. The second inequality is valid, if it is valid for $n = 0$, i.e. if

$$(u_0 - p)p \ge v_0(p^2 + u_0). \tag{4.6}$$

For $p = \sqrt{u_0 + 1} - 1$ this inequality turns over into

$$v_0 \le \tfrac{1}{2}\big(\sqrt{u_0+1} - 1\big). \tag{4.7}$$

Hence, (4.7) implies (4.5), i.e. in view of $u_0 = x_{-1}$ and $v_0 = x_0$, (4.2) implies (4.1).

Writing $\eta_n = x_{2n}$ and $\xi_n = x_{2n+1}$, system (1.6) is equivalent to

$$(1+\xi_n)\eta_{n+1} = \eta_n$$
$$(1+\eta_{n+1})\xi_{n+1} = \xi_n.$$

For $u_n = \eta_n$ and $v_n = \xi_n$ these equations coincide with (3.2) so that (4.5) turns over into

$$\left.\begin{aligned} \xi_n &\le \xi_0 t^n \\ \eta_n &\ge p + (\eta_0 - p)t^n \end{aligned}\right\} \tag{4.8}$$

with $t = \frac{1}{p+1}$ and $p = \sqrt{\eta_0 + 1} - 1$, and (4.8) is valid for $n \in N_0$ when $0 < \xi_0 \le \frac{1}{2}(\sqrt{\eta_0+1} - 1)$. According to $\eta_n = x_{2n}$ and $\xi_n = x_{2n+1}$ this means that (4.3) is valid when (4.4) is valid ∎

Remark.

1. In view of equation (1.7) condition (4.4) can be written as

$$x_{-1} \le \tfrac{1}{2}(x_0+1)\big(\sqrt{x_0+1} - 1\big) \tag{4.9}$$

and (4.2) by inversion as

$$4x_0(x_0+1) \le x_{-1}. \tag{4.10}$$

Hence, by positive initial values, Proposition 1 and Remark 1 of Section 3 imply $x_{2n} \to 0$, $\lim_{n\to\infty} x_{2n-1} \ge \sqrt{x_{-1}+1} - 1 > 0$ under (4.9), and $x_{2n-1} \to 0$ and $\lim_{n\to\infty} x_{2n} \ge \sqrt{x_0+1} - 1 > 0$ under (4.10).

2. The choice of p in the proof of Proposition 1 is optimal, since domain (4.6) in the first quadrant of the (u,v)-plane has the *envelope*

$$(v+1)p^2 - up + uv = 0$$
$$2(v+1)p - u = 0$$

so that $p = \frac{u}{2(v+1)}$ and $u = 4v(v+1)$, i.e. $p = 2v$ and $v = \frac{1}{2}\big(\sqrt{u+1} - 1\big)$.

5. Asymptotically three-periodic solutions

Looking for a three-periodic solution of equation (1.7) generated by $x_{-1} = p, x_0 = q, x_1 = r$, we have to solve the system of equations

$$\left.\begin{aligned} p &= (1+q)r \\ q &= (1+r)p \\ r &= (1+p)q \end{aligned}\right\}. \tag{5.1}$$

Not all solutions of this system can be positive, because every positive solution of equation (1.7) converges to a two-periodic solution (cf. [4: p. 60]). The non-trivial solutions of (5.1) are solutions of the polynomial equation $z^3+3z^2=3$, and if $p=z$ is one solution, then $q=\frac{3}{z^2-3}$ and $r=\frac{3(z+1)}{z^2-3}$. Hence, e.g.,

$$\begin{aligned} p &= 2\cos\tfrac{\pi}{9}-1 &&\approx 0.879385\\ q &= -2\sin\tfrac{\pi}{18}-1 &&\approx -1.347296\\ r &= -2\cos\tfrac{2\pi}{9}-1 &&\approx -2.532089. \end{aligned}$$

For the first terms of an asymptotically three-periodic solution we expect, as in Section 3, the structure

$$\left.\begin{aligned} x_{3n-1} &= p+at^n\\ x_{3n} &= q+bt^n\\ x_{3n+1} &= r+ct^n \end{aligned}\right\} \tag{5.2}$$

up to an $O(t^{2n})$-term where the coefficients must satisfy the equations

$$\begin{aligned} p+at^n &= (1+q+bt^n)(r+ct^n)\\ q+bt^n &= (1+r+ct^n)(p+at^{n+1})\\ r+ct^n &= (1+p+at^{n+1})(q+bt^{n+1}) \end{aligned}$$

again up to an $O(t^{2n})$-term, i.e. besides of (5.1),

$$\left.\begin{aligned} (1+q)c+rb &= a\\ (1+r)ta+pc &= b\\ (1+p)tb+qta &= c \end{aligned}\right\}. \tag{5.3}$$

This homogeneous system has a non-trivial solution, if its determinant

$$\begin{vmatrix} -1 & r & 1+q\\ (1+r)t & -1 & p\\ qt & (1+p)t & -1 \end{vmatrix} = t^2+9t-1 \tag{5.4}$$

vanishes. Since it must be $|t|<1$, we expect the existence of an asymptotically three-periodic solution of equation (1.7) with asymptotic approximations (5.2) and $t=\frac{1}{2}\big(-9+\sqrt{85}\big)\approx 0.109772$. The corresponding solution of system (5.3) reads, up to a constant factor,

$$\begin{aligned} a &= 11z^2+5z-14\\ b &= -z^2(t+5)-2z+t+6\\ c &= z^2(2-t)+z(1-t)-2. \end{aligned}$$

Now, we could proceed as in Section 3, but we resign from doing this. Note that the existence of a second zero of (5.4) with $t<-1$ indicates that the three-periodic solution p,q,r is unstable.

6. Bounded solutions

Next, we verify a generalization of three conjectures concerning bounded solutions.

Conjecture [4: 11.4.1]. Assuming that all coefficients in equation (1.1) are positive, show that every of its positive solution is bounded.

Even in the case that all coefficients of equation (1.1) are non-negative an analogous conjecture comes true if there exists a constant M satisfying $\alpha \le MA, \beta \le MB, \gamma \le MC$ because then every non-negative solution of (1.1) satisfies $x_n \le M$ for $n \in \mathbb{N}$. If all coefficients in the denominator of equation (1.1) are positive whereas the coefficients in its numerator can remain non-negative, then we can choose $M = \max\left(\frac{\alpha}{A}, \frac{\beta}{B}, \frac{\gamma}{C}\right)$. This means, in particular, that the preceding conjecture comes true.

The case $\gamma = 0$ was already treated in [4: Theorem 9.2.2]. The case $\beta = 0$ verifies Conjecture [4: 9.5.2], and the case $\alpha = 0$ verifies Conjecture [4: 9.5.3].

7. Global behaviour

Finally, we refer to

Conjecture [4: 11.4.11]. Show that difference equation (1.11) has the following trichotomy character:

(i) When $p > 1$, every positive solution converges to the positive equilibrium.

(ii) When $p = 1$, every positive solution converges to a period-five solution.

(iii) When $p < 1$, there exist positive unbounded solutions.

In the elementary case $p = 0$ the conjecture turns out to be true. Otherwise, for $p > 0$ we supplement it by an open problem.

Preliminarily, we make the ansatz

$$x_n = \sum_{j=0}^{\infty} c_j a^j z^{nj} \tag{7.1}$$

with an arbitrary a and put it into equation (1.11) in the form

$$x_n = x_{n+3} x_{n+2} - p. \tag{7.2}$$

Comparing coefficients we obtain

$$\left.\begin{aligned} c_0 &= c_0^2 - p \\ a c_1 \big(1 - c_0(z^3 + z^2)\big) &= 0 \end{aligned}\right\} \tag{7.3}$$

and for $k \ge 2$ the recursions

$$c_k = \frac{z^{2k}}{1 - c_0 z^{2k}(z^k + 1)} \sum_{j=1}^{k-1} c_j c_{k-j} z^j \tag{7.4}$$

provided that the denominator is different from zero. The first equation of (7.3) means that c_0 is an equilibrium of (7.2), we choose the solution

$$c_0 = \tfrac{1}{2}\big(1 + \sqrt{1+4p}\big). \tag{7.5}$$

As a function of p it is strictly increasing with $c_0 \geq \frac{1}{2}$ for $p \geq -\frac{1}{4}$. The second equation yields either $ac_1 = 0$ which leads to the stationary solution $y_n = c_0$, or it leaves ac_1 undetermined. Without loss of generality we choose $c_1 = 1$, and it remains to study the solutions of the equation

$$z^3 + z^2 = \tfrac{1}{c_0} \tag{7.6}$$

for $c_0 \geq \frac{1}{2}$, which is the characteristic equation of the linearized equation associated with equation (7.2). The solution $z = 1$ of equation (7.6) with $c_0 = \frac{1}{2}$ is useless since then all denominators in (7.4) vanish. For $c_0 > \frac{1}{2}$ there exists always a positive solution with $z < 1$. For $c_0 = \frac{27}{4}$, i.e. for $p = \frac{621}{16}$, there exists also the twofold negative solution $-\frac{2}{3}$, and for $c_0 > \frac{27}{4}$ there exist two different solutions z with $-1 < z < 0$. For $\frac{1}{2} < c_0 < \frac{27}{4}$ there exist two conjugate complex solutions to which we come back later on. In particular, for $c_0 = \frac{1}{2}\big(\sqrt{5}+1\big)$, i.e. for $p = 1$, the solutions of equation (7.6) are

$$z_1 = e^{\frac{4\pi i}{5}}, \quad z_2 = e^{\frac{6\pi i}{5}}, \quad z_3 = \tfrac{1}{2}\big(\sqrt{5} - 1\big). \tag{7.7}$$

In order to construct further solutions of equation (7.2) we extend ansatz (7.1) to

$$x_n = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} c_{jk} a^j z^{nj} b^k w^{nk} \tag{7.8}$$

with $w \neq z$. The recursions for the coefficients are the two-dimensional generalizations of (7.4). It turns out that w must be also a solution of equation (7.6), that $c_{j0} = c_j$, and replacing z by w we obtain c_{0k} from c_k. More generally, c_{jk} arises from c_{kj} by exchanging z and w. Hence $c_{jk} = \bar{c}_{kj}$ when $w = \bar{z}$. Some special cases are

$$\begin{aligned}
c_{20} &= \frac{z^5}{1 - c_0 z^4(z^2+1)} \\
c_{11} &= \frac{z^2 w^2 (z+w)}{1 - c_0 z^2 w^2 (zw+1)} \\
c_{02} &= \frac{w^5}{1 - c_0 w^4 (w^2+1)} \\
c_{30} &= \frac{c_{20} z^7 (z+1)}{1 - c_0 z^6 (z^3+1)} \\
c_{21} &= \frac{z^4 w^2 \big(c_{20}(z^2+w) + c_{11} z (w+1)\big)}{1 - c_0 z^4 w^2 (z^2 w + 1)}.
\end{aligned}$$

In the case $p \neq \frac{621}{16}$ the most general ansatz for a solution of equation (7.2) reads

$$x_n = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} c_{jkl} a^j z^{nj} b^k w^{nk} c^l t^{nl} \tag{7.9}$$

with three different solutions z, w, t of equation (7.6). There are analogous recursions, symmetries and relations as before, in particular $c_{ij0} = c_{ij}$. The recursions for c_{jkl} contain the denominator

$$D = 1 - c_0 z^{2j} w^{2k} t^{2l} (z^j w^k t^l + 1) \tag{7.10}$$

which has the following property:

Lemma. *Let z, w, t be three pairwise different solutions of equation* (7.6)*, let be $c_0 > \frac{1}{2}(\sqrt{5}+1)$ and $j+k+l \geq 2$ $(j,k,l \in \mathbb{N}_0)$. Then D from* (7.10) *is different from zero.*

Proof. For $j+k+l = 1$ we have $D = 0$ in view of (7.6). If the solutions z, w, t are real, then they have absolute values less then 1, and the powers of these values diminish. Hence, $D > 0$ for $j+k+l \geq 2$.

Now, let z be complex and $w = \overline{z}$, and assume that $D = 0$. For fixed j, k, l we introduce the notation $z^j w^k t^l = \rho e^{i\vartheta}$. The assumption $D = 0$ implies

$$\tfrac{1}{c_0} = \rho^3 \cos 3\vartheta + \rho^2 \cos 2\vartheta, \quad \rho^3 \sin 3\vartheta + \rho^2 \sin 2\vartheta = 0,$$

and elimination of ϑ yields

$$c_0 = \tfrac{1}{2\rho^4}\big(1 + \sqrt{1+4\rho^2}\big). \tag{7.11}$$

Since the right-hand side herein is strictly decreasing, there exists exactly one ρ satisfying (7.11) for given c_0, namely $\rho = |z|$. For $c_0 > \frac{1}{2}(\sqrt{5}+1)$ we have $\rho < 1$, and the powers of $|z|, |w|, t$ again diminish, so that $D \neq 0$ ∎

The lemma implies that all coefficients c_{jkl} exist for $c_0 > \frac{1}{2}(\sqrt{5}+1)$, i.e. for $p > 1$, where $|z| < 1$ for all solutions of equation (7.6). However, for $\frac{1}{2} < c_0 \leq \frac{1}{2}(\sqrt{5}+1)$, i.e. for $-\frac{1}{4} < p \leq 1$, we have $|z| = |w| \geq 1$ and $t < 1$, so that $D = 0$ is possible. E.g., for $z = z_1$ and $w = z_2$ from (7.7) we get $zw = 1$ and therefore $D = 0$ in (7.10) for $j = 2, k = 1, l = 0$, but then the numerator in c_{21} also vanishes, and $c_{21} = c_{210}$ exists nevertheless.

In the case $p = 0$ it can easily be seen that $x_n = e^{az^n + b\overline{z}^n + ct^n}$, where z is a complex and t the real solution of equation (7.6) with $c_0 = 1$, is the general complex solution of equation (7.2) when a, b, c are arbitrary, and x_n is the general positive solution of equation (7.2) when c is real and $b = \overline{a}$ (cf. [4: Section 3.3]). For $a \neq 0$, x_n is indeed unbounded as conjectured in (iii), and obviously, it can be expanded into form (7.9) with $c_{jkl} = \frac{1}{j!\,k!\,l!}$.

After these preparations there remains the following

Open Problem. What are the precise conditions for the existence of c_{jkl}, for the convergence of (7.9), and for the (unique) determination of the parameters a, b, c out of given positive initial values x_{-2}, x_{-1}, x_0?

For $p > 1$ series (7.9) are simultaneously asymptotic expansions as $n \to \infty$ and conform (i).

In the case $p = 1$ we can modify ansatz (7.9) for solutions (7.7) of equation (7.6) in the following way. With the notations $z = z_1$ and $t = z_3$ we have $w = z_2 = \overline{z}$ so that $z^j w^k = z^{j+4k}$ and, in view of $z^5 = 1$, we can replace ansatz (7.9) by

$$x_n = \sum_{m=0}^{4} \sum_{l=0}^{\infty} b_{ml} z^{nm} c^l t^{nl} \tag{7.12}$$

with

$$b_{ml} = \sum_{j+4k \equiv m \bmod 5} c_{jkl} a^j b^k. \tag{7.13}$$

But it is simpler to determine the coefficients b_{ml} out of (7.2). This possibility shows that they can exist even if the right-hand side of (7.13) does not make sense. The special case of equation (7.12) with $c = 0$, i.e.

$$x_n = \sum_{m=0}^{4} b_{m0} z^{nm} \tag{7.14}$$

yields the 5-periodic solution of (7.2) with $p = 1$ generated by

$$x_0 = r, \quad x_1 = s, \quad x_2 = \frac{r+1}{rs-1}, \quad x_3 = rs - 1, \quad x_4 = \frac{s+1}{rs-1}. \tag{7.15}$$

Here r and s are arbitrary positive parameters satisfying $rs > 1$, if we look for positive x_n.

Since (7.14) is a discrete Fourier transform we easily find by inversion

$$b_{m0} = \frac{1}{5} \sum_{k=0}^{4} x_m z^{-mk}$$

with x_m from (7.15). The coefficients contain the arbitrary parameters r and s instead of a and b in (7.13), they determine the further coefficients b_{ml} in (7.12) recursively. For $r = s = \frac{1}{2}(\sqrt{5} + 1)$ the 5-periodic solution degenerates to the equilibrium, to which the solution (7.12) converges in the case $a = b = 0$, cf. (ii). For the initial values $x_{-2} = x_0$ and $x_{-1} = \frac{1}{x_0}$ the olution x_n of equation (7.2) with $p = 1$ continues to smaller n by

$$\left.\begin{aligned} x_{2-5n} = x_{1-5n} &= 0 \\ x_{-5n} = x_{-1-5n} = x_{-2-5n} &= -1 \end{aligned}\right\} \qquad (n \in \mathbb{N}).$$

For $p < 0$ it is not possible to choose the initial values for the solutions of equation (7.2) arbitrarily (cf. [4: p. 17]). Moreover, for $-\frac{1}{4} < p < 0$ besides of (7.5) also the second equilibrium $c_0 = \frac{1}{2}(1 - \sqrt{1+4p})$ is positive and must be taken into consideration.

The ansatz (7.9) can be transferred to the excluded case $p = \frac{621}{16}$, where $z = w = -\frac{2}{3}$, replacing w^{nk} by $(nz^n)^k$ and letting c_{jkl} depend on a and b.

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