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Pattern rigidity in hyperbolic spaces: duality and PD subgroups

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This paper is dedicated to the memory of Kalyan Mukherjea

Abstract. For $i = 1, 2$, let G_i be cocompact groups of isometries of hyperbolic space $\mathbf{H}^n$ of real dimension n, $n \geq 3$. Let $H_i \subset G_i$ be infinite index quasiconvex subgroups satisfying one of the following conditions:

(1) The limit set of H_i is a codimension one topological sphere.
(2) The limit set of H_i is an even dimensional topological sphere.
(3) H_i is a codimension one *duality group*. This generalizes (1). In particular, if $n = 3$, H_i could be *any* freely indecomposable subgroup of G_i.
(4) H_i is an odd-dimensional Poincaré duality group PD$(2k+1)$. This generalizes (2).

We prove pattern rigidity for such pairs extending work of Schwartz who proved pattern rigidity when H_i is cyclic. All this generalizes to quasiconvex subgroups of uniform lattices in rank one symmetric spaces satisfying one of the conditions (1)–(4), as well as certain special subgroups with disconnected limit sets. In particular, pattern rigidity holds for all quasiconvex subgroups of hyperbolic 3-manifolds that are not virtually free. Combining this with results of Mosher, Sageev, and Whyte, we obtain quasi-isometric rigidity results for graphs of groups where the vertex groups are uniform lattices in rank one symmetric spaces and the edge groups are of any of the above types.

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1. Introduction

1.1. Statement of results. In [Gro93] Gromov proposed the project of classifying finitely generated groups up to quasi-isometry. A class of groups where any two members are quasi-isometric if and only if they are commensurable is said to be quasi-isometrically rigid. However, in certain classes of groups, for instance uniform lattices G in some fixed hyperbolic space $\mathbf{H}$, all members of the class are quasi-isometric

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to **H** and hence to each other. In this context (or in a context where quasi-isometric rigidity is not known) it makes sense to ask a relative version of Gromov's question. Here, (almost as a rule) additional restriction is imposed on the quasi-isometries by requiring that they preserve some additional structure given by a 'symmetric pattern' of subsets. A 'symmetric pattern' of subsets roughly means a G-equivariant collection $\mathcal{J}$ of subsets in **H**, each of which in turn is invariant under a conjugate of a fixed subgroup H of G, such that the quotient of an element of $\mathcal{J}$ by its stabilizer is compact. Then the relative version of Gromov's question for classes of pairs (G, H) becomes:

Question 1.1. Given a quasi-isometry q of two such pairs (G_i, H_i) $(i = 1, 2)$ pairing a (G_1, H_1)-symmetric pattern $\mathcal{J}_1$ with a (G_2, H_2)-symmetric pattern $\mathcal{J}_2$, does there exist an isometry I which performs the same pairing? Further, does q lie within a bounded distance of I?

Formulated in these terms, the phenomenon addressed by Question 1.1 is called *pattern rigidity* (See [MSW04] where this terminology was first used. See Section 1.2 for more on the genesis of the problem and the techniques used, particularly work of Mostow [Mos68] and Sullivan [Sul81].)

One of the first papers to come out in the subject of quasi-isometric rigidity was by Schwartz [Sch95], and even here, the problem can be formulated (in part) as a pattern rigidity question for symmetric patterns of horoballs in **H**. The next major piece of work on pattern rigidity was for subgroups $H = \mathbb{Z}$ by Schwartz [Sch97] again. In a certain sense, [Sch95] deals with symmetric patterns of convex sets whose limit sets are single points, and [Sch97] deals with symmetric patterns of convex sets (geodesics) whose limit sets consist of two points. In this paper we initiate the study of pattern rigidity for symmetric patterns of convex sets whose limit sets are infinite.

For $i = 1, 2$, let G_i be cocompact groups of isometries in a rank one symmetric space $\mathbf{H}^n$ of real dimension n, $n \geq 3$. Let $H_i \subset G_i$ be an infinite index quasiconvex subgroup satisfying *any* of the following conditions:

(1) The limit set of H_i is a codimension one topological sphere.
(2) The limit set of H_i is an even dimensional topological sphere.
(3) H_i is a codimension one *duality group*. This generalizes (1).
(4) H_i is an odd-dimensional Poincaré duality group PD$(2k + 1)$. This generalizes (2).

In this paper, we prove pattern rigidity for such pairs (See Theorem 1.4 below for a precise statement).

Examples of (1) above include quasi-Fuchsian surface subgroups of closed hyperbolic 3-manifold groups corresponding to immersed surfaces. A special case of this would correspond to *embedded totally geodesic* surfaces (cf. [Fri04]). Examples of (3) include all freely indecomposable subgroup of closed hyperbolic 3-manifold groups. (Note that these need not correspond to embedded submanifolds with boundary.) See also [Bel10], [Bis09] and [Mj09] for related work.

Definition 1.2. A *symmetric pattern* of closed convex (or quasiconvex) sets in a rank one symmetric space $\mathbf{H}$ is a G-invariant countable collection $\mathcal{J}$ of closed convex (or quasiconvex) sets such that the following holds:

(1) G acts cocompactly on $\mathbf{H}$.

(2) The stabilizer H of $J \in \mathcal{J}$ acts cocompactly on J.

(3) $\mathcal{J}$ is the orbit of some (any) $J \in \mathcal{J}$ under G.

This definition is slightly more restrictive than Schwartz' notion of a symmetric pattern of geodesics, in the sense that he takes $\mathcal{J}$ to be a finite union of orbits of geodesics, whereas condition (3) above forces $\mathcal{J}$ to consist of one orbit. All our results go through with the more general definition, where $\mathcal{J}$ is a finite union of orbits of closed convex (or quasiconvex) sets, but we restrict ourselves to one orbit for expository ease.

Suppose that (X_1, d_1), (X_2, d_2) are metric spaces. Let $\mathcal{J}_1$, $\mathcal{J}_2$ be collections of closed subsets of X_1, X_2 respectively. Then d_i induces a non-negative function (which, by abuse of notation, we continue to refer to as d_i) on $\mathcal{J}_i \times \mathcal{J}_i$ for $i = 1, 2$. This is just the ordinary (not Hausdorff) distance between closed subsets of a metric space.

In particular, consider two hyperbolic groups G_1, G_2 with quasiconvex subgroups H_1, H_2 and Cayley graphs Γ_1, Γ_2. Let $\mathcal{L}_j$ for $j = 1, 2$ denote the collection of translates of limit sets of H_1, H_2 in $\partial G_1, \partial G_2$ respectively. Individual members of the collection $\mathcal{L}_j$ will be denoted as L_i^j. Let $\mathcal{J}_j$ denote the collection $\{J_i^j = J(L_i^j) : L_i^j \in \mathcal{L}_j\}$ of joins of limit sets. Recall that the join of a limit set Λ_i is the union of bi-infinite geodesics in Γ_i with end-points in Λ_i. This is a uniformly quasiconvex set and lies at a bounded Hausdorff distance from the Cayley graph of the subgroup H_i Following Schwartz [Sch97], we define:

Definition 1.3. A bijective map ϕ from $\mathcal{J}_1 \to \mathcal{J}_2$ is said to be uniformly proper if there exists a function $f : \mathbb{N} \to \mathbb{N}$ such that

(1) $d_{G_1}(J(L_i^1), J(L_j^1)) \leq n \implies d_{G_2}(\phi(J(L_i^1)), \phi(J(L_j^1))) \leq f(n)$,

(2) $d_{G_2}(\phi(J(L_i^1)), \phi(J(L_j^1))) \leq n \implies d_{G_1}(J(L_i^1), J(L_j^1)) \leq f(n)$.

When $\mathcal{J}_i$ consists of all singleton subsets of Γ_1, Γ_2, we shall refer to ϕ as a uniformly proper map from Γ_1 to Γ_2.

Our first main theorem (combining Theorems 4.3, 4.7, 4.8, 4.17, 5.1 in the paper) is:

Theorem 1.4. *Let $n \geq 3$. Suppose that $\mathcal{J}_i$ (for $i = 1, 2$) are symmetric patterns of closed convex (or quasiconvex) sets in hyperbolic space $\mathbf{H}^n = \mathbf{H}$, or more generally uniform lattices in rank one symmetric spaces of dimension n, $n \geq 3$. For $i = 1, 2$, let G_i be the corresponding cocompact group of isometries. Let $H_i \subset G_i$ be an infinite index quasiconvex subgroup stabilizing the limit set of some element of $\mathcal{J}_i$ and satisfying* ***one*** *of the following conditions:*

(1) *The limit set of H_i is a codimension one topological sphere.*
(2) *The limit set of H_i is an even dimensional topological sphere.*
(3) *H_i is a codimension one* ***duality group****. This generalizes* (1)*. In particular, if $n = 3$, H_i could be* ***any*** *freely indecomposable subgroup of G_i.*
(4) *H_i is an odd-dimensional Poincaré duality group* PD$(2k+1)$. *This generalizes* (2).

Then any uniformly proper bijection between $\mathcal{J}_1$ and $\mathcal{J}_2$ is induced by a hyperbolic isometry.

To prove cases (1) and (2) we shall use the classical Brouwer and Lefschetz fixed point theorems respectively. To generalize these to cases (3) and (4) we shall use tools from the algebraic topology of generalized (or homological) manifolds.

Next suppose $\mathcal{J}$ is a symmetric pattern of closed convex sets in $\mathbf{H}$ as in Theorem 1.4. For convenience suppose that elements of $\mathcal{J}$ are ε-neighborhoods of convex hulls of limit sets of elements of $\mathcal{J}$, so that they are strictly convex and G-equivariant. Let ϕ be a uniformly proper bijection from $\mathcal{J}$ to itself. Then Theorem 1.4 shows that ϕ is induced by a hyperbolic isometry f. Consider the pattern of geodesic segments perpendicular to elements of $\mathcal{J}$ at their end-points. This collection is invariant under G and there can only be a finite number of such segments of bounded total length inside any bounded ball. Hence the subgroup of isometries of Isom$(\mathbf{H})$ preserving this pattern is discrete and contains G as a finite index subgroup. This proves the following corollary that is by now (after [Sch97]) a standard consequence of such pattern rigidity statements as Theorem 1.4.

Corollary 1.5. *Suppose $\mathcal{J}$ is a symmetric pattern of closed convex sets in hyperbolic space $\mathbf{H}^n = \mathbf{H}$ or more generally uniform lattices in rank one symmetric spaces of dimension n, $n \geq 3$, as in Theorem 1.4 and G the associated cocompact group of isometries. Then the subgroup of the quasi-isometry group* QI$(\mathbf{H})$ *that coarsely preserves $\mathcal{J}$ contains G as a subgroup of finite index.*

More generally, the pattern rigidity Theorem 1.4 goes through for quasiconvex subgroups with disconnected limit sets, at least one of whose components has a stabilizer H' of the form (1), (2), (3) or (4) in Theorem 1.4 above. For details, see Corollary 5.3 in this paper. Theorem 1.4 combined with Corollary 5.3 implies further that *pattern rigidity holds for all quasi convex subgroups of hyperbolic* 3*-manifolds that are not virtually free*.

Similar extensions hold for quasiconvex subgroups H when some finite intersection of conjugates $\bigcap_{i=1,\dots,k} g_i H g_i^{-1}$ is of the form (1), (2), (3) or (4) in Theorem 1.4 above. For details see Corollary 5.4 in this paper.

Combining this with the main theorem of Mosher–Sageev–Whyte [MSW04] (to which we refer for the terminology) we get the following QI-rigidity theorem.

Theorem 5.5. *Let $\mathcal{G}$ be a finite, irreducible graph of groups with associated Bass–Serre tree T of spaces such that no depth zero raft of T is a line. Further suppose that the vertex groups are fundamental groups of compact hyperbolic n-manifolds, or more generally uniform lattices in rank one symmetric spaces of dimension n, $n \geq 3$, and edge groups are all of exactly one the following types:*

- (a) *a **duality group** of codimension one in the adjacent vertex groups; in this case we require in addition that the crossing graph condition of Theorems* 1.5, 1.6 *of* [MSW04] *be satisfied and that $\mathcal{G}$ is of finite depth,*
- (b) *an odd-dimensional Poincaré duality group* $\mathrm{PD}(2k+1)$ *with* $2k+1 \leq n-1$.

If H is a finitely generated group quasi-isometric to $G = \pi_1(\mathcal{G})$ then H splits as a graph $\mathcal{G}'$ of groups whose depth zero vertex groups are commensurable to those of $\mathcal{G}$ and whose edge groups and positive depth vertex groups are respectively quasi-isometric to groups of type (a), (b).

1.2. Outline and sketch. *Outline*. In Section 2, we describe some general properties of limit sets of quasiconvex subgroups of hyperbolic groups and recall some theorems from [Mj08]. In Section 3, we recall some of the foundational work of Schwartz from [Sch97] and describe some generalizations that we shall use in this paper. Section 4 is the heart of the paper. We reduce the problem of pattern rigidity to finding fixed points of certain maps, and then proceed to apply classical fixed point theorems (Brouwer and Lefschetz) to limit sets that are either spheres of codimension one, or of even dimension. We generalize these results to quasiconvex Duality subgroups of dimension $n-1$ and quasiconvex $\mathrm{PD}(2k+1)$ subgroups. For this we need some tools from the algebraic topology of homology manifolds. In Section 5, we describe further generalizations of these results to quasiconvex subgroups with disconnected limit sets as well as subgroups with certain intersection properties. We also combine these results with the main theorem of [MSW04] by Mosher–Sageev–Whyte to obtain QI-rigidity results.

Sketch of proof. We describe in brief the various steps involved in the proof.

1) Uniformly proper pairings come from quasi-isometries [Mj08].

2) Use Mostow–Sullivan–Schwartz zooming in (cf. Lemma 3.1) at a point of differentiability and non-conformality to get an 'eccentric' map A on the boundary pairing limit sets. The 'eccentric' map is obtained by pre- and post-composing a linear map of Euclidean space (thought of as a sphere minus the North pole) with conformal maps of the sphere. This 'zoom-in, zoom-out' step is really quite classical

and goes back to Mostow [Mos68]. This was refined by Sullivan [Sul81] and adapted to the present context by Schwartz [Sch95] [Sch97].

3) Fix a particular limit set which is taken to another fixed limit set under the pairing. Zoom in using the stabilizer of the first limit set, act by A and zoom out using the stabilizer of the second limit set. This step differs from the corresponding step in [Sch97] as follows. Though we can by the Generalized Eccentricity Lemma 3.5, zoom in using powers of the same element, we cannot necessarily zoom out using powers of the same element (as in [Sch97]). This makes the step technically more complicated and we use a *generalized zoom-in zoom-out lemma* (Lemma 3.3) to address this difficulty.

4) Get a sequence of rational functions that leave invariant a finite collection of limit sets.

5) Apply Brouwer's Fixed Point Theorem (in the codimension one sphere case) to get fixed points in the ball bounded by the sphere; and the Lefschetz Fixed Point Theorem (in the even dimensional sphere limit set case) to get a fixed point on the sphere limit set itself.

6) Use some generalizations of the Lefschetz Fixed Point Theorem going back to work of Lefschetz (himself), Felix Browder, R. Thompson, R. Knill, R. Wilder along with a theorem of Bestvina and Bestvina–Mess to generalize Step 5 to duality and Poincaré duality groups.

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2. Preliminaries

Remark 2.1. A folklore fact that we shall be using is that for discrete subgroups G of isometries of real hyperbolic space $\mathbb{H}^n$, convex cocompactness is equivalent to quasiconvexity. The forward implication is a consequence of the Milnor–Svarc Lemma (cf. [Gro85]).

One way to prove the converse implication is to use the fact that $\mathbb{H}^n$ is projectively flat. Hence the convex hull of a finite set of points is the union of the convex hulls of its $n+1$-tuples. Now let Λ be the limit set of G and let CH_0 denote the union of all ideal geodesic $n+1$-simplices with vertices in Λ. Then CH_0 is dense in the (closed) convex hull CH of Λ. Now, if an orbit $G \cdot x$ is quasiconvex in $\mathbb{H}^n$, then the inclusion of $G \cdot x$ into $\mathbb{H}^n$ extends continuously to a homeomorphism from the (Gromov-)hyperbolic boundary ∂G to Λ. Hence, by δ-hyperbolicity of G and quasiconvexity, it follows that for any $\{x_0, \ldots, x_n\} \subset \Lambda$ and any p in the ideal simplex with vertices $\{x_0, \ldots, x_n\}$, $d(p, G \cdot x)$ is uniformly bounded, where d is the usual metric on $\mathbb{H}^n$. It follows that CH_0 and hence CH lies in a bounded Hausdorff neighborhood of $G \cdot x$.

Therefore the quotient CH/G is compact.

Limit sets and pairings. Let G be a hyperbolic group. ∂G will denote its boundary equipped with a visual metric. Any fixed point of a hyperbolic element on ∂G is called a pole. $\partial^2 G$ will denote the set of unordered pairs of distinct points on ∂G with the topology inherited from ∂G. A *pole-pair* is a pair of points $(x, y) \in \partial^2 G$ corresponding to the fixed points of a hyperbolic element of G.

Lemma 2.2 (Gromov [Gro85], 8.2G, p. 213). *Pole-pairs are dense in $\partial^2 G$ and more generally, if H be a finitely generated group of isometries acting on hyperbolic space* $\mathbf{H}$ *with limit set Λ then pole-pairs are dense in $\partial^2 \Lambda$.*

The next lemma is a consequence of the fact that the action of a finitely generated group of isometries of a hyperbolic metric space $\mathbf{H}$ acting on the limit set is a convergence group action.

Lemma 2.3. *Let H be a finitely generated group of isometries acting on hyperbolic space* $\mathbf{H}$ *with limit set Λ. Then for all $(x, y) \in \partial^2 \Lambda$, there exists a sequence of hyperbolic isometries $T_i \in H$ with attracting (resp. repelling) fixed points x_i (resp. y_i) such that $x_i \to x$, $y_i \to y$, and the translation length of T_i tends to ∞.*

Proof. We choose pole-pairs (x_i, y_i) converging to $(x, y) \in \partial^2 \Lambda$ by Lemma 2.2. Let T_i be a hyperbolic isometry in G with attracting fixed point x_i and repelling fixed point y_i. Choosing appropriately large powers $T_i^{n_i}$ of T_i, we are through. □

Since the orbit of an open subset of ∂G under G is the whole of ∂G, it follows that the limit set L_H of any infinite index quasiconvex subgroup H of G is nowhere dense in ∂G. Assume for simplicity that $H = \mathrm{Stab}(L_H)$. Then for all $g \in G \setminus H$, $gHg^{-1} \cap H$ is an infinite index quasiconvex subgroup of H (by a theorem of Short [Sho91]) and hence its limit set is nowhere dense in L_H. As g ranges over $g \in G \setminus H$, we get a countable collection of nowhere dense subsets of L_H. The next lemma follows.

Lemma 2.4. *Suppose that $H = \mathrm{Stab}(L_H)$ is a quasiconvex subgroup of a hyperbolic group G. For all $p \in L_H$ and all $\varepsilon > 0$ there exists $x \in L_H$ such that $d(p, x) < \varepsilon$ and L_H is the unique translate of L_H to which x belongs, i.e., if $x \in L_H \cap gL_H$ then $g \in H$.*

Lemma 2.5. *Suppose that $H = \mathrm{Stab}(L_H)$ is a quasiconvex subgroup of a hyperbolic group G. Let $U \subset \partial G$ be an open subset and let $\varepsilon > 0$. Then there exists a finite collection of points $x_1, \dots, x_n \in U$ such that*

(1) *$\{x_1, \dots, x_n\}$ is an ε-net in U,*
(2) *$x_i \in L_i = g_i L_H$ for some $g_i \in G \setminus H$ and L_i is the unique translate of L_H to which x belongs.*

Proof. This follows from Lemma 2.4 and the fact that the union of all the translates of L_H under G is dense in ∂G. □

Definition 2.6. A point that belongs to a unique translate of L_H will be called a *unique point*.

In [Mj08] we showed the following:

Theorem 2.7 (Theorem 3.5 of [Mj08]). *Let ϕ be a uniformly proper (bijective, by definition) map from $\mathcal{J}_1 \to \mathcal{J}_2$. There exists a quasi-isometry q from Γ_1 to Γ_2 which pairs the sets $\mathcal{J}_1$ and $\mathcal{J}_2$ as ϕ does.*

Proposition 2.8 (Characterization of quasiconvexity, Proposition 2.3 of [Mj08]). *Let H be a subgroup of a hyperbolic group G with limit set Λ. Let $\mathcal{L}$ be the collection of translates of Λ (counted with multiplicity) by elements of distinct cosets of H (one for each coset). Then H is quasiconvex if and only if $\mathcal{L}$ is a discrete subset of $C_c^0(\partial G)$, where $C_c^0(\partial G)$ denotes the collection of compact subsets of ∂G with more than one point equipped with the Hausdorff metric.*

Finally, combining Lemmas 2.2 and 2.4 along with Proposition 2.8, we get

Corollary 2.9 (Generic pole-pairs). *Suppose that $H = \mathrm{Stab}(L_H)$ is a quasiconvex subgroup of a hyperbolic group G. Identify L_H with the boundary ∂H of H. For all $(p, q) \in \partial^2 H$ and all $\varepsilon > 0$ there exists a pole-pair $(x, y) \in \partial^2 H$ such that $d(p, x) < \varepsilon$, $d(q, y) < \varepsilon$ and L_H is the unique translate of L_H to which x (or y) belongs, i.e., if x (or) $y \in L_H \cap gL_H$ then $g \in H$.*

Small homotopies. We shall have need for the following fact [BT90].

Lemma 2.10. *Given a closed Riemannian manifold (M, d), there exists $\varepsilon_1 > 0$ such that the following holds:*

If f is a self-homeomorphism such that $d(f(x), x) < \varepsilon_1$ for all $x \in M$, then f is homotopic to the identity.

Sketch of proof for smooth maps. Since M is a compact Riemannian manifold, there is an $\varepsilon_0 >$ such that (modulo the natural identification of the normal bundle of the diagonal, $D_M \subset (M \times M)$, with the tangent bundle of M) tangent vectors of length ε_0 map via the exponential map diffeomorphically onto an open tubular neighborhood of D_M. If $f : M \to M$ is sufficiently near to the identity 1_M in the compact-open topology, then the image of the graph of f will lie in this tubular neighborhood. The inverse of the exponential map identifies the graph of F with a section of the tangent bundle of M and since any section is homotopic to the zero section (which corresponds to the graph of the identity map) we are through. □

Boundaries. We shall have need for the following theorem of Bestvina and Mess [BM91].

Theorem 2.11. *Boundaries ∂G of* PD(n) *hyperbolic groups G are locally connected homological manifolds* (*over the integers*) *with the homology of a sphere of dimension* $(n-1)$. *Further, if G acts freely, properly, cocompactly on a contractible complex X, then the natural compactification $X \cup \partial G$ is an absolute retract* (*AR*).

Remark 2.12. In particular, $X \cup \partial G$ has the fixed point property, i.e., any continuous map from $X \cup \partial G$ to itself has a fixed point.

3. Differentiability principles and eccentric maps

3.1. Differentiability principles. Let $\mathbf{H}^{n+1} = \mathbf{H}$ denote the hyperbolic $(n+1)$-space and let $\partial\mathbf{H}^{n+1} = S^n_\infty$ denote the boundary sphere at infinity with the standard conformal structure (preserved by isometries of $\mathbf{H}^{n+1}$). Let $\mathbb{E}^n = \mathbb{E}$ denote the Euclidean space obtained from S^n_∞ by removing the point at infinity.

We recall a certain *differentiability principle* from Schwartz's paper [Sch97]. Suppose that $h\colon S^n_\infty \to S^n_\infty$ is a homeomorphism fixing $0, \infty$ such that $dh(0)$ exists. Let T_1, T_2 be two contracting similarities (with possible rotational components) of $\mathbb{E}$ both fixing 0. For each pair k_1, k_2 of positive integers, Schwartz defines the map

$$h[k_1, k_2] = T_2^{-k_2} \circ h \circ T_1^{k_1}$$

and shows

Lemma 3.1 (Lemma 5.3 of [Sch97]). *Suppose that K_1, $K_2 \subset \mathbb{E}$ are compact subsets. Suppose that $(k_{11}, k_{21}), (k_{12}, k_{22}), (k_{13}, k_{23}), \dots$ is a sequence of pairs such that*

(1) $k_{1n} \to \infty$,
(2) $h[k_{1n}, k_{2n}](K_1) \cap K_2 \neq \emptyset$.

Then some subsequence of $h[k_{1n}, k_{2n}]$ converges, uniformly on compact sets, to a linear map.

Remark 3.2. Lemma 3.1 can be slightly generalized by replacing the maps $T_1^{k_1}$ and $T_2^{k_2}$ by maps T_{1k_1} and T_{2k_2} such that the translation lengths of T_{1k_1} and T_{2k_2} tend to infinity as $k_1, k_2 \to \infty$. This is all Schwartz uses in his proof.

We shall need a generalization and weakening of this to continuously differentiable functions.

Lemma 3.3 (Generalized zoom-in zoom-out). *Suppose $h\colon S^n_\infty \to S^n_\infty$ is continuously differentiable. Let T_{1n}, T_{2n} be sequences of hyperbolic Möbius transformations such that their fixed point sets $\{x_{1n}, y_{1n}\}$, $\{x_{2n}, y_{2n}\}$ satisfy*

(1) *x_{in} is the attracting fixed point of T_{in} for $i = 1, 2$,*
(2) *y_{in} is the repelling fixed point of T_{in} for $i = 1, 2$.*

We further assume that $x_{1n} = 0$ and $y_{1n} = \infty$ for all n.

Let

$$h_n = T_{2n}^{-1} \circ h \circ T_{1n}.$$

Suppose that $(T_{11}, T_{21}), (T_{12}, T_{22}), (T_{13}, T_{23}), \ldots$ is a sequence of pairs such that

(a) *the translation lengths of $T_{1n} \to \infty$,*
(b) *there exists $\varepsilon > 0$ such that*

$$\inf_n (\min\{d(h_n(0), h_n(1)), d(h_n(\infty), h_n(1)), d(h_n(0), h_n(\infty))\}) \geq \varepsilon.$$

Then some subsequence of h_n converges, uniformly on compact sets, to a linear map post-composed with a conformal map.

Proof. The sequence $g_n = T_{1n}^{-1} \circ h \circ T_{1n}$ converges (up to sub-sequencing) to a linear map by Lemma 3.1 and Remark 3.2. Condition (b) in the hypothesis guarantees that the ratio of the translation lengths of T_{1n} and T_{2n} is bounded away from both 0 and ∞. Hence, (extracting a further subsequence if necessary) $T_{2n}^{-1} \circ T_{1n}$ converges to a conformal map. The result follows. □

3.2. Eccentric maps

Definition 3.4 ([Sch97]). Let T be a real linear map of the Euclidean space $\mathbb{E}^n = \mathbb{E}$. Let g_i (for $i = 1, 2$) be two conformal maps of $\partial \mathbf{H}^{n+1} = S^n_\infty$. The map $\mu = g_2 \circ T \circ g_1^{-1}$ is said to be an *eccentric* map if

(1) μ preserves E and fixes 0,
(2) μ is differentiable at 0,
(3) μ is *not* a real linear map.

Then the *Eccentricity Lemma* (Lemma 2.2) of Schwartz [Sch97] generalizes to

Lemma 3.5 (Generalized Eccentricity Lemma). *Let G_1, G_2 be two groups acting freely, properly discontinuously by isometries and cocompactly on $\mathbf{H}^{n+1} = \mathbf{H}$. Let H_1^0, H_2^0 be quasiconvex subgroups of G_1, G_2 respectively with limit sets Λ_1^0, Λ_2^0. Let $\mathcal{J}_i^0$ (for $i = 1, 2$) be the set of translates of joins (or convex hulls) of Λ_i^0. Let q^0 be a quasi-isometry pairing $\mathcal{J}_1^0$ with $\mathcal{J}_2^0$. Assume that $h^0 = \partial q^0$ is not conformal. Then there exist symmetric pattern of joins (or convex hulls) $\mathcal{J}_i$ (of limit sets Λ_i abstractly homeomorphic to Λ_1^0, Λ_2^0) and a quasi-isometry $q \colon \mathbf{H} \to \mathbf{H}$ such that*

(1) *q pairs the elements of $\mathcal{J}_1$ with those of $\mathcal{J}_2$,*
(2) *$\mu = \partial q$ is an eccentric map,*

(3) *the geodesic* $\gamma = \overline{0\infty}$ *is a subset of some* $J_i \in \mathcal{J}_i$ *for* $i = 1, 2$; *further, the* (*translates of the*) *limit sets* Λ_1 *and* Λ_2 *in which the end-points* $0, \infty$ *lie is unique,*

(4) $0, \infty$ *are poles for the action of the stabilizer* $\mathrm{Stab}(\Lambda_1)$ *on* Λ_1.

Proof. The difference with Lemma 2.2 of [Sch97] is in conditions (3) and (4) above. Schwartz' proof proceeds by zooming in at a point of differentiability and non-conformality (taken to be the origin) of the quasiconformal map h^0 to obtain a linear map h' from $\mathbb{E}$ to itself in the limit. The sequence of maps used in zooming in come by conjugating h^0 by D^n where D is a dilatation map with $0, \infty$ as fixed points. Further h' is the boundary value of some quasi-isometry q' which pairs some symmetric pattern of joins $\mathcal{J}_1'$ with $\mathcal{J}_2'$. This step goes through verbatim.

Next, by Lemma 2.4 there exist pairs of points α, β on some limit set Λ_1 of an element of $\mathcal{J}_1'$ such that Λ_1 is unique, i.e., α, β do not belong to any other limit set Λ_1' of an element of $\mathcal{J}_1'$. Also, by Corollary 2.9 the pair (α, β) can be taken as pole-pair for the action of the stabilizer $\mathrm{Stab}(\Lambda_1)$ on Λ_1. Since q' pairs the symmetric pattern of joins $\mathcal{J}_1'$ with $\mathcal{J}_2'$, $h'(\alpha) = \alpha'$, $h'(\beta) = \beta'$ belong to some unique Λ_2, i.e., $h'(\alpha)$, $h'(\beta)$ do not belong to any other limit set Λ_2' of an element of $\mathcal{J}_2'$. Let g_j be chosen in such a way that g_1 (resp. g_2) maps $0, \infty$ to α, β (resp. α', β') respectively.

Then $\mu = g_2 \circ h' \circ g_1^{-1}$ and $q = g_2 \circ q' \circ g_1^{-1}$ are the required maps. □

We shall need the following 'Zariski-density' property of eccentric maps due to Schwartz [Sch97].

Lemma 3.6 (Corollary 5.2 of [Sch97]). *Let* $U \subset \mathbb{E}$ *be an open subset. Then there is a constant* $\delta = \delta(U) > 0$ *such that if two eccentric maps agree on a* δ*-dense subset of* U*, then they agree everywhere.*

4. Pattern rigidity

4.1. Scattering. For $i = 1, 2$, let F_i be a (compact) fundamental domain for the action of $H_i = \mathrm{Stab}(L_{H_i})$ on the domain of discontinuity Ω_i of H_i (see Remark 2.1). Let Q_i be the quotient of Ω_i by H_i. Let $\Pi_i : \Omega_i \to Q_i$ be the covering map. Recall, by Lemma 3.5 that $0, \infty$ form a pole-pair for the action of H_1 on L_{H_1}. Next, suppose that we have

(1) an eccentric map μ,

(2) a subset $\Sigma \subset Q_1$,

(3) a neighborhood $S \subset E$ of 0.

Define $\Psi(\mu, \Sigma, S) = \Pi_2 \circ \mu(S \cap \Pi_1^{-1}(\Sigma)) \subset Q_2$.

Lemma 4.1 (Scattering Lemma). *Independent of μ there is a constant $\delta_0 > 0$ such that if $S \subset E$ is any neighborhood of 0, and $\Sigma \subset Q_1$ is δ_0-dense, then $\Psi(\mu, \Sigma, S) \subset Q_2$ is an infinite set.*

Proof. Though we shall follow the broad scheme of the proof of Lemma 2.3 of Schwartz [Sch97], technically our proof will be quite a bit more involved as we shall first use the Generalized Zoom-in Zoom-out Lemma 3.3 and then Lemma 3.1 (and not Lemma 3.1 directly as in [Sch97]). In particular, steps (1) and (2) below will be different, while step (3) will be the same as in [Sch97].

Let $\Sigma_0 = \Pi_1^{-1}(\Sigma) \cap F_1$. Let S be an open neighborhood of 0.

There exists by the Generalized Eccentricity Lemma 3.5 a sequence of hyperbolic Möbius transformations $T_{1n} \in H_1$ such that the fixed point sets $\{x_{1n}, y_{1n}\}$, satisfy

(1) $x_{1n} = 0$ is the attracting fixed point of T_{1n},
(2) $y_{1n} = \infty$ is the repelling fixed point of T_{1n},
(3) the hyperbolic isometries corresponding to T_{1n}, form an unbounded set in $\mathrm{PSL}_2(C)$,
(4) $T_{1n}(F_1) \subset S$.

Condition (4) follows from (1) and substituting T_{1n} by large enough powers of T_{1n} if necessary.

Step 1. Choosing $T_{2n} \in H_2$.

The first step is to choose $T_{2n} \in H_2$. $T_{1n}(\Sigma_0) \subset \Pi_1^{-1}(\Sigma) \cap S$. First choose a point $w \in F_1$. Let $P(w)$ denote the foot of the perpendicular from w to $(0, \infty) = (x_{1n}, y_{1n})$. Then $T_{1n}(w) \in S$. Map the tripod with vertices $(0, \infty, T_{1n}(w))$ over by μ. Recall that $\mu(0) = 0$, $\mu(\infty) = \infty$. Then $(0, \infty, \mu \circ T_{1n}(w))$ form the vertices of a *uniform* K-quasitripod in $\mathbf{H}$ with centroid $\mu \circ T_{1n}(P(w))$. Choose $T_{2n} \in H_2$ such that $T_{2n}^{-1} \circ \mu \circ T_{1n}(P(w))$ lies in a fixed fundamental domain for the action of H_2 on the convex hull $\mathrm{CH}(L_{H_2})$ of the limit set L_{H_2}. (We could equally well have chosen the join $J(L_{H_2})$ of the limit set L_{H_2}.) Then, automatically, the three points $T_{2n}^{-1}(0)$, $T_{2n}^{-1}(\infty)$, $T_{2n}^{-1} \circ \mu \circ T_{1n}(w)$ satisfy the hypotheses of Lemma 3.3, i.e., there exists $\varepsilon > 0$ such that the three points $T_{2n}^{-1}(0)$, $T_{2n}^{-1}(\infty)$, $T_{2n}^{-1} \circ \mu \circ T_{1n}(w)$ are at a distance of at least ε from each other on the sphere (uniformly for all n). By Lemma 3.3, (up to extracting a subsequence) $T_{2n}^{-1} \circ \mu \circ T_{1n}$ converges to a map $\psi \circ L$, where ψ is conformal and L is linear.

Step 2. The sequence $T_{2n}^{-1} \circ \mu \circ T_{1n}$ consists of infinitely many distinct elements.

We would like to conclude that there are *infinitely many distinct maps* in the sequence $T_{2n}^{-1} \circ \mu \circ T_{1n}$ converging to a map $\psi \circ L$. Suppose not. Then the sequence of maps $T_{2n}^{-1} \circ \mu \circ T_{1n}$ is eventually constant and equal to $\psi \circ L$. In particular, since μ, T_{1n}, L all fix $0, \infty$, it follows that $T_{2n}^{-1}(0) = \psi(0)$ and $T_{2n}^{-1}(\infty) = \psi(\infty)$ for all n. But then

(1) $T_{2m} \circ T_{2n}^{-1}(0) = 0$, and $T_{2m} \circ T_{2n}^{-1}(\infty) = \infty$ for all m, n,
(2) $T_{2n} \in H_2$ for all n.

Since the collection of elements of the form $T_{2m} \circ T_{2n}^{-1}$ is infinite, it follows that the set of elements in H_2 fixing $0, \infty$ is virtually infinite cyclic. Thus, $(0, \infty)$ form a pole-pair for the action of H_2 on L_{H_2}. Let $\mathcal{C}$ be an infinite cyclic subgroup of H_2 fixing $0, \infty$. In this case, we modify the sequence T_{2n} by choosing these to be elements of $\mathcal{C}$ satisfying the hypotheses of Lemma 3.1. Then $\mu_n = T_{2n}^{-1} \circ \mu \circ T_{1n}$ converges to a linear map by Lemma 3.1. Since μ is eccentric, so is μ_n and we may assume that $\mu_n \to \mu'$, a linear map. Hence in either case, we can conclude that the sequence $\mu_n = T_{2n}^{-1} \circ \mu \circ T_{1n}$ of maps consists of infinitely many distinct elements converging either to a map of the form $\psi \circ L$ (with ψ conformal and L linear) or simply a linear map L.

Step 3. Using Zariski density.

The rest of the proof follows that of [Sch97]. Define $V = \bigcup_{n=1}^{\infty} \mu_n(\Sigma_0)$.

Claim 4.2. *V is infinite and $\overline{V} \subset \Omega_2$.*

Proof of the claim. Since $\mu_n \to \mu$, V is bounded and $\overline{V} \subset \Omega_2$. In particular, $\overline{V}$ is contained in the union of finitely many translates of the compact fundamental domain F_2. By step (2) there are infinitely many distinct maps in the sequence. If V is finite, then only finitely many choices are there for $\mu_n(\Sigma_0)$ and hence by Lemma 3.6, there are only finitely many choices for μ_n. This contradiction proves that V is infinite. □

Since V is infinite and $\overline{V}$ is contained in the union of finitely many translates of the compact fundamental domain F_2, it follows that $\Pi_2(V)$ is infinite. But $\Pi_2(V) \subset \Psi(\mu, \Sigma, S) \subset Q_2$. Hence $\Psi(\mu, \Sigma, S)$ is infinite. □

4.2. Pattern rigidity: topological spheres. In this section we shall prove pattern rigidity for symmetric patterns of closed convex (or quasiconvex) sets in hyperbolic space $\mathbf{H}^{n+1} = \mathbf{H}$ such that the limit sets are topological spheres (of either codimension one or of even dimension). The techniques used are from fixed point theory. In the next two subsections, we shall generalize this, to quasiconvex subgroups of 3-manifolds with connected limit sets, to codimension one quasiconvex duality subgroups, and to closed limit sets whose stabilizers are PD$(2n + 1)$ quasiconvex subgroups. These will be generalizations of Theorem 4.3 (a) and Theorem 4.3 (b), respectively. The technicalities for these generalizations are postponed for ease of exposition.

Recall that a point that belongs to a unique translate of L_H is called a *unique point*.

Theorem 4.3. *Let $n \geq 2$. Let $\mathcal{J}_i^0$ (for $i = 1, 2$) be symmetric patterns of closed convex (or quasiconvex) sets in hyperbolic space $\mathbf{H}^{n+1} = \mathbf{H}$ such that the limit sets of $\mathcal{J}_i^0$ are either*

(a) *topological spheres of dimension $(n - 1)$,*

or

(b) *even-dimensional topological spheres.*

Then any proper bijection ϕ between $\mathcal{J}_1^0$ and $\mathcal{J}_2^0$ is induced by a hyperbolic isometry.

Proof. By Theorem 2.7, there is a quasi-isometry q^0 that pairs the convex (or quasi-convex) sets $\mathcal{J}_i^0$ as ϕ does.

Suppose that $h^0 = \partial q^0$ is not conformal. Then by the Generalized Eccentricity Lemma 3.5 there exist, for $i = 1, 2$, symmetric patterns of convex, or quasiconvex sets $\mathcal{J}_i$ (with limit sets Λ_1 abstractly homeomorphic to Λ_2) and a quasi-isometry $q : \mathbf{H} \to \mathbf{H}$ such that

(1) q pairs the elements of $\mathcal{J}_1$ with those of $\mathcal{J}_2$,
(2) $\mu = \partial q$ is an eccentric map,
(3) the geodesic $\gamma = \overline{0\infty}$ is a subset of some $J_i \in \mathcal{J}_i$ for $i = 1, 2$. Further, the (translates of the) limit sets Λ_1 and Λ_2 in which the end-points $0, \infty$ lie is unique.

Let δ_0 be as in Lemma 3.6. Pick points as per Lemma 2.5 to get a δ_0-net Σ in the interior of the fundamental domain F_1 of the action of H_1 on its domain of discontinuity Ω_1, consisting of unique points. Let S be an open neighborhood of 0. Then, by the Scattering Lemma 4.1, $\Psi(\mu, \Pi_1(\Sigma), S) = \Pi_2 \circ \mu(S \cap \Pi_1^{-1}(\Pi_1(\Sigma))) \subset Q_2$ is infinite.

However, since $\Sigma = \{x_1, \ldots, x_n\}$ is finite, and since its points belong to unique limit sets, ($x_i \in L_i$ say) there is an upper bound on the distance of $J(L_{H_1})$ from $J(L_i)$. Since q is a quasi-isometry, there is an upper bound on the distance of $J(L_{H_2})$ from $\phi(J(L_i))$. Hence, modulo the action of H_2, there are only finitely many choices for $\phi(J(L_i))$.

Since $\Psi(\mu, \Pi_1(\Sigma), S) = \Pi_2 \circ \mu(S \cap \Pi_1^{-1}(\Pi_1(\Sigma))) \subset Q_2$ is infinite, it follows that there exists (after subsequencing again) $T_{in} \in H_i$ for $i = 1, 2$ such that:

(1) If $\mu_n = T_{2n}^{-1} \circ \mu \circ T_{1n}$, then $\mu_n(L_{H_1}) = L_{H_2}$ and for some $L_i = L_1$ (say, without loss of generality) $\mu_n(L_1) = L_2$ is a fixed limit set. This follows from the fact that the x_i's are unique points. Also note that we can arrange that the visual diameters of L_1, L_2 are smaller than any pre-assigned ε_0.
(2) The attracting (resp. repelling) fixed point x_{1n} (resp. y_{1n}) is 0 (resp. ∞).
(3) The hyperbolic isometries corresponding to T_{1n} form an unbounded set in $\mathrm{PSL}_2(C)$.
(4) The maps μ_n restricted to L_1 are distinct since $\Psi(\mu, \Pi_1(\Sigma), S)$ is infinite. In particular, the μ_n's are distinct maps.
(5) $\mu_n \to \mu'$, where μ' is either a real linear map or a real linear map post-composed with a conformal map where the linear factor of the map μ' is not a similarity but continues to satisfy property (1).

Further, by Proposition 2.8, if we fix any *finite* collection of translates, L_{11}, …,

L_{1m}, of the limit set L_{H_1}, then the (ordered tuple) $\mu_n(L_{11}), \dots, \mu_n(L_{1m})$ is eventually constant. Hence for n, l sufficiently large, $\mu_l^{-1} \circ \mu_n$ maps L_{1j} to itself for $j = 1, \dots, m$.

The argument so far does not use any special topological property of the limit sets. We summarize our conclusions in the remark below.

Remark 4.4. We have shown that given an eccentric map pairing symmetric patterns $\mathcal{J}_i$ of convex (or quasiconvex) sets, there exists

(1) a sequence of eccentric maps $\mu_j \to \mu'$ uniformly on compact sets, where μ' is
a) either a linear map that is *not* a similarity, or
b) a real linear map post-composed with a conformal map where the linear factor of the map is not a similarity;
(2) the μ_j's pair $\mathcal{J}_1$ with $\mathcal{J}_2$;
(3) for any finite collection $\mathcal{L}$ of limit sets of elements of $\mathcal{J}_1$, there exists a positive integer N, such that $\mu_n(L) = \mu_l(L)$ for all $L \in \mathcal{L}$ and $n, l \geq N$;
(4) the μ_j's are *distinct* eccentric maps.

We now deal with the two cases of the theorem separately.

Case (a). Limit sets of $\mathcal{J}_i^0$ are topological spheres of dimension $(n-1)$.

Since each L_{1i} is a topological sphere of codimension one and J_{1i} is a convex set (for quasiconvex sets, we take the convex hull), the compactification of a small ε-neighborhood, $N_\varepsilon(J_{1i})$ obtained by adjoining L_{1i} is a strong deformation retract of the whole compactified ball $\mathbf{D} = \mathbf{H} \cup S^n_\infty$. In particular, if Ω is one of the two components of the domain of discontinuity of Stab(L_{1i}), then $\Omega \cup L_{1i} = D_{1i}$ is an AR by Theorem 2.11 and hence satisfies the fixed-point property (Remark 2.12).

For n, l sufficiently large, $\mu_l^{-1} \circ \mu_n$ maps D_{1i} to itself for $i = 1, \dots, m$. By Brouwer's Fixed Point Theorem (Remark 2.12), there exist $x_{1i} \in D_{1i}$ such that $\mu_n(x_{1i}) = \mu_l(x_{1i})$ for $i = 1 \dots m$. Now, by Remark 4.4 above, we can choose L_{1i} of sufficiently small diameter such that for any $x_{1i} \in D_{1i}$, the collection $\{x_{11}, \dots, x_{1m}\}$ is an ε_0-net in S^n, where ε_0 is as in Lemma 3.6. Hence, by Lemma 3.6, $\mu_n = \mu_l$. This contradicts condition (4) of Remark 4.4 above and proves case (a) of the theorem.

Case (b). Limit sets of $\mathcal{J}_i^0$ are topological spheres of even dimension.

By Lemma 2.10 and Remark 4.4 it follows that given any finite collection $\mathcal{L}$ of limit sets, there exists a positive integer N such that for all $n, l \geq N$ and all $L_i \in \mathcal{L}$,

(1) $\mu_l^{-1} \circ \mu_n(L_i) = L_i$,
(2) $\mu_l^{-1} \circ \mu_n$ restricted to L_i is homotopic to the identity; hence, the Lefschetz number of $\mu_l^{-1} \circ \mu_n$ restricted to L_i is equal to the Euler characteristic of L_i.

Since each $L_i \in \mathcal{L}$ is an even-dimensional sphere, the Euler characteristic of L_i is 2, in particular non-zero. By the Lefschetz Fixed Point Theorem there exists $x_i \in L_i$ such that $\mu_n(x_i) = \mu_l(x_i)$.

The rest of the proof is as in case (a) above. By Remark 4.4, we can choose L_i of sufficiently small diameter such that for any $x_i \in L_i$, the collection $\{x_1, \dots, x_m\}$ is an ε_0-net in S^n, where ε_0 is as in Lemma 3.6. Hence, by Lemma 3.6, $\mu_n = \mu_l$. This contradicts condition (4) of Remark 4.4 above and proves case (b) of the theorem. □

Remark 4.5. The proof of case (b) when specialized to dimension zero (i.e., S^0 limit sets) is exactly the one given by Schwartz in [Sch97]. To see this note that the existence of a fixed point of a map from S^0 to itself that is 'close to the identity' (hence equal to the identity) is clear.

4.3. 3-manifolds and codimension one duality subgroups

Remark 4.6. In our proof of Theorem 4.3, we have used the fact that the limit sets are spheres in a mild way. In case (a) we used them to construct invariant absolute retracts bounded by these spheres. After this, the proof of both Case (a) and Case (b) end up using the Lefschetz Fixed Point Theorem. We have used the following facts:

(1) the Euler characteristic of each invariant limit set L is non-zero;
(2) a map that moves each point of L through a small distance is homotopic to the identity;
(3) the Lefschetz Fixed Point Theorem holds for L.

We generalize Theorem 4.3 (a) now to quasiconvex subgroups of 3-manifolds with connected limit sets.

Theorem 4.7. *Let $n = 3$. Let $\mathcal{J}_i^0$ (for $i = 1, 2$) be symmetric patterns of closed convex (or quasiconvex) sets in hyperbolic space $\mathbf{H}^3 = \mathbf{H}$ such that the limit sets of $\mathcal{J}_i^0$ are connected. Then any proper bijection ϕ between $\mathcal{J}_1^0$ and $\mathcal{J}_2^0$ is induced by a hyperbolic isometry.*

Proof. Since limit sets are connected, we may assume by the Scott core theorem [Sco73] that each $J_{1i} \in \mathcal{J}_1^0$ is the (Gromov compactified) universal cover of a compact hyperbolic 3-manifold with incompressible boundary. In particular, its limit set L_{1i} shares a boundary circle C_{1i} with the unbounded component of its complement. Adjoining all the bounded components of $S^2_\infty \setminus L_{1i}$ to L_{1i} we obtain 2-disks D_{1i} invariant under $\mu_l^{-1} \circ \mu_n$ as in Theorem 4.3 (a). Again, by Brouwer's Fixed Point Theorem $\mu_l^{-1} \circ \mu_n$ has fixed points in D_{1i}. The rest of the proof is as in Theorem 4.3 (a). □

We next generalize Theorem 4.3 (a) to symmetric patterns of codimension one closed convex (or quasiconvex) sets with connected limit sets such that their stabilizers are duality groups. This is similar to Theorem 4.7 above.

Theorem 4.8. *Let $n \geq 3$. Suppose $\mathcal{J}_i^0$ (for $i = 1, 2$) are symmetric patterns of closed convex (or quasiconvex) sets in hyperbolic space $\mathbf{H}^n = \mathbf{H}$ such that the limit sets of*

$\mathcal{J}_i^0$ are connected of dimension $(n-2)$ and assume that the stabilizers of elements of $\mathcal{J}_i^0$ (freely indecomposable codimension one quasiconvex subgroups of G by the restriction on limit sets) are duality groups. Then any proper bijection ϕ between $\mathcal{J}_1^0$ and $\mathcal{J}_2^0$ is induced by a hyperbolic isometry.

Proof. Since limit sets $\mathcal{L}_i^0$ of $\mathcal{J}_i^0$ have codimension one, it follows that their stabilizers are codimension one in the big group G (the group acting on $\mathbf{H}$ cocompactly).

The argument in this paragraph is similar to an argument of Kapovich and Kleiner [KK05]. Let G_1 denote a stabilizer of (some) $L \in \mathcal{L}_1^0$. Since G_1 is a duality group, it follows that elements of $\mathcal{L}_i^0$ have the same homology as a wedge of $(n-2)$-spheres. By Alexander duality, each component of the domain of discontinuity (i.e., the complement of the limit set) $\Omega(G_1) = S_\infty^{n-1} \setminus \bigcup_{L \in \mathcal{L}_i^0} L$ is acyclic. Since G_1 is quasiconvex (and hence convex-cocompact), there are only finitely many G_1-orbits of such components and the stabilizers H_i, $i = 1, \ldots, k$, of such components act on them cocompactly. Therefore each H_i is a PD$(n-1)$-group.

Since each H_i is a PD$(n-1)$-group, the limit set of each H_i is an $(n-2)$ homology sphere S_i by Theorem 2.11. By Alexander duality again, S_i separates S_∞^n into two acyclic components (so the domain of discontinuity of H_i has two components). Adjoining either of these to S_i gives an absolute retract (AR).

Since the Lefschetz Fixed Point Theorem holds for AR's (Remark 2.12), the proof of Theorem 4.3 (a) goes through as before. □

4.4. Local homology and PD$(2k+1)$-subgroups. Bestvina [Bes96] shows that Gromov boundaries of Poincaré duality (PD(m)) hyperbolic groups are homology spheres (Theorem 2.11). Thus, if one knew some homology analogues of properties (2), (3) in Remark 4.6 above for such spaces, pattern rigidity would follow for subgroups that are PD$(2k+1)$.

We connect the work we have done so far in this paper to local homology properties of boundaries of hyperbolic groups and classical techniques in algebraic topology and fixed-point theory.

Homotopies and coarse topology. Lemma 2.10 goes through for topological manifolds and more generally, ANR's. But more important for us, it generalizes to the coarse category, where the coarse topology used is that of Schwartz [Sch95], Farb–Schwartz [FS96], as refined and generalized by Kapovich–Kleiner [KK05]. To see this, first recall the following consequence of a theorem of Bestvina–Mess [BM91].

Theorem 4.9. *For a* PD(n) *hyperbolic group acting properly and cocompactly on a proper finite dimensional simplicial complex X with metric inherited from the simplicial structure, there exists a compact exhaustion by compact sets B_n such that the natural inclusion map of $X \setminus B_{n+1}$ into $X \setminus B_n$ induce isomorphisms on homology.*

We shall also be using the following theorem which is a result that follows from work of Bestvina–Mess [BM91] and Bestvina [Bes96]. (See also Swenson [Swe99], Bowditch [Bow98] and Swarup [Swa96].)

Theorem 4.10. *Let G be a* PD(n) *hyperbolic group acting freely, properly, cocompactly on a contractible complex X. Then $H_n^{LF}(X) = H_{n-1}\partial G$.*

Remark 4.11. The isomorphism I (say) of Theorem 4.10 is functorial with respect to quasi-isometries, i.e., if f is a simplicial quasi-isometry of X to itself, and $q = \partial f$ is the induced map from ∂G to itself, then $q_* \circ I = I \circ f_*$.

Approximating quasi-isometries by Lipschitz and smooth maps. Let X be a convex contractible manifold (possibly with boundary) of pinched negative curvature equipped with a cocompact G-action (in particular, G is Gromov-hyperbolic). Triangulating X/G and lifting the triangulation to X, we have a proper finite dimensional, locally finite simplicial complex structure on X equipped with a proper simplicial cocompact G-action. Let f be a (K, ε)- quasi-isometry of X. Let f^0 be the restriction of f to the zero-skeleton of the triangulation. Let $v_0, \dots, v_k$ be vertices of a top dimensional simplex $\Delta \subset X$. Let d_i be the distance function from v_i. Let $\alpha_0, \dots, \alpha_k$ be the barycentric co-ordinates of a point $x \in \Delta \subset X$. We define (following Kleiner [Kle99]) $\hat{f}(x)$ to be the unique point in X minimizing $\sum \alpha_i d_i^2$. It follows from work of Kleiner [Kle99] that $\hat{f}$ is a Lipschitz self-map of X uniformly close to f. Hence $\hat{f}$ is also a quasi-isometry with quasi-isometry constants depending on (K, ε) and the pinching constants of X. Since X is itself smooth, we can further approximate $\hat{f}$ by a smooth Lipschitz self-map of X uniformly close to f. For the purposes of this paper, X will typically be a closed ε-neighborhood $\overline{N_\varepsilon(\mathrm{CH}(\Lambda))}$ of the convex hull of a limit set Λ of a convex cocompact G. Thus we can approximate each $\hat{f}_n$ arbitrarily closely by a smooth map homotopic to f_n. Since our concern in this paper will be with convex hulls of limit sets of quasiconvex subgroups in rank one symmetric spaces, we can therefore assume that each f_n is a smooth map. Note that approximating by Lipschitz maps is a considerably less delicate issue than approximating by bi-Lipschitz maps (cf. [Why99]).

Now consider a sequence f_n of uniform quasi-isometries of (the vertex set of the Cayley graph) of G acting freely and cocompactly by isometries on a convex contractible manifold X (possibly with boundary) of pinched negative curvature. By the above discussion, we may approximate f_n by smooth uniformly Lipschitz uniform self-quasi-isometries.

Then the coarse version of Lemma 2.10 is

Lemma 4.12. *Given* (a) *a proper finite dimensional, locally finite simplicial complex structure on a smooth convex manifold X* (*possibly with boundary*) *of pinched negative curvature with auxiliary simplicial metric inherited from the simplicial structure,* *equipped with a proper simplicial cocompact action by a Poincaré duality Gromov-hyperbolic group G, and* (b) *a sequence f_n of simplicial* ***uniform*** (K, ε) *quasi-isometries of X converging uniformly on compact sets to the identity, there exists a positive integer N such that for all $n \geq N$ and all $k \geq 0$, f_n induces the identity on the locally finite homology $H_k^{LF}(X)$.*

Proof. By the discussion preceding this lemma, we can assume, without loss of generality, that each f_n can be approximated by smooth maps. Further, if f_n converges uniformly on compact sets to the identity, it follows (from the barycentric simplex construction of Kleiner [Kle99] outlined above) that so do the smooth approximants. Hence without loss of generality we may assume that f_n's are smooth uniform self quasi-isometries of X.

Also, since the f_n's are *uniform* (K, ε) quasi-isometries, there exist $A_1 \geq A_2 \geq 10$ (say) and a positive integer N such that for $n \geq N$ large enough, no point outside a ball of radius A_1 is mapped inside a ball of radius A_2 under f_n. Further (taking N larger if necessary), we may assume by Lemma 2.10 (since each f_n is sufficiently close to the identity map on the ball B_{10} of radius 10 about a fixed origin 0) that f_n restricted to B_{10} is homotopic to the identity with small tracks. By using a homotopy on a slightly smaller ball (of radius 9, say) we may assume that each f_n *is* the identity on B_9, and using straightening homotopies, we may also assume that no point of the complement B_9^c gets mapped to B_9 under f_n. But then the degree of the map induced by f_n on locally finite homology $H_k^{LF}(X)$ is the same as the degree of the map $f_* \colon H_k(X, X \setminus A_1) \to H_k(X, X \setminus A_2)$, by Theorem 4.9. This is the same as the local degree of f_n (see [Hat02] for the local degree formula) at 0, which in turn is 1. Note that if G is a hyperbolic Poincaré duality group of dimension m, $H_k(X, X \setminus A_1)$ vanishes for $k \neq m$ and A_1 sufficiently large. This proves the lemma. □

Now if $q_n = \partial f_n$ is a sequence of boundary values of *uniform* quasi-isometries f_n, such that q_n converges uniformly to the identity map, then we may assume that there is a point 0 such that each f_n moves 0 through a uniformly bounded amount. By composing with bounded track homotopies if necessary, we may homotope f_n to maps which satisfy the hypotheses of Lemma 4.12.

Corollary 4.13. *Let $L = \partial X$ be the boundary of a* PD(n) *hyperbolic group.*

(1) *Then L is a compact homology manifold with the singular* (*co*)*homology groups of a sphere S^d.*

(2) *Let q_i be a uniformly Cauchy sequence of homeomorphisms of $L = \partial X$* (*i.e., for all $\varepsilon > 0$ there exists N such that for all $x \in L$, $d(q_m(x), q_k(x)) < \varepsilon$ for all $m, k \geq N$*) *induced by* (*uniform*) *K, ε quasi-isometries f_i of X such that* (*for a fixed base-point o*) *$f_i(o)$ lies in a uniformly bounded neighborhood of o. Then there exists a positive integer N such that q_i and q_j induce the same isomorphism on homology groups of L for all $i, j \geq N$.*

Proof. Assertion (1) of the corollary follows from Theorem 2.11. We provide some details here. One of the main results of [BM91] asserts that the (reduced) Čech cohomology groups of L vanish except in dimension $(n-1)$. Bestvina [Bes96] also shows that the (reduced) Steenrod homology groups of L vanish except in dimension $(n-1)$. Since L is compact metrizable, Steenrod homology coincides with Čech homology (see for instance [Mil95]). Further, for locally connected metrizable compacta such as L, the Čech (co)-homology groups coincide with singular (co)-homology groups (see p. 107 of [Lef42b]). Hence both the singular (as well as Čech) homology and cohomology of L coincide with that of a sphere of dimension $(n-1)$.

Using the functoriality of Remark 4.11, the second assertion of the corollary now follows from the first assertion and Lemma 4.12. □

Scheme. Our strategy to extend the techniques of Theorem 4.3 (b) beyond spheres to Poincaré duality PD$(2n+1)$ groups (to ensure even dimensional boundary) is as follows:

1) Recall a consequence of an old theorem of Lefschetz [Lef34], [Lef37], [Lef42a], p. 324 (for what Lefschetz calls quasicomplexes that partly generalize ANR's) generalized by Thompson [Tho67] (to weak semicomplexes that embrace quasicomplexes, ANR's and homology manifolds in the sense of Wilder [Wil79]) and also Knill [Kni72], Corollary 4.3 (in the general context of what Knill calls Q-simplicial complexes) that in modern terminology says that the Lefschetz Fixed Point Theorem holds for generalized co(homological) manifolds (in the sense of Wilder [Wil79]).

2) Use Theorem 2.11 (or Corollary 4.13, assertion 1) due to Bestvina that the boundary of a hyperbolic PD(m)-group over the integers is a homological manifold (in fact a homology sphere) with locally connected boundary.

3) Finally use Corollary 4.13, assertion 2, to conclude that the homeomorphisms of the homological manifolds we have, moving points through very small distances, induce the identity map on homology.

We shall need the following result.

Theorem 4.14 (Lefschetz [Lef34], [Lef34], [Lef42a], p. 324, Thompson [Tho67], and Corollary 4.3 of Knill [Kni72]). *If Y is a compact locally connected generalized homology manifold then for any continuous map $f : Y \to Y$, if the Lefschetz number $A(f) \neq 0$, then $f(y) = y$ for some $y \in Y$.*

Combining Theorem 4.14 with assertion (1) of Corollary 4.13, we get

Corollary 4.15. *If Y is the boundary of a* PD(m) *Gromov-hyperbolic group, then for any continuous map $f : Y \to Y$ if the Lefschetz number $A(f) \neq 0$, then $f(y) = y$ for some $y \in Y$.*

Corollary 4.15 and Corollary 4.13 combine to give the following.

Proposition 4.16. *Let $L = \partial X$ be the boundary of a* PD(n) *hyperbolic group. Let q_i be a uniformly Cauchy sequence of homeomorphisms of $L = \partial X$ induced by (uniform) K, ε quasi-isometries f_i of X such that (for a fixed base-point o) $f_i(o)$ lies in a uniformly bounded neighborhood of o. Then there exists a positive integer N such that $q_i^{-1} \circ q_j : L \to L$ for all $i, j \geq N$ and also $q_i : L \to L$ has a fixed point.*

Proof. The second assertion of Corollary 4.13 shows that the there exists a positive integer N such that for all $i \geq N$, the maps q_i induce the identity map on the singular homology group.

Using the pairing between top dimensional singular homology and cohomology, the Lefschetz number of q_i or $q_i^{-1} \circ q_j$ (for all $i, j \geq N$) computed via singular cohomology (or equivalently, via Čech cohomology) is also the Euler characteristic. Corollary 4.15 now furnishes the desired conclusion. □

The conclusion of Theorem 4.3 (b) for (symmetric patterns of convex hulls of limit sets of) PD$(2k+1)$ subgroups now follows exactly along the lines of Theorem 4.3 (b):

Theorem 4.17. *Let $n \geq 3$. Suppose that $\mathcal{J}_i^0$ (for $i = 1, 2$) are symmetric patterns of closed convex (or quasiconvex) sets in hyperbolic space $\mathbf{H}^n = \mathbf{H}$ such that the stabilizers of limit sets of $\mathcal{J}_i^0$ are* PD$(2k+1)$ *quasiconvex subgroups of G. Then any proper bijection ϕ between $\mathcal{J}_1^0$ and $\mathcal{J}_2^0$ is induced by a hyperbolic isometry.*

5. Consequences and questions

5.1. Rank one symmetric spaces. As explained by Schwartz in Section 8 (specifically Lemma 8.1) of [Sch95], Lemmas 3.1, 3.5 and 3.6 generalize to complex hyperbolic space. So do Lemmas 3.3 and 4.1 (which are generalizations of lemmas of Schwartz [Sch97]). Thus Theorem 1.4 generalizes to the following.

Theorem 5.1. *Suppose that $\mathcal{J}_i$ (for $i = 1, 2$) are symmetric patterns of closed convex (or quasiconvex) sets in complex hyperbolic space $\mathbf{H}$ of (real) dimension n, $n > 2$. For $i = 1, 2$, let G_i be the corresponding uniform lattices. Let $H_i \subset G_i$ be infinite index quasiconvex subgroups stabilizing the limit set of some element of $\mathcal{J}_i$ and satisfying one of the following conditions:*

(1) *H_i is a codimension one* ***duality group****.*

(2) *H_i is an odd-dimensional Poincaré duality group* PD$(2k+1)$ *with $2k+1 \leq n-1$.*

Then any proper bijection between $\mathcal{J}_1$ and $\mathcal{J}_2$ is induced by a hyperbolic isometry.

Remark 5.2. For other rank one symmetric spaces (quaternionic and Cayley hyperbolic spaces), any quasi-isometry is a bounded distance from an isometry by work of Pansu [Pan89]. Hence Theorem 5.1 goes through for all rank one symmetric spaces.

5.2. Special disconnected limit sets and intersections. All of what we have done so far goes through with minor modifications for disconnected limit sets, at least one of whose components has a stabilizer H of the form (1) or (2) in Theorem 5.1 above. To see this, let us retrace the argument in Theorems 4.3. There we showed that for large enough m, n, $\mu_n^{-1} \circ \mu_m$ preserves limit sets that are spheres. The same argument shows that for large enough m, n, $\mu_n^{-1} \circ \mu_m$ preserves components of limit sets of diameter bigger than (some fixed) ε. Since the limit set of H has components whose stabilizers are of the form (1) or (2), the arguments for Theorems 4.8 and 4.17 go through to prove the existence of fixed points for $\mu_n^{-1} \circ \mu_m$. This is enough to show the following.

Corollary 5.3. *Let $n \geq 3$. Suppose $\mathcal{J}_i$ (for $i = 1, 2$) are symmetric patterns of closed convex (or quasiconvex) sets in hyperbolic space $\mathbf{H}^n = \mathbf{H}$ or more generally a rank one symmetric space $\mathbf{H}$ of (real) dimension n. For $i = 1, 2$, let G_i be the corresponding uniform lattices in $\mathbf{H}$. Let $H_i \subset G_i$ be an infinite index quasiconvex subgroup stabilizing the (possibly disconnected) limit set of some element of $\mathcal{J}_i$ and satisfying the condition that the limit set of H_i has components whose stabilizers H_i' (obviously containing H_i) are of one of the following forms:*

(1) *H_i' is a codimension one **duality group**.*
(2) *H_i' is an odd-dimensional Poincaré duality group* PD$(2k+1)$ *with $2k+1 \leq n-1$.*

Then any proper bijection between $\mathcal{J}_1$ and $\mathcal{J}_2$ is induced by a hyperbolic isometry.

We next state a generalization of Theorem 1.4 when the intersection of some finitely many conjugates of $H_i \subset G_i$ is of the form (1) or (2) above.

Corollary 5.4. *Let $n \geq 3$. Suppose $\mathcal{J}_i$ (for $i = 1, 2$) are symmetric patterns of closed convex (or quasiconvex) sets in hyperbolic space $\mathbf{H}^n = \mathbf{H}$ or more generally a rank one symmetric space $\mathbf{H}$ of (real) dimension n. For $i = 1, 2$, let G_i be the corresponding uniform lattices in $\mathbf{H}$. Let $H_i \subset G_i$ be an infinite index quasiconvex subgroup and $g_1, \dots g_m \in G$ be finitely many elements such that $H_i' = \bigcap_{j=1,\dots,m} g_j H_i g_j^{-1}$ is of one of the following forms:*

(1) *H_i' is a codimension one **duality group**.*
(2) *H_i' is an odd-dimensional Poincaré duality group* PD$(2k+1)$ *with $2k+1 \leq n-1$.*

Then any proper bijection between $\mathcal{J}_1$ and $\mathcal{J}_2$ is induced by a hyperbolic isometry.

Sketch of proof. The condition $H_i' = \bigcap_{j=1\dots m} g_j H_i g_j^{-1}$ implies (by theorems of Short [Sho91] and Gitik–Mitra–Rips–Sageev [GMRS98]) that H_i' is quasiconvex and that $\Lambda_i' = \bigcap_{j=1\dots m} g_j \Lambda_i$, where Λ_i' (resp. Λ_i) represents the limit sets of H_i' (resp. H_i). Since the maps $\mu_n^{-1} \circ \mu_m$ preserve limit sets and hence their intersections it follows that the collection of translates of joins of limit sets of H_i' is a symmetric pattern preserved by $\mu_n^{-1} \circ \mu_m$. The rest of the argument proving pattern rigidity is as in Theorem 4.3. □

5.3. Quasi-isometric rigidity. Let $\mathcal{G}$ be a graph of groups with Bass–Serre tree of spaces $X \to T$. Let $G = \pi_1 \mathcal{G}$.

(We refer the reader to [MSW04] specifically for the following notions: depth zero raft, crossing graph condition, coarse finite type and coarse dimension, finite depth.)

Combining Theorems 1.5, 1.6 of [MSW04] with the Pattern Rigidity Theorem 1.4 we have the following QI-rigidity theorem along the lines of Theorem 7.1 of [MSW04].

Theorem 5.5. *Let $\mathcal{G}$ be a finite, irreducible graph of groups such that for the associated Bass–Serre tree T of spaces no depth zero raft of T is a line. Further suppose that the vertex groups are fundamental groups of compact hyperbolic n-manifolds, or, more generally, uniform lattices in rank one symmetric spaces of dimension n, $n \geq 3$, and edge groups are all of exactly one the following types:*

(a) *A **duality group** of codimension one in the adjacent vertex groups. In this case we require in addition that the crossing graph condition of Theorems* 1.5, 1.6 *of* [MSW04] *be satisfied and that $\mathcal{G}$ is of finite depth.*
(b) *An odd-dimensional Poincaré duality group* PD$(2k+1)$ *with* $2k+1 \leq n-1$.

If H is a finitely generated group quasi-isometric to $G = \pi_1(\mathcal{G})$ then H splits as a graph $\mathcal{G}'$ of groups whose depth zero vertex groups are commensurable to those of $\mathcal{G}$ and whose edge groups and positive depth vertex groups are respectively quasi-isometric to groups of type (a), (b).

Proof. By the restrictions on the vertex and edge groups, it automatically follows that all vertex and edge groups are PD groups of coarse finite type. In case (b), $\mathcal{G}$ is automatically finite depth, because an infinite index subgroup of a PD(n) groups has coarse dimension at most $n-1$. Also the crossing graph is empty in this case hence the crossing graph condition of Theorems 1.5 and 1.6 of [MSW04] is automatically satisfied.

Then by Theorems 1.5 and 1.6 of [MSW04], H splits as a graph of groups $\mathcal{G}'$ with depth zero vertex spaces quasi-isometric to $\mathbf{H} = \mathbf{H}^n$ and edge groups quasi-isometric to the edge groups of $\mathcal{G}$ and hence respectively type (a), (b). Further, the quasi-isometry respects the vertex and edge spaces of this splitting, and thus the quasi-actions of the vertex groups on the vertex spaces of $\mathcal{G}$ preserve the patterns of edge spaces.

By Corollary 1.5 the depth zero vertex groups in $\mathcal{G}'$ are commensurable to the corresponding groups in $\mathcal{G}$. □

Using Theorem 5.1 or Corollary 5.3, we could get the corresponding generalizations to quasiconvex subgroups covered by these theorems.

5.4. Questions. Note that our proof of Lemma 4.13 does not answer the following.

Question 5.6. Let G be a PD(m) hyperbolic group. Let ∂G be its (Gromov) boundary equipped with a visual metric d. Does there exist $\varepsilon > 0$ such that if f is a homeomorphism of ∂G satisfying $d(x, f(x)) < \varepsilon$ for all $x \in \partial G$, then f induces the identity map on homology?

Kapovich [Kap07a] constructs convex projective representations of the Gromov–Thurston examples; it is conceivable that these may be realized as convex cocompact subgroups of uniform hyperbolic lattices. But there is a dearth of examples of higher dimensional Kleinian groups in general (see [Kap07b] for a survey). In particular, there is a dearth of examples in higher dimensional Kleinian groups to which Theorem 5.1 applies.

There are non-ANR examples of hyperbolic Coxeter group boundaries coming from work of Davis [Dav83]. These boundaries are not locally simply connected. Doubling some of these examples (in dimension ≥ 5) along their boundaries gives the standard topological sphere S^n. Thus exotic (non-ENR) homology spheres might conceivably arise as limit sets. The following seems interesting in its own right.

Question 5.7. Does there exist a convex cocompact (i.e., geometrically finite) PD(n) hyperbolic group G with non-ENR Gromov boundary acting on $\mathbf{H} = \mathbf{H}^{n+1}$? Can such a G appear as a codimension one quasiconvex subgroup of a uniform lattice in $\mathbf{H}$?

Fischer [Fis03] has further investigated these examples.

Observation 5.8. Note that Theorem 1.4 combined with Corollary 5.3 implies that pattern rigidity holds for all quasiconvex subgroups of hyperbolic 3-manifolds that are not virtually free. Hence a test-case not covered by the work in this paper is that of symmetric patterns in hyperbolic 3-manifolds corresponding to free quasiconvex subgroups. This is the subject of work in progress.

Another test case is the case of symmetric patterns of quasiconvex surface subgroups in hyperbolic 4-manifolds, or, at the level of limit sets, copies of S^1 in S^3.

Remark 5.9. Much of what has been done in the context of Poincaré duality groups might as well have been done in group-free language in the context of coarse Poincaré duality spaces. (See Kapovich–Kleiner [KK05].)

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