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Groups, Geometry, and Dynamics

The fully residually F quotients of $F * \langle x, y \rangle$

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Abstract. We describe the fully residually F groups, or limit groups relative to F, that are quotients of $F * \langle x, y \rangle$. We use the structure theory of finitely generated fully residually free groups to produce a finite list of possible types of cyclic JSJ decompositions modulo F that can arise. We also give bounds on uniform hierarchical depth.

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Contents

1. Introduction

Systems of equations over free groups have been a very important and fruitful subject of study in the field of combinatorial and geometric group theory. A major achievement was the algorithm due to Makanin and Razborov [Mak82], [Raz87] which produces a complete description of the solution set of an arbitrary finite system of equations over a free group. The method, however, is algorithmic and uses surprisingly little algebra.

For a free group F, the classical algebraic geometry viewpoint given by Baumslag, Myasnikov and Remeslennikov in [BMR99] established fully residually F groups as

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the key algebraic structures in the theory of systems of equations with coefficients over F.

These groups still remained rather intractable until the work of Kharlampovich and Myasnikov [KM98a], [KM98b], [KM05], and independently Sela [Sel01], which shows that finitely generated fully residually free, or limit, groups in fact have a very nice structure. We apply this structure theory to prove the following result:

Theorem 1.1 (The Main Theorem). *Let $F_{R(S)}$, a fully residually F quotient of $F * \langle x, y\rangle$, be freely indecomposable modulo F. The underlying graph X of its cyclic JSJ decomposition modulo F has at most 3 vertices, and X has at most two cycles. All vertex groups except maybe the vertex group $F \leq \widetilde{F} \leq F_{R(S)}$ are either free of rank 2 or free abelian. In all cases the vertex group $\widetilde{F}$ itself can be generated by F and two other elements. Finally $F_{R(S)}$ has uniform hierarchical depth relative to F at most 4.*

This result can be seen as a generalization of Chiswell and Remeslennikov's classification in [CR00] of the fully residually F quotients of $F * \langle x\rangle$, which enabled them to finally give a proof of a result on the solution sets of equations in one variable over F claimed independently by Appel and Lorenc [App68], [Lor68]. The fully residually free groups generated by at most three elements were classified in [FGM+98]. We will give examples which show that the class of fully residually F quotients of $F * \langle x, y\rangle$ is considerably richer.

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1.1. Definitions and notation. We will denote the commutator $x^{-1}y^{-1}xy = [x, y]$. For conjugation we will use the following notation:

$$x^w = w^{-1}xw, \quad {}^w x = wxw^{-1}.$$

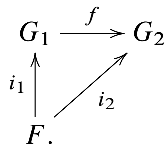
We use this convention since ${}^x({}^y w) = {}^{xy}w$. We shall also denote by $\mathcal{G}(X)$ a graph of groups with underlying graph X. We shall denote by $\operatorname{rank}(G)$ the minimal cardinality among all generating sets of G.

1.1.1. Fully residually *F* groups. Throughout this paper F will denote a fixed free group of rank N.

Definition 1.2. A group G equipped with a distinguished monomorphism

$$i : F \hookrightarrow G$$

is called an *F-group* we denote this (G, i). Given F-groups (G_1, i_1) and (G_2, i_2), we define an F-homomorphism to be a homomorphism of groups f such that the following diagram commutes:



We denote by $\mathrm{Hom}_F(G_1, G_2)$ the set of F-homomorphisms from (G_1, i_1) to (G_2, i_2).

In the rest of the paper the distinguished monomorphisms will not be explicitly mentioned, seeing as the inclusions will always be obvious.

Definition 1.3. Let G and H be groups. A collection of homomorphism Φ from G to H *discriminates* G if for every finite subset $P \subset G$ there is some $f \in \Phi$ such that the restriction $f|_P$ is injective.

Definition 1.4. A group G is *fully residually* F if $\mathrm{Hom}_F(G, F)$ discriminates G.

This definition is slightly non-standard in that we are *requiring* fully residually F groups to be discriminated by F*-homomorphisms*. More generally we shall say that a group G is *fully residually free* to mean that it is discriminated by some arbitrary set of homomorphisms into some free group.

For the rest of the paper $F_{R(S)}$ shall denote a fully residually F quotient of $F * \langle x, y \rangle$. The notation reflects the fact that $F_{R(S)}$ is the coordinate group of system of equations $S(x, y)$ over F. Since $F_{R(S)}$ is fully residually F it is in fact the coordinate group of an irreducible system of equations with coefficients in F and variables x, y. We refer the reader to [BMR99] for a complete treatment of the interpretation of fully residually F groups in algebraic geometry.

The following facts will be used throughout this paper and follow easily from the definitions:

Theorem 1.5. *Let G be fully residually free.*

- *G is torsion free.*
- *Two elements g, h either commute or generate a free subgroup of rank* 2.
- *G is a CSA group, i.e., maximal abelian subgroups groups are malnormal* (*which implies commutation transitivity*).

Convention 1.6. The elements $\bar{x}, \bar{y} \in F_{R(S)}$ denote the images of x, y respectively via the epimorphism $F * \langle x, y \rangle \to F_{R(S)}$.

1.1.2. Generalized JSJ decompositions and uniform hierarchical depth. We assume some familiarity with Bass–Serre theory and cyclic JSJ theory for groups (as introduced in [RS97].) If $F \neq F_{R(S)}$ then it follows, for example from Corollary 5.16, that it has an essential cyclic or free splitting modulo F. If $F_{R(S)} \neq F$ is freely indecomposable modulo F then it has a non-trivial cyclic JSJ decomposition modulo F. As a matter of terminology, *MQH stands for "maximal quadratically hanging". QH* (*i.e., "quadratically hanging"*) *subgroups are the surface-type subgroups that arise from hyperbolic-hyperbolic pairs of splittings.*

Convention 1.7. Unless stated otherwise, instead of saying the *cyclic JSJ decomposition of $F_{R(S)}$ modulo F*, we will simply say the *JSJ of $F_{R(S)}$*.

Definition 1.8. Let $F_{R(S)} = H_0 * H_1 * \cdots * H_m * F(Y)$ be a Grushko decomposition of $F_{R(S)}$ modulo F, with $F \leq H_0$. A *generalized JSJ* of $F_{R(S)}$ is a graph of groups decomposition $F_{R(S)} = \pi_1(\mathcal{G}(X))$ modulo F, where $\mathcal{G}(X)$ is a graph of groups, such that:

- Edge groups are either trivial or cyclic.
- If $e_1, \ldots, e_k$ are the edges of X with trivial edge group, and $X_0, X_1, \ldots, X_{m+1}$ are the connected components of $X \setminus \left(e_1 \cup \cdots \cup e_k\right)$ then, up to reordering, each graph of groups $\mathcal{G}(X_i)$ is the JSJ of H_i for $i = 1, \ldots, m$, $\pi_1(\mathcal{G}(X_{m+1})) = F(Y)$, and $\mathcal{G}(X_0)$ is the JSJ of $\widetilde{F}$ modulo F

Convention 1.9. The JSJ of $F_{R(S)}$ will always have a vertex group containing F, we shall always denote this vertex group by $\widetilde{F}$.

Further details on JSJ theory are deferred to Section 5.1.1. Uniform hierarchical depth is a very natural notion, however the definition is complicated by the fact that subgroups of fully residually F groups need not themselves be fully residually F.

Definition 1.10. For a finitely generated fully residually free group G we define its *uniform hierarchical depth*, denoted $\mathrm{uhd}(G)$, as follows:

- If G is trivial, abelian, free or a surface group, then $\mathrm{uhd}(G) = 0$.
- Otherwise let $G_1, \ldots, G_n$ be the vertex groups of the generalized JSJ of G. We set
$$\mathrm{uhd}(G) = \max(\mathrm{uhd}(G_1), \ldots, \mathrm{uhd}(G_n)) + 1.$$

If G is a finitely generated fully residually F group then the vertex groups of its generalized JSJ are not necessarily F-groups. We define its *uniform hierarchical depth relative to F*, denoted $\mathrm{uhd}_F(G)$, as follows:

- If $G = F$ then $\mathrm{uhd}_F = 0$.
- Otherwise let $F \leq \widetilde{F}, H_1, \ldots, H_n$ be the vertex groups of G's generalized JSJ modulo F, then
$$\mathrm{uhd}_F(G) = \max(\mathrm{uhd}_F(\widetilde{F}), \mathrm{uhd}(H_1), \ldots, \mathrm{uhd}(H_n)) + 1.$$

For a finitely generated fully residually free group G, $\mathrm{uhd}(G)$ is essentially the number of levels of the *canonical analysis lattice* defined in §4 of [Sel01]. The main difference is that a free product of uhd 0 groups has uhd 1, whereas the analysis lattice of such a group has only a 0-level. It follows that $\mathrm{uhd}(G)$ is at least l, the number of levels of an analysis lattice, and at most $l+1$. By Theorem 4.1 of [Sel01] $\mathrm{uhd}(G) < \infty$ for a finitely generated fully residually free group. $\mathrm{uhd}_F(F_{R(S)}) < \infty$ is an easy consequence of Theorem 4 of [KM98b] (it is restated in this paper as Theorem 5.15.)

1.1.3. Relative presentations. Let $G_1, \ldots, G_n$ be groups with presentations $\langle X_1 \mid R_1\rangle, \ldots, \langle X_n \mid R_n\rangle$ respectively and $t_1, \ldots, t_k$ a set of letters. Let R denote a set of words in $\bigcup X_i^{\pm 1} \cup \{t_1, \ldots, t_k\}^{\pm 1}$ then we will define the *relative presentation*

$$\langle G_1, \ldots, G_n, t_1, \ldots t_k \mid R\rangle$$

to be the group defined by the presentation

$$\langle X_1, \ldots, X_n, t_1, \ldots, t_k \mid R_1, \ldots R_n, R\rangle.$$

We assume that the reader is familiar with the relative presentation that can be given to the fundamental group of a graph of groups $\mathcal{G}(X)$ which depends on some maximal spanning tree $T \subset X$ (see §5 of [Ser03] for details).

Convention 1.11. The "generators" of the relative presentations will be the vertex groups of $\mathcal{G}(X)$ and *stable letters* corresponding to edges of $X \setminus T$, where $T \subset X$ is a maximal spanning tree. Vertex groups will always be written using capital letters, and stable letters will always be denoted in lower case.

The "relations" will always involve stable letters. Moreover we will abuse notation and, for example, abbreviate the HNN extension associated to the isomorphism $\psi\colon A_1 \to A_2$, $A_1 \le G \ge A_2$, simply as $\langle G, t \mid A_1^t = A_2\rangle$; $A_1, A_2 \le G$.

For each edge $e \in T$ we will consider the images of the corresponding edge group in the vertex groups to be identified. This means that the corresponding edge group will be given as the intersection of two vertex groups.

2. The classification of the fully residually F quotients of $F * \langle x, y\rangle$

2.1. Auxiliary results. So far the only comprehensive classification theorems of fully residually free groups in terms of the number of generators are the following, recall that the *rank* of a group is the minimum cardinality of its generating sets.

Theorem 2.1 ([FGM+98]). *If G is fully residually free group, then:*

- *If* $\mathrm{rank}(G) = 1$ *then G is infinite cyclic.*
- *If* $\mathrm{rank}(G) = 2$ *then G is free or free abelian of rank* 2*.*

- *If* $\operatorname{rank}(G) = 3$ *then* G *is either free, free abelian of rank* 3*, or* G *is isomorphic to a rank* 1 *centralizer extension of a free group of rank* 2*.*

Let $\mathrm{Ab}(x, y)$ denote the free abelian group with basis $\{x, y\}$.

Theorem 2.2 ([CR00]). *The fully residually* F *quotients of* $F * \langle x \rangle$ *are*

$$F_{R(S)} = \begin{cases} F, \\ F * \langle x \rangle, \\ F *_{\langle \alpha \rangle} \mathrm{Ab}(\alpha, r). \end{cases}$$

Our proofs build on these previous classifications. This next result, which is proved in Section 5.1.4, is interesting in its own right is also important for proving some of the corollaries of our classification.

Proposition 2.3. *Let* F *be a free group of rank* N *and let* G *be a fully residually* F *group. Then* $b_1(G) = N$ *if and only if* $G = F$*.*

If G is the fundamental group of a graph of groups $\mathcal{G}(X)$ with cyclic edge groups, then it is easy to estimate $b_1(G)$, the first Betti number of G, in terms of the first Betti numbers of its vertex groups. We have the following lower bound: for $T \subset X$ a maximal spanning tree we have

$$b_1(G) \geq \sum_{v \in T} b_1(G_v) - E, \tag{1}$$

where E is the number of edges in T. If there is an epimorphism $G \to H$ then $b_1(G) \geq b_1(H)$. This fact is used to derive many of the corollaries regarding first Betti numbers.

2.2. When $F_{R(S)}$ is freely decomposable modulo F.

This proposition is proved in Section 4.

Proposition 2.4. *If* $F_{R(S)}$ *is freely decomposable modulo* F *then*

$$F_{R(S)} = \begin{cases} F * \langle t \rangle, \\ F * H \text{ where } H \text{ is fully residually free of rank } 2, \\ (F *_{\langle \alpha \rangle} \mathrm{Ab}(\alpha, r)) * \langle s \rangle. \end{cases}$$

Corollary 2.5. *If* $F_{R(S)}$ *is freely decomposable modulo* F*, then* $\mathrm{uhd}_F(F_{R(S)}) \leq 1$.

2.3. When all the vertex groups of the JSJ of $F_{R(S)}$ except $\widetilde{F}$ are abelian. The proof of this proposition takes up Section 5 and Section 7.2.

Proposition 2.6. *If the JSJ of $F_{R(S)}$ has abelian vertex groups, but only the non-abelian vertex group $F \leq \widetilde{F}$, then the possible underlying graphs of the JSJ are:*

$$u\bullet \text{———} \bullet v\ ,\quad v\bullet \text{———} \bullet u \text{———} \bullet w\ ,\quad \text{or}\quad \circlearrowleft\bullet u \text{———} \bullet v\,.$$

Moreover if there are two abelian vertex groups, or one of the abelian vertex groups has rank 3 then $\widetilde{F} = F$. In all cases $\widetilde{F}$ is generated by F and at most two other elements and $b_1(\widetilde{F}) < b_1(F_{R(S)})$.

This corollary follows from Proposition 2.3 and an easy estimation of the first Betti number.

Corollary 2.7. *If $F_{R(S)}$ is as in Proposition 2.6 and $b_1(F_{R(S)}) = N + 1$ then $F_{R(S)} = F *_{\langle\alpha\rangle} \mathrm{Ab}(\alpha, r)$. In particular* $\mathrm{uhd}_F(F_{R(S)}) = 1$.

The next proposition is proved in Section 7.5.

Proposition 2.8. *If the JSJ of $F_{R(S)}$ is as in Proposition 2.6, then* $\mathrm{uhd}_F(F_{R(S)}) \leq 3$.

2.4. When the JSJ of $F_{R(S)}$ has at least two non-abelian vertex groups. This situation is covered in Section 6. We now give a list of the possible JSJs for $F_{R(S)}$. Throughout this section let $\widetilde{F} = F *_{\langle u\rangle} \mathrm{Ab}(u, r)$ be a rank 1 centralizer extension of F (see Definition 5.14) and let H be a free group of rank 2.

Definition 2.9. A collection of elements $\alpha_1, \ldots, \alpha_n \in H$ are *almost conjugate in H* if there exists a cyclic subgroup $\langle\gamma\rangle \leq H$ and elements $g_1, \ldots, g_n \in H$ such that $\langle g_i^{-1}\alpha_i g_i\rangle \leq \langle\gamma\rangle$ for $i = 1, \ldots, n$.

2.4.1. When all the vertex groups are free non-abelian

A. If the underlying graph of the JSJ has only *one edge* the possibilities are:

(1) $F_{R(S)} = F *_{\langle\alpha\rangle} H$.

(2) $F_{R(S)} = F *_{\langle\alpha\rangle} Q$ with Q a QH subgroup so that in fact

$$F_{R(S)} = \langle F, s, t \mid [s,t] = \alpha\rangle, \quad \alpha \in F.$$

B. If the underlying graph of the JSJ has *two edges* the possibilities are:

(1)

$$F_{R(S)} = \begin{cases} \langle F, H, t \mid \beta^t = \beta'\rangle, \\ \beta, \beta' \in H,\ \langle\alpha\rangle = \widetilde{F} \cap H, \end{cases}$$

where α is not almost conjugate to β or β' in H. The subgroup $\langle H, t\rangle$ is also free of rank 2.

(2)

$$F_{R(S)} = \begin{cases} \langle F, H, t \mid \beta^t = \gamma \rangle, \\ \beta \in F,\ \gamma \in H,\ \langle \alpha \rangle = F \cap H. \end{cases}$$

C. If the underlying graph of the JSJ has *three edges* the possibilities are:

(1)

$$F_{R(S)} = \begin{cases} \langle F, H, t, s \mid \beta^t = \gamma, \delta^s = \delta' \rangle, \\ \beta \in F,\ \gamma, \delta, \delta' \in H,\ \langle \alpha \rangle = F \cap H, \end{cases}$$

where $\langle H, s \rangle$ is also free of rank 2, moreover α and γ are not almost conjugate in H.

(2)

$$F_{R(S)} = \begin{cases} \langle F, H, t, s \mid \beta^t = \delta, \gamma^s = \epsilon \rangle, \\ \beta, \gamma \in F,\ \delta, \epsilon \in H,\ \langle \alpha \rangle = F \cap H, \end{cases}$$

where H is generated by α, δ, ϵ.

2.4.2. When there is an abelian vertex group

A. If the underlying graph of the JSJ has *two edges* the only possibility is

$$F_{R(S)} = F *_{\langle \alpha \rangle} H *_{\langle \beta \rangle} \mathrm{Ab}(\beta, r).$$

B. If the underlying graph of the JSJ has *three edges* the possibilities are:

(1)

$$F_{R(S)} = \begin{cases} \langle F, H, \mathrm{Ab}(p, r), t \mid \alpha^t = \alpha' \rangle, \\ \alpha, \alpha' \in H,\ \langle u \rangle = \widetilde{F} \cap H, \\ \langle p \rangle = H \cap \mathrm{Ab}(p, r); \end{cases}$$

moreover u, p are not almost conjugate to either α or α' in H.

(2)

$$F_{R(S)} = \begin{cases} \langle F, H, \mathrm{Ab}(\delta, r), t \mid \beta^t = \gamma \rangle, \\ \beta \in F,\ \gamma \in H,\ \langle \alpha \rangle = F \cap H,\ \langle \delta \rangle = H \cap \mathrm{Ab}(\delta, r); \end{cases}$$

moreover γ and α are not almost conjugate in H.

2.4.3. When $\widetilde{F}$ is a rank 1 centralizer extension of F. The possibilities for the JSJ are

(1) $F_{R(S)} = \widetilde{F} *_{\langle \alpha \rangle} H$.

(2)

$$F_{R(S)} = \begin{cases} \langle \widetilde{F}, H, t \mid \beta^t = \beta' \rangle, \\ \beta, \beta' \in H,\ \langle \alpha \rangle = \widetilde{F} \cap H, \end{cases}$$

where α is not almost conjugate to β or β' in H. The subgroup $\langle H, t \rangle$ is also free of rank 2.

(3)

$$F_{R(S)} = \begin{cases} \langle \widetilde{F}, H, t \mid \beta^t = \gamma\rangle, \\ \beta \in \widetilde{F},\ \gamma \in H,\ \langle\alpha\rangle = \widetilde{F} \cap H. \end{cases}$$

Proposition 2.10. *If the JSJ of $F_{R(S)}$ has more than one non-abelian vertex group, then its JSJ is one of those given in Sections 2.4.1, 2.4.2 or 2.4.3.*

Corollary 2.11. *Suppose the JSJ of $F_{R(S)}$ has more than one non-abelian vertex group then:*

- *If the JSJ of $F_{R(S)}$ has three vertex groups then* $\mathrm{uhd}_F(F_{R(S)}) = 1$.
- $\mathrm{uhd}_F(F_{R(S)}) = 2$ *if and only if the vertex group $F \leq \widetilde{F}$ is a centralizer extension of F. In particular $F_{R(S)}$ must have a non-cyclic abelian subgroup.*
- *If $b_1(F_{R(S)}) = N + 1$, then it has no non-cyclic abelian subgroups; therefore* $\mathrm{uhd}_F(F_{R(S)}) = 1$.

Corollary 2.12. *If the JSJ of $F_{R(S)}$ has a QH subgroup, then it follows that $F_{R(S)} = \langle F, s, t \mid [s, t] = \alpha\rangle$, $\alpha \in F$.*

2.5. When the JSJ of $F_{R(S)}$ has one vertex group. The proof of this proposition takes up Section 7. It should also be noted that the arguments rely heavily upon the results of Sections 2.2 through 2.4.

Proposition 2.13. *If the JSJ of $F_{R(S)}$ has only one vertex group $\widetilde{F}$, then $\widetilde{F} \neq F$ and it is generated by F and two other elements. Moreover we have the following possibilities:*

(I) *The JSJ of $F_{R(S)}$ has two edges, $\widetilde{F}$ does not contain any non-cyclic abelian subgroups and* $\mathrm{uhd}_F(F_{R(S)}) = 2$.

(II) *The JSJ of $F_{R(S)}$ has one edge and* $\mathrm{uhd}_F(F_{R(S)}) \leq 4$.

2.6. Proof of the main theorem

Proof of Theorem 1.1. $F_{R(S)}$ must fall into one of the situations of Sections 2.3 through 2.5. It therefore follows that $F_{R(S)}$ must conform to one of the descriptions given by propositions 2.6, 2.10, and 2.13. □

2.7. Examples. We give some examples of the groups given in Section 2. We first note that it is very easy to construct examples that are freely decomposable modulo F. Examples of Proposition 2.6 are easy to construct by taking centralizer extensions of F or $F * \langle r\rangle$ or by taking iterated centralizer extensions of height 2. The next few examples are more delicate.

Example 2.14 (Example of a group in Section 2.4.3 of type 1). Let $F = F(a, b)$. It is proved in [Tou09] that the group

$$\begin{aligned} G &= \langle F, x, y \mid [a^{-1}ba[b,a][x,y]^2x, a] = 1\rangle \\ &= \langle F, x, y, t \mid [x,y]^2x = [a,b]a^{-1}b^{-1}at, [t,a] = 1\rangle \end{aligned}$$

is freely indecomposable modulo F. Let $u = [a,b]a^{-1}b^{-1}at$, and $w(x,y) = [x,y]^2x$ then

$$G = \widetilde{F} *_{\langle u = w(x,y)\rangle} \langle x, y\rangle,$$

where $\widetilde{F}$ is a rank 1 centralizer extension of F. Moreover G is shown to be fully residually F by the F-embedding into the iterated centralizer extension

$$F_2 = \langle F, t, s \mid [t,a] = 1, [s,u] = 1\rangle$$

via the mapping $x \mapsto s^{-1}(b^{-1}t)s$ and $y \mapsto s^{-1}(b^{-1}ab)s$.

This example above is also interesting since it is a one relator fully residually F group that cannot embed in a rank 1 centralizer extension of F. See Corollary 7.12.

Example 2.15 (Example of a group in Section 2.4.2 of type B.1). Let $F = F(a, b)$ and consider the iterated centralizer extension

$$F_2 = \langle F, s, t \mid [s,a] = 1, [t, (a^2(b^{-1}ab)^2)^s] = 1\rangle.$$

One can check that the subgroup $K \leq \langle F, s^{-1}bs, t\rangle$ has induced splitting

$$K = \begin{cases} \langle F, H, \mathrm{Ab}(p,t), r \mid \gamma'^r = \gamma\rangle, \\ \gamma, \gamma' \in H, \ \langle a\rangle = F \cap H, \ \langle p\rangle = H \cap \mathrm{Ab}(p,t), \end{cases}$$

where $H = s^{-1}\langle a, b^{-1}ab\rangle s$, $\gamma = s^{-1}as$, $\gamma' = s^{-1}b^{-1}abs$, $r = s^{-1}bs$, and $p = (a^2(b^{-1}ab)^2)^s$. Moreover it is freely indecomposable, fully residually F and generated by two elements modulo F.

Example 2.16 (Example of a group in Section 2.4.3 of type 3). We modify Example 2.14. Let $F = F(a, b)$ and let

$$F_1 = \langle F, s, t, r \mid [t,a] = 1, [s, b^{-1}ab] = 1, [u,r] = 1\rangle,$$

where $u = [a,b]a^{-1}b^{-1}at$. F_1 in iterated centralizer extension. Let $x' = b^{-1}t$, $y' = b^{-1}ab$ and let $G = \langle F, r^{-1}x'r, sr\rangle$. Let $H = r^{-1}\langle x', y'\rangle r$ and consider $G \cap H$. We see that $(sr)^{-1}b^{-1}ab(sr) = r^{-1}b^{-1}abr$ so $H \leq G$, on the other hand letting $y = (sr)$ and $x = r^{-1}x'r$ and by Britton's lemma we have a splitting

$$G = \begin{cases} \langle \widetilde{F}, H, y \mid (b^{-1}ab)^y = \gamma\rangle, \\ b^{-1}ab \in \widetilde{F}, \ \gamma \in H, \ \langle u\rangle = \widetilde{F} \cap H, \end{cases}$$

with $\widetilde{F} = \langle F, t\rangle = F(a,b) *_{\langle a\rangle} \mathrm{Ab}(a,t)$, $\gamma = r^{-1}b^{-1}abr$, $u = [a,b]a^{-1}b^{-1}at$, and H free of rank 2 and not freely decomposable modulo edge groups.

2.8. A conjecture and a question. Conspicuously absent from the list is an example of a fully residually F quotient of $F * \langle x, y\rangle$ whose JSJ has only one vertex group. Which leads to the following conjecture:

Conjecture 2.17. There are no fully residually F quotients of $F * \langle x, y\rangle$ whose JSJ has only one vertex group.

A related question is the following:

Question 2.18. Is there a finitely generated fully residually free group whose JSJ has a single vertex group that is free?

By Theorem 2.1 the answer to Question 2.18 is “no” in the case of three-generated fully residually free groups.

3. Graphs of groups and folding processes

3.1. Graphs of groups. The main result of Bass–Serre theory is that minimal actions of groups (without inversions) on simplicial trees correspond to splittings as fundamental groups of graphs of groups. The account we give here is only to fix the notation. We refer the reader to [Ser03] for a full treatment of the subject.

Definition 3.1. A *graph of groups* $\mathcal{G}(A)$ consists of a connected directed graph A with vertex set VA and edges EA. A is directed in the sense that to each $e \in EA$ there are functions $i: EA \to VA$, $t: EA \to VA$, corresponding to the *initial and terminal* vertices of edges. To A we associate the following:

- To each $v \in VA$ we assign a *vertex group* A_v.
- To each $e \in EA$ we assign an *edge group* A_e.
- For each edge $e \in EA$ we have monomorphisms

$$i_e: A_e \to A_{i(e)}, \quad t_e: A_e \to A_{t(e)},$$

  we call the maps i_e, t_e *boundary monomorphisms* and the images of these maps *boundary subgroups*.

For each $e \in EA$ we also formally define the following expressions:

$$(e^{-1})^{-1} = e, \quad i(e^{-1}) = t(e), \quad t(e^{-1}) = i(e), \quad i_{e^{-1}} = t_e, \quad t_{e^{-1}} = t_e.$$

We denote by $\pi_1(\mathcal{G}(A))$ the fundamental group of a graph of groups.

Definition 3.2. We say that a group *splits* as the fundamental group as a graph of groups if $G = \pi_1(\mathcal{G}(A))$ and refer to the data $D = (G, \mathcal{G}(A))$ as a *splitting*. A *cyclic splitting* is a splitting such that the edge groups are all cyclic. A $(\leq \mathbb{Z})$-*splitting* is a splitting whose edge groups are trivial or infinite cyclic.

Definition 3.3. A sequence of the form

$$a_0,\ e_1^{\epsilon_1},\ a_1,\ e_2^{\epsilon_2},\ \ldots,\ e_n^{\epsilon_n},\ a_n,$$

where $e_1^{\epsilon_1}, \ldots, e_n^{\epsilon_n}$ is an edge path of A and where $a_i \in A_{i(e_{i+1}^{\epsilon_i+1})} = A_{t(e_1^{\epsilon_i})}$ is called a $\mathcal{G}(A)$-path.

Definition 3.4. We denote by $\pi_1(\mathcal{G}(A), u)$ the group generated by $\mathcal{G}(A)$-paths based at the vertex u equipped with the obvious multiplication (i.e., concatenation and reduction) rules.

As usual, if A is connected the isomorphisms class of $\pi_1(\mathcal{G}(A), u)$ does not depend on u.

3.2. Induced splittings and $\mathcal{G}(A)$-graphs

Definition 3.5. Suppose that G has a splitting D as the fundamental group of a graph of groups and let H be a subgroup of G. Then G acts on a tree T and H acts on the minimal H-invariant subtree $T(H) \subset T$. Therefore H also splits as a graph of groups. We call this splitting of H the *induced splitting of* H.

We now present the folding machinery developed in [KWM05], which is a more combinatorial version of Stallings–Bestvina–Feighn–Dunwoody folding sequences. We will use it to find induced splittings. This gives an alternative to normal forms when dealing with fundamental groups of graphs of groups which simplifies and unifies the arguments of Sections 4, 6 and 7.

3.2.1. Basic definitions.

We follow [KWM05].

Definition 3.6. Let $\mathcal{G}(A)$ be a graph of groups. A $\mathcal{G}(A)$-graph $\mathcal{B}$ consists of an underlying graph B with the following data:

- A graph morphism $[\,\cdot\,]\colon B \to A$.
- For each $u \in VB$ there is a group $\mathcal{B}_u$ with $\mathcal{B}_u \leq A_{[u]}$, called a $\mathcal{B}$-vertex group. We give u the *label* $(\mathcal{B}_u, [u])$.
- To each edge $e \in EB$ there are two associated elements $e_i \in A_{[i(f)]}$ and $e_t \in A_{[t(f)]}$. If we flip the orientation of e we have the convention $(e^{-1})_i = (e_t)^{-1}$. We give the edge e the *label* $(e_i, [e], e_t)$.

Convention 3.7. We will usually denote $\mathcal{G}(A)$-graphs by $\mathcal{B}$ and will assume that the underlying graph of $\mathcal{B}$ is some graph B.

Definition 3.8. Let $\mathcal{B}$ be a $\mathcal{G}(A)$-graph and suppose that $e_1^{\epsilon_1}, \ldots, e_n^{\epsilon_n}$, where $e_j \in EB$, $\epsilon_j \in \{\pm 1\}$, is an edge path of B. A sequence of the form

$$b_0,\ e_1^{\epsilon_1},\ b_1,\ e_2^{\epsilon_2},\ \ldots,\ e_n^{\epsilon_n},\ b_n,$$

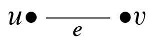
where $b_j \in \mathcal{B}_{t(e_j^{\epsilon_j})}$ is called a $\mathcal{B}$-path. To each $\mathcal{B}$-path p we associate a label

$$\mu(p) = a_0[e_1]^{\epsilon_1}a_1[e_2]^{\epsilon_2}\ldots[e_n]^{\epsilon_n}a_n,$$

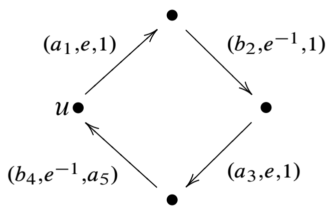
where $a_0 = b_0(e_1^{\epsilon_1})_i$, $a_j = (e_j^{\epsilon_1})_t b_j (e_{j+1}^{\epsilon_1})_i$ and $a_n = (e_n^{\epsilon_n})_t b_n$, which is a $\mathcal{G}(A)$-path (see Definition 3.3).

Definition 3.9. Let $\mathcal{B}$ be a $\mathcal{G}(A)$-graph with a basepoint u. Then we define the subgroup $\pi_1(\mathcal{B}, u) \leq \pi_1(\mathcal{G}(A), [u])$ to be the subgroup generated by the $\mu(p)$ where p is a $\mathcal{B}$-loop based at u.

Example 3.10. Let $G = \pi_1(\mathcal{G}(X), u) = A *_C E$. The underlying graph is



with $X_u = A$, $X_v = E$ and $X_e = C$. Let $g = a_1b_2a_3b_4a_5$ be a word in normal form where $a_i \in A$ and $b_j \in E$. Then the $\mathcal{G}(X)$-graph $\mathcal{B}$ given by



whose $\mathcal{B}$-vertex groups are all trivial is such that $\pi_1(\mathcal{B}, u) = \langle g \rangle$

This example motivates a definition.

Definition 3.11. Let $g = b_0, e_1^{\epsilon_1}, b_1, e_2^{\epsilon_2}, \ldots, e_n^{\epsilon_n}, b_n$ be an element of $\pi_1(\mathcal{G}(X), u)$. Then we call the based $\mathcal{G}(X)$-graph $\mathcal{L}(g; u)$ a *g-loop* if $\mathcal{L}(g; u)$ consists of a cycle starting at u whose edges have labels

$$(b_0, e_1^{\epsilon_1}, 1),\ (b_1, e_2^{\epsilon_2}, 1),\ \ldots, (b_{n-2},\ e_{n-1}^{\epsilon_{n-1}}, 1),\ (b_{n-1}, e_n^{\epsilon_n}, b_n).$$

Definition 3.12. $(\mathcal{G}(A), v_0)$ be a graph of groups decomposition of $F_{R(S)}$. Let the *$\bar{x}, \bar{y}$-wedge*, $\mathcal{W}(F, \bar{x}, \bar{y}; u)$, be the based $\mathcal{G}(A)$-graph formed from a vertex v with label (F, v_0) and attaching the loops $\mathcal{L}(\bar{x}; v_0)$ and $\mathcal{L}(\bar{y}; v_0)$.

It is clear that $\pi_1(\mathcal{W}(F, \bar{x}, \bar{y}; u), u) = \langle F, \bar{x}, \bar{y} \rangle = F_{R(S)}$.

3.2.2. Folding moves on $\mathcal{G}(A)$-graphs. Let $\mathcal{B}$ be a $\mathcal{G}(A)$ graph, with underlying graph B. We now briefly define the moves on $\mathcal{B}$ given in [KWM05] that we will use, we will sometimes replace an edge e by e^{-1} to shorten the descriptions:

(A0) *Conjugation at* v. For some vertex v of and some $g \in A_{[v]}$ do the following: replace $\mathcal{B}_v$ by $g\mathcal{B}_v g^{-1}$, up to changing e by e^{-1} we may assume that each edge e incident to v are such that $i(e) = v$. Such an edge has label $(e_i, [e], e_t)$. Replace e_i by ge_i.

(A1) *Bass–Serre move at* e. For some edge e, replace its label $(a, [e], b)$ by $(ai_e(c), [e], t_e(c^{-1})b)$ for some c in $A_{[e]}$.

(A2) *Simple adjustment at* u *on* e. For some vertex u and some edge e such that w.l.o.g $i(e) = u$, we replace the label $(a, [e], b)$ by $(ga, [e], b)$ where $g \in \mathcal{B}_u$.

(F1) *Simple fold of* e_1 *and* e_2 *at the vertex* u. For a vertex u and edges e_1, e_2 such that w.l.o.g. $i(e_1) = i(e_2) = u$ but $t(e_1) = v_1 \neq t(e_2) = v_2$ and $[v_1] = [v_2]$, if e_1 and e_2 have the same label, then make a new graph by identifying the edges e_1 and e_2. The resulting edge has the same label as e_1 and the $\mathcal{B}$-vertex group associated to the result of the identification of v_1 and v_2 is $\langle \mathcal{B}_{v_1}, \mathcal{B}_{v_2} \rangle$.

(F4) *Double edge fold (or collapse) of* e_1 *and* e_2 *at the vertex* u. For edges e_1, e_2 such that w.l.o.g. $i(e_1) = i(e_2) = u$, $t(e_2) = t(e_2) = v$, and $[e_1] = [e_2] = f$ if they have labels (a, f, b_1) and (a, f, b_2) respectively. Then we can identify the edges e_1 and e_2, the resulting edges has label (a, f, b_1) and the group $\mathcal{B}_v$ is replaced by $\langle \mathcal{B}_v, b_1^{-1} b_2 \rangle$. *We will also call such a fold a collapse from* u *towards* v.

The moves (F2) and (F3) in [KWM05] are analogous to (F1) and (F4), respectively only they involve simple loops. However, because these moves only show up implicitly in Section 7.1, we do not describe them explicitly. We also introduce three new moves:

(T1) *Transmission from* u *to* v *through* e. For an edge e such that $i(e) = u$ and $t(e) = v$ with label $(a, [e], b)$. Let $g \in A_{[e]}$ be such that $ai_{[e]}(g)a^{-1} = c \in \mathcal{B}_u$, then replace $\mathcal{B}_v$ by $\langle \mathcal{B}_v, b^{-1} t_{[e]}(g) b \rangle$. Transmissions are assumed to be *proper*, i.e., they result in a change of the $\mathcal{B}$-vertex groups.

(L1) *Long range adjustment*. Perform a sequence of transmissions through edges $e_1, \ldots, e_n$ followed by a simple adjustment at some vertex v that changes the label of some edge f but leaves unchanged the labels of the edges $e_1, \ldots, e_n$. Finally replace all the modified $\mathcal{B}$-vertex groups by what they were before the sequence of transmissions (see Figure 1).

(S1) *Shaving move*. Suppose that u is a vertex of valence 1 such that $u = t(e)$ and $v = i(e)$, e has label $(a, [e], b)$ and $\mathcal{B}_u = b^{-1}(t_{[e]}(C))b$, where $C \leq A_{[e]}$. Then collapse the edge e to its endpoint v and replace $\mathcal{B}_v$ by $\langle \mathcal{B}_v, a(i_{[e]}(C))a^{-1} \rangle$.

Convention 3.13. Although formally, applying a move to a $\mathcal{G}(A)$-graph $\mathcal{B}$ gives a new graph $\mathcal{B}'$. Unless noted otherwise we will denote this new $\mathcal{G}(A)$-graph as $\mathcal{B}$ as well.

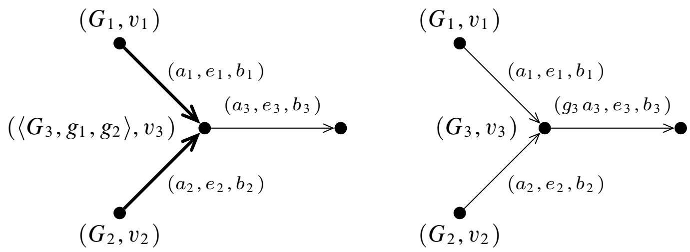


Figure 1. An example of an (L1) long-range adjustment. First make (T1) transmissions through the thickened edges labeled (a_1, e_1, b_1) and (a_2, e_2, b_2). These change the $\mathcal{B}$-vertex group G_3 to $\langle G_3, g_1, g_2 \rangle$. We then perform an (A2) simple adjustment which changes (a_3, e_3, b_3) to $(g_3 a_3, e_3, b_3)$ for some $g_3 \in \langle G_3, g_1, g_2 \rangle$ (but maybe not in G_3.) Finally we change $\langle G_3, g_1, g_2 \rangle$ back to G_3. The end result is the $\mathcal{G}(A)$-graph on the right.

We regard the (T1) transmission as the group $\mathcal{B}_u$ sending the element c to $\mathcal{B}_v$ through the edge e. In this paper, since all the edge groups are finitely generated abelian, we can use finitely many (T1) transmission moves instead of the edge equalizing moves (F5)–(F6) in [KWM05]. The moves (T1), (L1), and (S1) do not change the group $\pi_1(\mathcal{B}, u)$. Note moreover that vertices v of valence 1 with $\mathcal{B}_v = \{1\}$ can be shaved off.

3.2.3. The folding process

Definition 3.14. A $\mathcal{G}(A)$-graph such that it is impossible to apply any of the above moves other than (A0)–(A2) is called *folded*.

This next important result is essentially a combination of Proposition 4.3, Lemma 4.16 and Proposition 4.15 of [KWM05].

Theorem 3.15. [KWM05] *Applying the moves* (A0)–(A2), (F1)–(F4)*, and* (T1) *to* $\mathcal{B}$ *does not change* $H = \pi_1(\mathcal{B}, u)$; *moreover if* $\mathcal{B}$ *is folded, then the associated data* (*see Definition* 3.6) *gives the graph of groups decomposition of* H *induced by* $H \leq \pi_1(\mathcal{G}(A))$.

This theorem implies the existence of a folding process. Consider the three following classes of moves:

- *Adjustment*: Apply a sequence of moves (A0)–(A2), (L1), and (S1).
- *Folding*: Apply moves (F1) or (F4).
- *Transmission*: Apply move (T1).

First note that each folding decreases the number of edges in the graph, and that adjustments (except for shavings) are essentially reversible. In the folding process there is therefore a finite number of foldings and between foldings there are adjustments and transmissions.

4. When $F_{R(S)}$ is freely decomposable modulo F

This next proof is essentially the proof of Theorem 6.2 of [KWM05].

Proposition 4.1. *Suppose that* $F_{R(S)} = \widetilde{F} * H$*, then* $\widetilde{F}$ *is generated by* F *and* $(2 - \operatorname{rank}(H))$ *elements.*

Proof. First note that the underlying graph of the splitting $F_{R(S)} = \widetilde{F} * H$ consists of an edge and two distinct vertices. Let $\mathcal{G}(A)$ denote this graph of groups and let $\mathcal{B}$ be any $\mathcal{G}(A)$-graph. Only the moves (A0)–(A3), (F1),and (F4) can be applied.

Take $\mathcal{W}$ to be the wedge $\mathcal{W}(F, \bar{x}, \bar{y})$. Since $\pi_1(\mathcal{W}) = F_{R(S)}$ we have by Theorem 3.15 that $\mathcal{W}$ can be brought to a graph with a single edge and two distinct vertices. The underlying graph of $\mathcal{W}$ has two cycles and A has no cycles, which means that two (F4) collapses must occur. Moreover each collapse, maybe after applying (F1) moves, either contributes a generator to H or to $\widetilde{F}$. The result now follows. □

Corollary 4.2. *If* $F_{R(S)}$ *is freely decomposable modulo* F *then either it is one generated modulo* F *or*

$$F_{R(S)} = \begin{cases} F * \langle x, y \rangle, \\ F *_{\langle u \rangle} \operatorname{Ab}(u, t) * \langle x \rangle, \\ F * \operatorname{Ab}(x, y). \end{cases}$$

Proof. Apply Proposition 4.1 and Theorems 2.1 and 2.2. □

5. When all the vertex groups of the JSJ of $F_{R(S)}$ except $\widetilde{F}$ are abelian

We consider the case where the JSJ of $F_{R(S)}$ has abelian vertex groups but only one non-abelian vertex group $\widetilde{F} \geq F$. Before we continue we need some extra machinery.

5.1. Preliminaries

5.1.1. The (generalized) JSJ decomposition. As noted earlier $F_{R(S)}$ always has a generalized JSJ as given in Definition 1.8. We give some more details that will be necessary to our work.

Definition 5.1. Let G act on a simplicial tree T without inversions. An element or subgroup of G is called *elliptic* if it fixes a point of T. Otherwise it is called *hyperbolic*. If D_1 and D_2 are two splittings of G then D_1 is hyperbolic with respect to D_2 if an edge group of D_1 is hyperbolic with respect to the action of G on the Bass–Serre tree associated to D_2.

Consider now the following moves that can be made on a graph of groups.

Definition 5.2 (Moves on $\mathcal{G}(A)$). We have the following moves on a graph of groups $\mathcal{G}(A)$ that do not change the fundamental group.

- *Conjugate boundary monomorphisms*: Replace i_e by $\gamma_g \circ i_e$ where γ_g denotes conjugation by g and $g \in A_{i(e)}$.
- *Slide*: If there are edges e, f such that $i_e(A_e) \geq i_f(A_f)$ then we change A by redefining $i : EA \to VA$ so that $f \mapsto t(e)$ and replacing the homomorphisms i_f by $t_e \circ i_e^{-1} \circ i_f$.
- *Folding*: If $i_e(A_e) \leq A \leq A_{i(e)}$, then replace $A_{t(e)}$ by $A_{t(e)} *_{t_e(A_e)} A$, replace A_e by a copy of A and change the boundary monomorphism accordingly. (We remark that the name "folding" comes from the effect of the move on the Bass–Serre tree, we could also call it "edge group enlargement" or "vertex group expansion".)
- *Collapse an edge* e: For some edge $e \in EA$, let $A \to A'$ be the quotient obtained by collapsing e to a point $[e] \in A'$. We get a new graph of groups with underlying graph A' as follows: if $i(e) \neq t(e)$ we set $A_{[e]}$ to be the free product with amalgamation $A_{i(e)} *_{A_e} A_{t(e)}$, if $i(e) = t(e)$ we set $A_{[e]}$ to be the HNN extension $A_{i(e)} *_{A_e}$. The boundary monomorphisms are the natural ones.

Definition 5.3. A splitting D is *almost reduced* if vertex groups of vertices of valence one and two properly contain the images of edge groups, except possibly the vertex groups of vertices between two MQH subgroups that may coincide with one of the edge groups.

A splitting D of $F_{R(S)}$ is *unfolded* if D cannot be obtained from another splitting D' via a folding move (see Definition 5.2).

Definition 5.4. An *elementary splitting* is a splitting whose underlying graph has one edge.

Theorem 5.5 (Proposition 2.15 of [KM05]). *Suppose that $F_{R(S)}$ is freely indecomposable modulo F. Then there exists an almost reduced unfolded cyclic splitting D called the* **cyclic JSJ splitting of $F_{R(S)}$ modulo** *F with the following properties:*

(1) *Every MQH subgroup of $F_{R(S)}$ can be conjugated into a vertex group in D; every QH subgroup of $F_{R(S)}$ can be conjugated into one of the MQH subgroups of $F_{R(S)}$; non-MQH (vertex) subgroups in D are of two types: maximal abelian and non-abelian (rigid), every non-MQH vertex group in D is elliptic in every cyclic splitting of H modulo F.*

(2) *If an elementary cyclic splitting $F_{R(S)} = A *_C B$ or $F_{R(S)} = A *_C$ is hyperbolic in another elementary cyclic splitting, then C can be conjugated into some MQH subgroup.*

(3) *Every elementary cyclic splitting $F_{R(S)} = A *_C B$ or $F_{R(S)} = A *_C$ modulo F which is elliptic with respect to any other elementary cyclic splitting modulo F of $F_{R(S)}$ can be obtained from D by a sequence of moves given in Definition 5.2.*

(4) *If D_1 is another cyclic splitting of $F_{R(S)}$ modulo F that has properties* (1)–(2) *then D_1 can be obtained from D by a sequence of slidings, conjugations, and modifying boundary monomorphisms by conjugation* (*see Definition* 5.2).

Definition 5.6. An action of G on a simplicial tree T is said to be *k-acylindrical* if the diameter of a subset of T fixed by a non-trivial element of G is at most k. A *splitting* of G is said to be k-acylindrical if the action of G on the induced Bass–Serre tree is k-acylindrical.

Convention 5.7. It is, for example, possible to write an amalgam $G *_u \mathrm{Ab}(u,t)$ as an HNN extension $\langle G, t \mid t^{-1}ut = u\rangle$. *We will always chose our JSJ so that non-cyclic abelian subgroups of $F_{R(S)}$ are elliptic.* This is necessary to ensure 2-acylindricity of the splitting, and in our situation this is always possible.

Definition 5.8. Let D be the JSJ of $F_{R(S)}$ modulo F. If e is a an edge ending in a vertex v of valence 1 in A, the graph underlying D, and A_v is cyclic, then e is called a *hair*. Let D' be splitting of $F_{R(S)}$ obtained by collapsing hairs into the adjacent vertex groups. Then D' is called the *hairless JSJ of $F_{R(S)}$*.

We note that since we require D to be almost reduced, it is a simple exercise involving the use of commutation transitivity to see that after removing all the hairs of D, the hairless splitting D' will indeed not have any hairs. We also note that passing to a hairless splitting does not change the group of canonical (or modular) automorphisms (see Section 2.15 of [KM05] or Definition 5.2 of [Sel01] for details.)

Convention 5.9. Unless stated otherwise, we will always replace the JSJ by the hairless JSJ.

5.1.2. Strict resolutions. Strict resolutions are an extremely useful tool for studying the vertex groups of a JSJ.

Definition 5.10. An epimorphism $\rho\colon F_{R(S)} \to F_{R(S')}$ of fully residually F groups is called (weakly) *strict* if it satisfies the following conditions on the (generalized) JSJ modulo F.

(1) For each abelian vertex group A, ρ is injective on the subgroup $A_1 \leq A$ generated by the boundary subgroups in A.
(2) ρ is injective on edge groups.
(3) The images of QH subgroups are non-abelian.
(4) For every rigid subgroup (as defined in (1) of Theorem 5.5) R, ρ is injective on R.
(5) Distinct factors of the Grushko decomposition of $F_{R(S)}$ modulo F are mapped onto distinct free factors a free decomposition of $F_{R(S')}$ modulo F.

Convention 5.11. Weakly strict differs from strict as defined in [Sel01] only in item (4). We have simplified the definition for the convenience of the reader. Throughout this paper we shall say *strict* instead of *weakly strict.*

Definition 5.12. A strict resolution of $F_{R(S)}$,

$$\mathcal{R}: \ F_{R(S_0)} = F_{R(S)} \xrightarrow[\pi_1]{} \cdots \xrightarrow[\pi_p]{} F_{R(S_p)} \xrightarrow[\pi_{p+1}]{} F * F(Y) \ ,$$

is a sequence of proper epimorphisms of fully residually F groups such that all the epimorphism are strict.

Theorem 5.13. *If $F_{R(S)}$ is fully residually F then it admits a strict resolution $\mathcal{R}$.*

As formulated, Theorem 5.13 is an easy corollary of the definitions of the canonical (or modular) automorphisms and the fact that $\mathrm{Hom}_F(F_{R(S)}, F)$ can be encoded in a finite Hom (also called a Makanin–Razborov) diagram (see Theorem 5.12 of [Sel01] or Theorem 11.2 of [KM05] for details.)

5.1.3. Iterated centralizer extensions

Definition 5.14. A (rank n) centralizer extension of F is an amalgam $F *_{\langle u \rangle} A_u$ where $u \in F$ is malnormal and A_u is free abelian (of rank $n+1$). G is an iterated centralizer extension of F if either

- G is a centralizer extension of F, or
- $G = H *_{\langle w \rangle} A_w$ where H is an iterated centralizer extension, the centralizer of w in H is $\langle w \rangle$ and A_w is free abelian.

We say it is *finite* if it is obtained from F by finitely many centralizer extensions.

Theorem 5.15 (Theorem 4 of [KM98b]). *G is finitely generated and fully residually F if and only if it embeds in a finite iterated centralizer extension of F.*

Corollary 5.16. *Any subgroup $F < \widehat{F} \leq F_{R(S)}$ has a non-trivial $(\leq \mathbb{Z})$-splitting D modulo F. Moreover any element $\beta' \in \widehat{F}$ that is conjugate in $F_{R(S)}$ to some $\beta \in F$ is elliptic in this splitting of $\widehat{F}$.*

Proof. Let $F_{R(S)} \leq G$ be the embedding of $F_{R(S)}$ into an iterated centralizer extension of F. The cyclic JSJ of G modulo F is very simple: it is a star of groups with a vertex group containing F at its center and all the other vertex groups are free abelian. The central vertex group is itself either F or an iterated centralizer extension of F.

If $\widehat{F}$ is elliptic in the JSJ of G then we can replace G by its central vertex group. Since $\widehat{F} \neq F$ eventually there some iterated centralizer extension $\widehat{F} \leq H \leq G$ in which $\widehat{F}$ has a non-trivial induced splitting D.

Claim: β' is conjugate to β in H. Suppose towards a contradiction that this is not the case. Obviously β is conjugate to β' in G, so if $H = G$ we have a contradiction. We have $G = G' *_{\langle u \rangle} A_u$ with A_u free abelian and the centralizer of $\langle u \rangle$ in G' cyclic. Now by hypothesis there is some element $W(G', A_u) \in G$ such that $W(G', A_u)\beta W(G', A_u)^{-1} = \beta'$. Looking at normal forms, we immediately see that if such a product is to lie in G' we must have that β and β' are conjugate to u in G', so β, β' are conjugate in G'.

We repeat replacing G by G'. Continuing in this fashion, we eventually get that β, β' are conjugate in H, contradiction. The claim is therefore proved.

It thus follows that β' is conjugate to β in H, so β' must be elliptic in the induced splitting of $\widehat{F}$ modulo F, which has either cyclic or trivial edge groups. □

5.1.4. The first Betti number. The following useful fact is obvious from a relative presentation:

Lemma 5.17. *Let $H < G$ be a rank n centralizer extension of H (see Definition 5.14), then $b_1(G) = b_1(H) + n$.*

Lemma 5.18. *The subgroup $F \leq F_{R(S)}$ has the property CC (conjugacy closed), that is to say, for $f, f' \in F$, if there exists $g \in F_{R(S)}$ such that $f^g = f'$, then there exists $k \in F$ such that $f^k = f'$.*

Proof. Let f, f' and g be as in the statement of the lemma. Let $r\colon F_{R(S)} \to F$, be a retraction. Then $k = r(g) \in F$ has the desired property. □

Proof of Proposition 2.3. Suppose towards a contradiction that $G \neq F$ but $b_1(G) = N$. Then, by being fully residually F, there is a retraction $G \to F$. It follows that $b_1(G) \geq N$. If G is freely decomposable modulo F, then one of its free factors retracts onto F and any other free factor maps onto an infinite cyclic group so $b_1(G) > N$, a contradiction.

It follows that G is freely indecomposable and since $G \neq F$, G has D, a non-trivial JSJ decomposition. Let $F \leq \widetilde{F} \leq G$ be the vertex group containing F; obviously $\widetilde{F}$ is also fully residually F. By formula (1), $b_1(G) \geq b_1(\widetilde{F})$ and if D has more than one vertex group then the inequality is proper which forces $b_1(G) > N$, a contradiction. D is therefore a bouquet of circles with a single vertex group $\widetilde{F}$. By Lemma 5.18, if $\widetilde{F} = F$, then the stable letters of the splitting D in fact extend centralizers of elements of F, so by Lemma 5.17, $b_1(G) > b_1(F)$, a contradiction.

It follows that we cannot have $\widetilde{F} = F$. We therefore look for a JSJ of $\widetilde{F}$. Again it must have a unique vertex group $\widetilde{F}^1$. Since $\widetilde{F}^1 \neq F$, it must have an essential cyclic splitting. Since $\mathrm{uhd}_F(G)$ is finite, we have a terminating sequence

$$\widetilde{F} > \widetilde{F}^1 > \cdots > \widetilde{F}^r > \widetilde{F}^{r+1} = F,$$

where $\widetilde{F}^{i+1}$ is the unique vertex group of the JSJ of $\widetilde{F}^i$. Now, by assumption, $N = b_1(\widetilde{F}) \geq b_1(\widetilde{F}^1) \geq \cdots \geq b_1(\widetilde{F}^{r+1}) = N$, but $\widetilde{F}^r$ has a splitting D^r that is a bouquet of circles with vertex group F, so it is a centralizer extension of F. It follows that $b_1(\widetilde{F}^r) > b_1(F)$, again a contradiction. $\square$

5.1.5. Splittings with two cycles. Suppose $F_{R(S)} = \langle F, \bar{x}, \bar{y}\rangle$ can be collapsed to a graph of groups modulo F with one vertex and two edges, i.e.,

$$F_{R(S)} = s \begin{cases} \langle \widehat{F}, t, s \mid A^t = A', B^s = B'\rangle, \\ A, A', B, B' \le \widehat{F}, \end{cases} \tag{2}$$

where $F \le \widehat{F}$.

Definition 5.19. For a generating set X and a word $W = W(X)$ in X, for a letter $x \in X$ we denote the exponent sum of x in W by $\sigma_x(W)$.

Definition 5.20. Let $\langle X, t \mid R\rangle$ be some relative presentation for G where t is a stable letter. Any $g \in G$ can be expressed as a word $g = W(X, t)$. We define the *exponent sum* $\sigma_t(g)$ of t in g as

$$\sigma_t(g) = \sigma_t(W(X, t)).$$

By Britton's lemma this is well defined.

This next lemma follows immediately from abelianizing.

Lemma 5.21. *Suppose that $F_{R(S)}$ splits as in (2). If some word $W(F, \bar{x}, \bar{y})$ lies in $\widehat{F}$, then it must have exponent sum* 0 *in both $\bar{x}$ and $\bar{y}$.*

Definition 5.22. Let $F_{R(S')}$ be a fully residually F group and let A be a subgroup of an abelian vertex group $\hat{A}$ of $F_{R(S')}$'s JSJ. A direct summand $A' \le A$, where $A = A' \oplus A''$, that does not intersect the images of the edge groups incident to $\hat{A}$ is said to be *exposed*.

Lemma 5.23. *Let $A \le F_{R(S)}$ be a non-cyclic abelian subgroup, and let*

$$\mathfrak{R}: \ F_{R(S_0)} = F_{R(S)} \xrightarrow[\pi_1]{} \cdots \xrightarrow[\pi_p]{} F_{R(S_p)} \xrightarrow[\pi_{p+1}]{} F * F(Y)$$

be a strict resolution of $F_{R(S)}$. A must eventually be mapped monomorphically inside an abelian vertex group $\hat{A}$ of the (generalized) JSJ of some quotient $F_{R(S_i)}$ occurring in $\mathfrak{R}$ and the isomorphic image of A should have an exposed direct summand.

Proof. By Convention 5.7 non-cyclic abelian groups are always elliptic in a JSJ. Suppose first that throughout $\mathfrak{R}$ the subgroup A is always mapped inside a rigid vertex group, or inside the subgroup generated by the boundary subgroups of some abelian vertex group. Since the terminal group of a strict resolution is always free non-abelian, A is eventually mapped to a cyclic group. This means that some strict quotient is not injective on a rigid vertex group or a subgroup of an abelian vertex group generated by the boundary subgroups, which is a contradiction.

It therefore follows that at some point A is mapped inside some abelian vertex group $\hat{A}$ and is not contained in the subgroup generated by the incident edge groups. The result now follows. $\square$

Lemma 5.24. *Suppose $A \leq F_{R(S)}$ is a non-cyclic abelian subgroup of $F_{R(S)}$. Then A cannot be generated by words $W_i(F, \bar{x}, \bar{y})$ such that for each W_i the exponent sums in $\bar{x}$ and $\bar{y}$ are zero.*

Proof. Let $\mathcal{R}$ be a strict resolution with fully residually F groups $\{F_{R(S_i)}\}$. By Lemma 5.23 in some $F_{R(S_i)}$ A is mapped monomorphically into an abelian vertex group $\hat{A}$ and has an exposed cyclic summand $\langle r \rangle$. Let $\widehat{A'}$ be the subgroup of $\hat{A}$ generated by the incident edge groups and let $\langle \eta \rangle \leq \hat{A}$ be the maximal cyclic subgroup containing $\langle r \rangle$, so that $r = \eta^n$. Then we can extend the projection $\hat{A} \to \widehat{A'} \oplus \langle \eta \rangle$ to $F_{R(S_j)}$. The resulting quotient can be seen as an HNN extension:

$$\overline{F_{R(S_i)}} = \langle G, \eta \mid [\eta, a] = 1 \text{ for all } a \in \widehat{A'} \rangle.$$

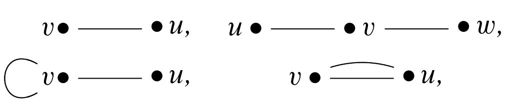
On one hand $\bar{x}$ and $\bar{y}$ are sent to elements with normal forms $\bar{x}'(G, \eta)$, $\bar{y}'(G, \eta)$; on the other hand, η^n is in the image of A, and by hypothesis we can write $r = R(F, \bar{x}'(G, \eta), \bar{y}'(G, \eta))$ where R has exponent sum zero in $\bar{x}'(G, \eta)$ and $\bar{y}'(G, \eta)$, but $R(F, \bar{x}'(G, \eta), \bar{y}'(G, \eta)) = r = \eta^n, n \neq 0$, must have exponent sum zero in η, which is a contradiction. $\square$

Corollary 5.25. *If $F_{R(S)}$ has a cyclic splitting modulo F with two cycles, then none of the vertex groups of this splitting can contain non-cyclic abelian subgroups.*

Proof. By Lemma 5.21 any element conjugable into a vertex group of the splitting must be expressible by words in F, $\bar{x}$, $\bar{y}$ that have exponent sum zero in $\bar{x}$ and $\bar{y}$, Lemma 5.24 now gives a contradiction. $\square$

5.2. The possible JSJs when all the vertex groups of the JSJ of $F_{R(S)}$ except $\widetilde{F}$ are abelian. In this section we prove that the JSJs given in Proposition 2.6 are the only possibilities.

Lemma 5.26. *If the cyclic JSJ decomposition of $F_{R(S)}$ modulo F contains only one non-abelian vertex group then the JSJ is given by the graph of groups $\mathcal{G}(X)$ where X is one of the following*



with $\widetilde{F} \leq X_v$ and X_w, X_u abelian.

Proof. By CSA and commutation transitivity $\mathcal{G}(X)$ cannot have subgraphs of groups

$$u \bullet \text{———} \bullet w \quad \text{or} \quad u \bullet\!\supset$$

with X_u, X_w non-cyclic abelian. Each abelian vertex group A contributes $\operatorname{rank}(A)-1$ to $b_1(F_{R(S)})$, so by Proposition 2.3 there are at most two of them of rank 2, or one of them of rank at most 3. By Corollary 5.25 the resulting underlying graph of groups cannot have two cycles. So far the only possibilities are the graphs of groups given in the statement of the lemma and

$$u\bullet \text{———} v\bullet \overset{\frown}{\text{———}} \bullet w$$

with $\widetilde{F} \le X_v$ and X_w, X_u abelian. But note that X_u must have rank 2 and since we can rewrite $G *_{\langle \alpha \rangle} A$ as $\langle G, t \mid \alpha^t = \alpha \rangle$ if A is free abelian of rank 2. We can again apply Corollary 5.25 to get a contradiction to the fact that X_w is non-cyclic abelian. □

Proposition 5.27. *If $F_{R(S)}$ is as in the statement of Lemma 5.26, then it cannot have the JSJ with underlying graph*

$$X = v\bullet \overset{\frown}{\text{———}} \bullet u\,.$$

Proof. Suppose, on the contrary, that $F_{R(S)}$ had the JSJ

$$\begin{cases} \langle \widetilde{F}, A, t \mid \alpha^t = \beta \rangle, \\ \alpha \in \widetilde{F},\ \beta \in A,\ \langle \delta \rangle = \widetilde{F} \cap A, \end{cases}$$

with $F \le \widetilde{F}$ and A abelian. Let $A' = \langle \beta, \delta \rangle \le A$. *Claim:* $F \ne \widetilde{F}$. Suppose this is not the case, then α and δ lie in F but to discriminate A', either β or δ must be sent to arbitrarily high powers via retractions $F_{R(S)} \to F$, which is impossible since β is conjugate to α and $\delta \in F$.

Since $\widetilde{F} \ne F$, $\widetilde{F}$ must have a $(\le \mathbb{Z})$-splitting modulo F, but because our splitting of $F_{R(S)}$ is a JSJ, α or δ must be hyperbolic in any $(\le \mathbb{Z})$-splitting of $\widetilde{F}$. We apply Lemma 5.23 to $A' \le F_{R(S)}$. Let $F_{R(S_i)}$ be the quotient in the strict resolution where A' has an exposed summand. Since in the initial segment $F_{R(S)} \xrightarrow{\pi_1} \cdots \xrightarrow{\pi_p} F_{R(S_i)}$ of the strict resolution A' is always elliptic, so are the elements α, δ, which means that $\widetilde{F}$ never splits and hence is always mapped monomorphically.

Let $\hat{A}$ be the abelian vertex group of $F_{R(S_i)}$ containing A' and let $E \le \hat{A}$ be the subgroup generated by incident edge groups. Since A' has an exposed summand, w.l.o.g. $\delta \notin E$, which means that in the Bass–Serre tree T for the JSJ of $F_{R(S_i)}$, the minimal invariant subtree $T(\langle \delta \rangle)$ consists of a vertex whose stabilizer is abelian. $T(\langle \beta \rangle)$, on the other hand, consists of a vertex whose stabilizer is non-abelian, but $\delta \in \widetilde{F}$, a contradiction. □

Proposition 5.28. *If we have the JSJs*

- $F_{R(S)} = \widetilde{F} *_{\langle\alpha\rangle} A_1$ *and* $\mathrm{rank}(A_1) \geq 3$*, or*
- $F_{R(S)} = A_2 *_{\langle\beta\rangle} \widetilde{F} *_{\langle\gamma\rangle} A_3$*,*

with A_1, A_2, A_3 *free abelian, then* $\widetilde{F} = F$.

Proof. This follows immediately by estimating $b_1(F_{R(S)})$ and applying Proposition 2.3. □

The proof that $\widetilde{F}$ is 2-generated modulo F is deferred to Section 7.2.

6. When the JSJ of $F_{R(S)}$ has at least two non-abelian vertex groups

6.1. Preliminaries. The approach here is to see what kind of graphs of groups we can obtain as images of $F * \langle x, y\rangle$. To make the problem tractable we first consider a coarser splitting. It will turn out that this is an effective way to get started.

6.1.1. Maximal abelian collapse. Suppose that $D = (\mathcal{G}(X), F_{R(S)})$ and the JSJ of $F_{R(S)}$ contains at least two non-abelian vertex groups. Take D and do the following (see Definition 5.2):

(i) If any boundary subgroup $\langle\alpha\rangle = i_e(X_e)$ is a proper subgroup of a maximal abelian subgroup A, then do a folding move (as in Definition 5.2) where we replace X_e by a copy of A.

(ii) Ensuring that the resulting graph of groups always has at least two non-abelian vertex groups, perform sliding and collapsing moves until it is no longer possible to decrease the number of vertices or edges

In the end the resulting graph of groups $\mathcal{G}(X)$ will have one of three possible forms:

$$\widehat{F} \;\text{——}\; H, \qquad \widehat{F} \;\overset{\frown}{\text{——}}\; H, \qquad \text{or} \qquad \widehat{F} \;\equiv\; H, \tag{3}$$

where the vertex group $\widehat{F}$ contains F and boundary subgroups are maximal abelian in their vertex groups.

Definition 6.1. We will call such a splitting a *maximal abelian collapse of* $F_{R(S)}$.

Lemma 6.2. *The maximal abelian collapse of* $F_{R(S)}$ *is a* 1*-acylindrical splitting.*

Proof. On one hand by item (ii) of our construction, distinct edges have distinct boundary subgroups. On the other hand we also have that they are maximal abelian in their vertex groups, and maximal abelian subgroups are malnormal in fully residually free groups. The result now follows. □

Although we may have sacrificed some information in passing to a maximal abelian collapse, we now have a 1-acylindrical splitting. This will enable us to use the very useful Lemmas 6.13 and 6.18.

6.1.2. Balancing folds and adjoining roots. It may happen that the image of an edge group is not maximal cyclic in one of the adjacent vertex groups.

Definition 6.3. Let $\mathcal{G}(X)$ be a $(\leq \mathbb{Z})$-graph of groups. An edge group X_e is said to be *balanced* if its images are maximal cyclic in the vertex groups. A graph of group is called *balanced* if all its edge groups are balanced.

The main technical advantage of having a balanced $(\leq \mathbb{Z})$-graph of groups is that if all the edge groups are maximal abelian (i.e., they do not lie in non-cyclic abelian subgroups) *then our graph of groups is* 1*-acylindrical.*

By commutation transitivity one of the image of an edge group must be maximal in the vertex groups. Our graphs of groups are not always balanced, however we may do the following.

Definition 6.4. Let X_e be a non-balanced edge group of a $(\leq \mathbb{Z})$-graph of group $\mathcal{G}(X)$, then the folding move (as in Definition 5.2) which enlarges X_e so that its images in the adjacent vertex groups is maximal cyclic is called a *balancing fold.*

The result of a balancing fold on the "enlarged" vertex group is the adjunction of a proper root. We will want to make balancing folds, but we also want to keep the rank of the vertex groups under control. In [Wei06] Weidmann proves the following:

Theorem 6.5 (Theorem 1 of [Wei06]). *Let G be a group, $g \in G$ be an element of order $n \in \mathbb{N} \cup \{\infty\}$ and $k \geq 2$ an integer. Then*

$$\operatorname{rank}(G *_{\langle g=z^k\rangle} \langle z \mid z^{nk}\rangle) \geq \operatorname{rank}(G).$$

We will need the following variant of this result that is not an immediate corollary.

Lemma 6.6. *Let $F_{R(S)}$ be a finitely generated fully residually F group and let*

$$\widehat{F_{R(S)}} = F_{R(S)} *_{\langle z\rangle} \langle \sqrt[n]{z}\rangle,$$

where $(\sqrt[n]{z})^n = z$, be a fully residually F quotient of $F * \langle x_1, \dots, x_m\rangle$. Then $F_{R(S)}$ is also a fully residually F quotient of $F * \langle x_1, \dots, x_m\rangle$.

Sketch of proof. The argument used to prove Theorem 1 of [Wei06] is in fact perfectly suitable for our purposes. Let $\mathcal{G}(X)$ be the graph of groups for $F_{R(S)} *_{\langle z\rangle} \langle \sqrt[n]{z}\rangle$ and let $\mathcal{G}(X')$ be the graph of groups for $F_{R(S)} *_{\langle z\rangle} \langle z\rangle = F_{R(S)}$. We start a folding sequence for $\widehat{F_{R(S)}}$ with $\mathcal{B}_0$ the $\mathcal{G}(X)$-graph that consists of a vertex u with $\mathcal{B}_{0u} = F$ and

$\bar{x}_i$-loops $\mathcal{L}(\bar{x}_i; u)$, i.e., the graph underlying $\mathcal{B}_0$ is a bouquet of m-circles. After this initial setup the arguments of the proof Theorem 6.5, that consider different types of folds in the folding sequence, go through and we find that $F_{R(S)}$ is also generated by F and m other elements. □

6.1.3. Weidmann–Nielsen normalization for groups acting on trees. We present some of the techniques developed by Weidmann in [Wei02]. Let G act on a simplicial tree T.

Definition 6.7. Let $M \subset G$ be partitioned as

$$M = S_1 \sqcup \cdots \sqcup S_p \sqcup \{h_1, \ldots, h_s\}.$$

We say that M has a *marking* $(S_1, \ldots, S_p; \{h_1, \ldots, h_s\})$. The elementary Weidmann–Nielsen transformations on *marked sets* are:

(WN1) Replace some S_i by $g^{-1} S_i g$, where $g \in M - S_i$.

(WN2) Replace some element $h_i \in \{h_1, \ldots, h_s\}$ by $g_1 h_i g_2$, where $g_1, g_2 \in M - \{h_i\}$.

Definition 6.8. For a subgroup $K \leq G$ we denote the minimal K-invariant subtree $T(K)$ and we denote

$$T_{\langle S_i \rangle} = \{x \in T \mid \text{there exists } g \in \langle S_i \rangle \setminus \{1\} \text{ such that } gx = x\} \cup T(\langle S_i \rangle).$$

We now formulate the main results in [Wei02].

Theorem 6.9. *Let M be a set with markings $(S_1, \ldots, S_p; \{h_1, \ldots, h_s\})$. Then either*

$$\langle M \rangle = \langle S_1 \rangle * \cdots * \langle S_p \rangle * F(\{h_1, \ldots, h_s\})$$

or by successively applying transformations (WN1) *and* (WN2) *we can bring* $(S_1, \ldots, S_p; \{h_1, \ldots, h_s\})$ *to a* normalized *marked set*

$$\widetilde{M} = (\widetilde{S}_1, \ldots, \widetilde{S}_p; \{\widetilde{h}_1, \ldots, \widetilde{h}_s\})$$

such that one of the following must hold:

(1) $T_{\langle \widetilde{S}_i \rangle} \cap T_{\langle \widetilde{S}_j \rangle} \neq \emptyset$ *for some* $i \neq j$.

(2) *There exists* $\widetilde{h}_i \in \{\widetilde{h}_1, \ldots, \widetilde{h}_s\}$ *such that* $\widetilde{h}_i T_{\langle \widetilde{S}_i \rangle} \cap T_{\langle \widetilde{S}_i \rangle} \neq \emptyset$.

(3) *There is some* $\widetilde{h}_i \in \{\widetilde{h}_1, \ldots, \widetilde{h}_s\}$ *that fixes a point of* T.

Notice that in passing from a marked set M to a normalized marked set $\widetilde{M}$ as in Theorem 6.9 the subgroups $\langle S_i \rangle$ and $\langle \widetilde{S}_i \rangle$ differ only in that there is some $g \in G$ such that $\langle \widetilde{S}_i \rangle = g^{-1} \langle S_i \rangle g$. From this it is not hard to see that we can chose our sequence of transformations (WN1) and (WN2) so that one of the subsets S_j in M remains invariant.

Convention 6.10. Suppose $F \leq \langle S_i \rangle$. When we apply Weidmann–Nielsen transformations to a marked generating set of $(S_1, \ldots, S_p; \{h_1, \ldots, h_s\})$ of $F_{R(S)}$ we want to make sure that the elements of F remain fixed pointwise. We therefore do not use any moves of type (WN1) that will change S_1. This restriction does not alter the applicability of Theorem 6.9.

Definition 6.11. *Weidmann–Nielsen normalization* is the process of using moves (WN1) and (WN2) to bring a marked generating set to a *normalized* generating set as in Theorem 6.9.

6.1.4. Elements with small translation lengths. We always act on a tree T from the left, i.e., for all $g, h \in F_{R(S)}$ and for all $v \in T$ we have

$$ghv = g(hv)$$

it follows that for any point $v \in T$, and for any $g \in F_{R(S)}$ we will have $\mathrm{stab}(gv) = g(\mathrm{stab}(v))g^{-1} = {}^{g}\,\mathrm{stab}(x)$. Recall the definition of T_F given in Definition 6.8.

Lemma 6.12. *Let $F_{R(S)}$ act on a simplicial tree T with edge stabilizers maximal abelian in their vertex groups and with F elliptic. If there is some edge $e \subset T_F$ and some $g \in F_{R(S)}$ such that $ge \subset T_F$, then there is some $f \in F$ such that $f^{-1}ge = e$.*

Proof. Let $\langle\gamma\rangle = \mathrm{stab}(e) \cap F$ and let $\langle\beta\rangle = \mathrm{stab}(ge) \cap F = {}^{g}\mathrm{stab}(e) \cap F$. Since edge stabilizers are maximal abelian to prove the claim it is enough to find some $f \in F$ such that $[{}^{f}\gamma, \beta] = 1$ as this would imply ${}^{f}\mathrm{stab}(e) = \mathrm{stab}(ge)$. By hypothesis $[{}^{g}\gamma, \beta] = 1$. Let $\psi\colon F_{R(S)} \to F$ be a retraction. Then $[{}^{\psi(g)}\gamma, \beta] = 1$, so $f = \psi(g)$ is the desired element. □

If the action of $F_{R(S)}$ on its Bass–Serre tree T is as in Lemma 6.12 then the action is 1-acylindrical. Suppose F is elliptic and let $\mathrm{fix}(F) = v_0$, suppose also that ρ is hyperbolic. Of particular interest to us is the situation where $T_F \cap \rho T_F \neq \emptyset$, as in item (2) of Theorem 6.9.

We will focus on the situation where $d(v_0, \rho v_0) = 2$ and where the path $[v_0, \rho v_0]$ intersects two $F_{R(S)}$-vertex orbits. Let w_0 be a vertex such that $d(w_0, v_0) = 1$ and suppose that $H = \mathrm{stab}(w_0) \cap F \neq \{1\}$. Let w_1 be the vertex in $[v_0, \rho v_0]$ that is an $F_{R(S)}$-translate of w_0. Consider the two relative sub-presentations (i.e., they are "subgraphs of groups of" $F_{R(S)}$)

$$\begin{cases} \langle \widehat{F}, H, t \mid A^t = B \rangle, \\ A \leq \widehat{F},\ B \leq H,\ C = \widehat{F} \cap H, \end{cases} \quad \text{or} \quad \widehat{F} *_C H, \tag{4}$$

with $F \leq \widehat{F} = \mathrm{stab}(v_0)$.

Lemma 6.13. *Let the action of $F_{R(S)}$ on T be as in the statement of Lemma 6.12. Let $v_0 = \mathrm{fix}(F)$ and suppose that there is some ρ such that $T_F \cap \rho T_F \neq \emptyset$, $d(v_0, \rho v_0) = 2$, and the path $[v_0, \rho v_0]$ intersects two $F_{R(S)}$-vertex orbits. Let $H = \mathrm{stab}(w_0)$ be as described above. Using the notation introduced in (4), there exist $f_1, f_2 \in F$ such that either*

1. *the path $[v_0, \rho v_0]$ intersects one $F_{R(S)}$-edge orbit and $f_2 \rho f_1 = h \in H$; or*
2. *the path $[v_0, \rho v_0]$ contains two $F_{R(S)}$-edge orbits and $f_2 \rho f_1 = th$ for some $h \in H$, which could be assumed to be the stable letter t if we change the presentation by conjugating a boundary monomorphism.*

Proof. We consider case (1). By hypothesis T_F cannot be a point, $w_0 \in T_F$, and $H \cap F \neq \{1\}$. Consider Figure 2. Then we must have

$$\rho = a_2 b_1 a_1, \quad a_i \in \widehat{F},\ b_1 \in H,$$

with $w_1 = a_2 w_0$. Now $a_2 e \in T_F$, so by Lemma 6.12 there exists an $f_2 \in F$ such that $f_2 a_2 e = e$ so that $f_2 a_2 \in \mathrm{stab}(e) = H \cap \widehat{F}$. So up to replacing b_1 by $f_2 a_2 b_1 \in H$ we may now assume that $\rho = b_1 a_1$ and that $w_0 \in [v_0, \rho v_0]$.

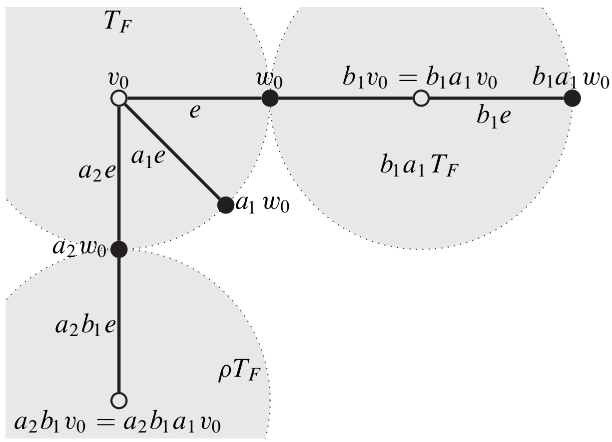


Figure 2. The path from v_0 to ρv_0 only intersects one $F_{R(S)}$-edge orbit.

Looking again at Figure 2 we see that $T_F \cap b_1 a_1 T_F \neq \emptyset$ but more specifically that $b_1 e \subset b_1 a_1 T_F$. This gives ${}^{b_1}\mathrm{stab}(e) \cap {}^{b_1 a_1} F \neq \{1\}$, from which it follows that $\mathrm{stab}(e) \cap {}^{a_1} F \neq \{1\}$ and so ${}^{a_1^{-1}}\mathrm{stab}(e) \cap F \neq \{1\}$, which implies that $a_1^{-1} e \subset T_F$. By Lemma 6.12 there is some $f_1^{-1} \in F$ such that if $f_1^{-1} a_1^{-1} e = e$ then $f_1^{-1} a_1^{-1} = b' \in H \cap F$. It follows that $\rho f_1 = b_1 a_1 f_1 = b_1 b'^{-1} \in H$. Case (2) is proved similarly. □

6.1.5. Avoiding transmissions. Folding sequences are difficult to analyse. The goal of this section is to prove some lemmas that give us some control over folding sequences, making them more tractable.

The only moves applied to a $\mathcal{G}(A)$-graph $\mathcal{B}$ that may increase the number of non-trivial $\mathcal{B}$-vertex groups are (T1) transmissions and (F4) folds, however an (F4) fold decreases the number of cycles in the underlying graph B by 1. Whenever there is a proper transmission to some $\mathcal{B}$-vertex group $\mathcal{B}_v$, it gets enlarged to $\langle \mathcal{B}_v, \alpha\rangle$ for some $\alpha \in A_{[v]}$, this in a sense increases the complexity of $\mathcal{B}$. We will give conditions that enable us to perform a maximal number of (F1) and (F4) foldings without having to resort to transmissions. This enables us to keep most $\mathcal{B}$-vertex groups of our $\mathcal{G}(A)$-graphs trivial or cyclic.

Convention 6.14. Throughout this section we will assume that in the graph of groups $\mathcal{G}(A)$, the images of edge groups are maximal abelian and malnormal in the vertex groups. We will also require our splittings to be 1-acylindrical. These conditions imply that if there are distinct edges e, f in A such that $i(e) = u = i(f)$ then the images of $i_e \colon A_e \to A_u$ and $i_f \colon A_f \to A_u$ cannot be conjugated into one another inside A_u.

Definition 6.15. Let u be a vertex in a $\mathcal{G}(A)$-graph $\mathcal{B}$ such that $\mathcal{B}_u$ is either abelian or trivial. Suppose that we have a subgraph

$$l = v \bullet \xrightarrow{(a,e,b)} u \bullet \xrightarrow{(a',e^{-1},b')} \bullet\, w$$

such that we are able to perform a sequence of transmissions t_1, t_2, where t_1 and t_2 are through the different edges of l, and suppose afterwards that $\mathcal{B}_u$ is still abelian. Then l is called a *cancellable path centered at u*.

Lemma 6.16. *Let $A, B \leq G$ be two abelian subgroups of a fully residually free group G such that for some $a \in A$, $b \in B$ we have $[a, b] \neq 1$. Then we have*

$$\langle A, B\rangle = A * B$$

Proof. Let $w = a_1 b_2 a_3 \ldots b_n$ be product of non-trivial factors $a_i \in A$ and $b_j \in B$ with perhaps the exception that a_1 or b_n are trivial. Since G is fully residually free there exists a map of G into F such that all the non-trivial a_i, b_j as well as some commutator $[a, b]$, $a \in A$, $b \in B$, do not vanish. We have that the a_i are sent to powers of some element $u \in F$ and the b_j are sent to powers of some $v \in F$. It follows that the homomorphic image of w is sent to a freely reduced word in u and v, and since $u, v \in F$ do not commute they freely generate a free subgroup of F. It follows that w is not sent to a trivial element. □

Lemma 6.17. *Let $\mathcal{B}$ be a $\mathcal{G}(A)$-graph and suppose it has a cancellable path centered at u. Then it is possible to perform a folding* (F1) *or* (F4) *at u in $\mathcal{B}$ using only a Bass–Serre* (A1) *move, and maybe a conjugation* (A0) *move.*

Proof. W.l.o.g. the edges in the cancellable path are edges e, e' with $i(e) = u = i(e')$ and with labels $(a, [e], b)$, $(a', [e'], b')$, respectively. We have $\mathcal{B}_u \le A_{[u]}$ and we have injections $i_{[e]}\colon A_{[e]} \hookrightarrow A_{[u]}$ and $i_{[e']}\colon A_{[e']} \hookrightarrow A_{[u]}$. Denote the images $i_{[e]}(A_{[e]}) = A$ and $i_{[e']}(A_{[e']}) = A'$.

Let $\mathcal{B}_2$ be the $\mathcal{G}(A)$-graph obtained from $\mathcal{B}$ after applying the two transmissions that witness the fact that e, u, e' is a cancellable path, then

$$(\mathcal{B}_2)_u \cap aAa^{-1} \neq \{1\} \neq (\mathcal{B}_2)_u \cap a'A'a'^{-1}.$$

By Lemma 6.16, $\langle aAa^{-1}, a'Aa'^{-1}\rangle$ is either $(aAa^{-1}) * (a'Aa'^{-1})$, or free abelian. Since $(\mathcal{B}_2)_u$ is abelian, $\langle aAa^{-1}, a'Aa'^{-1}\rangle$ is free abelian. The subgroups aAa^{-1} and $a'A'a'^{-1}$ therefore lie inside a maximal abelian group C. It therefore follows that A and A' are not conjugacy separated in $A_{[u]}$. Convention 6.14 forces $A = A'$ and hence $[e] = [e']$. This also means that, by malnormality,

$$a'^{-1}aAa^{-1}a' = A \Rightarrow a'^{-1}a = i_{[e]}(\alpha) \in A$$

for some $\alpha \in A_{[e]}$.

Therefore if we consider $\mathcal{B}$ before any transmissions were performed, using a Bass–Serre (A1) move we can change the label of e' as follows,

$$(a', [e'], b') \Rightarrow (a' i_{[e]}(\alpha), [e], t_{[e]}(\alpha^{-1})b') = (a, [e], b''),$$

and then either perform an (F4) fold if $t(e) = t(e')$ or an (A0) conjugation at $t(e)$ and then an (F1) fold at u. □

Lemma 6.18. *Let $\mathcal{B}$ be a $\mathcal{G}(A)$-graph. Suppose that $\mathcal{B}_{v_0}, \dots, \mathcal{B}_{v_m}$ are the non-trivial $\mathcal{B}$-vertex groups. Suppose that for some $u \notin \{v_0, \dots, v_m\}$, after a sequence $t_1, \dots, t_n$ of transmissions yielding a $\mathcal{G}(A)$-graph $\mathcal{B}_n$, the $\mathcal{B}$-vertex group $(\mathcal{B}_n)_u$ is abelian or trivial and after maybe making some* (A0)–(A2) *adjustments it is possible to perform either move* (F1) *or* (F4) *at u. Then it is also possible to perform a folding move* (F1) *or* (F4) *at u after only applying a sequence of* (A0)–(A2) *and* (L1) *adjustments to $\mathcal{B}$.*

Proof. By Lemma 6.17 we may assume that there are no cancellable paths in $\mathcal{B}$. We observe that the (A0) conjugation and the (A1) Bass–Serre moves which can be made at (an edge incident to) u do not depend on $\mathcal{B}_u$.

Suppose first that $(\mathcal{B}_n)_u$ is trivial. Then there are no new possible (A2) simple adjustments at u in $\mathcal{B}_n$, so the result holds.

Suppose now that $(\mathcal{B}_n)_u$ is non-trivial abelian and that in $(\mathcal{B}_n)$ we can perform an (A2) move changing the label of e, where $i(e) = u$ and then after performing moves (A1), (A0) at $t(e)$ we can do either move (F1) or (F4) identifying e and e'. In particular e and e' had labels $(a, [e], b)$ and $(a', [e], b')$ respectively and after doing move (A2) at u the labels became either $(a'c, [e], b)$ and $(a', [e], b')$ or $(a, [e], b)$ and $(ac, [e], b')$, for some c in the edge group, respectively.

If there were no transmission through either e or e', then w.l.o.g. there were no transmissions through e', and we can use an (L1) long-range adjustment in $\mathcal{B}$ to change $(a', [e'], b')$ to $(ac, [e'], b')$. In both cases after applying an (A1) Bass–Serre move, we can then apply an (F4) or (after the appropriate (A0) conjugation) an (F1) folding move.

Otherwise there are transmissions through e and e', so e, u, e' is a cancellable path in $\mathcal{B}_n$. Lemma 6.17 implies that we only needed (A0) and (A1) moves to change the labels of e, e' to enable an (F1) or (F4) fold. In particular, we could have made these moves before in $\mathcal{B}$ before any transmissions were used. The result now follows. □

6.1.6. The strategy. For each possibility given in (3) in Section 6.1.1 we will do the following:

(1) Get the maximal abelian collapse of the JSJ of $F_{R(S)}$, and use this splitting as the underlying graph of groups $\mathcal{G}(X)$. This must be one of the graphs given in (3).
(2) We prove using Theorem 6.9 and Lemma 6.13, that we can always arrange so that $\bar{x}$ is somehow simple, i.e., $\bar{x}$ lies in $\widehat{F} \cup H$ or is a stable letter, up to conjugating boundary monomorphisms.
(3) We then take the wedge $\mathcal{B} = \mathcal{W}(F, \bar{x}, \bar{y})$, with the loop $\mathcal{L}(\bar{x})$ of length at most 2. Now $\mathcal{B}$ must fold down to $\mathcal{G}(X)$.
(4) We will then apply folding moves to simplify the graph as much as possible *while avoiding transmissions*. It will turn out using the results of Section 6.1.5 that the resulting $\mathcal{G}(X)$-graph $\mathcal{B}$ will have the underlying graph as X.
(5) All that will then remain to get a folded graph is to make some transmission moves, keeping track of these will tell us how the vertex groups are generated.
(6) Finally, by arguing algebraically we will recover the original cyclic JSJ decomposition of $F_{R(S)}$ modulo F.

6.2. The one edge case. We consider the case where the maximal abelian collapse of $F_{R(S)}$ is a free product with amalgamation

$$F_{R(S)} = \widehat{F} *_A H \tag{5}$$

with $F \leq \widehat{F}$, A maximal abelian in both factors and H non-abelian. Throughout this section $\widehat{F}$, A, H will denote these groups.

6.2.1. Arranging so that $\bar{x}$ lies in H

Lemma 6.19. *Let $F_{R(S)}$ be freely indecomposable modulo F with a maximal abelian collapse (5). After Weidmann–Nielsen normalization on $(F, \bar{x}, \bar{y})$ we can arrange, conjugating boundary monomorphisms if necessary, so that $\bar{x}$ lies in either $\widehat{F}$ or H.*

Proof. Since we are assuming free indecomposability of $F_{R(S)}$ modulo F, we can apply Theorem 6.9. Let T be the Bass–Serre tree induced from the splitting (5). Let $v_0 = \mathrm{fix}(F)$. We start by looking at the marked generating set $(F; \{\bar{x}, \bar{y}\})$. We consider different cases.

Case I: T_F *is a point*. Since $F_{R(S)}$ is not freely decomposable by Theorem 6.9, w.l.o.g. after Weidmann–Nielsen normalization $\bar{x}$ must be elliptic. $T_{\langle \bar{x} \rangle}$ is either a vertex or an edge.

Case I.I: $T_F \cap T_{\langle \bar{x} \rangle} = \emptyset$. Consider the marked generating set $(F, \langle \bar{x} \rangle; \{\bar{y}\})$ and apply Theorem 6.9 again. We now find that after Weidmann–Nielsen normalization either, w.l.o.g. $\bar{y} T_{\langle \bar{x} \rangle} \cap T_{\langle \bar{x} \rangle} \neq \emptyset$, or $\bar{y}$ is also elliptic.

We always have the latter possibility. Indeed, by 1-acylindricity $T_{\langle \bar{x} \rangle}$ is either an edge or a point so for $\bar{y} T_{\langle \bar{x} \rangle} \cap T_{\langle \bar{x} \rangle} \neq \emptyset$ we must have that $\bar{y}$ fixes one of the endpoints of $T_{\langle \bar{x} \rangle}$ which implies that $\bar{y}$ is elliptic.

If $\bar{y}$ is also elliptic then the trees $T_F, T_{\langle \bar{x} \rangle}, T_{\langle \bar{y} \rangle}$ cannot all be disjoint, otherwise $F_{R(S)}$ would be freely decomposable modulo F. If $T_{\langle \bar{y} \rangle} \cap T_F \neq \emptyset$ then we can switch $\bar{x}$ and $\bar{y}$ and pass to case I.II Otherwise $T_{\langle \bar{y} \rangle} \cap T_{\langle \bar{x} \rangle} \neq \emptyset$ then the tree $T_{\langle \bar{y}, \bar{x} \rangle}$ is either a point, an edge, or has radius 1. Passing to the marking $(F, \langle \bar{x}, \bar{y} \rangle; \emptyset)$ and applying Theorem 6.9 implies that $T_{\langle \bar{y}, \bar{x} \rangle}$ can be taken so that $T_F \cap T_{\langle \bar{y}, \bar{x} \rangle} \neq \emptyset$. Which means that, conjugating boundary monomorphisms in $\widehat{F}$ if necessary, both $\bar{x}$ and $\bar{y}$ can be brought into H.

Case I.II: $T_F \cap T_{\langle \bar{x} \rangle} \neq \emptyset$. We are assuming that $T_F = v_0$. Since $\bar{x}$ fixes v_0 we have $\bar{x} \in \widehat{F}$.

Case II: T_F *is not a point*. Conjugating boundary monomorphisms, we can arrange for some generator α of A to lie in F. We apply Theorem 6.9 and find that either $\bar{x} T_F \cap T_F \neq \emptyset$ or $\bar{x}$ is elliptic. In the former case we have $d(v_0, \bar{x} v_0) = 2$ and we can apply Lemma 6.13 to make $\bar{x} \in H$ and we are done. Otherwise $\bar{x}$ is elliptic and we consider the next case.

Case II.I: $T_F \cap T_{\langle \bar{x} \rangle} = \emptyset$. We consider the marked set $(F, \bar{x}; \{\bar{y}\})$ and we see that applying Theorem 6.9 we can either arrange for $\bar{y}$ to be elliptic or get that either $T_F \cap \bar{y} T_F \neq \emptyset$ or $T_{\langle \bar{x} \rangle} \cap \bar{y} T_{\langle \bar{x} \rangle} \neq \emptyset$. In case I.I the latter possibility was seen to be impossible unless $\bar{y}$ is elliptic. If $\bar{y} T_F \cap T_F \neq \emptyset$, then we can apply Lemma 6.13 as for the previous case and obtain that $\bar{y} \in H$, and we are done.

It therefore remains to verify the case where $\bar{y}$ is elliptic and $T_{\langle \bar{y} \rangle} \cap T_F = \emptyset$. For our group not to be freely decomposable modulo F we must have that $T_{\langle \bar{y} \rangle} \cap T_{\langle \bar{x} \rangle} \neq \emptyset$. Moreover since both $\bar{x}, \bar{y}$ are elliptic the tree $T_{\langle \bar{x}, \bar{y} \rangle}$ must have radius 1. This is dealt with exactly as in the end of case I.I.

Case II.II: $T_F \cap T_{\langle \bar{x} \rangle} \neq \emptyset$. This means that $\bar{x}$ either fixes v_0 or some vertex in T_F. In all cases, applying Lemma 6.12 and conjugating boundary monomorphisms if necessary, this implies that $\bar{x} \in H$ or $\bar{x} \in \widehat{F}$. □

Lemma 6.20. *Let $A \leq F_{R(S)}$ be abelian and let c be such that $c^{-1} A c \cap A = \{1\}$. Then $c \notin K = \langle A, c^{-1} A c \rangle$ and the centralizer of A in K, $Z_K(A) = A$.*

Proof. By Lemma 6.16, $K = A * B$ where $B = c^{-1}Ac$. If there is some word $U(A, B)$ such that we have the relation

$$U(A, B)^{-1}a^{-1}U(A, B)a = 1$$

for some $a \in A$, then the free product structure implies that $U(A, B) = U(A)$, so the second claim holds. It also follows that $Z_K(B) = B$. Suppose now that $c \in K$. Then for each $a \in A$ we have that $c^{-1}ac \in B$, which is impossible from the free product structure. □

Lemma 6.21. *Suppose that $F_{R(S)}$ has a maximal abelian collapse (5). If $F_{R(S)}$ is generated as $\langle F, \bar{x}, \bar{y}\rangle$ with $\bar{x} \in \widehat{F}$, then $F_{R(S)}$ is freely decomposable modulo F.*

Proof. Let $\mathcal{G}(X)$ be the graph of groups representing the splitting (5). Then $F_{R(S)}$ can be represented by a $\mathcal{G}(X)$-graph $\mathcal{B}$ which consists of a vertex v with $\mathcal{B}_v = \langle F, \bar{x}\rangle$ and a $\bar{y}$-loop $\mathcal{L}(\bar{y}, v)$.

Since $\pi_1(\mathcal{B}, v) = F_{R(S)}$, by Theorem 3.15 we should be able to bring $\mathcal{B}$ to $\mathcal{G}(X)$ using the moves of Section 3.2.2. Now only the $\mathcal{B}$-vertex group $\mathcal{B}_v$ is non-trivial. We do our folding process using only moves (A0)–(A3), (F1), (F4), (L1) and (S1). If an (F4) collapse occurs then $\mathcal{B}$ does not have any cycles, but it may have an extra non-trivial cyclic $\mathcal{B}$-vertex group. By doing (S1) shaving moves we can assume that either $\mathcal{B}$ is a line with endpoints the vertices v, u with $\mathcal{B}_v = \langle F, \bar{x}\rangle$, $\mathcal{B}_u$ cyclic and all other $\mathcal{B}$-vertex groups trivial, or $\mathcal{B}$ is a loop with only $\mathcal{B}_v = \langle F, \bar{x}\rangle$ non-trivial.

By Lemma 6.18 and Lemma 6.17 we see that we can always avoid using transmissions unless one of the three following possibilities occurs:

Case I: $\mathcal{B}$ has two vertices v, u and one edge e. At this point we have $\mathcal{B}_u = H'$ which is cyclic. After a transmission $\mathcal{B}_u = \langle A, H'\rangle$, where A is conjugable into the image of an edge group. The graph is then folded, but we see by Lemma 6.16 that $H = A * H'$ which implies free decomposability of $F_{R(S)}$ modulo F.

Case II: $\mathcal{B}$ has no cycles, three vertices and two edges. We assume that all (S1) shaving moves were performed. Then the only possibility for $\mathcal{B}$ is that it has endpoints v and u, with $\mathcal{B}_u$ cyclic, and the other $\mathcal{B}$-vertex group $\mathcal{B}_w$ is trivial. If it is possible to transmit from $\mathcal{B}_u$ then u can be shaved off. If it is possible to transmit from v to w and then from w to u, then there is a cancellable path centered at w and Lemma 6.17 applies. So in both cases we can continue folding $\mathcal{B}$ without using transmissions.

Case III: $\mathcal{B}$ consists of a cycle of length 2, with vertices v and u. This means that the $\mathcal{B}$-vertex group $\mathcal{B}_u$ is trivial. For $\mathcal{B}$ to be folded, after a transmission we must be able to perform an (F4) collapse move.

If the collapse is towards u we distinguish two possibilities. Either no transmissions are needed and $\mathcal{B}_u$ will be generated by some element and the edge group, which implies free decomposability modulo F; or there is a transmission from v to u through one edge followed by a transmission from u to v through the other edge, but this gives a cancellable path so Lemma 6.17 applies, and we get a collapse from u towards v which we immediately deal with below.

The remaining possibility is that the collapse is towards v. If no transmissions were needed, then $\mathcal{B}_u$ is generated by the edge group and we have $F_{R(S)} = \widehat{F}$, a contradiction. Otherwise by Lemma 6.18, before the collapse, we must do *two* transmissions from v to u so that, after an (A0) conjugation move, we have w.l.o.g. $\mathcal{B}_u = \langle b_1 A' b_1^{-1}, A'' \rangle$ where $A', A'' \subset A$ and $b_1 \not\in A$. We now either do a simple adjustment and collapse towards v or some non-trivial transmission from u to v. For a collapse we need $b_1 \in \langle b_1 A' b_1^{-1}, A'' \rangle$, for a non-trivial transmission we need either A'' or $b_1 A' b_1^{-1}$ to have a proper centralizer in $\mathcal{B}_u$. Both of these are forbidden by Lemma 6.20. □

6.2.2. Arranging so that $\bar{y}$ lies in H as well

Lemma 6.22. *Let $A \leq F_{R(S)}$ be a maximal non-cyclic abelian subgroup, then either*

- *$\langle F, A \rangle = F * A$, or*
- *there is some $p \in F$ such that $\langle F, A \rangle = F *_{\langle p \rangle} A$, or*
- *$\langle F, A \rangle = \langle F, r, A \mid [r, p] = 1 \rangle$ where $p \in F$ and $\langle a \rangle = \langle F, r \rangle \cap A$. Moreover $\langle F, r \rangle = \langle F, a \rangle$. In particular no conjugate of an element of F is centralized by A.*

Proof. Since $\langle F, A \rangle \leq F_{R(S)}$ it is fully residually F and since $F \neq \langle F, A \rangle$ it has an essential cyclic or free splitting modulo F. Since A is non-cyclic abelian it is forced to be elliptic. Suppose that $\langle F, A \rangle \neq F * A$. Then it has an essential cyclic splitting and since it is generated by elliptic elements the underlying graph of groups has no cycles. The only possibilities are that there is some $a \in A$ such that $\langle F, A \rangle = \langle F, a \rangle *_{\langle a \rangle} A$ or that there is some $p \in F$ such that $\langle F, A \rangle = F *_{\langle p \rangle} H$ with $H = \langle p, A \rangle$.

We first consider the former case, since $\langle F, A \rangle$ is assumed to be freely indecomposable, Theorem 2.2 implies that $\langle F, a \rangle = \langle F, r \mid [r, p] \rangle$ for some $p \in F$, but if we had $[a, f] = 1$ for some $f \in F$ by normal forms we see $[f, A] \neq 1$ contradicting commutation transitivity. A is therefore the centralizer of an element that is hyperbolic in the cyclic JSJ of $\langle F, r \mid [r, p] \rangle$ modulo F.

The remaining case is that $\langle F, A \rangle = F *_{\langle p \rangle} H$ with $H = \langle p, A \rangle$. By Lemma 6.16, $H \neq \langle p \rangle * A$ only if $p \in A$. The result therefore holds. □

Corollary 6.23. *Let $A \leq F_{R(S)}$ be a non-cyclic maximal abelian subgroup that centralizes some $\alpha \in F$, and let $a \in F_{R(S)}$ be such that there is no $f \in F$ with $fa \in A$. Then $\langle F, aAa^{-1} \rangle = F * aAa^{-1}$ and $a \not\in \langle F, aAa^{-1} \rangle$.*

Proof. By Lemma 6.22 $\langle F, aAa^{-1} \rangle = F * aAa^{-1}$ or $F *_{\langle p \rangle} aAa^{-1}$ for some $p \in F$. Suppose towards a contradiction that the latter possibility holds. Then $p = a\alpha'' a^{-1}$, $\alpha'' \in A$. A is discriminated by F-retractions $F_{R(S)} \to F$. This means that every element of $aAa^{-1} \setminus F$ can be sent to arbitrarily high powers of α via such retractions. It follows that $\alpha'' = \alpha^n$. Since F has property CC there is some $f \in F$ such that

$f(a\alpha^n a^{-1})f^{-1} = \alpha^n$ which implies that if $[fa, \alpha] = 1$ then $fa \in A$ since A is maximal abelian. This contradicts the hypothesis.

We therefore have $\langle F, aAa^{-1}\rangle = F * aAa^{-1}$, and $a \notin \langle F, aAa^{-1}\rangle$ follows from the fact that the free product structure implies that the centralizer of α lies in F. □

Lemma 6.24. *Suppose $F_{R(S)}$ is freely indecomposable modulo F and has a maximal abelian collapse (5) and is generated as $\langle F, \bar{x}, \bar{y}\rangle$ with $\bar{x} \in H$. Then $F_{R(S)}$ is generated by $\langle F, \bar{x}, \bar{y}'\rangle$ where $\bar{y}'$ also lies in H.*

Proof. We start with the $\mathcal{G}(X)$-graph $\mathcal{B}$ with one edge e and two vertices v, u with $\mathcal{B}_v = F$ and $\mathcal{B}_u = \langle \bar{x}\rangle$, then at v attach the $\bar{y}$-loop $\mathcal{L}(\bar{y}, v)$. Start our adjustment-folding process applying only moves (A0)–(A3), (F1), (F4), (L1), (S1), as much as possible, but avoiding transmissions.

Case I: We brought $\mathcal{B}$ to a graph with two vertices u, v and one edge without having to use transmissions. We then either have $\mathcal{B}_u = F$ and $\mathcal{B}_v = \langle \bar{x}, \bar{y}'\rangle$, in which case the result follows; or $\mathcal{B}_u = \langle F, \bar{y}'\rangle$ and $\mathcal{B}_v = \langle \bar{x}\rangle$, which by Lemma 6.21 implies free decomposability of $F_{R(S)}$ modulo F, which is a contradiction.

Case II: An (F4) collapse occurred. In this case $\mathcal{B}$ is a line with one endpoint either u or v and the other endpoint is some vertex w with $\mathcal{B}_w = \langle \bar{y}'\rangle$. We see that if any transmissions from w were possible then could use an (S1) shaving move and remove w. On one hand this graph should fold down to $\mathcal{G}(X)$, on the other hand by Lemma 6.18, if w is at distance at least 2 from either u or v we can apply an (F1) move without using transmissions.

We can therefore assume that $\mathcal{B}$ has two edges, and three vertices with w as an endpoint. The last fold will be an (F1) fold. We note that it is impossible for there to be a transmission *to* w followed by a transmission *from* w. Indeed, suppose this were the case, then after the first transmission we have by Lemma 6.16 that $\mathcal{B}_w = \langle \bar{y}', A'\rangle = \langle \bar{y}'\rangle * \langle A'\rangle$, where A' is conjugable into A. Now for there to be a transmission back from w we need A' to have a proper centralizer in $\langle \bar{y}'\rangle * \langle A'\rangle$, which is impossible.

So any transmissions preceding a simple adjustment that enables the (F1) fold will be through the edge connecting u and v. It follows that instead we can make a long-range adjustment (L1) on the edge adjacent to w, and then apply the (F1) fold. Since no transmissions were used we have reduced this to case I.

Case III: No collapses occurred. In this case $\mathcal{B}$ contains a cycle and the only non-trivial $\mathcal{B}$-vertex groups are $\mathcal{B}_u$ and $\mathcal{B}_v$. We note that the cycle must be of even length so by Lemma 6.18 we can assume that the cycle has length 2 and contains u or v. We distinguish two subcases.

Case III.I: $\mathcal{B}$ has three vertices u, v, w with $B_w = \{1\}$ and the cycle in $\mathcal{B}$ consists of two edges e, f going from w to u or w to v. First note that it is impossible for them to be a transmission to w through e followed by a transmission from w through f since then we would have a cancellable path and could make a collapse at w contrary to our assumptions.

Suppose now that it is possible to transmit to w through the edges e and f. Then by Lemma 6.20 it is impossible to perform a simple adjustment at w preceding an (F4) collapse at w and it is impossible to make a new transmission from w back through e or f. It therefore follows that the next fold is preceded only by transmission through the edge between u and v, so again we can make an (L1) long-range adjustment to change the label of either e or f (but not both) and then perform either an (F4) collapse, which brings us to case I, or an (F1) fold which brings us to case III.II.

Case III.II: $\mathcal{B}$ has two vertices, two edges and one cycle. Then we can represent $\mathcal{B}$ as the $\mathcal{G}(X)$-graph

$$v \bullet \underset{(a',e,b')}{\overset{(a,e,b)}{\rightrightarrows}} \bullet\, u, \tag{6}$$

with $a, a' \in \widehat{F}$ and $b, b' \in H$ and $\mathcal{B}_u = F, \mathcal{B}_v = \langle \bar{x} \rangle$. Now if a transmission from u were possible then w.l.o.g. we could express $F_{R(S)}$ as the same $\mathcal{G}(X)$-graph but with $\mathcal{B}_v = \langle F, ab\bar{x}b^{-1}a^{-1} \rangle$ and $\mathcal{B}_u = \{1\}$, which by Lemma 6.21 implies free decomposability modulo $F_{R(S)}$.

It follows that if no transmissions from v are possible then no transmissions at all are possible and we can therefore make an (F1) collapse without transmission and reduce to the case I.

We may therefore assume that a transmission from $\mathcal{B}_v$ is possible which implies that $F \cap A = \langle \alpha \rangle \neq \{1\}$ and that there is some $f \in F$ such that $fa' \in A$. This means that using an (A2) simple adjustment, an (A1) Bass–Serre move and an (A0) conjugation we may assume that $a' = b' = 1$ in (6). We can therefore put $\mathcal{B}_v = F$ and $\mathcal{B}_u = \langle \bar{x}, \alpha \rangle$. Now note that if it were also possible to transmit from v through the other edge then before any transmissions we could use an (A2) simple adjustment an (A1) Bass–Serre move to change the label (a, e, b) to $(1, e, b'')$ in (6), which means that we can reduce to the case I. Hence,

(†) we may assume that there is no $f \in F$ such that $fa \in A$.

If no further transmissions are possible then we must be able to perform a collapse from u to v. This means that $b \in \langle \bar{x}, \alpha \rangle$ so that after collapsing we get a $\mathcal{G}(X)$-graph $\mathcal{B}$ with one edge labeled $(1, e, 1)$ and $\mathcal{B}$-vertex groups $\mathcal{B}_v = \langle F, a \rangle$ and $\mathcal{B}_u = \langle \bar{x}, \alpha \rangle$. After transmissions we have $\mathcal{B}_u = \langle A, \bar{x} \rangle$ which by Lemma 6.16 implies that $F_{R(S)}$ is freely decomposable modulo F.

We can therefore assume that $b \notin \langle \bar{x}, \alpha \rangle$ but that there is a transmission from u to v through the edge labeled (a, e, b). This means that there is some $\alpha' \in A$ such that $b^{-1}\alpha' b \in \langle \bar{x}, \alpha \rangle$. So after the transmission we have that $\mathcal{B}_v = \langle F, a\alpha' a^{-1} \rangle$. By Corollary 6.23 and (†) $\mathcal{B}_v \leq F * aAa^{-1}$ and since $\langle \alpha' \rangle \leq A$, it follows that $\mathcal{B}_v = F * a\langle \alpha' \rangle a^{-1}$. So no further transmissions from v to u are possible since by the free product structure $Z_{\mathcal{B}_v}(\alpha)$, the centralizer of α in $\mathcal{B}_v$ is $\langle \alpha \rangle$.

Since $\mathcal{B}$ must fold down to a graph with one edge we must be able to perform a simple adjustment to change the label $(1, e, 1)$ to $(a, e, 1)$ and fold $\mathcal{B}$ down to

$$v \bullet \xrightarrow{(a,e,1)} \bullet\, u.$$

But to do this we would need $a \in \mathcal{B}_v$, but we saw in the previous paragraph that after the only possible transmission $\mathcal{B}_v \le F * aAa^{-1}$, so by Corollary 6.23 this is impossible. Having exhausted all the possibilities the result follows. □

From the previous lemmas we get

Proposition 6.25. *If $F_{R(S)}$ is freely indecomposable modulo F and has a maximal abelian collapse (5), then, conjugating boundary monomorphisms if necessary, $F_{R(S)}$ can be generated by F and two elements $\bar{x}, \bar{y} \in H$.*

6.2.3. Recovering the original cyclic splitting from the maximal abelian collapse. This next proposition enables us to revert to a cyclic splitting.

Proposition 6.26. *Suppose that $F_{R(S)}$ is freely indecomposable modulo F and has a maximal abelian collapse (5) then $F_{R(S)}$ admits a cyclic splitting*

$$\widetilde{F}' *_{\langle\alpha\rangle} H'$$

where either:

(1) *$\widetilde{F}' = F$ and H' is generated by $\langle \alpha, \bar{x}, \bar{y}\rangle, \alpha \in F$. Hence H' is a 3-generated fully residually free group (see Theorem 2.1).*

(2) *$\widetilde{F}' = \langle F, \alpha\rangle$ and $H' = \langle \bar{x}, \bar{y}\rangle$ with $\alpha \in H'$, i.e., H' is free of rank 2.*

Proof. We first consider when the amalgamating maximal abelian subgroup A in (5) is cyclic. We write $A = \langle\alpha\rangle$ and $\widetilde{F}' = \widetilde{F}, H' = H$, then this splitting is in fact already a maximal abelian collapse so we can apply Proposition 6.25 and we get that $\bar{x}, \bar{y} \in H$. Looking at normal forms we have that $\widehat{F} = \langle F, \alpha\rangle$ and $H = \langle \alpha, \bar{x}, \bar{y}\rangle$ if $\alpha \in F$ then $\widehat{F} = F$ and H is 3-generated fully residually free. If $\alpha \notin F$ then we must have $\alpha \in \langle \bar{x}, \bar{y}\rangle$ and it follows that $H = \langle \bar{x}, \bar{y}\rangle$.

We now consider the case where A in (5) is not cyclic. By Proposition 6.25 we can assume that $\bar{x}, \bar{y} \in H$ and it follows that $\widehat{F} = \langle F, A\rangle$. First suppose that some conjugate of A centralizes some element of F. Then by Lemma 6.22 and by free indecomposability of $F_{R(S)}$ modulo F we have

$$\widehat{F} = F *_{\langle\alpha\rangle} A,$$

which means that $F_{R(S)}$ admits the splitting

$$F_{R(S)} = F *_{\langle\alpha\rangle} (A *_A H),$$

and as before we get that $H' = \langle \bar{x}, \bar{y}, \alpha\rangle$ is a 3-generated fully residually free group.

The remaining possibility is that no conjugate of A centralizes an element of F. So by Lemma 6.22 and by free indecomposability of $F_{R(S)}$ we have $\langle F, A\rangle = \widehat{F}' *_{\langle a\rangle} A$, where $\widehat{F}'$ is a rank 1 centralizer extension of F. This means that we can unfold the splitting (5) to get the cyclic splitting

$$\widehat{F}' *_{\langle a\rangle} (A *_A H)$$

and by Proposition 6.25, $F \leq \widehat{F}'$ and $\bar{x}, \bar{y} \in H$. Since no conjugate of F intersects A, we need w.l.o.g. $\langle x, y\rangle \cap A = \langle a^n\rangle$ for some $n \in \mathbb{Z}$. Now by free indecomposability and by Theorem 2.2, $\langle F, a^n\rangle$ must be a centralizer extension of F. Note however that the centralizer of a^n in $\langle F, a^n\rangle$ must be $\langle a^n\rangle$, so there are no transmissions from $\widehat{F}'$ back to $(A *_A H)$. Therefore

$$F_{R(S)} = \widehat{F}' *_{\langle a^n\rangle} (\langle x, y\rangle),$$

which contradicts the assumption that $\langle a\rangle$ is contained in a non-cyclic abelian subgroup. □

An element in a free group is called *primitive* if it belongs to a basis. For the next result, we need:

Theorem 6.27 (Main Theorem of [Bau65]). *Let $w = w(x_1, x_2, \ldots, x_n)$ be an element of a free group F freely generated by $x_1, x_2, \ldots, x_n$ which is neither a proper power nor a primitive. If $g_1, g_2, \ldots, g_n, g$ are elements of a free group connected by the relation*

$$w(g_1, g_2, \ldots, g_n) = g^n \quad (m > 1),$$

then the rank of the group generated by $g_1, g_2, \ldots, g_n, g$ is at most $n - 1$.

Lemma 6.28. *Suppose that $F_{R(S)}$ splits as*

$$F *_{\langle\alpha\rangle} A *_{\langle\gamma\rangle} \langle\bar{x}, \bar{y}\rangle,$$

where A is non-cyclic abelian and $\langle\gamma\rangle \cap \langle\alpha\rangle = \{1\}$. Then $F_{R(S)}$ is freely decomposable modulo F. In particular, $\langle\gamma\rangle$ must be a free factor of $\langle\bar{x}, \bar{y}\rangle$.

Proof. First note that $\alpha \in F$ and $\gamma \in \langle\bar{x}, \bar{y}\rangle$ are not proper powers (otherwise, looking at normal forms, we would find a contradiction to commutation transitivity). Suppose on the contrary that $\gamma \in \langle\bar{x}, \bar{y}\rangle$ is not primitive.

Since $\langle\bar{x}, \bar{y}\rangle$ is a 2-generated non-abelian subgroup of a fully residually free group, there are retractions $f: F_{R(S)} \to F$ such that the restriction of f to $\langle\bar{x}, \bar{y}\rangle$ is a monomorphism. Since we also have that $\alpha \in F$, there are retractions that are monomorphic on $\langle\bar{x}, \bar{y}\rangle$ that send γ to arbitrarily high powers of α. We fix f so that it is injective on $\langle\bar{x}, \bar{y}\rangle$ and such that $f(\gamma)$ is a proper power. Denote by $K \leq F$ the image of $\langle\bar{x}, \bar{y}\rangle$. K is free of rank 2, but we see that in the ambient group $f(\gamma)$

is a proper power. Theorem 6.27 applies and the assumption that $\gamma \in \langle \bar{x}, \bar{y} \rangle$ is not a proper power and not primitive forces the image of $\langle \bar{x}, \bar{y} \rangle \to F$ to be cyclic, contradicting injectivity of f on $\langle \bar{x}, \bar{y} \rangle$. □

Lemma 6.29 is a restatement of Lemma 2.10 in [Tou09]. Unfortunately upon re-reading I noticed that the lemma as stated is false; we give here a corrected version of the lemma. (In the original version the line (7) is $H = \langle G, t \mid t^{-1} p t = q \rangle$, $p, q \in G - \{1\}$.) Thankfully this mistake does not have an impact on the results of [Tou09] in a significant way.

Lemma 6.29. *Let H be a free group of rank 2 and let $w \in H$ be non primitive, and not a proper power. Then the only possible almost reduced (see Definition 5.3), hairless non-trivial cyclic splitting of H modulo w is*

$$H = \langle G, t \mid t^{-1} p^n t = q \rangle, \quad p, q \in H - \{1\}, \tag{7}$$

where $w \in G$, G is a free group of rank 2 and $n \in \mathbb{Z}$. Moreover, $G = \langle p \rangle * \langle q \rangle$ so that $H = \langle p, t \rangle$.

Consider a splitting of a free group H of rank 2 such as the one given in (7). We can apply a balancing fold (see Definition 6.4) to replace the vertex group G by

$$\hat{G} = \langle p \rangle * (\langle q \rangle *_{\langle q \rangle} (t^{-1} \langle p \rangle t)). \tag{8}$$

Letting $\hat{q} = t^{-1} p t$ we have $\hat{q}^n = q$, $\hat{G} = \langle p \rangle * \langle \hat{q} \rangle$ and $H = \langle \hat{G}, t \mid t^{-1} p t = \hat{q} \rangle = \langle p, t \rangle$.

Proposition 6.30. *Suppose that $F_{R(S)}$ is freely indecomposable modulo F and splits as*

$$F *_{\langle \alpha \rangle} H, \tag{9}$$

and that $H = G *_{\langle \beta \rangle} \mathrm{Ab}(\beta, r)$ is a rank 1 free extension of a centralizer of a free group G of rank 2. Then (9) refines to

$$F *_{\langle \alpha \rangle} G *_{\langle \beta \rangle} \mathrm{Ab}(\beta, r).$$

Proof. The hypothesis arises as item (1) of Proposition 6.26. We need to show that α is conjugable into G. Suppose towards a contradiction that this is impossible.

We first consider the possible cyclic JSJ splittings of $H = G *_{\langle \beta \rangle} \mathrm{Ab}(\beta, r)$. These correspond to cyclic splittings of G modulo β. By Lemma 6.29, the two non-trivial possibilities, after possibly applying a balancing fold, are

$$G = \langle \beta \rangle * \langle \beta' \rangle \quad \text{or} \quad G = \langle G', s \mid s^{-1} \gamma s = \gamma' \rangle \quad \text{with } \beta \in G'.$$

Since $F_{R(S)}$ is not freely decomposable, and replacing G by G' if necessary, we have by hypothesis that either α is conjugable into $\mathrm{Ab}(\beta, r)$ or that $G *_{\langle \beta \rangle} \mathrm{Ab}(\beta, r)$ has no cyclic splittings modulo α. We note that in all cases, by commutation transitivity β cannot be a proper power in G.

Consider the case where $G *_{\langle\beta\rangle} \mathrm{Ab}(\beta, r)$ has no cyclic splittings modulo α. Let $\mathcal{R}$ be a strict resolution of $F_{R(S)}$. Since $\alpha \in F$ is always forced to be elliptic, the image of the subgroup $G *_{\langle\beta\rangle} \mathrm{Ab}(\beta, r)$ in all the quotients of $F_{R(S)}$ in $\mathcal{R}$ is always forced to be elliptic, which implies that it is isomorphic to a subgroup of a free group, which is impossible.

Suppose now that $\alpha \in \mathrm{Ab}(\beta, r)$. Since $\alpha \in \mathrm{Ab}(\beta, r)$ is not a proper power, $\langle\alpha\rangle$ is a direct summand of $\mathrm{Ab}(\beta, r)$ if $\langle\beta\rangle \cap \langle\alpha\rangle \neq \{1\}$. Then $\alpha = \beta$, and we are done. Otherwise Lemma 6.28 applies, and $F_{R(S)}$ is freely decomposable, a contradiction. □

Proposition 6.31. *Suppose that $F_{R(S)}$ is freely indecomposable modulo F and that it splits as*

$$\widetilde{F}' *_{\langle\alpha\rangle} H',$$

where $F \leq \widetilde{F}'$ and H' is free of rank 2, and suppose moreover that H' splits further as an HNN extension

$$H' = \langle G, t \mid t^{-1}\gamma^n t = \gamma'\rangle$$

modulo α. Then α cannot be almost conjugate (as in Definition 2.9) to either γ or γ' in G

Proof. Obviously by Lemma 6.29 γ and γ' are not almost conjugate in G. If α is almost conjugate to either γ or γ' in G then this is still true after performing a balancing fold to get (8) and writing γ' again instead of $\hat{\gamma}'$ and writing G again instead of $\hat{G}$.

So we may now assume that $H' = \langle G, t \mid \gamma^t = \gamma'\rangle$ with $G = \langle\gamma\rangle * \langle\gamma'\rangle$ and that w.l.o.g. α and γ are almost conjugate in G. This means that there are some $\eta, g \in G$ such that $\langle\alpha^g\rangle, \langle\gamma\rangle \leq \langle\eta\rangle$. On the other hand γ is not a proper power in G so $\langle\gamma\rangle = \langle\eta\rangle$ which implies that $\langle\alpha^g\rangle \leq \langle\gamma\rangle$ so, conjugating boundary monomorphisms, we get a relative presentation

$$F_{R(S)} = \begin{cases} \langle \widetilde{F}', G, t \mid t\gamma t = \gamma' \rangle, \\ \langle\alpha\rangle = \widetilde{F}' \cap G, \ \gamma, \gamma' \in G, \end{cases}$$

with $G = \langle\gamma'\rangle * \langle\gamma\rangle$ and $\langle\alpha\rangle \leq \langle\gamma\rangle$, which means that we can rewrite this relative presentation using a Tietze transformation as

$$F_{R(S)} = \langle \widetilde{F}', \gamma, t \mid \rangle$$

with $\gamma^m = \alpha \in F$ for some $m \in \mathbb{Z}$. This mean that $F_{R(S)}$ is freely decomposable modulo F, a contradiction. □

The same argument yields:

Proposition 6.32. *Suppose that $F_{R(S')}$ splits as*

$$F *_{\langle\alpha\rangle} H *_{\langle\beta\rangle} \mathrm{Ab}(\beta, t)$$

with H free of rank 2 and suppose moreover that H splits further as an HNN extension

$$H = \langle G, t \mid t^{-1}\gamma^n t = \gamma'\rangle$$

modulo α and β, then we cannot have that α, β and γ are almost conjugate in H.

Lemma 6.33. *Suppose that $F_{R(S)}$ has a QH subgroup Q. Then its JSJ must be of the form*

$$F_{R(S)} = \tilde{F} *_{\langle q\rangle} H_1 *_{\langle a_2\rangle} \cdots *_{\langle a_k\rangle} H_k *_{\langle p\rangle} Q. \tag{10}$$

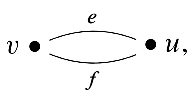
Proof. Consider the quotient $F_{R(S)} \to F_{R(S)}/R = K$ obtained by killing the vertex group $\tilde{F}$ as well as all the edge groups. On one hand K is generated by the image of $\langle x, y\rangle$ on the other hand it contains a closed surface group $\bar{Q}$ as a free factor. The only possibility is $\bar{Q} = \mathbb{Z} \oplus \mathbb{Z}$, and we immediately see that the underlying graph of the JSJ must be simply connected. It moreover follows that $\bar{Q} = K$, hence there are no other QH subgroups.

Now a non-cyclic fully residually free group L is either free abelian or maps onto a free group of rank 2. This means that for all $\gamma \in L$ we have that the quotient $L/\mathrm{ncl}(\gamma)$, where ncl denotes the normal closure, is non-trivial. So if $K = \bar{Q}$ then JSJ of $F_{R(S)}$ must be as in (10). □

Proof of Corollary 2.12. Let $F_{R(S)}$ have a MQH subgroup Q. Propositions 6.26, 6.30 and Lemma 6.33 together imply that the JSJ of $F_{R(S)}$ cannot have an abelian vertex group. And must either be $\tilde{F} *_{\langle\alpha\rangle} Q$, with $\tilde{F}$ a rank 1 centralizer extension of F, or $F *_{\langle\alpha\rangle} Q$. The latter case implies the result. Suppose, on the contrary, the former case, i.e., (2) of Proposition 6.26 holds. Then $\bar{x}$, $\bar{y}$ freely generate Q and so α must be in the commutator subgroup of Q. Now α can be written as a word in $\bar{x}, \bar{y}$ with exponent sum 0 in $\bar{x}$, $\bar{y}$. It follows that $\tilde{F}$ is generated by elements of exponent sum 0 in $\bar{x}$, $\bar{y}$ and therefore cannot contain a non-cyclic abelian subgroup by Lemma 5.24, a contradiction. □

Corollary 6.34. *If $F_{R(S)}$ is freely indecomposable and the maximal abelian collapse of its cyclic JSJ decomposition modulo F is a free product with amalgamation. Then all the possibilities for the JSJ of $F_{R(S)}$ are to be found in the descriptions given Sections 2.4.1, 2.4.2, and 2.4.3.*

6.3. The two edge case. We now consider the case where the maximal abelian collapse of $F_{R(S)}$ has underlying graph

$X =$  (11)

to which we give the relative presentation

$$\begin{cases} \langle \widehat{F}, H, t \mid B^t = C \rangle, \\ B \le \widehat{F},\ C \le H,\ A = \widehat{F} \cap H, \end{cases} \tag{12}$$

where $F \le \widehat{F} = X_u$, $H = X_v$ and A, B, C are maximal abelian and conjugacy separated in their vertex groups. Throughout this section the groups $\widehat{F}$, H, A, B, C are as above.

6.3.1. Arranging so that either $\bar{x} \in H$ or $\bar{x} = t$

Lemma 6.35. *Let $F_{R(S)}$ be freely indecomposable modulo F and have a maximal abelian collapse (12). After Weidmann–Nielsen normalization on $(F, \bar{x}, \bar{y})$ modulo F we can arrange, conjugating boundary monomorphisms if necessary, so that $\bar{x}$ either lies in $\widehat{F} \cup H$ or $\bar{x} = t$.*

Proof. We first observe that $F_{R(S)}$ cannot be generated by elliptic elements with respect to the splitting (12). We apply Theorem 6.9 to the marked generating set $(F; \{\bar{x}, \bar{y}\})$. Let T be the Bass–Serre tree of this splitting. Let $v_0 \in T$ be the vertex fixed by $F \le F_{R(S)}$.

Suppose that T_F is a point. Then after Weidmann–Nielsen normalization, $\bar{x}$ must be brought to an elliptic element. We can then arrange $T_{\langle \bar{x} \rangle} \cap T_F \neq \emptyset$ or $\bar{y} T_F \cap T_F \neq \emptyset$. Since $\bar{y}$ cannot also be elliptic, we must have that if $T_{\langle \bar{x} \rangle} \cap T_F \neq \emptyset$ then $\bar{x} \in \widehat{F}$. Suppose now that T_F is not a point. Then after Weidmann–Nielsen normalization we can get either:

Case I: $T_F \cap \bar{x} T_F \neq \emptyset$. Then we can apply Lemma 6.13 and get that $\bar{x}$ is either in H or $\bar{x} = th, h \in H$.

Case II: $\bar{x}$ is elliptic. Then we use Theorem 6.9 on the marked set $(F, \bar{x}; \{\bar{y}\})$. $\bar{y}$ cannot also be elliptic.

If $T_F \cap T_{\langle \bar{x} \rangle} \neq \emptyset$ then if $v_0 \in T_{\langle \bar{x} \rangle}$ then we can assume that $\bar{x} \in \widehat{F}$. If $\bar{x}$ fixes a vertex w' adjacent to v_0, then we can assume that $\bar{x} \in \mathrm{stab}(w')$. By Lemma 6.13 we either have that $\bar{x}$ can be brought into H or tHt^{-1}, after possibly changing the relative presentation, the result follows.

If $T_F \cap \bar{y} T_F \neq \emptyset$ then as before we can arrange to that $\bar{y} = th$ and interchanging $\bar{x}$ and $\bar{y}$ the result will follow. The remaining case is $T_{\langle \bar{x} \rangle} \cap \bar{y} T_{\langle \bar{x} \rangle} \neq \emptyset$. We note, however, that $T_{\langle \bar{x} \rangle}$ intersects at most one $F_{R(S)}$-edge orbit, but since $\bar{y}$ have exponent sum 1 in the stable letter, we find that the path ρ connecting $T_{\langle \bar{x} \rangle}$ and $\bar{y} T_{\langle \bar{x} \rangle}$ must intersect both $F_{R(S)}$-edge orbits. It follows that $T_{\langle \bar{x} \rangle} \cap \bar{y} T_{\langle \bar{x} \rangle} = \emptyset$. So we must have $T_F \cap \bar{y} T_F \neq \emptyset$, and the result follows from Lemma 6.13. □

Lemma 6.36. *If the vertex group H in the splitting (12) is generated by conjugates of its boundary subgroups, i.e., $H = \langle A^{h_1}, C^{h_2} \rangle$, $h_i \in H$, then $F_{R(S)}$ is freely decomposable modulo F.*

Proof. W.l.o.g. by conjugating boundary subgroups if necessary we may assume that $h_1, h_2 = 1$. By Lemma 6.16, $H = A * C$ and we have the relative presentation $\langle \widehat{F}, H, t \mid t^{-1}Bt = C\rangle$ with $A, B \leq \widehat{F}$. Using Tietze transformations we can rewrite this as $\widehat{F} * \langle t\rangle$. □

Lemma 6.37. *Suppose that $F_{R(S)}$ has a maximal abelian collapse (12). If $F_{R(S)}$ is generated as $\langle F, \bar{x}, \bar{y}\rangle$ with $\bar{x} \in \widehat{F}$, then $F_{R(S)}$ is freely decomposable modulo F.*

Proof. Consider the $\mathcal{G}(X)$-graph $\mathcal{B}$ obtained by attaching a $\bar{y}$-loop to the vertex v where $\mathcal{B}_v = \langle F, \bar{x}\rangle$. All other vertices in $\mathcal{B}$ have trivial $\mathcal{B}$-vertex group. In the folding process (F4) collapses are impossible since the final graph should have a cycle. By Lemma 6.18 we can perform our folding sequence without using transmissions as long as the underlying graph is not the graph with two vertices u, v, two edges e, f, and one cycle. When we do reach this point all that remains to be done to get a folded graph is to do transmissions. By hypothesis the boundary subgroups $C, A \leq H$ are conjugacy separated and so the first transmission from v to u through e cannot be followed by a transmission form u to v through f. It therefore follows that after the first two transmissions $\mathcal{B}_v = \langle C', A'\rangle$, with $C' \leq C$ and $A' \leq A$ respectively.

By Lemma 6.16, $\mathcal{B}_v = C' * A'$ and so no further transmissions are possible from u to v. $\mathcal{B}$ is therefore folded and free decomposability now follows from Lemma 6.36. □

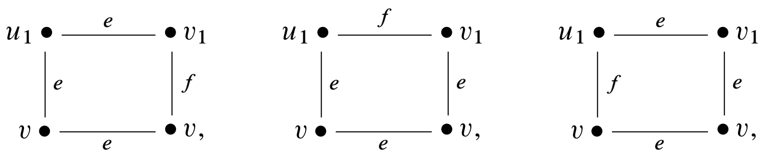
6.3.2. Arranging so that w.l.o.g. $\bar{x} = t$ and $\bar{y} \in H$

Lemma 6.38. *Suppose that $F_{R(S)}$ has a maximal abelian collapse (12) and is freely indecomposable modulo F. If $F_{R(S)}$ is generated as $\langle F, \bar{x}, \bar{y}\rangle$ with $\bar{x} \in H$, then, conjugating boundary monomorphisms if necessary, we can arrange so that $F_{R(S)}$ is generated as $\langle F, \bar{x}, \bar{y}'\rangle$ with $\bar{y}' = ath$, $h \in H$, $a \in \widehat{F}$.*

Proof. We first note that if $\bar{x}$ is conjugate into an edge group, then we can assume that $\bar{x} \in \widehat{F}$, which by Lemma 6.37 leads to a contradiction. The hypotheses imply that $F_{R(S)}$ is the fundamental group of a $\mathcal{G}(X)$-graph $\mathcal{B}$ obtained by taking an edge labelled, say $(1, e, 1)$, with endpoints v and u, attaching the $\bar{y}$-loop $\mathcal{L}(\bar{y}, v)$, and setting $\mathcal{B}_v = F$, $\mathcal{B}_u = \langle \bar{x}\rangle$.

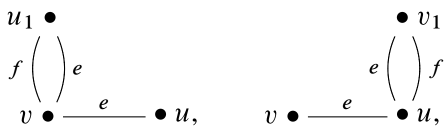
We start our folding process using only moves (A0)–(A3), (F1), (F4), (L1), (S1). Note that if a (F4) collapse occurs, then the underlying graph will be simply connected, which is impossible. Any cycle must have even length and by Lemma 6.18, as long as there are more than four vertices we can continue our folding process while avoiding transmissions.

Suppose that $\mathcal{B}$ has only four vertices, noting that this must fold to a graph like (11) using (F1) moves we see (exchanging the labels e and f, if necessary) that the only possibilities after doing (S1) moves are



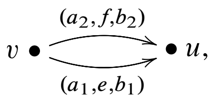
where the edges are labelled by their image in X via the map $[\,]$ (recall Definition 3.6). If we consider all possible sequences of transmissions we see that either there is a cancellable path at u_1 or v_1 or that $\mathcal{B}_{u_1}$ and $\mathcal{B}_{v_1}$ are contained in conjugates of their edge groups. In either case $\mathcal{B}_u$, $\mathcal{B}_v$ remain unchanged and we can make an (F1) fold without transmissions.

Suppose now that $\mathcal{B}$ has only three vertices, then the only possibilities after doing (S1) moves are



with $\mathcal{B}_{u_1}$ or $\mathcal{B}_{v_1}$ trivial. The only folding that can occur is an (F1) fold at v or u. By Lemma 6.20 $\mathcal{B}_u$ or $\mathcal{B}_v$ can only be changed by transmissions through the edge between u and v it follows that we only need to make (A0)–(A4), (L1) moves before our (F1) fold.

$\mathcal{B}$ can therefore be brought to a graph of the form



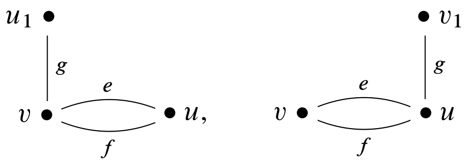
with $\mathcal{B}_v = F$ and $\mathcal{B}_u = \langle \bar{x} \rangle$. Moreover we see that it is possible to leave the label $(1, e, 1)$ of the edge between u and v unchanged throughout the folding process, so we may assume that $a_1 = b_1 = 1$. $F_{R(S)}$ is therefore generated by $F, \bar{x} \in H$ and $\bar{y}' = a_2 f, b_2, e^{-1}$ with $a_2 \in \widehat{F}$ and $b_2 \in H$, i.e., in the relative presentation we can write $\bar{y}' = a_2 t b_2$. $\square$

Lemma 6.39. *Suppose that $F_{R(S)}$ has a maximal abelian collapse (12) and is freely indecomposable modulo F. If $F_{R(S)}$ is generated as $\langle F, \bar{x}, \bar{y} \rangle$ with $\bar{x} = t$, then we can arrange so that $F_{R(S)}$ is generated as $\langle F, \bar{x}, \bar{y}' \rangle$ with $\bar{y}' \in H$.*

Proof. The hypotheses imply that $F_{R(S)}$ is the fundamental group of a $\mathcal{G}(X)$-graph $\mathcal{B}$ obtained by taking two edges with labels $(1, e, 1)$ and $(1, f, 1)$ with common endpoints v and u, setting $\mathcal{B}_v = F$, $\mathcal{B}_u = \{1\}$, and attaching the $\bar{y}$-loop $\mathcal{L}(\bar{y}, v)$. We start our folding process using only moves (A0)–(A3), (F1), (F4), (L1), (S1), but avoiding transmissions.

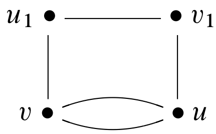
Case I: Suppose we were able to bring $\mathcal{B}$ to a graph with two vertices and two edges with $\mathcal{B}_v = \langle F, \bar{y}' \rangle$, $\mathcal{B}_u = \{1\}$ or $\mathcal{B}_v = F$, $\mathcal{B}_u = \{\bar{y}'\}$. To get a folded graph, all that remains are transmissions, but note that in the former case, Lemma 6.36 will imply free decomposability modulo F, in the latter case the result follows.

Case II: Suppose first that an (F4) collapse occurred then $\mathcal{B}$ has one cycle and two non-trivial $\mathcal{B}$-vertex groups. By Lemma 6.18 unless we have one of the two possibilities



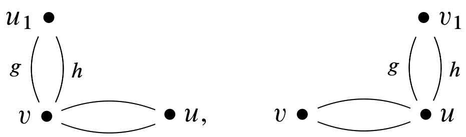
we can continue folding without using transmissions. Here $\mathcal{B}_{u_1}$ or $\mathcal{B}_{v_1}$ is cyclic. The next fold is an (F1) fold at u or v, or shaving off u_1 or v_1. By Lemma 6.16 unless u_1 or v_1 can be shaved off, there can be no transmissions to u_1 or v_1 and then back again through g it therefore follows that prior to the (F1) fold no transmissions through the edge g are needed so we can change its label with an (A2) simple adjustment or an (L1) long-range transmission. This brings us to case I.

Case III: Suppose that no collapses occurred. By Lemma 6.18 $\mathcal{B}$ has at most four vertices. The only possibility with four vertices is something of the form



with $\mathcal{B}_u$, $\mathcal{B}_{u_1}$, $\mathcal{B}_{v_1}$ trivial. If there are no cancellable paths, then the next fold is of type (F1) at u or v, and no transmissions are needed. If the fold is at v_1 or u_1 Lemma 6.18 ensures that no transmissions are needed to make the fold.

Case III.I: Suppose now that after shaving $\mathcal{B}$ has three vertices then the possibilities are



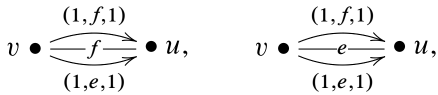
with $\mathcal{B}_u$, $\mathcal{B}_{u_1}$, $\mathcal{B}_{v_1}$ trivial.

Case III.I.I: If $[g] \neq [h]$ then since the edge groups are conjugacy separated and by Lemma 6.16 after any sequence of transmissions there can be no transmissions from u_1 or v_1 back to u or v respectively. The next fold is an (F1) fold and can therefore be done using an (A2) simple adjustment or an (L1) long-range adjustment to change the label of g or h. This brings us to case III.II.

Case III.I.II: If $[g] = [h]$ then either we can (F4) collapse at u_1 or v_1 towards v or u respectively which after shaving off u_1 or v_1 reduces to the case I. Otherwise by

Lemma 6.20 after any sequence of transmissions there can be no transmissions from u_1 or v_1 back to u or v respectively and no (F4) collapse at u_1 or v_1. The subsequent fold can therefore be made without any transmissions as in the previous paragraph. A collapse at this point reduces to case II, otherwise we are in case III.II.

Case III.II: Suppose now that $\mathcal{B}$ has two vertices and three edges, the possibilities are



where $\mathcal{B}_u = \{1\}$ and the remaining edge is marked only by its image in X via map $[\ ]$. Note that these cases are symmetric. Suppose the middle edge has label (a, e, b) with $a \in \widehat{F}$ and $b \in H$. First note that if it is possible to transmit through both e-type edges from v to u, this means because F has property CC that there is some $f \in F$ such that $fa \in i_e(X_e)$, which in turn means that after an (A2) simple adjustment and an (A1) Bass–Serre move we can make a (F4) collapse towards u and the result follows. We may therefore assume that

(†) there is no $f \in F$ such that $fa \in i_e(X_e)$.

If only one transmission from v to u is possible before the (F4) collapse, then we could use an (L1) long-range adjustment instead, and reduce to the case I. The final remaining possibility is that there are transmissions from v to u through the edges labeled $(1, e, 1)$ and $(1, f, 1)$. In particular we can assume that

(‡) some conjugate of X_e centralizes $\alpha \in F$.

Let $\mathcal{B}_u = \langle \alpha, \beta \rangle$ be the $\mathcal{B}$-vertex group after these transmissions. If $b \in \mathcal{B}_u$ and we can make a simple adjustment on the label of the edge labeled (a, e, b) and then do an (F4) collapse, then we could have used an (L1) long-range adjustment instead, and reduce to case I.

We may therefore assume that there is a transmission from u through the edge labeled (a, e, b) so that $\mathcal{B}_v$ is now $\langle F, aAa^{-1} \rangle$ where $A \leq i_e(X_e)$. Since we A centralizes some conjugate of an element of F and by (†), (‡), we can apply Corollary 6.23 to obtain $\langle F, aAa^{-1} \rangle = F * aAa^{-1}$, $a \notin \mathcal{B}_v$, so there can be no collapse at v and by the free product structure there can be no more transmissions. The graph is therefore already folded, contradicting the fact that $F_{R(S)} = \pi_1(\mathcal{G}(X))$ as in (11). $\square$

All these lemmas combine to give:

Proposition 6.40. *If $F_{R(S)}$ is freely indecomposable modulo F and has a maximal abelian collapse (12), then $F_{R(S)}$ is generated by F, the stable letter t, and some element $\bar{y} \in H$.*

6.3.3. Recovering the original cyclic splitting from the maximal abelian collapse.

The next proposition enables us to revert to a cyclic splitting.

Proposition 6.41. *Suppose that $F_{R(S)}$ is freely indecomposable modulo F and has a maximal abelian collapse (12), then $F_{R(S)}$ admits one of the two possible cyclic splittings:*

(1)
$$F_{R(S)} = \begin{cases} \langle F, H', \bar{x} \mid \beta^{\bar{x}} = \beta' \rangle, \\ \beta \in F,\ \beta' \in H',\ \langle \alpha \rangle = F \cap H', \end{cases}$$

where $H' = \langle \alpha, \beta', \bar{y} \rangle$ is a 3-generated fully residually free group.

(2)
$$F_{R(S)} = \begin{cases} \langle \widetilde{F}', H', \bar{x} \mid \beta^{\bar{x}} = \beta' \rangle, \\ \beta \in \widetilde{F}',\ \beta' \in H',\ \langle \alpha \rangle = \widetilde{F} \cap H', \end{cases}$$

where $\widetilde{F}'$ is a rank 1 free extension of a centralizer of F and H' is generated by α, $\bar{y}$. Moreover H' may not split further as an HNN extension.

Proof. Let $\widehat{F}$, A, B, C, H, t be as in (12). By Proposition 6.40 we can assume that $\bar{x} = t$ and $\bar{y} \in H$. We can always assume that $F \cap A \neq \{1\}$ (otherwise we could derive free decomposability modulo F.)

Suppose first that $F \cap A = \langle \alpha \rangle$ and $F \cap B = \{1\}$. To ensure free indecomposability modulo F we need there to be some $\gamma \in \langle \alpha, \bar{y} \rangle$ such that $\bar{x}\gamma\bar{x}^{-1} = \beta \in B \leq \widehat{F}$. Now by Theorem 2.2 if $\langle F, \beta \rangle \neq F * \langle \beta \rangle$ then we must have $\langle F, \beta \rangle = \langle F, t | [p, t] = 1 \rangle$ for some $p \in F$. If p is not conjugate to α in F then $F_{R(S)}$ has a cyclic splitting as in item 2. If p and $\alpha^{\pm 1}$ are conjugate in F, then we can assume that $\alpha = p$, so then the group A in (12) is non-cyclic abelian of rank 2. We study the maximal abelian subgroup $C \leq H$ we already had that $\gamma(\bar{y}, \alpha) \in C$ if $C \leq H$ is not cyclic then there must be some $\gamma_1 \in \langle \bar{y}, A \rangle$ such that γ and γ_1 do not lie in a common cyclic subgroup and which satisfies the relation

$$[\gamma, \gamma_1] = 1 \tag{13}$$

however by Lemma 6.16 we have

$$\langle A, \bar{y} \rangle = A * \langle \bar{y} \rangle,$$

which means that (13) is impossible. It follows that $\widehat{F} = \langle F, \beta \rangle$. This gives the cyclic splitting

$$F_{R(S)} = \begin{cases} \langle \widehat{F}, H, \bar{x} \mid \beta^{\bar{x}} = \gamma \rangle, \\ \beta \in \widehat{F},\ \gamma \in H,\ \langle \alpha \rangle = \widehat{F} \cap H, \end{cases}$$

with $H = \langle \alpha, \bar{y} \rangle$.

Suppose now towards a contradiction that H split further as an HNN extension:

$$H = \langle K, t \mid \delta^t = \delta' \rangle, \quad \delta, \delta' \in K,$$

modulo α, γ, then we have

$$F_{R(S)} = \begin{cases} \langle \widehat{F}, K, \bar{x}, t \mid \beta^{\bar{x}} = \gamma, \delta^t = \delta' \rangle, \\ \beta \in \widehat{F}, \ \gamma, \delta, \delta' \in K, \ \langle \alpha \rangle = \widehat{F} \cap K. \end{cases}$$

Then we can collapse this splitting to a double HNN and by Corollary 5.25 $\widehat{F}$ cannot contain any non-cyclic abelian subgroups, a contradiction.

Suppose now that $F \cap A = \langle \alpha \rangle$ and $F \cap B = \langle \beta \rangle$. If both A and B are cyclic we are done: by Proposition 6.40 H is generated by three elements. We therefore assume w.l.o.g. that A is not cyclic. First note that by Lemma 6.22, $\langle F, A \rangle$ has the JSJ $F *_{\langle \alpha \rangle} A$. Suppose first that the B is not cyclic. We have a surjection

$$A *_{\langle \alpha \rangle} F *_{\langle \beta \rangle} B \to \langle F, A, B \rangle,$$

which is injective on $F *_{\langle \beta \rangle} B$ as well. Suppose this map wasn't injective, then some element w lies in the kernel, moreover $A *_{\langle \alpha \rangle} F *_{\langle \beta \rangle} B$ should not have any essential cyclic splittings modulo w, F. On the other hand $\langle F, A, B \rangle$ should have a non-trivial JSJ modulo F but the triviality of the image w implies that $\langle F, A, B \rangle$ has only one vertex group, a contradiction.

Thus, whether or not B is cyclic, we always have that $\langle F, A, B \rangle = A *_{\langle \alpha \rangle} F *_{\langle \beta \rangle} B$. This means we have the cyclic splitting

$$F_{R(S)} = \begin{cases} \langle F, H', t \mid \beta^t = \gamma \rangle, \\ \beta \in F, \ \gamma \in C \leq H', \ \langle \alpha \rangle = F \cap H', \end{cases}$$

where $H' = \langle H, A, C \rangle$. Now by Proposition 6.40, even with the new splitting, we still have that $\bar{x} = t$ and that $\bar{y} \in H \leq H'$ so considering a folding sequence starting at this point we have that F is a full vertex group and all that remains are transmissions from F to H' to get $F_{R(S)}$. It follows that $H' = \langle \bar{y}', \alpha, \gamma \rangle$, so it is 3-generated. □

Proposition 6.42. *Suppose that $F_{R(S)}$ splits as*

$$\begin{cases} \langle F, H, t \mid \beta^t = \gamma \rangle, \\ \beta \in F, \ \gamma \in H, \ \langle \alpha \rangle = F \cap H. \end{cases}$$

Then α and γ cannot be almost conjugate H. Moreover α and β cannot be almost conjugate in F.

Proof. Suppose on the contrary that α, γ were almost conjugate in H. Then after conjugating boundary monomorphisms, making folding moves (as in Definition 5.2) which keep the splitting cyclic and making a sliding move we would get a splitting

$$\begin{cases} \langle F, H, t \mid \beta^t = \alpha \rangle, \\ \alpha, \beta \in F, \ \langle \alpha \rangle = F \cap H, \end{cases}$$

whose maximal abelian collapse is as considered in Section 6.2, a contradiction. Similarly, if α and β were almost conjugate in F, we can similarly derive a contradiction to the choice of maximal abelian collapse. □

Proposition 6.43. *Suppose we have the splitting*

$$F_{R(S)} = \begin{cases} \langle F, H', \bar{y} \mid \beta^{\bar{y}} = \gamma \rangle, \\ \beta \in F,\ \gamma \in H',\ \langle \alpha \rangle = F \cap H', \end{cases}$$

where $H' = H *_{\langle \delta \rangle} \mathrm{Ab}(\delta, r)$ is a rank 1 free extension of a centralizer of the free group H of rank 2. Then this splitting can be refined to

$$F_{R(S)} = \begin{cases} \langle F, H, \mathrm{Ab}(\delta, r), \bar{y} \mid \beta^{\bar{y}} = \gamma \rangle, \\ \beta \in F,\ \gamma \in H,\ \langle \alpha \rangle = F \cap H,\ \langle \delta \rangle = H \cap \mathrm{Ab}(\delta, r), \end{cases}$$

such that δ, α, γ are not almost conjugate in H. *This cyclic splitting is the JSJ of $F_{R(S)}$.*

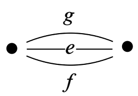
Proof. If α and γ are almost conjugate, then we can derive a contradiction arguing as in the proof of Proposition 6.42.

If H could split further as an HNN extension modulo α, γ, δ then we could apply Corollary 5.25 contradicting the existence of a non-cyclic abelian subgroup of $F_{R(S)}$. It follows that the cyclic JSJ splitting of H' modulo α, γ either consists of one vertex group H' or is an amalgam $H *_{\langle u \rangle} \mathrm{Ab}(\delta, r)$. In the former case since α, γ are conjugable into F their images will be elliptic in every term of a strict resolution of $F_{R(S)}$ which implies that $H *_{\langle u \rangle} \mathrm{Ab}(\delta, r)$ is free, a contradiction.

Suppose now that either α or γ, say w.l.o.g. γ, is conjugable in H' into $\mathrm{Ab}(\delta, r)$ but that γ is not almost conjugate to δ. We can conjugate boundary monomorphisms so that $\gamma \in \mathrm{Ab}(\delta, r)$ and replace $\mathrm{Ab}(\delta, r)$ with $\langle \gamma, \delta \rangle$. This gives an F-subgroup $G \le F_{R(S)}$ that has a splitting with vertex groups F, $\langle \gamma, \delta \rangle$, and H. We see that in every term of a strict resolution of G the images of the elements α, β, γ, δ are forced to be elliptic, this means that H is always mapped monomorphically, so γ and δ will always be conjugable into some non-abelian vertex group, this means that $\langle \gamma, \delta \rangle$ will never have an exposed direct summand (see Definition 5.22) contradicting Lemma 5.23. □

Corollary 6.44. *If $F_{R(S)}$ is freely indecomposable has a maximal abelian collapse (12). Then all the possibilities for the JSJ of $F_{R(S)}$ are to be found in the descriptions given Sections 2.4.1, 2.4.2, and 2.4.3.*

6.4. The three edge case. We now consider the case where the maximal abelian collapse of $F_{R(S)}$ has underlying graph

$X = v$ 

u, (14)

to which we give the relative presentation

$$\begin{cases}\langle \widehat{F}, H, s, t \mid A^s = B,\ D^t = E\rangle, \\ A, D \leq \widehat{F},\ B, E \leq H,\ C = \widehat{F} \cap H, \end{cases} \tag{15}$$

where $F \leq \widehat{F} = X_v$, $H = X_u$ and A, B, C, D, E are maximal abelian and conjugacy separated in their vertex groups. Note that by Corollary 5.25, $F_{R(S)}$ cannot contain any non-cyclic abelian subgroups, in particular the subgroups A, B, C, D, E must all be cyclic.

Lemma 6.45. *Let $F_{R(S)}$ be freely indecomposable modulo F with a maximal abelian collapse (15). After Weidmann–Nielsen normalization on $(F, \bar{x}, \bar{y})$ modulo F we can arrange, conjugating boundary monomorphisms if necessary, for $\bar{x} = t$.*

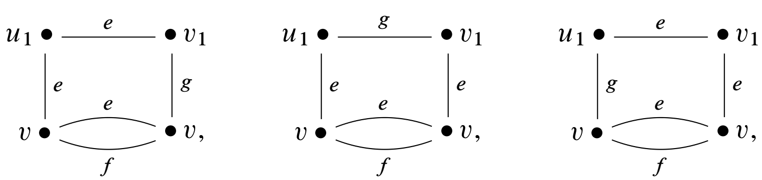
Proof. Since we are assuming free indecomposability of $F_{R(S)}$ modulo F, we can apply Theorem 6.9 to the marked generating set $(F; \{\bar{x}, \bar{y}\})$. Let T be the Bass–Serre tree corresponding to the splitting (14). We note that neither $\bar{x}$ nor $\bar{y}$ can be brought to elliptic elements with respect to the splitting (14). Let $v_0 \in T$ be the vertex fixed by F. W.l.o.g. after Weidmann–Nielsen normalization we have $T_F \cap \bar{x}T_F \neq \emptyset$. 1-acylindricity implies that $d(v_0, \bar{x}v_0) = 2$. It follows w.l.o.g. that $\bar{x}$ as a $\mathcal{G}(X)$-path is of the form a_1, f, b_1, g^{-1}, a_2, where $b_1 \in H, a_1, a_2 \in \widehat{F}$. By Lemma 6.13 we can arrange so that $\bar{x} = f, b, g^{-1}$, and conjugating boundary monomorphisms enables us to assume that $\bar{x} = f, g^{-1} = t^{-1}$ in terms of the relative presentation (15). □

Lemma 6.46. *Let $F_{R(S)}$ be freely indecomposable modulo F with a maximal abelian collapse (15) and with $\bar{x} = t$, then $F_{R(S)}$ is generated by F, t, and s.*

Proof. The hypotheses imply that $F_{R(S)}$ is the fundamental group of a $\mathcal{G}(X)$-graph $\mathcal{B}$ obtained by taking two edges with labels $(1, e, 1)$ and $(1, f, 1)$ with common endpoints v and u, setting $\mathcal{B}_v = F$, $\mathcal{B}_u = \{1\}$, and attaching the $\bar{y}$-loop $\mathcal{L}(\bar{y}, v)$.

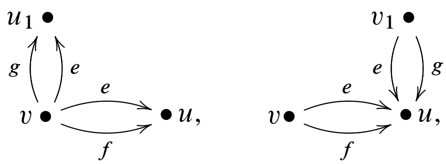
Again we start our adjustment-folding process, using only moves (A0)–(A2), (L1), (F1), (S1). (F4) collapses are forbidden since they reduce the number of cycles in the underlying graph. As long as after (S1) shavings there are strictly more than four vertices, we see by Lemma 6.18 that we can always perform a folding move without using transmissions. It follows that we can bring $\mathcal{B}$ to a graph with four vertices such that $\mathcal{B}_v = F$ is the only non-trivial $\mathcal{B}$-vertex group.

Interchanging e and f if necessary, and noting that the exponent sum $\sigma_s(\bar{y}) = 1$, w.l.o.g. the only possibilities are

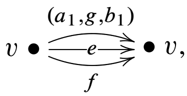


where the edges are marked by their images in X via the map [ ]. In all three cases we see that after applying transmissions the groups $\mathcal{B}_{u_1}, \mathcal{B}_{v_1}$ are cyclic. Again Lemma 6.18 applies and we can continue our adjustment-folding process.

If there are only three vertices then the only possibilities are:



and the last fold is of type (F1) at either u or v and in particular no transmissions are needed. We get that $\mathcal{B}$ is given by

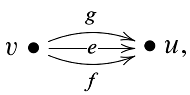


where the edges labelled e and f have labels $(1, e, 1)$ and $(1, f, 1)$ respectively. In the end we have that $F_{R(S)}$ is generated by F, t and some element $\bar{y}'$ represented by the $\mathcal{G}(X)$ path a_1, g, b_1, e^{-1} where $b_1 \in H$ and $a_1 \in \widehat{F}$. After conjugating boundary monomorphisms, we may assume that $\bar{y}' = s$. □

Corollary 6.47. *If $F_{R(S)}$ is freely indecomposable and the maximal abelian collapse of its cyclic JSJ decomposition modulo F has three edges. Then the JSJ of $F_{R(S)}$ is of type* (C2) *in Section* 2.4.1.

Proof. All we need to show is that the vertex groups are F and a free group of rank 2.

By the two previous lemmas we have that $F_{R(S)}$ is the fundamental group of the $\mathcal{G}(X)$-graph

$\mathcal{B} =$ 

with $\mathcal{B}_v = F$ and $\mathcal{B}_u = \{1\}$. To get a folded graph, all that are needed are transmissions. We also saw that the edge groups are cyclic. Suppose first that the only possible transmission is from v to u through e, then by conjugacy separability of the edge groups it is impossible for there to be any further transmissions from u back to v through the other edges.

Suppose that now there were transmissions possible only from v to u through edges e and f. So as not to have free decomposability modulo F, we must have a transmission from u to v through g. We note that the boundary subgroups associated to the edges e, f must be maximal cyclic because they lie in F. It then follows that there are no further possible transmissions and the graph is folded. In particular we find that $\mathcal{B}_u = \widetilde{F} = \langle F, \alpha\rangle$ where α is the element transmitted from H to $\widetilde{F}$. $F_{R(S)}$

is freely indecomposable modulo F only if $\widetilde{F} \neq F * \langle \alpha \rangle$, but by Theorem 2.2 the only other possibility for $\langle F, \alpha \rangle$ is $F *_u \mathrm{Ab}(u, t)$, which is impossible since $F_{R(S)}$ has no non-cyclic abelian subgroups.

It therefore follows that $\widetilde{F} = F$ and H is a free group of rank 2 generated by its boundary subgroups. □

6.5. The proof of the Proposition 2.10. If the JSJ of $F_{R(S)}$ has more than one non-abelian vertex group then it falls into the premises of Corollaries 6.34, 6.44, or 6.47 so our list of possible JSJs given in Section 2.4 is complete.

7. When the JSJ of $F_{R(S)}$ has one vertex group

We now consider the situation where $F_{R(S)}$ has a cyclic JSJ decompositions modulo F, with only one vertex group. We have the relative presentations

$$\langle \widetilde{F}, t \mid \beta^t = \beta' \rangle, \quad \beta, \beta' \in \widetilde{F}, \tag{16}$$

$$\langle \widetilde{F}, s, t \mid \beta^t = \beta', \alpha^s = \alpha' \rangle, \quad \alpha, \alpha', \beta, \beta' \in \widetilde{F}. \tag{17}$$

7.1. $\widetilde{F}$ is 2-generated modulo F. We first need some further auxiliary results.

Lemma 7.1. *Let the JSJ of $F_{R(S)}$ be either (16) or (17). Then β and β' cannot be conjugate in $\widetilde{F}$.*

Proof. Suppose the contrary. Then β' and β are conjugate in $\widetilde{F}$ so conjugating boundary monomorphisms gives us

$$\langle \widetilde{F}, t \rangle = \langle \widetilde{F}, r \mid [r, \beta] = 1 \rangle,$$

which implies that the JSJ of $F_{R(S)}$ has an abelian vertex group, a contradiction. □

This corollary now follows immediately from the fact that $F \leq F_{R(S)}$ has property CC.

Corollary 7.2. *Let the $F_{R(S)}$ split as in (16) or (17) then $\widetilde{F} \neq F$.*

Lemma 7.3. *Suppose $F_{R(S)}$ splits as a double HNN extension:*

$$\begin{cases} \langle \widehat{F}, t, s \mid \alpha^s = \alpha', \ \beta^t = \beta' \rangle, \\ \alpha, \alpha', \beta, \beta' \in \widetilde{F}, \end{cases}$$

where $\widehat{F}$ has no cyclic or free splittings modulo F, α, α', β, β'. Then w.l.o.g. either $\langle \alpha \rangle$ or $\langle \beta \rangle$ is conjugable into F, but not both.

Proof. We may consider $F_{R(S)}$ as a double HNN extension. Suppose on the contrary that neither α, α' nor β, β' were conjugable into F in $\widehat{F}$. Then T_F is a point, which means that $T_F \cap gT_F = \emptyset$ for any hyperbolic $g \in F_{R(S)}$. Since $F_{R(S)}$ is a double HNN, looking at exponent sums of stable letters, we see it must be generated by F and *at least* two hyperbolic elements. It now follows from Theorem 6.9 on the marked generating set $(F; \{\bar{x}, \bar{y}\})$ that $F_{R(S)}$ is freely decomposable modulo F, a contradiction.

Suppose now towards a contradiction that both β and α were conjugable into F. Then by Corollary 5.16 there is a splitting of $\widehat{F}$ modulo $\alpha, \beta, \alpha', \beta', F$, with either trivial or cyclic edge groups, which is again a contradiction. □

We can now say something about the vertex groups of the JSJs (16) and (17).

Lemma 7.4. *Suppose that $F_{R(S)}$ has the JSJ (17). Then:*

(1) *$\widetilde{F}$ has no non-cyclic abelian subgroups.*
(2) *One of the edge groups, say $\langle\alpha\rangle$, is conjugate into F.*
(3) *The elements α and β are not conjugate in $F_{R(S)}$.*
(4) *The centralizers of α, β are cyclic in $\widetilde{F}$. In particular after making balancing folds, the splitting is 1-acylindrical.*
(5) *After balancing folds, where $\widetilde{F}_f$ denotes the resulting vertex group, we have $\widetilde{F}_f = \langle F, \alpha', \beta'\rangle$ with α' conjugable into F and β' conjugable into $\langle F, \alpha'\rangle$. In particular $\widetilde{F}$ is also 2-generated modulo F.*

Proof. By Corollary 5.25, $\widetilde{F}$ has no non-cyclic abelian subgroups. Items (2) and (3) follow from Lemma 7.3. Item (4) now follows from the previous items.

To prove item (5) we first replace the vertex group $\widetilde{F}$ by $\widetilde{F}_f$, which is obtained by performing balancing folds. By Lemma 6.6, if $\widetilde{F}_f$ is 2-generated modulo F so is $\widetilde{F}$. We now have a 1-acylindrical splitting and the Bass–Serre tree T has two edge orbits. We now use Theorem 6.9. By items (3) and (4), T_F has radius 1 and contains edges from only one orbit. We also have that $\bar{x}$ and $\bar{y}$ cannot be elliptic. It therefore follows that after Weidmann–Nielsen normalization we have w.l.o.g. $T_F \cap \bar{x}T_F \neq \emptyset$ and if $\mathrm{fix}(F) = v_0$ then $d(v_0, \bar{x}v_0) \leq 2$. From this we may assume that there are no symbols t in the normal form of $\bar{x}$ with respect to the relative presentation (17), which means that $\bar{x}$ is forced to have exponent sum 1 in s, we therefore have $d(v_0, \bar{x}v_0) = 1$ so that $\bar{x} = f_1 s f_2$, $f_i \in \widetilde{F}$. Now conjugating boundary monomorphisms if necessary we have w.l.o.g. $\bar{x}^{-1}\alpha\bar{x} = \alpha' \in \widetilde{F}$. W.l.o.g. some conjugate of β lies in $\langle F, \alpha'\rangle$, otherwise $\langle F, \bar{x}, \bar{y}\rangle = \langle F, \bar{x}\rangle * \langle\bar{y}\rangle$ since $\bar{y}$ must have exponent sum 1 in t.

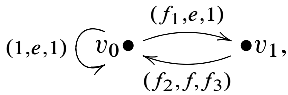
After Weidmann–Nielsen normalization $T_{\langle F,\bar{x}\rangle} \cap \bar{y}T_{\langle F,\bar{x}\rangle} \neq \emptyset$. Let E denote the edges in the same $F_{R(S)}$-orbit as the edges in $\mathrm{Axis}(s) \subset T$. The minimal invariant trees $T(\langle F, \bar{x}\rangle)$ and $\bar{y}T(\langle F, \bar{x}\rangle)$ do not contain edges from E. The shortest path η between the minimal invariant subtrees $T(\langle F, \bar{x}\rangle)$ and $\bar{y}T(\langle F, \bar{x}\rangle)$ therefore must contain some edge in E, since $\bar{y}v_0 \in \bar{y}T(\langle F, \bar{x}\rangle)$. By 1-acylindricity η has length at most 2. Let η have endpoints $v_1 \in T(\langle F, \bar{x}\rangle)$ and $v_2 \in \bar{y}T(\langle F, \bar{x}\rangle)$. Let $\rho_1, \rho_2 \langle F, \bar{x}\rangle$ be such that $\rho_1 v_1 = v_0, (\bar{y}\rho_2\bar{y}^{-1})\bar{y}v_0 = v_2$. Then $d(v_0, \rho_1\bar{y}\rho_2 v_0) \leq 2$. Replacing $\bar{y}$ by $\rho_1\bar{y}\rho_2$ if necessary we have w.l.o.g. either $\bar{y} = f_1 t f_2$, $f_i \in \widetilde{F}$ or $\bar{y} = f_1 s f_2 t f_3$.

Case I: If $\bar{y} = f_1 t f_2$, $f_i \in \widetilde{F}_f$. Then for $u, w \in \langle F, \bar{x}\rangle$ the stable letters t cancel in products

$$(f_1 t f_2) u (f_1 t f_2)^{-1} \quad \text{or} \quad (f_1 t f_2)^{-1} w (f_1 t f_2)$$

if and only if either ${}^{f_2}u \in \langle \beta' \rangle$ or $w^{f_1} \in \langle \beta \rangle$. W.l.o.g. we may assume $w \in \langle \beta \rangle$ and conjugating boundary monomorphisms we may assume $\bar{y}^{-1} w \bar{y} \in \langle \beta' \rangle$. Now either $\langle F, \alpha', \bar{y}^{-1} w y\rangle = \widetilde{F}_f$ or $\bar{y}^{-1} w y = (\beta')^m$ and $\langle F, \alpha', (\beta')^m \rangle \cap \langle \beta \rangle \geq \langle F, \alpha' \rangle \cap \langle \beta \rangle$. Then we can make another transmission to get $\widetilde{F}_f \geq \langle F, \alpha', (\beta')^{m_1}\rangle$ with $|m_1| < |m|$. Either we have equality or we have $\langle F, \alpha', (\beta')^{m_1}\rangle \cap \langle \beta \rangle \geq \langle F, \alpha', (\beta')^m \rangle \cap \langle \beta \rangle$ etc. This cannot go on indefinitely and we find finally that $\widetilde{F}_f$ is 2-generated modulo F.

Case II: Suppose that $\bar{y} = f_1 s f_2 t f_3$. Then $F_{R(S)}$ is the fundamental group of the $\mathcal{G}(X)$-graph $\mathcal{B}$,



with $\mathcal{B}_{v_0} = \langle F, \alpha' \rangle$ and $\mathcal{B}_{v_1} = \{1\}$. This folds down to a bouquet of two circles. Since it is impossible to increase $\mathcal{B}_{v_0}$ via transmissions, there must be some $f' \in \langle F, \alpha' \rangle$ such that $f' f_1 \in \langle \beta \rangle$ so that we can fold together the edges labeled $(1, e, 1)$ and $(f_1, e, 1)$. We may now assume that $\bar{y} = f_2 t f_3$ so we have reduced to case I. □

Lemma 7.5. *If $F_{R(S)}$ has the JSJ (16), then w.l.o.g. $\beta, \beta' \notin F \leq \widetilde{F}$. Moreover, the centralizers of β and β' are cyclic in $\widetilde{F}$, so after balancing folds the splitting is 1-acylindrical. After balancing folds, where $\widetilde{F}_f$ denotes the resulting vertex group, we have w.l.o.g. $\widetilde{F}_f = \langle F, \bar{x}, \beta' \rangle$. In particular $\widetilde{F}$ is also 2-generated modulo F.*

Proof. Let $\pi: F_{R(S)} \to F_{R(S')}$ be a composition of strict epimorphisms that is injective on $\widetilde{F}$. Suppose there is a one edge splitting of $F_{R(S')}$ modulo F with either trivial or cyclic edge group such that D, the induced splitting of $\widetilde{F}$, is non-trivial and β is elliptic. β' is then also elliptic so we can refine the splitting (16), contradicting the fact that it is a JSJ. It follows that β, β' must be hyperbolic elements in the generalized JSJ of $\widetilde{F}$ and therefore have cyclic centralizers in $\widetilde{F}$. By Corollary 5.16, β is not conjugable in $F_{R(S)}$ into F. 1-acylindricity after balancing folds now follows.

Suppose we performed our balancing folds and the resulting splitting of $F_{R(S)}$ has the unique vertex group $\widetilde{F}_f$. Let T the Bass–Serre tree corresponding to this splitting and consider the marked generating set $(F; \{\bar{x}, \bar{y}\})$ we have that T_F must be a point, so by Theorem 6.9 after Weidmann–Nielsen normalization $\bar{x}$ can be sent into $\widetilde{F}$. It follows that we must have $\beta \in \langle F, \bar{x}\rangle$. And since we must have $T_{\langle F, \bar{x}\rangle} \cap \bar{y}T_{\langle F, \bar{x}\rangle} \neq \emptyset$ by 1-acylindricity we easily conclude $\bar{y} = f_1 t^{\pm 1} f_2$ for $f_1, f_2 \in \widetilde{F}$, conjugating boundary monomorphisms, we may therefore assume that $\bar{y} = s$ and arguing exactly as in case I of the proof of Lemma 7.4 we have $\widetilde{F}_f = \langle F, \bar{x}, \beta'\rangle$. The 2-generation of $\widetilde{F}$ modulo F now follows from Lemma 6.6. □

7.2. When all the vertex groups of the JSJ of $F_{R(S)}$ except $\widetilde{F}$ are abelian (revisited). We are now able to prove that when all the vertex groups of the JSJ of $F_{R(S)}$ except $\widetilde{F}$ are abelian, then $\widetilde{F}$ is also 2-generated modulo F.

Proposition 7.6. *If $F_{R(S)}$ is as in Proposition 5.27, then $\widetilde{F}$ is generated by F and at most two other elements.*

Proof. By Proposition 5.28 we may assume $F_{R(S)}$ has the vertex groups $\widetilde{F}$ and A, which is abelian.

Case I: If the JSJ of $F_{R(S)}$ is $\widetilde{F} *_{\langle\alpha\rangle} A$ then, perhaps after making a balancing fold which replaces $\widetilde{F}$ by $\widetilde{F}_f$, we have a retraction $\widetilde{F}_f *_{\langle\alpha\rangle} A \to \widetilde{F}_f$. Thus, by Lemma 6.6, $\widetilde{F}$ is 2-generated modulo F.

Case II: Suppose now that the JSJ of $F_{R(S)}$ is

$$\begin{cases}\langle \widetilde{F}, A, t \mid \alpha^t = \alpha'\rangle,\\ \alpha, \alpha' \in \widetilde{F},\ \langle\beta\rangle = \widetilde{F} \cap A.\end{cases}$$

By Proposition 7.3, w.l.o.g., conjugating boundary monomorphisms if necessary, either β or α, but not both lie in F.

Case II.I: Suppose that β lies in F. Then, by Corollary 5.16, the JSJ of $\langle \widetilde{F}, t\rangle$ is $\langle \widetilde{F}, t \mid \alpha^t = \alpha'\rangle$. Again we have a retraction $F_{R(S)} \to \langle \widetilde{F}, t \mid \alpha^t = \alpha'\rangle$ (balancing folds are not necessary since we cannot add proper roots to elements of F), so that $\langle \widetilde{F}, t \mid \alpha^t = \alpha'\rangle$ is 2-generated modulo F. The result now follows from Lemma 7.5.

Case II.II: Suppose finally that α lies in F. Then by Corollary 5.16 $\langle \widetilde{F}, t \mid \alpha^t = \alpha'\rangle$ admits a non-trivial ($\leq \mathbb{Z}$)-splitting modulo α, α'. We first assume that all possible balancing folds were applied and, abusing notation, we do not change the notation for the vertex groups; by Lemma 6.6 this will not affect the result. Again we have a retraction $F_{R(S)} \to \langle \widetilde{F}, t \mid \alpha^t = \alpha'\rangle$ so that $\langle \widetilde{F}, t \mid \alpha^t = \alpha'\rangle$ is 2-generated modulo F. Note moreover that $b_1(\langle \widetilde{F}, t \mid \alpha^t = \alpha'\rangle) < b_1(F_{R(S)})$ which implies (since $\widetilde{F} \neq F$) that $b_1(\langle \widetilde{F}, t \mid \alpha^t = \alpha'\rangle) = N + 1$.

Case II.II.I: Suppose first that $\langle \widetilde{F}, t \mid \alpha^t = \alpha'\rangle$ is freely decomposable modulo F, say as $\widehat{F} * H$ with $F \leq \widehat{F}$. Then $\widetilde{F}$ cannot be elliptic with respect to this splitting since otherwise $\widetilde{F} \leq \widehat{F}$, which means that we must have $t \in \widehat{F}$ as well, a contradiction. Thus $\widetilde{F}$ splits into $\widetilde{F}' * K$ with $\alpha \in \widetilde{F}' \leq \widehat{F}$ and $\alpha' \in K$, and we have $\langle \widetilde{F}, t \mid \alpha^t = \alpha'\rangle = (\widetilde{F}' *_{\langle\alpha\rangle} {}^tK) * \langle t\rangle$. Since $b_1(\widetilde{F}) = N + 1$, we must have $\widetilde{F}' *_{\langle\alpha\rangle} {}^tK = F$ so that $\widetilde{F} = F * t^{-1}\langle\alpha\rangle t$, and the result holds.

Case II.II.II: Suppose now that the JSJ of $\langle \widetilde{F}, t \mid \alpha^t = \alpha'\rangle$ has only one vertex group. Then the only possibility is that the JSJ of $\langle \widetilde{F}, t \mid \alpha^t = \alpha'\rangle$ is $\langle \widetilde{F}', t, s \mid \alpha^t =$

$\alpha', \delta^s = \delta'\rangle$ with $\alpha, \alpha', \delta, \delta' \in \widetilde{F}'$. In particular we have $\widetilde{F} = \langle \widetilde{F}', s\rangle$. We may again assume that this splitting is balanced. By Lemma 7.4 we have $\widetilde{F}' = \langle F, \alpha', \delta'\rangle$, which means that $\widetilde{F} = \langle F, \alpha', s\rangle$, and the result holds.

Case II.II.III: Suppose finally that the JSJ of $\langle \widetilde{F}, t \mid \alpha^t = \alpha'\rangle$ has more than two vertex groups. Then the JSJ can be obtained by refining the splitting $\langle \widetilde{F}, t \mid \alpha^t = \alpha'\rangle$. In particular the underlying graph of the JSJ is not simply connected. Since $b_1(\langle \widetilde{F}, t \mid \alpha^t = \alpha'\rangle) = N + 1$, Corollaries 2.7 and 2.11 imply that $\langle \widetilde{F}, t \mid \alpha^t = \alpha'\rangle$ has no non-cyclic abelian subgroups. It therefore follows from Proposition 6.41 (1) and Corollary 6.47 that $\widetilde{F} = F *_\alpha H$ or $\langle F, H, s \mid \gamma^s = \epsilon\rangle$ with $\langle\alpha\rangle = F \cap H$ and $\gamma \in F, \epsilon \in H$, both of which are generated by two elements modulo F. The result therefore holds. □

Proof of Proposition 2.6. The result now follows from Propositions 5.27, 5.28, and 7.6. □

7.3. Uniform hierarchical depth does not increase for finitely generated subgroups. The purpose of this section is to show that uniform hierarchical depth is well behaved when passing to finitely generated subgroups. This is necessary to bound the uhd of $\widetilde{F}$ when JSJ of $F_{R(S)}$ has one vertex and one edge. This next lemma is essentially an application of Theorem 3.15 coupled with the observation that if edge groups are cyclic or trivial, there are only finitely many adjustments and transmissions that can be applied to a $\mathcal{G}(X)$-graph $\mathcal{B}$ that actually change the $\mathcal{B}$-vertex groups.

Lemma 7.7. *Let G be finitely generated. If G is the fundamental group of a graph of groups with edge groups either cyclic or trivial, then all the vertex groups are finitely generated.*

Lemma 7.8. *Let $\mathcal{G}(X)$ be $(\leq \mathbb{Z})$-splitting (see Definition 3.2) of G (modulo F) and let $\mathcal{G}(Y)$ be the generalized JSJ of G (modulo F). Every rigid vertex group of $\mathcal{G}(Y)$ is conjugable into a vertex group of $\mathcal{G}(X)$.*

Proof. Let Y_u be a rigid (i.e., non-QH, non-abelian) vertex group of $\mathcal{G}(Y)$ and suppose towards a contradiction that Y_u is not elliptic in $\mathcal{G}(X)$. In such a case we can collapse $\mathcal{G}(X)$ to some elementary $(\leq -\mathbb{Z})$ splitting D such that Y_u is hyperbolic.

We may first suppose that G is freely indecomposable (modulo F). Let $\langle c\rangle$ be the edge group of D. Then by items (2) and (3) of Theorem 5.5, we can obtain D from $\mathcal{G}(Y)$ by perhaps first refining $\mathcal{G}(Y)$ by further splitting a MQH subgroup along a simple closed curve (if c is conjugable into a MQH subgroup) and then performing a sequence of slides, foldings and collapses as described in Definition 5.2. All these moves preserve the ellipticity of Y_u, a contradiction.

Suppose now that G is freely decomposable (modulo F). We have by definition of a generalized JSJ that Y_u is a rigid vertex group of the cyclic JSJ decomposition of

G_i (modulo F), where G_i is a free factor of the Grushko decomposition of G (modulo F), but not a free group itself . The elementary splitting D induces a non-trivial *cyclic* splitting of G_i (modulo F) with Y_u hyperbolic. We can now derive a contradiction as in the previous paragraph arguing with G_i in place of G. □

Theorem 7.9. *Let G be finitely generated fully residually free group and let $H \leq G$ be a finitely generated subgroup, then* $\mathrm{uhd}(H) \leq \mathrm{uhd}(G)$.

Proof. We proceed by induction on uniform hierarchical depth. If G is a finitely generated fully residually free group such that $\mathrm{uhd}(G) = 0$ then the same is true for any finitely generated subgroup of G.

Suppose now that $\mathrm{uhd}(G) = n + 1$ and that the theorem holds for $m \leq n$. Let $H \leq G$ be a finitely generated subgroup. Let E denote the generalized JSJ of G. If H is conjugable into a vertex group then by induction hypothesis, $\mathrm{uhd}(H) \leq n$, and the result holds.

Suppose now that H is hyperbolic with respect to E. Then H has an induced $(\leq \mathbb{Z})$-splitting D as a finite graph of groups with vertex groups conjugable into the vertex groups of E. On one hand the vertex groups of D are conjugable into the vertex groups of E. On the other hand by Lemma 7.8 the rigid vertex groups of the generalized JSJ of H are conjugable into the vertex groups of D and by Lemma 7.7 the rigid vertex groups are finitely generated. It follows that we can apply the induction hypothesis so for each rigid vertex group H_i of the generalized JSJ of H we have $\mathrm{uhd}(H_i) \leq n$. Noting that QH and abelian vertex groups have uhd $= 0$, we can now conclude that $\mathrm{uhd}(H) \leq n + 1$. So the result holds by induction. □

Lemma 7.10. *If G is fully residually F then* $\mathrm{uhd}(G) \leq \mathrm{uhd}_F(G)$.

Proof. We proceed by induction on $\mathrm{uhd}_F(G)$. If $\mathrm{uhd}_F(G) = 0$, then the result holds. Let $\widetilde{F}_G$ be the vertex group of the generalized JSJ of G(modulo F containing F and let E be the generalized cyclic JSJ splitting of G (*not* modulo F).

Suppose that for all $m \leq n$, if $\mathrm{uhd}_F(K) \leq m$ then $\mathrm{uhd}(K) \leq \mathrm{uhd}_F(K)$, where K is fully residually F. Let $\mathrm{uhd}_F(G) = n + 1$ and let $F \leq H \leq G$ be a finitely generated subgroup. By definition, $\mathrm{uhd}_F(\widetilde{F}_G) \leq n$, so by induction hypothesis, $\mathrm{uhd}(\widetilde{F}_G) \leq \mathrm{uhd}_F(\widetilde{F}_G)$. Now the vertex groups G_i of E or are either finitely generated subgroups of $\widetilde{F}_G$ or finitely generated subgroups of other vertex groups of the generalized JSJ of G modulo F. In both cases by Theorem 7.9 and by induction hypothesis, $\mathrm{uhd}(G_i) \leq n$ for each vertex group G_i, so $\mathrm{uhd}(G) \leq n + 1$. The result now follows by induction. □

Corollary 7.11. *Let $F \leq H \leq G$ where G is fully residually F and H is finitely generated, then* $\mathrm{uhd}_F(H) \leq \mathrm{uhd}_F(G)$.

Proof. We proceed by induction on $\mathrm{uhd}_F(G)$. If $\mathrm{uhd}_F(G) = 0$ then the result holds.

Suppose the corollary is true for all G such that $\mathrm{uhd}_F(G) \leq n$. Let $\mathrm{uhd}_F(G) = n+1$. Let E be the generalized JSJ of G modulo F and let D be the generalized JSJ of H modulo F. Let $\widetilde{F}_G$ and $\widetilde{F}_H$ be the vertex groups of E and D respectively containing F.

As argued in Theorem 7.9, the rigid vertex groups of D are conjugable into the vertex groups of E. It therefore follows that $\widetilde{F}_H \leq \widetilde{F}_G$, and by induction hypothesis $\mathrm{uhd}_F(\widetilde{F}_H) \leq \mathrm{uhd}_F(\widetilde{F}_G) \leq n$. As for the other rigid vertex groups $H_1, \dots, H_m$ in D, Lemma 7.10 implies that $\mathrm{uhd}(G_i) \leq n$ for each vertex group G_i of E, so by Theorem 7.9, $\mathrm{uhd}(H_i) \leq n$. It follows that $\mathrm{uhd}(H) \leq n+1$. The result now follows by induction. □

Corollary 7.12. *There is a one relator fully residually F group which does not embed in a centralizer extension of F.*

Proof. The group $F_{R(S)}$ given in Example 2.14 is freely indecomposable modulo F and has a cyclic splitting modulo F with a vertex group $F \leq \widetilde{F}_1 = \langle F, t \mid [t,u] = 1\rangle$, which does not split modulo F and the incident edge group. Moreover, $\mathrm{uhd}_F(\widetilde{F}_1) = 1$, which means that $\mathrm{uhd}_F(F_{R(S)}) = 2$. On the other hand any centralizer extension of F has $\mathrm{uhd}_F = 1$, so by Corollary 7.11, $F_{R(S)}$ cannot embed into any rank 1 centralizer extension. □

7.4. The two edge case. In this section we describe the structure of $\widetilde{F}$ when the JSJ of $F_{R(S)}$ has one vertex and two edges. In particular we will explicitly bound its uniform hierarchical depth. Suppose that $F_{R(S)}, \widetilde{F}, \alpha, \alpha', \beta, \beta', s, t$ are given in as (17). By Lemma 7.4 we can assume that $\alpha \in F$ and β, β' are hyperbolic in the JSJ of $\widetilde{F}$.

Definition 7.13. For each edge e in the JSJ of $F_{R(S)}$ we define the *edge class associated to e* to be the conjugacy class in $F_{R(S)}$ corresponding to the edge group associated to e.

It is important to note that distinct edges of the JSJ of $F_{R(S)}$ may have the same edge class.

Lemma 7.14. *Let $F_{R(S)}$ has the JSJ (17). If $\widetilde{F}$ is freely indecomposable modulo F, then the JSJ of $\widetilde{F}_f$ (as given in Lemma 7.4) has more than one vertex group.*

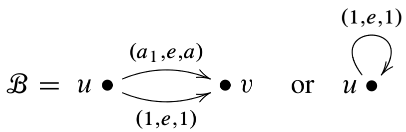
Proof. Suppose towards a contradiction that JSJ of $\widetilde{F}_f$ has only one vertex group. By Lemma 7.4, $\widetilde{F}_f$ is 2-generated modulo F, which means that the JSJ of $\widetilde{F}_f$ has at most two edges. The unique vertex group of $\widetilde{F}_f$ cannot be F itself by Corollary 7.2. Let $F_{R(S')}$ be the first term in a strict resolution of $F_{R(S)}$ where $\widetilde{F}$ splits. We first consider the case where $F_{R(S')}$ is freely indecomposable modulo F.

We can collapse the JSJ of $F_{R(S')}$ to an elementary splitting $F_{R(S')} = \pi_1(\mathcal{G}(Z))$ with edge group C and $\widetilde{F}$ hyperbolic. If $\widetilde{F}$ is hyperbolic in $F_{R(S')}$, then so is $\widetilde{F}_f$. Moreover by Lemma 7.4, after replacing α', β' by their proper roots if necessary, $\widetilde{F}_f$ is generated as $\langle F, \alpha', \beta' \rangle$.

The elementary splitting $\mathcal{G}(Z)$ is 2-acylindrical. If the JSJ of $\widetilde{F}_f$ has two edges, then $\widetilde{F}_f$ has two edge classes, but they cannot be conjugate in $F_{R(S')}$ because by Lemma 7.4 exactly one of the edge classes of the JSJ $\widetilde{F}_f$ will conjugable into F. It therefore follows that only one edge class of $\widetilde{F}_f$ is conjugable into C in $F_{R(S')}$. Therefore only one of the edge classes of the JSJ of $\widetilde{F}_f$ corresponds to edge groups of induced splitting of $\widetilde{F}_f$ in $F_{R(S')}$.

Although the JSJ of $\widetilde{F}_f$ has only one vertex group, it is still possible for the induced splitting of $\widetilde{F}_f$ to have more than one vertex group, but by the previous paragraph the induced graph of groups decomposition of $\widetilde{F}_f$ given by the $\mathcal{G}(Z)$-graph $\mathcal{B}$ has at most one non-cyclic $\mathcal{B}$ vertex group $\mathcal{B}_v$ and at most one cycle. Moreover, this splitting is non-trivial and must collapse to a cyclic HNN extension. Since $\mathcal{G}(Z)$ is 2-acylindrical, this means that the cycle in $\mathcal{B}$ has length at most 2, since when $\widetilde{F}_f$ acts on the Bass–Serre tree T corresponding to $\mathcal{G}(Z)$ the edge group of the JSJ of $\widetilde{F}_f$ that is conjugable into C should fix an arc between two translates of some $v_0 \in T$ with $\mathrm{fix}(\mathcal{B}_v) = v_0$.

Suppose first that $F_{R(S')}$ split as a cyclic HNN extension $\langle A, r \mid c^r = c' \rangle$ with c, c' not conjugate in A and $F \leq A$, and $\widetilde{F}$ not elliptic. Up to conjugating boundary monomorphisms (or equivalently replacing r by $g_1 r g_2$ for some $g_1, g_2 \in A$) if necessary, the possible induced splittings of $\widetilde{F}_f$ are given by the folded $\mathcal{G}(Z)$-graphs



(18)

with $B_u = \widetilde{F}_1 \leq \widetilde{F}_f, \mathcal{B}_v = \langle c' \rangle$ and $a \in A \setminus \langle c' \rangle$ but with $[a, c'] = 1$ and $a_1 \in A$. In particular the first possibility can only occur if $\langle c' \rangle$ is not malnormal and hence cyclic in A. Let $\mathcal{G}(Y)$ give the JSJ of $F_{R(S')}$. Since $\mathcal{G}(Z)$ is just a collapse of $\mathcal{G}(Y)$, $C = \langle c \rangle$ is an edge group of $\mathcal{G}(Y)$. By Corollary 7.16, C must be an edge group adjacent to an abelian vertex group of $\mathcal{G}(Y)$. Since the elementary splitting is an HNN extension, C is the edge group of a non-separating edge in $e \subset Y$ and has infinite intersection with an edge group that is incident to an abelian vertex group of $\mathcal{G}(Y)$. It follows that the JSJ of $F_{R(S')}$ has more than one vertex and by Lemma 7.15 the JSJ of $F_{R(S')}$ must have at least two non-abelian vertex groups.

Now $\mathcal{B}_u = \widetilde{F}_1$ lies in the subgroup corresponding to the "subgraph of groups" $\mathcal{G}(Y \setminus e)$, which has vertex groups F, some free group H, and an abelian vertex group A. It therefore follows that $\widetilde{F}_1$ has an induced splitting D' with F as a vertex group, moreover since $\langle c' \rangle, \langle c \rangle$ are elliptic in $\mathcal{G}(Y)$ we can refine the splitting of $\widetilde{F}_f$

so that it has F as a vertex group, contradicting the fact that its JSJ had only one vertex group, and this vertex group is not F.

If $\langle c\rangle$ and $\langle c'\rangle$ are malnormal in A then the second possibility of (18) must occur. By Lemma 7.4, $\widetilde{F}_f = \langle F, \alpha', \beta'\rangle$ with $\beta \in \langle F, \alpha'\rangle$ and α' conjugate to F. It therefore follows that in $F_{R(S')} = \langle A, r \mid c^r = c'\rangle$, β has exponent sum 0 in r. Now β is conjugate to β' in $F_{R(S')}$, which means that β' also has exponent sum 0 in r. In other words $\widetilde{F}_f$ in $F_{R(S')}$ is generated by elements with exponent sum 0 in r which is impossible because the induced splitting on the right of (18) implies that $\widetilde{F}_f$ contains an element with exponent sum 1 in r.

Suppose now that $F_{R(S')}$ can split as a cyclic free product with amalgamation $A *_C B$, with $F \leq A$ and B non-abelian and $\widetilde{F}$ not elliptic. This means that $F_{R(S')}$ has a one edge maximal abelian collapse. It therefore follows from Proposition 2.10 that the induced splitting of $\widetilde{F}$ has one edge class and one vertex group $\widetilde{F}_1$. More specifically, since $A *_C B$ is 2-acylindrical, the induced splitting will be given by some $\mathcal{G}(X)$-graph

$$\mathcal{B} = u \bullet \underset{(1,e,1)}{\overset{(b,e^{-}-1,a)}{\rightleftarrows}} \bullet\, v,$$

with $a \in A$, $b \in B$ and $\mathcal{B}_u = F_1$ and $\mathcal{B}_v = C$. We see that this is only possible if there is some element $b \in B$ that does not lie in C but centralizes the cyclic edge group C. By Lemma 7.15 and Proposition 2.10, $F_{R(S')}$ must have a cyclic splitting of the form $F *_{\langle u\rangle} (H *_{\langle u\rangle} K)$ with K abelian. So we could chose $A = F$ and still have $\widetilde{F}_f$ hyperbolic, but this would imply $\widetilde{F}_1 = F$, which is impossible.

If $F_{R(S')} = A *_C B$ with B abelian of rank 3 then $A = F$, which again yields a contradiction. Recall that we are assuming that the JSJ of $\widetilde{F}$ has only one vertex group. If $B = \langle c\rangle \oplus \langle r\rangle$ is free abelian of rank 2, then, by 2-acylindricity of the JSJ of $F_{R(S')}$, the only possible $\mathcal{G}(Z)$ graph is

$$\mathcal{B} = u \bullet \underset{(a_1,e,b_1)}{\overset{(a_2,e,b_2)}{\rightrightarrows}} \bullet\, v,$$

with $a_i \in A$, $b_j \in B$. Since this graph is folded, we must have $b_1 b_2^{-1} \notin \langle c\rangle$. Now note that we can also write $F_{R(S')}$ as an HNN extension $\langle A, r \mid c^r = c\rangle$. The fact that $a_1 b_1 b_2^{-1} a_2^{-1} \in \widetilde{F}$ implies that there is an element of $\widetilde{F}$ that has non-zero exponent sum in r. But by Lemma 7.4, $\widetilde{F} = \langle F, \alpha', \beta'\rangle$ with α' conjugate to $\alpha \in F$ and β' conjugate to $\beta \in \langle F, \alpha'\rangle$. So $\widetilde{F}$ is generated by elements with exponent sum 0 in r, a contradiction.

Finally, consider the case where $F_{R(S')}$ is freely decomposable modulo F. Since $\widetilde{F}_f$ is freely indecomposable modulo F, it must lie in one of the free factors H of a Grushko decomposition of $F_{R(S')}$ modulo F. On the other hand, we have the relations $\pi(s)^{-1}\alpha\pi(s) = \alpha'$ and $\pi(t)^{-1}\beta\pi(t) = \beta'$ which hold if and only if $s, t \in H$, which implies $F_{R(S')} = H$, which is a contradiction. □

Lemma 7.15. *If $F_{R(S)}$ is as in Proposition 5.27, then the centralizers of the edge groups that are incident to $\widetilde{F}$ are cyclic. Moreover, if the JSJ of $F_{R(S)}$ has two edges, then the incident edge groups are conjugacy separated in $\widetilde{F}$.*

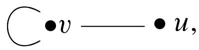
Proof. If the JSJ of $F_{R(S)}$ has underlying graph

$$u \bullet \!\!-\!\!\!-\!\!\!-\!\!\bullet v \!\!-\!\!\!-\!\!\!-\!\!\bullet w,$$

then $\widetilde{F} = F$ by Proposition 5.28 and the result follows. If the JSJ of $F_{R(S)}$ has underlying graph

$$u \bullet \!\!-\!\!\!-\!\!\!-\!\!\bullet v$$

and $\widetilde{F} \neq F$, then $\widetilde{F}$ must not have any splittings modulo F and the edge group. It follows that the edge group is hyperbolic in the generalized JSJ of $\widetilde{F}$, and therefore cannot be non-cyclic abelian. Finally if the JSJ of $F_{R(S)}$ has underlying graph



then the abelian vertex group has rank at most 2, so we may consider this group as double HNN extension of $\widetilde{F}$. The result now follows by applying Lemma 7.3. □

Corollary 7.16. *Let $F_{R(S)}$ be freely indecomposable modulo F and let $\langle c\rangle$ be one of the edge groups of the JSJ of $F_{R(S)}$. If $\langle c\rangle$ has a non-cyclic centralizer $Z(c)$, then $Z(c)$ is in fact an abelian vertex group of the JSJ of $F_{R(S)}$.*

Proof. If the JSJ of $F_{R(S)}$ has one vertex group then by Lemmas 7.4 and 7.5 $Z(c)$ is cyclic. If the JSJ $F_{R(S)}$ has at least two non-abelian vertex groups, the result follows by looking at the non-cyclic abelian subgroups that occur in Section 2.4. The remaining case follows from Lemma 7.15. □

Corollary 7.17. *Let $F_{R(S)}$ and $\widetilde{F}_f$ be as in Lemma 7.4. If $\widetilde{F}$ is freely indecomposable modulo F, then the JSJ of $\widetilde{F}_f$ has two vertex groups F and some free group H. It follows that in all cases* $\mathrm{uhd}(F_{R(S)}) \leq 2$.

Proof. By Lemma 7.4, $\widetilde{F}_f$ has no non-cyclic abelian subgroups. By Lemma 7.14 the JSJ of $\widetilde{F}_f$ has at least two vertex groups and since it cannot contain any non-cyclic abelian subgroups Corollary 2.11 implies that $\mathrm{uhd}_F(\widetilde{F}_f) = 1$. Now $\widetilde{F}$ is a finitely generated subgroup of $\widetilde{F}_f$ so by Corollary 7.11 $\mathrm{uhd}_F(\widetilde{F}) \leq 1$ as well and the result follows. □

7.5. The one edge case. In this section we bound the uniform hierarchical depth of $F_{R(S)}$ when its JSJ has one edge and one vertex. By Lemma 7.5, $\widetilde{F}$ is generated by two elements modulo F and the element $\beta \in \widetilde{F}$ must be hyperbolic in the JSJ of $\widetilde{F}$. The JSJ of $\widetilde{F}$ either has only one vertex group or is one of the groups described in Propositions 2.4 and Section 2.4.

Lemma 7.18. *Let $F_{R(S)}$, $\widetilde{F}$, t, β, β' be as in (16) and let $\pi : F_{R(S)} \to F_{R(S')}$be a proper strict epimorphism. If $\widetilde{F} \neq F$ is freely indecomposable modulo F, then there is no essential cyclic or free splitting of $F_{R(S')}$ modulo $\widetilde{F} = \pi(\widetilde{F})$.*

Proof. Suppose that $F_{R(S')} = A * B$ with $\widetilde{F} \leq A$. Then, since $F_{R(S')} = \langle \widetilde{F}, \pi(t)\rangle$, $\pi(t)$ written as a reduced word in A, B should have a syllable in B, we must, however, have the equality

$$\pi(t^{-1})\beta\pi(t) = \beta' \tag{19}$$

with $\beta, \beta' \in \widetilde{F} \leq A$, which is impossible.

We may therefore assume that $F_{R(S')}$ is freely indecomposable and that we can collapse its cyclic JSJ to either a free product with amalgamation or as an HNN extension modulo $\widetilde{F}$.

Suppose $F_{R(S')}$ splits as a cyclic HNN extension $\langle A, r \mid c^r = c'\rangle$, $c, c' \in A$, but not conjugate in A, with $\widetilde{F} \leq A$. Then $F_{R(S')} = \langle \widetilde{F}, \pi(t)\rangle$, so $\pi(t)$ must have exponent sum 1 in the stable letter r. Equality (19) implies that w.l.o.g. β is conjugate in A to $\langle c\rangle$ and β' is conjugate in A to $\langle c'\rangle$. So conjugating boundary monomorphisms (or, equivalently, replacing r by $a_1 r a_2$, $a_i \in A$) we can arrange so that $\beta \in \langle c\rangle$ and $\beta' \in \langle c'\rangle$, which means that $\pi(t)$ has a reduced form $r \dots r$. Now by Britton's Lemma any word in $\widetilde{F}, \pi(t)$ is reduced if and only if it does not contain the subword $\pi(t^{-1})\beta\pi(t)$ or $\pi(t)\beta'\pi(t^{-1})$. It therefore follows that (19) is the only non-trivial relation between $\widetilde{F}$ and $\pi(t)$. This implies that $F_{R(S)} \approx F_{R(S')}$, contradicting the fact that π is a proper epimorphism.

Suppose now that $F_{R(S')}$ splits as $A *_C B$ with $C = \langle c\rangle$, B abelian, and $\widetilde{F} \leq A$. If B has rank 3, then $A = F$ implies that $\widetilde{F} = F$, a contradiction. Suppose now that B is abelian of rank 2, i.e., $B = \langle c\rangle \oplus \langle b\rangle$. Then w.l.o.g. we may assume that $\langle \beta\rangle \leq \langle c\rangle$. We can also express $A *_C B$ as an HNN extensions $\langle A, r \mid c^r = c\rangle$. Note that $F_{R(S')} = \langle \widetilde{F}, \pi(t)\rangle$ and since $\widetilde{F} \leq A$, $\pi(t)$ has exponent sum 1 in r. This means that if $\pi(t)^{-1}\beta\pi(t) = \beta'$ then there must be some $g \in A \setminus \widetilde{F}$ such that $g\beta' g^{-1} = \beta$. So $\pi(t)g^{-1}$ centralizes β, which means by normal forms that $\pi(t)g^{-1} = b_1 \in B$, hence $\pi(t) = b_1 g$. But now looking at words in $\widetilde{F}, b_1 g$ we see, again that the only non-trivial relations are $(b_1 g)\beta'(g^{-1}\beta^{-1}) = \beta$ and $(g^{-1}b_1^{-1})\beta(b_1 g) = \beta'$, again forcing $F_{R(S)} \approx F_{R(S')}$.

Suppose now that $F_{R(S')}$ splits as $A *_C B$ with C cyclic, B non abelian and $\widetilde{F} \leq A$. Then $F_{R(S')}$ has a maximal abelian collapse with one edge. By Lemma 7.5, $F_{R(S')} = \langle F, \pi(\bar{x}), \pi(\bar{y})\rangle$ with $F, \pi(\bar{x})$ lying in the same vertex group of the maximal abelian collapse. By Lemma 6.21 this means $F_{R(S')}$ is freely decomposable modulo F and since $\widetilde{F}$ is freely indecomposable modulo F we have $F_{R(S')} = A * B$ with $\widetilde{F} \leq A$, which, as we saw, is impossible. □

Lemma 7.19. *If $F_{R(S')}$ as in Lemma 7.18 has a one edge cyclic splitting D and if $\widetilde{F}$ has an induced one edge cyclic splitting $D_{\widetilde{F}}$, then $F_{R(S')}$ can be obtained from $\widetilde{F}$ by adding an element $\sqrt[n]{\eta}$, an n^{th} root of the generator η of an edge group of $D_{\widetilde{F}}$.*

Proof. Consider the action of $F_{R(S')}$ and $\widetilde{F}$ on the Bass–Serre tree T corresponding to the splitting D. Abusing notation we use t to denote the image of $\pi(t)$ in $F_{R(S')}$.

We have hyperbolic elements $\beta, \beta' \in \widetilde{F}$ such that $\beta^t = \beta'$. This means that $t\,\mathrm{Axis}(\beta') = \mathrm{Axis}(\beta)$, that is, there are edges e', e in $\mathrm{Axis}(\beta')$ and $\mathrm{Axis}(\beta)$ respectively such that $te' = e$. On the other hand, $\mathrm{Axis}(\beta), \mathrm{Axis}(\beta') \subset T(\widetilde{F})$, the minimal $\widetilde{F}$-invariant subtree.

$D_{\widetilde{F}}$ has only one edge and $\widetilde{F}$ is transitive on the set of edges in $T(\widetilde{F})$. Let $g \in \widetilde{F}$ be such that $ge = e'$. Then $tg \in \mathrm{stab}_{\widetilde{F}}(e)$. Let $\mathrm{stab}_{F_{R(S')}}(e) \cap \widetilde{F} = \langle \eta\rangle$. Then we have $\langle \eta, tg\rangle \leq \mathrm{stab}_{F_{R(S')}}(e)$, which is cyclic, so $\langle \eta, tg\rangle = \langle \sqrt[n]{\eta}\rangle$. □

Corollary 7.20. *Suppose the JSJ of $F_{R(S)}$ has one edge and one vertex and let $\pi: F_{R(S)} \to F_{R(S')}$ be a strict epimorphism. If there is a one edge cyclic splitting of $F_{R(S')}$ such that the induced splitting of $\widetilde{F}$ only has one edge then $b_1(F_{R(S')}) < b_1(F_{R(S)})$.*

Proof. By Lemma 7.19, looking at abelianizations we immediately see that $b_1(F_{R(S')}) \leq b_1(\widetilde{F})$. Let $\langle\beta\rangle$, $\langle\beta'\rangle$ be boundary subgroups in $\widetilde{F}$. Consider the mapping

$$\widetilde{F} \to \widetilde{F}/[\widetilde{F}, \widetilde{F}] \otimes_{\mathbb{Z}} \mathbb{Q}, \quad g \mapsto [g].$$

We consider two cases, either $[\beta] - [\beta'] = 0$ or $[\beta] - [\beta'] \neq 0$. If $[\beta] = [\beta']$ then $b_1(F_{R(S)}) = b_1(\widetilde{F}) + 1$, which implies that $b_1(F_{R(S')}) < b_1(F_{R(S)})$.

If $[\beta] \neq [\beta']$ then $b_1(F_{R(S)}) = b_1(\widetilde{F})$, but from Lemma 7.19 it follows that $F_{R(S')}/[F_{R(S')}, F_{R(S')}] \otimes_{\mathbb{Z}} \mathbb{Q}$ is obtained from $\widetilde{F}/[\widetilde{F}, \widetilde{F}] \otimes_{\mathbb{Z}} \mathbb{Q}$ by adding $[\eta]/n$ (which does not increase the rank) and then forcing $[\beta] = [\beta']$ (since $\pi(t)^{-1}\beta\pi(t) = \beta'$ in $F_{R(S')}$), which decreases the rank by 1. So $b_1(F_{R(S')}) < b_1(\widetilde{F}) = b_1(F_{R(S)})$. □

Lemma 7.21. *Suppose that the JSJ of $F_{R(S)}$ has one vertex and one edge, and suppose that $b_1(F_{R(S)}) = N + 1$. Then $F_{R(S)}$ has uniform hierarchical depth relative to F at most 2. Moreover, if $\widetilde{F}$ is freely indecomposable modulo F, its JSJ has more than one vertex group.*

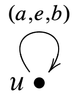
Proof. Abelianizing relative presentations, we immediately obtain that $b_1(\widetilde{F}) \leq b_1(F_{R(S)})$. Corollary 7.2 and Proposition 2.3 imply that $b_1(\widetilde{F}) = N + 1$.

If $\widetilde{F}$ is freely decomposable modulo F, then $\mathrm{uhd}_F(\widetilde{F}) = 1$, and the result holds.

Let $\pi: F_{R(S)} \to F_{R(S')}$ be a strict epimorphism. Then $b_1(F_{R(S')}) = m \leq N+1$. If $m = N$ then, by Proposition 2.3, $F_{R(S')} = F$ and since π is injective on $\widetilde{F}$ we have a contradiction to Corollary 7.2.

Consider the case where $F_{R(S')}$ is freely indecomposable modulo F and its JSJ has at least two vertex groups. Corollaries 2.7 and 2.11 imply $\mathrm{uhd}_F(F_{R(S')}) \leq 1$ so by Corollary 7.11, $\mathrm{uhd}_F(\widetilde{F}) \leq 1$ and the result holds. From this it also follows that unless the JSJ of $\widetilde{F}$ has only one vertex group, then $\mathrm{uhd}_F(\widetilde{F}) \leq 1$.

Otherwise the JSJ of $F_{R(S')}$ has one vertex group. By Lemmas 7.4 and 7.5 the centralizers of the edge groups of $F_{R(S')}$ are cyclic and distinct edges have distinct edge classes. By Lemma 7.18 any one edge cyclic splitting of $F_{R(S')}$ modulo F induces a cyclic of $\widetilde{F}$ modulo F, and this splitting of $F_{R(S')}$ is as an HNN extension with cyclic edge stabilizers. Since the JSJ of $\widetilde{F}$ also has only one vertex, the edge groups do not lie in non-cyclic abelian subgroups the vertex groups, so the only possible induced splitting of $\widetilde{F}$ (considering the argument used for Lemma 7.14) is of the form



.

Thus Corollary 7.20 applies and gives $b_1(F_{R(S')}) = N$. Therefore, $\widetilde{F} \leq F_{R(S')} = F$, a contradiction. □

We can now prove the following.

Proof of Proposition 2.8. We have that $m = b_1(\widetilde{F}) < b_1(F_{R(S)})$. If $m = N$ there is nothing to show, otherwise $m = N + 1$. If the JSJ of $\widetilde{F}$ has more than two vertex groups, Corollaries 2.7 and 2.11 imply that $\mathrm{uhd}_F(\widetilde{F}) \leq 1$. Otherwise $\widetilde{F}$ has one vertex group and Lemma 7.21 and Corollary 7.17 imply that $\mathrm{uhd}_F(\widetilde{F}) \leq 2$. □

Proposition 7.22. *If the JSJ of $F_{R(S)}$ has one edge and one vertex, then it has uniform hierarchical depth relative to F at most* 4.

Proof. If $\widetilde{F}$ is freely decomposable modulo F or if the JSJ of $\widetilde{F}$ does not have one vertex and one edge, then Corollary 2.11 and Corollary 7.17 and Proposition 2.8 imply that $\mathrm{uhd}_F(\widetilde{F}) \leq 3$.

By Corollary 7.11, the same bound holds if $\pi: F_{R(S)} \to F_{R(S')}$ is a strict epimorphism and the JSJ of $F_{R(S')}$ does not have one edge and one vertex.

The remaining possibility is that the JSJs of $\widetilde{F}$ and $F_{R(S')}$ both only have one edge and one vertex. But then, as in the proof Lemma 7.21, the induced splitting of $\widetilde{F}$ has only one edge and by Corollary 7.20, $b_1(F_{R(S')}) \leq N + 1$, which by Lemma 7.21 and Corollary 7.11 implies that $\mathrm{uhd}_F(\widetilde{F}) \leq 2$. □

Proof of Proposition 2.13. The result follows immediately from Corollary 7.17 and Proposition 7.22. □

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